Note

2013 unit vectors in the plane

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**A R T I C L E I N F O**

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**A B S T R A C T**

Given a norm in the plane and 2013 unit vectors in this norm, there is a signed sum of these vectors whose norm is at most one.

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Let \(B\) be the unit ball of a norm \(\| \cdot \|\) in \(\mathbb{R}^d\), that is, \(B\) is an 0-symmetric convex compact set with nonempty interior. Assume \(V \subseteq B\) is a finite set. It is shown in [2] that, under these conditions, there are signs \(\varepsilon(v) \in \{-1, +1\}\) for every \(v \in V\) such that \(\sum_{v \in V} \varepsilon(v)v \in dB\). That is, a suitable signed sum of the vectors in \(V\) has norm at most \(d\). This estimate is best possible: when \(V = \{e_1, e_2, \ldots, e_d\}\) and the norm is \(\ell_1\), all signed sums have \(\ell_1\) norm \(d\).

In this short note we show that this result can be strengthened when \(d = 2\), \(|V| = 2013\) (or when \(|V|\) is odd) and every \(v \in V\) is a unit vector. So from now onwards we work in the plane \(\mathbb{R}^2\).

**Theorem 1.** Assume \(V \subseteq \mathbb{R}^2\) consists of unit vectors in the norm \(\| \cdot \|\) and \(|V|\) is odd. Then there are signs \(\varepsilon(v) \in \{-1, +1\}\) (\(\forall v \in V\)) such that \(\| \sum_{v \in V} \varepsilon(v)v \| \leq 1\).

This result is best possible (take the same unit vector \(n\) times) and does not hold when \(|V|\) is even.

Before the proof some remarks are in order here. Define the convex polygon \(P = \text{conv}\{\pm v : v \in V\}\). Then \(P \subseteq B\), and \(P\) is again the unit ball of a norm, \(V\) is a set of unit vectors of this norm. Thus it suffices to prove the theorem only in this case.

A vector \(v \in V\) can be replaced by \(-v\) without changing the conditions and the statement. So we assume that \(V = \{v_1, v_2, \ldots, v_n\}\) and the vectors \(v_1, v_2, \ldots, v_n\), \(-v_1, -v_2, \ldots, -v_n\) come in this order on the boundary of \(P\). Note that \(n\) is odd. We prove the theorem in the following stronger form.

**Theorem 2.** With this notation \(\|v_1 - v_2 + v_3 - \cdots - v_{n-1} + v_n\| \leq 1\).

**Proof.** Note that this choice of signs is very symmetric as it corresponds to choosing every second vertex of \(P\). So the vector \(u = 2(v_1 - v_2 + v_3 - \cdots - v_{n-1} + v_n)\) is the same (or its negative) when one starts with another vector instead of \(v_1\). Define \(a_i = v_{i+1} - v_i\) for \(i = 1, \ldots, n - 1\) and \(a_n = -v_1 - v_n\) and set \(w = a_1 - a_2 + a_3 - \cdots + a_n\). It simply follows from the definition of \(a_i\) that

\[
-2(u_2 - u_3 + \cdots - u_{n-1} + u_n) = u.
\]

Consequently \(\|u\| = \|w\|\) and we have to show that \(\|w\| \leq 2\).

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Consider the line $L$ in direction $w$ passing through the origin. It intersects the boundary of $P$ at points $b$ and $-b$. Because of symmetry we may assume, without loss of generality, that $b$ lies on the edge $[v_1, -v_n]$ of $P$. Then $w$ is just the sum of the projections onto $L$ in direction parallel with $[v_1, -v_n]$, of the edge vectors $a_1, -a_2, a_3, -a_4, \ldots, a_n$. These projections do not overlap (apart from the endpoints), and cover exactly the segment $[-b, b]$ from $L$. Thus $\|w\| \leq 2$, indeed. □

Remark. There is another proof based on the following fact. $P$ is a zonotope defined by the vectors $a_1, \ldots, a_n$, translated by the vector $v_1$. Here the zonotope defined by $a_1, \ldots, a_n$ is simply

$$Z = Z(a_1, \ldots, a_n) = \left\{ \sum_{i=1}^{n} \alpha_i a_i : 0 \leq \alpha_i \leq 1 \left( \forall i \right) \right\}.$$  

The polygon $P = v_1 + Z$ contains all sums of the form $v_1 + a_{i_1} + \cdots + a_{i_k}$ where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. In particular with $i_1 = 2, i_2 = 4, \ldots, i_k = 2k$

$$v_1 + a_2 + a_4 + \cdots a_{2k} = v_1 - v_2 + v_3 - \cdots - v_{2k} + v_{2k+1} \in P.$$  

This immediately implies a strengthening of Theorem 1 (which also follows from Theorem 2).

Theorem 3. Assume $V \subset \mathbb{R}^2$ consists of $n$ unit vectors in the norm $\| \cdot \|$. Then there is an ordering $\{w_1, \ldots, w_n\}$ of $V$, together with signs $\epsilon_i \in \{-1, +1\} \left( \forall i \right)$ such that $\| \sum_{i=1}^{k} \epsilon_i w_i \| \leq 1$ for every odd $k \in \{1, \ldots, n\}$.

Of course, for the same ordering, $\| \sum_{i=1}^{k} \epsilon_i w_i \| \leq 2$ for every $k \in \{1, \ldots, n\}$. We mention that similar results are proved by Banaszczyk [1] in higher dimension for some particular norms.

In [2] the following theorem is proved. Given a norm $\| \cdot \|$ with unit ball $B$ in $\mathbb{R}^d$ and a sequence of vectors $v_1, \ldots, v_n \in B$, there are signs $\epsilon_i \in \{-1, +1\}$ for all $i$ such that $\| \sum_{i=1}^{k} \epsilon_i w_i \| \leq 2d - 1$ for every $k \in \{1, \ldots, n\}$. Theorem 1 implies that this result can be strengthened when the $v_i$s are unit vectors in $\mathbb{R}^2$ and $k$ is odd.

Theorem 4. Assume $v_1, \ldots, v_n \in \mathbb{R}^2$ is a sequence of unit vectors in the norm $\| \cdot \|$. Then there are signs $\epsilon_i \in \{-1, +1\}$ for all $i$ such that $\| \sum_{i=1}^{k} \epsilon_i w_i \| \leq 2$ for every odd $k \in \{1, \ldots, n\}$.

The bound 2 here is best possible as shown by the example of the max norm and the sequence $(-1, 1/2), (1, 1/2), (0, 1), (-1, 1), (1, 1)$.

The proof goes by induction on $k$. The case $k = 1$ is trivial. For the induction step $k \to k + 2$ let $s$ be the signed sum of the first $k$ vectors with $\|s\| \leq 2$. There are vectors $u$ and $w$ (parallel with $s$) such that $s = u + w$, $\|u\| = 1$, $\|w\| \leq 1$. Applying Theorem 1 to $u$, $v_{k+1}$ and $v_{k+2}$ we have signs $\epsilon(u)$, $\epsilon_{k+1}$ and $\epsilon_{k+2}$ with $\|\epsilon(u)u + \epsilon_{k+1}v_{k+1} + \epsilon_{k+2}v_{k+2}\| \leq 1$. Here we can clearly take $\epsilon(u) = +1$. Then

$$\|s + \epsilon_{k+1}v_{k+1} + \epsilon_{k+2}v_{k+2}\| \leq \|u + \epsilon_{k+1}v_{k+1} + \epsilon_{k+2}v_{k+2}\| + \|w\| \leq 2$$

finishing the proof. □

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