*FPSAC 2014, Chicago, USA* DMTCS proc. **AT**, 2014, 801–814

# *Stabilized-Interval-Free Permutations and Chord-Connected Permutations*

Natasha Blitvić[1*]

[1] *University of California at Berkeley, Department of Mathematics, Berkeley, CA 94720, USA*
[2] *Indiana University Bloomington, Department of Mathematics, Bloomington, IN 47405, USA*

**Abstract.** A *stabilized-interval-free (SIF) permutation* on $[n]$, introduced by Callan, is a permutation that does not stabilize any proper interval of $[n]$. Such permutations are known to be the irreducibles in the decomposition of permutations along non-crossing partitions. That is, if $s_n$ denotes the number of SIF permutations on $[n]$, $S(z) = 1 + \sum_{n\geq 1} s_n z^n$, and $F(z) = 1 + \sum_{n\geq 1} n! z^n$, then $F(z) = S(zF(z))$. This article presents, in turn, a decomposition of SIF permutations along non-crossing partitions. Specifically, by working with a convenient diagrammatic representation, given in terms of perfect matchings on alternating binary strings, we arrive at the *chord-connected permutations* on $[n]$, counted by $\{c_n\}_{n\geq 1}$, whose generating function satisfies $S(z) = C(zS(z))$. The expressions at hand have immediate probabilistic interpretations, via the celebrated *moment-cumulant formula* of Speicher, in the context of the *free probability theory* of Voiculescu. The probability distributions that appear are the exponential and the complex Gaussian.

**Résumé.** Tel que défini par Callan, une permutation sur $n$ chiffres est dite *à intervalle stabilisé* si elle ne stabilise pas d'intervalle propre de $[n]$. Ces permutations jouent le rôle des irréductibles dans la décomposition des permutations selon les partitions non-croisées. En d'autres mots, si $s_n$ dénombre les permutations à intervalle stabilisé sur $n$ chiffres, et si on considère les fonctions génératrices $S(z) = 1 + \sum_{n\geq 1} s_n z^n$ ainsi que $F(z) = 1 + \sum_{n\geq 1} n! z^n$, on obtient alors la relation $F(z) = S(zF(z))$. En revanche, le but de cet article est de décrire la décomposition analogue (toujours selon les partitions non-croisées) des permutations à intervalle stabilisé. Plus spécifiquement, partant d'une représentation diagrammatique formulée à partir des appariements sur des mots binaires à chiffres alternants, on introduit une nouvelle notion de *permutations connexe*. Si on dénote par $C(z)$ la fonction génératrice correspondante, alors cette dernière satisfait à son tour la relation $S(z) = C(zS(z))$. Les expressions dont il est question ont une interprétation naturelle dans le cadre des *probabilités libres* de Voiculescu, par le biais de la correspondance combinatorielle célèbre de Speicher entre *les moments et les cumulants libres* des variables aléatoires non-commutatives. Les variables aléatoires considérées à présent sont l'exponentielle et la gaussienne complexe.

**Keywords:** stabilized-interval-free permutations, chord-connected permutations, non-crossing partitions, complex Gaussian

*Email: `natasha.blitvic@indiana.edu`. Supported by Fonds Québecois de Recherche en Nature et Technologies.

# Contents



# 1 Introduction

**Definition 1** *A* **stabilized-interval-free (SIF) permutation** *of $[n]$ is a permutation $\sigma \in S_n$ that does not stabilize any proper interval of $[n]$, i.e. $\sigma(I) \neq I$ for all intervals $I \subsetneq [n]$.*

Introduced by Callan (2004), SIF permutations have recently attracted considerable interest. In particular, their role in positroid enumeration was discovered in Ardila et al. (2013), where for $n \geq 2$, the SIF permutations on $[n]$ were found to be in bijection to *connected positroids on* $[n]$. For other interesting connections, the reader is referred to e.g. Alman et al. (2013); Bergeron et al. (2006); Bonzom and Combes (2013); Salvatore and Tauraso (2009); Stanley (2005).

The number of SIF permutations on $[n]$ will be denoted $s_n$. In the present setting, the relevant fact about SIF permutations is that they provide a decomposition of a permutation into non-crossing components. Specifically, subject to the following notation, the factorials can be written as products of elements in $\{s_n\}_{n\geq 1}$ indexed by non-crossing partitions.

**Notation 1** *Let $NC(n)$ denote the lattice of non-crossing partitions on $[n]$. For each $\pi \in NC(n)$, let $|\pi|$ denote the number of blocks of $\pi$ and write $\pi = \{V_1, \ldots, V_{|\pi|}\}$ as a collection of its blocks.*

*Given a sequence $(a_n)_{n\geq 1}$, denote by $a_\pi$ the product $a_{|V_1|} \cdots a_{|V_{|\pi|}|}$, where $|V_i|$ denotes the cardinality of the block $V_i \in \pi$.*

**Theorem 1** *For any $n \in \mathbb{N}$, letting $s_n$ denote the number of SIF permutations in $S_n$,*

$$n! = \sum_{\pi \in NC(n)} s_\pi. \tag{1}$$

The decomposition (1) is well-known and implicit in Callan (2004) (see also the comment by Paul D. Hanna in OEI, A075834). For completeness, we include a brief bijective proof in Section 2. The resulting relationship between the factorials and the sequence $(s_n)_{n\geq 1}$ brings to mind classical exponential formula that relates the ordinary generating function of diagrams in terms of their "connected components", recalling that the underlying diagram decomposition induces set partitions. While (1) is instead expressed in terms *non-crossing* partitions, the corresponding analogue of the exponential formula is also known:

**Theorem 2 (Beissinger (1985); Speicher (1994))** *Let $A(z) = 1 + \sum_{n\geq 1} \alpha_n z^n$ be the ordinary generating function of diagrams and let $I(z) = 1 + \sum_{n\geq 1} i_n z^n$ be the generating function of their irreducibles with respect to non-crossing partitions, i.e. $\alpha_n = \sum_{\pi\in NC(n)} i_\pi$ ($n \in \mathbb{N}$). Then*

$$A(z) = I(zA(z)) \tag{2}$$

Theorem 2 follows from Beissinger (1985), which develops highly general analogues of the classical exponential formula, as well as independently from Speicher (1994), where it is given a deep probabilistic interpretation as the *free moment-cumulant formula*, one of the key combinatorial identities in the free probability theory of Voiculescu (e.g. Voiculescu et al. (1992)). Such expressions are also studied by Arizmendi (2011) in the more general case of $k$-divisible partitions of Edelman (1980).

The main goal of this article is to describe, in turn, a *decomposition of SIF permutations along non-crossing partitions*. That is, we exhibit a combinatorial sequence $\{c_n\}_{n\geq 1}$ with the property that

$$s_n = \sum_{\pi\in NC(n)} c_\pi. \tag{3}$$

Somewhat surprisingly, the sequence $(c_n)_{n\geq 1}$ will also count a subset of the permutations – namely, those permutations that we term *chord-connected*. In order to define a suitable notion of connectedness, we represent the permutation as a *parity-reversing pairing* on the *alternating bitstring*, which we define as follows.

**Definition 2** *For $n$ a positive integer, an* **alternating bitstring on** $[n]$ *is the word $* \circ * \circ \ldots * \circ$ of length $2n$, henceforth denoted $(*\circ)^n$. A* **parity-reversing pairing (PRP)** *of length $n$ is a perfect matching on $(*\circ)^n$ that matches only $\circ$'s with $*$'s. The collection of all such pairings of length $n$ is denoted $PRP(n)$.*

A motivation for considering these objects stems from probability theory and, in particular, the non-commutative viewpoint on classical stochastic processes. Concretely, such diagrams arise when computing the moments of a "non-commutative complex Gaussian" random variable via the so-called Wick formula. This intuition is further described in Section 3. In a related context, combinatorial questions surrounding *non-crossing* pairings on more general bitstrings of the form $(*)^{k_1}(\circ)^{k_1}\ldots(*)^{k_m}(\circ)^{k_m}$ were considered in Kemp et al. (2011); Schumacher and Yan (2013). In contrast, we emphasize that the parity-reversing pairings considered at present are allowed crossings.

Elements of $PRP(n)$ will be represented as chord diagrams, with all chords drawn below the word $(*\circ)^n$. For example, under this representation, the elements of $PRP(3)$ are shown in Figure 1.

At the combinatorial level, the PRP's of length $n$ are readily seen to be equinumerous to permutations on $n$ symbols. In the remainder of this paper, it will be convenient to identify the $PRP(n)$ with $S_n$ in the following manner: given $\sigma \in S_n$,

- assign to the alternating bitstring of length $n$ the decorated integer string

$$1^* \; 2^\circ \; 2^* \; 3^\circ \; 3^* \; 4^\circ \; 4^* \; \ldots n^\circ \; n^* \; 1^\circ,$$

- let each chord connect $i^\circ$ to $\sigma(i)^*$.

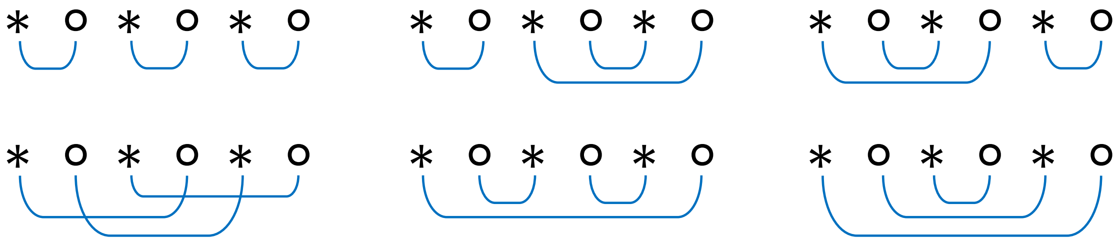

**Fig. 1:** Parity-reversing pairings on $(*\circ)^3$

In the present context, the salient properties of permutations will be intimately related to their *chord diagram* representations. Specifically, to each $\sigma \in S_n$, we associate the chord diagram of the corresponding PRP. For example, the permutations associated to the chord diagram in Figure 1 are:

$$\begin{array}{ccc} 3\,1\,2 & 2\,1\,3 & 3\,2\,1 \\ 2\,3\,1 & 1\,2\,3 & 1\,3\,2 \end{array}$$

**Definition 3** *A permutation $\sigma \in S_n$ is said to be* **chord-connected** *if the chord diagram of the corresponding PRP is connected (i.e. if its chord diagram cannot be factored into two or more disjoint non-crossing sub-diagrams). Let $c_n$ denote the number of chord-connected permutations on $[n]$.*

The first few terms of the sequence $(c_n)_{n\geq 1}$ read

$$1, 0, 1, 2, 13, 74, 544, 4458, 41221, 421412, 4722881, \ldots$$

To the author's knowledge, this sequence does not currently appear elsewhere in the literature. Note that the chord-connected permutations are not the same as the connected permutations (in the sense of A003319 OEI and the references therein), which are the irreducibles in the decomposition of a permutation along *interval partitions*. In contrast, the present development concerns the decompositions along *non-crossing partitions*.

The link between chord-connected permutations and SIF permutations is not immediately apparent. For instance, there are two SIF permutations in $S_3$, whereas, referring to Figure 1, there is a single chord-connected one. (Specifically, $\sigma_1 = 2\,3\,1$ and $\sigma_2 = 3\,1\,2$ are SIF; $\sigma_1$ is also chord-connected, whereas $\sigma_2$ is not.) Yet, the two types of permutations are intimately connected. Namely, just as the SIF permutations factorize a permutation along non-crossing partitions, the chord-connected permutations, in turn, factorize a SIF permutation. In particular:

**Theorem 3** *For any $n \in \mathbb{N}$, letting $s_n$ denote the number of SIF permutations in $S_n$,*

$$s_n = \sum_{\pi \in NC(n)} c_\pi, \tag{4}$$

*where $c_n$ is the number of connected permutations in $S_n$.*

In Section 2, we describe the diagrammatic decomposition that is at the core of Theorem 3, along with the mild (but useful) generalization to weighted chord diagrams. We also point out the manner in which (4) can be derived using generating functions.

Expressions (1) through (4) have ready probabilistic interpretations, which are outlined in Section 3. The expressions at hand concern the *moments* and *free cumulants* of familiar random variables and, in particular, link the complex Gaussian distribution to the exponential distribution. While there is a classical, well-understood link between these two probability distributions, the combinatorial correspondence obtained at present must instead be interpreted in terms of non-commutative probability (cf. Section 3) and is altogether more intriguing. This combinatorial correspondence appears to be related to a new transform describing the additive behavior of certain types of random variables in free probability (Bercovici et al.).

At the combinatorial level, the decomposition described in this paper reveals a new degree of structure present in SIF permutations and raises some new questions. In particular, expressions (1) and (4) together suggest that the present chord-diagram representation of a permutation behaves quite naturally with respect to two levels of factorization. Since the chord-diagram representation comes with a wealth of combinatorial statistics (e.g. the number of blocks, weighting of chords, or number of chord crossings), it is therefore natural to ask whether statistics of chord diagrams also have a broader meaning in any of the contexts in which SIF permutations have appeared thus far. For example, concerning the appearance of SIF permutations in non-commutative probability, the refinement in terms of chord crossings is already well understood – it corresponds to the passage from Gaussian random variables to the $q$-Gaussian random variables of Bożejko and Speicher (1991). (Note that the crossings considered here are structurally different from those of Corteel (2007).) A first application of the present decomposition to positroid enumeration is outlined in Section 2.3, where the action of weighting each chord by some scalar $\lambda^{-1}$ re-introduces the "rate" parameter into Theorem 11.1 of Ardila et al. (2013).

# 2 Decompositions along non-crossing partitions

The present section illustrates the decomposition of permutations along non-crossing partitions into SIF permutations, develops a decomposition of SIF permutations along non-crossing partitions into *chord-connected permutations* (cf. Theorem 3), and discusses some refinements.

## 2.1 Proof of Theorem 1

In the following, given a permutation $\sigma \in S_n$ containing a cycle $C$ of length $\ell$, we identify $C$ with the corresponding permutation on $[\ell]$. Under this identification, a cycle is SIF.

**Proof of Theorem 1:** For every permutation $\sigma \in S_n$ with the corresponding cycle decomposition $\sigma = C_1 \cdots C_m$, assign to it a set partition $\pi$ of $[n]$ in the obvious way, i.e. $\pi$ has the block decomposition $\pi = W_1 \cup \ldots \cup W_m$, where each $W_i$ is formed by the (unordered) elements of $C_i$. Next, send each $\pi$ to $\hat{\pi} \in NC(n)$, the minimal non-crossing partition containing $\pi$. Any two blocks $W_i$ and $W_j$ in $\pi$ will map into the same block of $\hat{\pi}$ if and only if they cross. Considering the resulting block decomposition $\hat{\pi} = V_1 \cup \ldots \cup V_{|\hat{\pi}|}$ (where $|\hat{\pi}|$ denotes the number of blocks in $\hat{\pi}$), let $\sigma_{V_i}$ denote the product of cycles corresponding in the above manner to $V_i$. It now suffices to note that the permutation $\sigma_{V_i}$ is SIF; for, otherwise, $\sigma_{V_i}$ would have a stabilized interval and therefore contain a proper, non-empty union of cycles that do not cross the remaining cycles in $\sigma_{V_i}$ – a contradiction. Clearly, this is a unique decomposition. This process can be reversed in the obvious way, starting with a non-crossing partition in $NC(n)$ and assigning to each block a SIF permutation, yielding an element of $S_n$. (Note this is injective.) $\square$

For example, consider the permutation $\sigma = C_1 \cdots C_5$, where $C_1 = (1, 7, 3)$, $C_2 = (2, 9, 8, 10)$, $C_3 = (4, 6)$, $C_4 = (5)$. The above decomposition then yields:

$$\begin{array}{lll} V_1 = \{1,2,3,7,8,9,10\}, & V_2 = \{4,6\}, & V_3 = \{5\} \\ \sigma_{V_1} = 4\,6\,1\,3\,7\,5\,2, & \sigma_{V_2} = 2\,1, & \sigma_{V_3} = 1. \end{array}$$

## 2.2 Proof of Theorem 3

Recall that for each permutation $\sigma \in S_n$ , we associate a chord diagram by assigning to the alternating bitstring of length $n$ the decorated integer string $1^*\ 2^\circ\ 2^*\ 3^\circ\ 3^*\ 4^\circ\ 4^*\ \ldots n^\circ\ n^*\ 1^\circ$, and drawing a chord between $i^\circ$ and $\sigma(i)^*$. A *connected component* of $\sigma$ refers to a connected component of the corresponding chord diagram and is denoted by the induced substring. We recall that the notion of "connectedness" in the present setting refers to *irreducibility with respect to non-crossing partitions*.

**Lemma 1** *Each connected component of $\sigma \in S_n$ can only have one of the following two forms:*

$$i(1)^\circ i(2)^* \ldots i(2r-1)^\circ i(2r)^* \tag{F1}$$

*for some $1 < i(1) < i(2) < \ldots < i(2r) \leq n$ or*

$$i(1)^* i(2)^\circ \ldots i(2r-1)^* i(2r)^\circ \tag{F2}$$

*for some $1 \leq i(1) < i(2) < \ldots < i(2r-1)$ with either $i(2r) = 1$ or $i(2r-1) < i(2r) \leq n$.*

**Proof:** It suffices to note that if the connected component contains $i(j)^\circ$ and $i(j+1)^\circ$ (i.e. if it contains two $\circ$'s in a row), then there must be another connected component containing $i(j)^*$. In this case, the two connected components cross – a contradiction. The case of $i(j)^*$ and $i(j+1)^*$ follows analogously. $\square$

**Lemma 2** *A permutation $\sigma \in S_n$ is SIF if and only each of its connected components is of the form (F2).*

**Proof:** Suppose that $\sigma$ has as a connected component of the form (F1). Since $\sigma$ maps $i^\circ \mapsto \sigma(i)^*$ and since the decomposition into the connected components induces non-crossing partitions, it follows that $\sigma$ stabilizes the interval $I = [i(1), \ldots, i(2r)]$. (For otherwise, if either $\sigma(i(j)) \neq I$ for some $i(j) \in I$ or $\sigma(k) \in I$ for some $k \neq I$, then $i(1)^\circ i(2)^* \ldots i(2r-1)^\circ i(2r)^*$ is contained in a strictly larger connected component, which is impossible. See Figure 2.) Furthermore, since the connected component is of the form (F1), it follows that $1 < i(1)$ and $I$ is therefore a strict subset of $[1, \ldots, n]$, i.e. $\sigma$ is not SIF.

Conversely, if $\sigma$ is not SIF, then $\sigma$ stabilizes some interval $I = [i(1), \ldots, i(2r)] \subsetneq [1, n]$. Suppose $i(1) \neq 1$ and consider the bitstring $i(1)^\circ i(2)^* \ldots i(2r-1)^\circ i(2r)^*$. Since $I$ is stabilized under $\sigma$, by a similar graphical argument to the one above, it follows that $i(1)^\circ i(2)^* \ldots i(2r-1)^\circ i(2r)^*$ is a (disjoint) union of connected components. Therefore, the connected component containing $\circ^{i(1)}$ is of the form (F1). If, instead, $i(1) = 1$, then (since $I$ is a strict subinterval) $i(2r) \neq n$. Therefore, the bitstring $i(2r+1)^\circ i(2r+1)^* \ldots n^\circ n^*$ is a (disjoint union) of connected components, at least one of which form (F1). $\square$

**Proof of Theorem 3:** Fix $\sigma \in S_n$ and let $\pi \in NC(2n)$ be the non-crossing partition induced by the decomposition into connected components. Write $\pi = \{V_1, \ldots, V_r\}$ (collection of disjoint blocks). By Lemma 2, $\sigma$ is SIF if and only if each $V_j$ ($j \in [r]$) has the form (F2). Thus, for a given block $V_j = i(1)^* i(2)^\circ \ldots i(2r-1)^* i(2r)^\circ$, $\sigma$ being SIF implies that $i(2) = i(1)+1$, $i(4) = i(3)+1$,. . .,$i(2r) = i(2r-1)+1$. (For, otherwise, in between $i(j)^*$ and $i(j+1)^\circ$, there will be a block of the form (F1)).

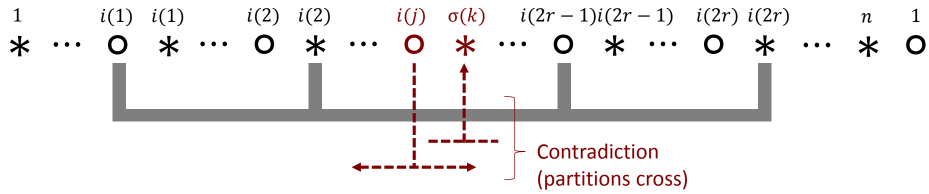


**Fig. 2:** Proof of Lemma 2.

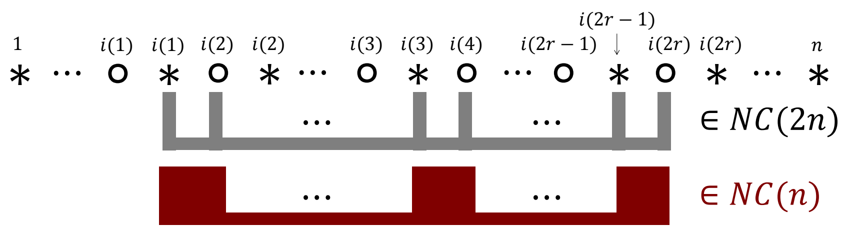


**Fig. 3:** Proof of Theorem 3.

Thus, a SIF permutation $\sigma$ decomposes along a subset of $NC(2n)$ that can be identified with $NC(n)$ by "collapsing into a single element" any two elements of the form $j^*(j+1)^\circ$. This is illustrated in Figure 3.

Conversely, an analogous argument yields that if $\sigma$ does not decompose along $NC(n)$ under the above identification, $\sigma$ will contain an interval of the form (F1) and, by Lemma 2, will not be SIF. $\square$

Note that the enumeration formula (4) may also be derived from generating functions. Namely, let $F(z) = \sum_{n=0}^{\infty} n! z^n$, $S(z) = 1 + \sum_{n=1}^{\infty} s_n z^n$, and $C(z) = 1 + \sum_{n \geq 1} c_n z^n$. By Theorem 2, (1) is equivalent to

$$F(z) = S(zF(z)). \tag{5}$$

Now, note that any permutation $\sigma \in S_n$ factorizes (along non-crossing partitions) into chord-connected permutations in the obvious manner, i.e. by passing to the corresponding PRP on $(*\circ)^n$. Specifically, letting $\hat{c}_n = c_{n/2}$ for $n$ even (and 0 otherwise), we have that

$$n! = \sum_{\pi \in NC(2n)} \hat{c}_\pi, \tag{6}$$

$$F(z) = C(zF^2(z)). \tag{7}$$

Concerning the passage from (6) to (7), an analogous expression for the case of arbitrary chord crossing diagrams was derived in Flajolet and Noy (2000). It is an instance of another analogue of the exponential formula, this time for partitions in $NC(2n)$ with blocks of even cardinality. A general lemma in Arizmendi (2011) shows that any two generating functions related as (5) and (7) must also satisfy

$$S(z) = C(zS(z)), \tag{8}$$

which, by Theorem 2, is equivalent to (4).

## 2.3 Further Refinements

We begin by addressing the question of the extent to which the correspondences between the chord-connected permutations and SIF permutations, i.e. (4), and between the chord-connected permutations and the factorials, i.e. (1) and (4), hinge on the exact form of the underlying bitstring.

**Remark 1** *Starting with a binary string* $*^n\circ^n := *\ldots*\circ\ldots\circ$ *instead of* $(*\circ)^n$, *one can analogously define parity-reversing pairings and use these to represent permutations on* $[n]$, *again letting each chord connect* $i^\circ$ *to* $\sigma(i)^*$. *Let* $d_n$ *denote the number of connected permutations with respect to this new representation. Direct enumeration yields the first few elements of the sequence* $(d_n)_{n\geq 1}$: $1, 1, 3, 13, \ldots$ *In turn, letting* $e_n := \sum_{\pi\in NC(n)} d_n$ *and performing the summation reveals the first few terms to be* $1, 2, 7, 28, \ldots$ *Finally, letting* $f_n := \sum_{\pi\in NC(n)} e_\pi$ *yields the sequence* $1, 3, 14, 65, \ldots$ *None of these sequences appears to be known at present and, observing the cardinalities, neither counts a subset of the permutations on* $[n]$.

We next consider the extent to which the decompositions in (1) and (4) are preserved under combinatorial refinements. The simplest such refinement is a $\lambda$-*weighted permutation*, which is a permutation whose each chord (in the corresponding PRP chord diagram) carries weight $\lambda^{-1}$. Letting $c_n^{(\lambda)}$ denote the number of connected $\lambda$-weighted connected permutations on $[n]$, it is immediate that $c_n^{(\lambda)} = \lambda^{-n}c_n$. Of course, a SIF permutation (as a permutation) also corresponds to a PRP chord diagram, so similarly letting $s_n^{(\lambda)}$ denote the number of connected $\lambda$-weighted SIF permutations on $[n]$, we have $s_n^{(\lambda)} = \lambda^{-n}s_n$. The observation is that the $\lambda$-weighting is consistent with (1) and (4).

**Corollary 1** *For any* $n \in \mathbb{N}$,

$$\lambda^{-n}n! = \sum_{\pi\in NC(n)} s_\pi^{(\lambda)}, \qquad s_n^{(\lambda)} = \sum_{\pi\in NC(n)} c_\pi^{(\lambda)}. \tag{9}$$

**Proof:** Each block $V_i \in \pi$ contributes a factor $s_{|V_i|}^{(\lambda)}$ or $c_{|V_i|}^{(\lambda)}$. Since $s_{|V_i|}^{(\lambda)} = \lambda^{-|V_i|}s_{|V_i|}$ and $c_{|V_i|}^{(\lambda)} = \lambda^{-|V_i|}c_{|V_i|}$, and since $|V_1| + \ldots + |V_{|\pi|}| = n$, the result follows. $\square$ Though the above corollary is simple, it is meaningful. In particular, recall the setting of Ardila et al. (2013), where the *positroids* on $[n]$ are enumerated by the sequence $(p_n)$ and the *connected positroids* on $[n]$ are enumerated by $k_1 = 2$ and $k_n = s_n$ for $n \geq 2$ (see their Theorem 10.7). Then, Theorem 11.1 of Ardila et al. (2013) shows that the $p_n$ and $k_n$ are, respectively, the moments and free cumulants of the random variable $1 + \mathrm{Exp}(1)$ (i.e. of the random variable $1 + X$, where $X$ is exponentially distributed with rate 1). Now, analogously defining the sequences $p_n^{(\lambda)}$ and $k_n^{(\lambda)}$, one may check that the factorization formula still holds, while the moments and free cumulants being enumerated are now those of the random variable $\lambda^{-1} + \mathrm{Exp}(\lambda)$ (where $\mathrm{Exp}(\lambda)$ has the probability density $\lambda e^{-\lambda x}dx$.)

Note that a more detailed inspection of the proof of Theorem 3 shows that any statistic of chord diagrams that is *multiplicative* with respect to decompositions along non-crossing partitions will behave well under the factorizations (1) and (4). Further consequences of this fact are presently under investigation.

# 3 Probabilistic Interpretations

The combinatorial results of the previous section can be interpreted through the lens of probability theory or, more specifically, by taking the non-commutative point of view on classical processes. A *non-commutative probability space* is a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital algebra whose elements are interpreted as "non-commutative random variables" and $\varphi$ a unital linear functional playing the role of "expectation". Two elementary examples are the following.

**Example 1** *Given some (classical) probability triple $(\Omega, \mathscr{B}, \mathbb{P})$, take $\mathcal{A}$ to be the algebra of (classical) random variables that have moments of all orders, i.e. $\mathcal{A} = \cap_{1 \leq p < \infty} L^p(\Omega, \mathscr{B}, \mathbb{P})$, and $\varphi = \mathbb{E}$ to be the usual expectation functional.*

**Example 2** *Let $\mathcal{A}$ be some algebra of $N \times N$ matrices and $\varphi = \frac{1}{N}Tr$. (Note that, unlike in Example 1, this algebra need not be commutative.) More broadly, one may think of non-commutative probability as endowing algebraic objects with probabilistic intuition.*

Given an element $a \in \mathcal{A}$, one can compute its *moment sequence* $(m_n)_{n \geq 1}$, where $m_n := \varphi(a^n)$. In many circumstances (e.g. when $a$ is a self-adjoint matrix)[(i)], the corresponding sequence is in fact a moment sequence of some probability measure on the real line. The *distribution* of a non-commutative random-variable is considered to be its moment sequence (whether or not there is a bona fide probability measure associated with it). However, that which distinguishes the various possible non-commutative probability theories is the notion of "non-commutative independence". We are at present concerned with *free probability*, where algebraic notion of *freeness* is the notion of independence for non-commutative random variables. For a beautiful (and combinatorics-oriented) introduction to *free probability*, the reader is referred to Nica and Speicher (2006).

## 3.1 SIFs as Free Cumulants

For our combinatorial purposes, given an element $a \in \mathcal{A}$ with moments of all orders, one can define the *free cumulant sequence* $(\kappa_n)_{n \geq 1}$ of $a$ by applying the Möbius inversion to the *free moment-cumulant formula* of Speicher (1994). The latter is given by

$$m_n = \sum_{\pi \in NC(n)} \kappa_\pi \tag{10}$$

following the convention of Notation 1 and for all $n \in \mathbb{N}$. Revisiting Theorem 1, the reader may recall that the moment sequence $(n!)_{n \geq 1}$ uniquely determines the *exponential probability distribution with rate parameter 1*, i.e. the probability measure $d\mu(x) = e^{-x}dx$ for $x \geq 0$. It follows that the SIF permutations are the free cumulants of the exponential random variable, i.e. $\kappa_n = s_n$. In fact, it is not too difficult to generalize this to the case of a general rate parameter $\lambda$, where the moments become $n!/\lambda^n$ and the corresponding free cumulants are $\kappa_n/\lambda^n$.

More surprising, however, is the "moment-cumulant formula" contained in Theorem 3. Specifically, the role of the moments is played by SIF permutations,[(ii)] whereas that of the cumulants now goes to the chord-connected permutations. In fact, as we discuss next, the sequence $(c_n)_{n \geq 1}$ is closely related to the free cumulants of the complex Gaussian random variable.

---

[(i)] More broadly, when $(\mathcal{A}, \varphi)$ admits a representation on a Hilbert space and $a \in \mathcal{A}$ is self-adjoint.

[(ii)] It should, however, be noted that a straightforward computation of Hankel determinants yields that the sequence $(s_n)_{n \geq 1}$ is not positive definite, i.e. it is not a moment sequence of a probability measure on the real line.

## 3.2 Chord-Connected Permutations as Free Cumulants

To understand the probabilistic interpretation of the sequence $(c_n)_{n\geq 1}$, one must pass to the notion of *joint* moments and *joint* free cumulants. Specifically, the *nth joint moment functional* $m_n$ is a multilinear functional on $\mathcal{A}^{\times n}$ such that $m_n(a_1, \ldots, a_n) = \varphi(a_1 \cdots a_n)$. (In particular, $m_n(a, \ldots, a)$ is simply the $n$th moment of $a$.) The corresponding joint cumulant is likewise defined by Möbius inversion of the *joint* free moment-cumulant formula (e.g. Nica and Speicher (2006)). Specifically, writing each $\pi \in NC(n)$ according to its block decomposition as $\pi = \{V_1, \ldots, V_{|\pi|}\}$ and writing, in turn, $V_i = \{i_1(i), \ldots, i_1(|V_i|)\}$ (where $|V_i|$ is the cardinality of $V_i$) for $i = 1, \ldots, |\pi|$, let

$$\kappa_\pi(a_1, \ldots, a_n) := \prod_{k=1}^{|\pi|} \kappa_{|V_k|}(a_{i_1(k)}, \ldots, a_{i_1(|V_i|)}).$$

The free moment-cumulant formula then reads

$$m_n(a_1, \ldots, a_n) = \sum_{\pi \in NC(n)} \kappa_\pi(a_1, \ldots, a_n). \tag{11}$$

Returning to the setting of classical probability theory, it is a basic fact that the exponential distribution with rate one is intimately related to the complex Gaussian distribution with zero mean and unit variance – namely, the modulus squared of a complex Gaussian random variable has the exponential distribution. More explicitly, adopting the notation of Example 1, let $a$ be a random variable with the complex Gaussian distribution of zero mean and unit variance, $a^*$ its complex conjugate, and $\varphi = \mathbb{E}$, the usual expectation functional. It is then well known (and readily verifiable) that:

$$\varphi((a^*a)^n) = n! \tag{12}$$

In other words, the random variable $|a|^2$ has the exponential distribution with rate parameter 1. Now recall expression (6), i.e. $n! = \sum_{\pi \in NC(2n)} \hat{c}_\pi$. By (11), it now follows that the sequence $(c_n)_{n\geq 1}$ gives certain free cumulants of the classical complex Gaussian random variable. Specifically:

$$c_n = \kappa_{2n}(a^*, a, \ldots, a^*, a). \tag{13}$$

**Remark 2** *While refraining from going into further detail, we point out that* (12) *and* (13) *having a natural representation in terms of chord diagrams is an instance of the* Wick formula*. For a sampling of recent work concerning generalized Wick formulas in non-commutative probability, the reader is referred to e.g. Blitvić (2012); Bożejko and Speicher (1991); Bożejko and Yoshida (2006); Guţă and Maassen (2002).*

Therefore, the present combinatorial framework, stemming from the pair of identities $n! = \sum_{\pi \in NC(n)} s_\pi$ and $s_n = \sum_{\pi \in NC(n)} c_\pi$, gives a deeply non-commutative link between the exponential distribution and the complex Gaussian.

**Remark 3** *As can be deduced from Remark 1, this new combinatorial correspondence between the above two probability distributions no longer holds if the underlying bitstring* $(*\circ)^n$ *is replaced by* $(*^n\circ^n)$*. In fact, the correspondence breaks down despite the fact that* $\varphi((a^*)^n a^n) = n!$ *(which follows as the complex Gaussian random variable is taken as an element of a commutative algebra, cf. Example 1).*

*More concretely, the analogous procedure conducted on the bitstring $(*^n\circ^n)$, and passing instead via the free cumulant $\kappa_{2n}(a^*,\ldots,a^*,a,\ldots,a)$, no longer yields $(n!)_{n\geq 1}$ as the ultimate sequence.*[(iii)]

The nature of this new combinatorial link between the exponential distribution and the complex Gaussian is intriguing. At the combinatorial level, such expressions feature in the framework of $k$-divisible elements of Arizmendi (2011), but the underlying interpretations are different. Specifically, there, the factorials are to be interpreted as the free cumulants of the random variable $Z^2$, whereas we are additionally interested in their role as the moments of $|Z|^2$. Rather, the probabilistic principle at hand may stem from a recent work (in progress) on the additive properties of the so-called R-diagonal elements in free probability, Bercovici et al.. Such connections are presently under investigation.

## Acknowledgements

The author is indebted to Hari Bercovici, Todd Kemp, and Lauren K. Williams for the interesting discussions that revived the author's interest in these ideas. Indeed, the main result dates back to 2009, in the context of the author's early doctoral work. Thanks are also due to the mathematics department at UC Berkeley for their hospitality in the Fall 2013.

## References

J. Alman, C. Lian, and B. Tran. Circular planar electrical networks i: The electrical poset ep_ $\{$n$\}$, 2013.

F. Ardila, F. Rincón, and L. Williams. Positroids and non-crossing partitions, 2013. To appear in the *Transactions of the AMS*.

O. Arizmendi. $k$-divisible random variables in free probability. arXiv:1203.4780, 2011.

J. S. Beissinger. The enumeration of irreducible combinatorial objects. *J. Combin. Theory Ser. A*, 38(2): 143–169, 1985. ISSN 0097-3165. doi: 10.1016/0097-3165(85)90065-2. URL `http://dx.doi.org/10.1016/0097-3165(85)90065-2`.

H. Bercovici, A. Nica, and M. Noyes. R-diagonal convolution and infinite divisibility for r-diagonal elements. in preparation.

N. Bergeron, C. Hohlweg, and M. Zabrocki. Posets related to the connectivity set of coxeter groups. *Journal of Algebra*, 303(2):831–846, 2006.

N. Blitvić. The $(q,t)$-Gaussian process. *J. Funct. Anal.*, 263(10):3270–3305, 2012. ISSN 0022-1236. doi: 10.1016/j.jfa.2012.08.006. URL `http://dx.doi.org/10.1016/j.jfa.2012.08.006`.

V. Bonzom and F. Combes. The calculation of expectation values in gaussian random tensor theory via meanders, 2013.

[(iii)] The alert reader may object to the cumulants $\kappa_{2n}(a^*,a,\ldots,a^*,a)$ and $\kappa_{2n}(a^*,\ldots,a^*,a,\ldots,a)$ being different when $a$ is an element of a commutative algebra. However, there is no contradiction; some hand calculations for small $n$ based on (11) may clarify this fact.

M. Bożejko and R. Speicher. An example of a generalized Brownian motion. *Comm. Math. Phys.*, 137 (3):519–531, 1991. ISSN 0010-3616. URL `http://projecteuclid.org/getRecord?id=euclid.cmp/1104202738`.

M. Bożejko and H. Yoshida. Generalized $q$-deformed Gaussian random variables. In *Quantum probability*, volume 73 of *Banach Center Publ.*, pages 127–140. Polish Acad. Sci. Inst. Math., Warsaw, 2006. doi: 10.4064/bc73-0-8. URL `http://dx.doi.org/10.4064/bc73-0-8`.

D. Callan. Counting stabilized-interval-free permutations. *J. Integer Seq.*, 7(1):Article 04.1.8, 7, 2004. ISSN 1530-7638.

S. Corteel. Crossings and alignments of permutations. *Advances in Applied Mathematics*, 38:149–163, 2007.

P. H. Edelman. Chain enumeration and noncrossing partitions. *Discrete Math.*, 31(2):171–180, 1980. ISSN 0012-365X. doi: 10.1016/0012-365X(80)90033-3. URL `http://dx.doi.org/10.1016/0012-365X(80)90033-3`.

P. Flajolet and M. Noy. Analytic combinatorics of chord diagrams. Technical Report No 3914, INRIA, Unité de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France), March 2000. Thème 2, Génie logiciel et calcul symbolique, Projet Algo.

M. Guţă and H. Maassen. Generalised Brownian motion and second quantisation. *J. Funct. Anal.*, 191 (2):241–275, 2002. ISSN 0022-1236. doi: 10.1006/jfan.2001.3855. URL `http://dx.doi.org/10.1006/jfan.2001.3855`.

T. Kemp, K. Mahlburg, A. Rattan, and C. Smyth. Enumeration of non-crossing pairings on bit strings. *Journal of Combinatorial Theory, Series A*, 118(1):129 – 151, 2011. ISSN 0097-3165. doi: http://dx.doi.org/10.1016/j.jcta.2010.07.002. URL `http://www.sciencedirect.com/science/article/pii/S0097316510001081`.

A. Nica and R. Speicher. *Lectures on the combinatorics of free probability*, volume 335 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2006. ISBN 978-0-521-85852-6; 0-521-85852-6.

P. Salvatore and R. Tauraso. The operad lie is free. *Journal of pure and applied algebra*, 213(2):224–230, 2009.

P. Schumacher and C. Yan. On the enumeration of non-crossing pairings of well-balanced binary strings. *Annals of Combinatorics*, 17(2):379–391, 2013. ISSN 0218-0006. doi: 10.1007/s00026-013-0186-5. URL `http://dx.doi.org/10.1007/s00026-013-0186-5`.

R. Speicher. Multiplicative functions on the lattice of noncrossing partitions and free convolution. *Math. Ann.*, 298(4):611–628, 1994. ISSN 0025-5831. doi: 10.1007/BF01459754. URL `http://dx.doi.org/10.1007/BF01459754`.

R. P. Stanley. The descent set and connectivity set of a permutation. *Journal of Integer Sequences*, 8(2): 3, 2005.

D. V. Voiculescu, K. J. Dykema, and A. Nica. *Free random variables*, volume 1 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 1992. ISBN 0-8218-6999-X.