Unification in the normal modal logic Alt_1

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1 Introduction

The unification problem in a logical system L can be defined in the following way: given a formula $\phi(x_1,\ldots,x_\alpha)$, determine whether there exists formulas ψ_1, ..., ψ_α such that $\phi(\psi_1,\ldots,\psi_\alpha)$ is in L. The research on unification for modal logics was originally motivated by the admissibility problem for rules of inference: given a rule of inference $\phi_1(x_1,\ldots,x_\alpha),\ldots,\phi_m(x_1,\ldots,x_\alpha)/\psi(x_1,\ldots,x_\alpha)$, determine whether for all formulas χ_1, ..., χ_α, if $\phi_1(\chi_1,\ldots,\chi_\alpha)$, ..., $\phi_m(\chi_1,\ldots,\chi_\alpha)$ are in L then $\psi(\chi_1,\ldots,\chi_\alpha)$ is in L [1]. Within the context of description logics, the main motivation for investigating the unification problem was to propose new reasoning services in the maintenance of knowledge bases like, for example, the elimination of redundancies in the descriptions of concepts [2].
Combining algebraic and model-theoretic methods, Rybakov [7] demonstrated that the admissibility problem and the unification problem in intuitionistic propositional logic and modal logic $S4$ are decidable. Later on, Ghilardi [4], proving that intuitionistic propositional logic has a finitary unification type, yielded a new solution of the admissibility problem, seeing that determining whether a given rule of inference preserves validity in intuitionistic propositional logic is equivalent to checking whether the finitely many maximal unifiers of its premises are unifiers of its conclusion. These results incited researchers to determine whether there exists finitely many admissible rules of inference of intuitionistic propositional logic and modal logic $S4$ so that the remaining admissible rules of inference would be derivable from them [5].
With respect to the issue of computational complexity, the admissibility problem and the unification problem were mostly unexplored before the work of Jerábek [6] who established the $coNEXPTIME$-completeness of the admissibility problem for several intuitionistic and modal logics extending $K4$ such as $S4$ and GL, in contrast with the satisfiability problem for these logics which is usually $PSPACE$-complete and in contrast with the unification problem for modal logics contained in $K4$ which is undecidable if one considers a language with the universal modality [8]. One may ask whether the situation is getting better if the language is restricted in one way or another. Recently, the admissibility problem in the negation-implication fragment of intuition-

istic propositional logic was proved to be $PSPACE$-complete [3].
Nevertheless, very little is known about the unification problem in some of the most important description and modal logics considered in Computer Science and Artificial Intelligence. For example, the decidability of the unification problem for the following description and modal logics remains open: description logic $\mathcal{ALC}$, modal logic K, multimodal variants of K, sub-Boolean modal logics. In the ordinary modal language, the modal logic Alt_1 is the least normal logic containing the formula $\Diamond x \rightarrow \Box x$. It is also the modal logic determined by the class of all frames (W, R) such that R is functional on W, i.e. for all $s, t, u \in W$, if sRt and sRu, then $t = u$. In this paper, we demonstrate that the unification problem in Alt_1 is in $PSPACE$.

2 Definitions

Syntax Let AF be a countable set of atomic formulas (denoted x, y, etc). The set F of all formulas (denoted ϕ, ψ, etc) is inductively defined as follows:

- $\phi ::= x \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid \Box\phi$.

We define the other Boolean constructs as usual. The formula $\Diamond\phi$ is obtained as an abbreviation: $\Diamond\phi ::= \neg\Box\neg\phi$. We adopt the standard rules for omission of the parentheses. The degree of a formula ϕ, in symbols $deg(\phi)$, and its atom-set, in symbols $var(\phi)$, are inductively defined as follows:

- $deg(x) = 0$, $var(x) = \{x\}$,
- $deg(\bot) = 0$, $var(\bot) = \emptyset$,
- $deg(\neg\phi) = deg(\phi)$, $var(\neg\phi) = var(\phi)$,
- $deg(\phi \vee \psi) = \max\{deg(\phi), deg(\psi)\}$, $var(\phi \vee \psi) = var(\phi) \cup var(\psi)$,
- $deg(\Box\phi) = deg(\phi) + 1$, $var(\Box\phi) = deg(\phi)$.

We shall say that a formula ϕ is atom-free iff $var(\phi) = \emptyset$. Let AFF be the set of all atom-free formulas.

Semantics For all $n \in \mathbb{N}$, an n-valuation is an $(n+1)$-tuple $(U_0, \ldots, U_n)$ of subsets of AF. We inductively define the truth of a formula ϕ in an n-valuation $(U_0, \ldots, U_n)$, in symbols $(U_0, \ldots, U_n) \models \phi$, as follows:

- $(U_0, \ldots, U_n) \models x$ iff $x \in U_n$,
- $(U_0, \ldots, U_n) \not\models \bot$,
- $(U_0, \ldots, U_n) \models \neg\phi$ iff $(U_0, \ldots, U_n) \not\models \phi$,
- $(U_0, \ldots, U_n) \models \phi \vee \psi$ iff $(U_0, \ldots, U_n) \models \phi$, or $(U_0, \ldots, U_n) \models \psi$,
- $(U_0, \ldots, U_n) \models \Box\phi$ iff if $n \neq 0$, then $(U_0, \ldots, U_{n-1}) \models \phi$.

Obviously, $(U_0, \ldots, U_n) \models \Diamond\phi$ iff $n \neq 0$ and $(U_0, \ldots, U_{n-1}) \models \phi$. A formula ϕ is said to be n-valid, in symbols $\models_n \phi$, iff for all n-valuations $(U_0, \ldots, U_n)$, $(U_0, \ldots, U_n) \models \phi$. The modal logic Alt_1 is the least normal logic containing the formula $\Diamond x \rightarrow \Box x$. It is also the modal logic determined by the class of all frames (W, R) such that R is functional on W, i.e. for all $s, t, u \in W$, if sRt and sRu, then $t = u$. Obviously, Alt_1 is equal to the set of all formulas ϕ such that for all $n \in \mathbb{N}$, $\models_n \phi$.

Unification In the sequel, we use $\phi(x_1, \ldots, x_\alpha)$ to denote a formula whose atomic formulas form a subset of $\{x_1, \ldots, x_\alpha\}$. We shall say that a formula $\psi(x_1, \ldots, x_\alpha)$ is unifiable iff there exists $\phi_1, \ldots, \phi_\alpha \in F$ such that $\psi(\phi_1, \ldots, \phi_\alpha) \in Alt_1$. The unification problem is the decision problem defined as follows: given a formula $\psi(x_1, \ldots, x_\alpha)$, determine whether $\psi(x_1, \ldots, x_\alpha)$ is unifiable.

3 Lemmas

Let $\psi(x)$ be a formula. The reader may easily verify that

Lemma 1 *For all $k \in \mathbb{N}$, the following conditions are equivalent: (1) $\psi(x)$ is unifiable; (2) there exists $\phi \in AFF$ such that $\psi(\phi) \in Alt_1$; (3) there exists $\phi \in AFF$ such that $\Box^k\bot \rightarrow \psi(\phi) \in Alt_1$ and $\Diamond^k\top \rightarrow \psi(\phi) \in Alt_1$.*

Remark that Lemma 1 still holds when one considers a formula $\psi(x_1, \ldots, x_\alpha)$ with more than one atomic formula. In this case, simply replace the "there exists $\phi \ldots$" by "there exists $\phi_1, \ldots, \phi_\alpha \ldots$". Concerning the remainder of the paper, the same remark is on as well. Hence, without loss of generality, we will always consider that ψ is a formula with at most one atomic formula. In this case, for all $n \in \mathbb{N}$, an n-valuation is comparable to an $(n + 1)$-tuple of bits. Let $k \in \mathbb{N}$ be such that $deg(\psi(x)) \leq k$. For all $\phi \in AFF$ and for all $n \in \mathbb{N}$, if $k \leq n$, then let $V_k(\phi, n, i) =$ "if $\models_{n-k+i} \phi, \text{then } 1, \text{else } 0$" for each $i \in \mathbb{N}$ such that $i \leq k$.

Lemma 2 *For all $\phi \in AFF$ and for all $n \in \mathbb{N}$, if $k \leq n$, then $\models_n \psi(\phi)$ iff $(V_k(\phi, n, 0), \ldots, V_k(\phi, n, k)) \models \psi(x)$.*

Proof: By induction on $\psi(x)$. $\dashv$

Lemma 3 *For all $\phi \in AFF$, $\Diamond^k\top \rightarrow \psi(\phi) \in Alt_1$ iff for all $n \in \mathbb{N}$, if $k \leq n$, then $(V_k(\phi, n, 0), \ldots, V_k(\phi, n, k)) \models \psi(x)$.*

Proof: Let $\phi \in AFF$. The following conditions are equivalent: (1) $\Diamond^k\top \rightarrow \psi(\phi) \in Alt_1$; (2) for all $n \in \mathbb{N}$, $\models_n \Diamond^k\top \rightarrow \psi(\phi)$; (3) for all $n \in \mathbb{N}$, if $\models_n \Diamond^k\top$, then $\models_n \psi(\phi)$; (4) for all $n \in \mathbb{N}$, if $k \leq n$, then $(V_k(\phi, n, 0), \ldots, V_k(\phi, n, k)) \models \psi(x)$. The reasons for these equivalences to hold are the following: the equivalence between (1) and (2) follows from the definition of Alt_1, the equivalence between (2) and (3) follows from the fact that $\phi \in AFF$ and the equivalence between (3) and (4) follows from Lemma 2. $\dashv$

For all $\phi \in AFF$ and for all $n \in \mathbb{N}$, if $k \leq n$, then let $\vec{V}_k(\phi, n) = (V_k(\phi, n, 0), \ldots, V_k(\phi, n, k))$. For all $\phi \in AFF$, let $f_k(\phi) = \{\vec{V}_k(\phi, n) \colon n \in \mathbb{N}$ is such that $k \leq n\}$. The atom-free formulas ϕ' and ϕ'' are said to be k-equivalent, in symbols $\phi' \equiv_k \phi''$, iff $f_k(\phi') = f_k(\phi'')$.

Lemma 4 *$\equiv_k$ is an equivalence relation on AFF possessing finitely many equivalence classes.*

Proof: By definitions of $\equiv_k$ and f_k, knowing that for all $\phi \in AFF$, $f_k(\phi)$ is a nonempty set of $(k+1)$-tuples of bits. $\dashv$

Lemma 5 *For all $\phi', \phi'' \in AFF$, if $\phi' \equiv_k \phi''$, then $\Diamond^k\top \rightarrow \psi(\phi') \in Alt_1$ iff $\Diamond^k\top \rightarrow \psi(\phi'') \in Alt_1$.*

Proof: By definitions of $\equiv_k$ and f_k and Lemma 3. $\dashv$

For all $\phi \in AFF$ and for all $n \in \mathbb{N}$, let $\vec{a}_k(\phi, n) = \vec{V}_k(\phi, n \cdot (k+1) + k)$. For all $\phi \in AFF$, let $g_k(\phi) = \{(\vec{a}_k(\phi, n), \vec{a}_k(\phi, n+1)) \colon n \in \mathbb{N}\}$. We shall say that the atom-free formulas ϕ' and ϕ'' are k-congruent, in symbols $\phi' \cong_k \phi''$, iff $g_k(\phi') = g_k(\phi'')$.

Lemma 6 *$\cong_k$ is an equivalence relation on AFF possessing finitely many equivalence classes.*

Proof: By definitions of $\cong_k$ and g_k, knowing that for all $\phi \in AFF$, $g_k(\phi)$ is a nonempty set of pairs of $(k+1)$-tuples of bits. $\dashv$

Lemma 7 *For all $\phi', \phi'' \in AFF$, if $\phi' \cong_k \phi''$, then $\phi' \equiv_k \phi''$.*

Proof: Let $\phi', \phi'' \in AFF$. Suppose $\phi' \cong_k \phi''$ and $\phi' \not\equiv_k \phi''$. Hence, $g_k(\phi') = g_k(\phi'')$ and $f_k(\phi') \neq f_k(\phi'')$. Thus, there exists $n' \in \mathbb{N}$ such that $k \leq n'$ and $\vec{V}_k(\phi', n') \notin f_k(\phi'')$, or there exists $n'' \in \mathbb{N}$ such that $k \leq n''$ and $\vec{V}_k(\phi'', n'') \notin f_k(\phi')$. Without loss of generality, assume there exists $n' \in \mathbb{N}$ such that $k \leq n'$ and $\vec{V}_k(\phi', n') \notin f_k(\phi'')$. By the division algorithm, there exists $m, l \in \mathbb{N}$ such that $n' = m \cdot (k+1) + l$ and $l < k+1$.
Case $m = 0$**.** Since $k \leq n'$, $n' = m \cdot (k+1) + l$ and $l < k+1$, then $n' = k$. Hence, $\vec{V}_k(\phi', n') = \vec{a}_k(\phi', 0)$. Since $g_k(\phi') = g_k(\phi'')$, then there exists $n'' \in \mathbb{N}$ such that $(\vec{a}_k(\phi', 0), \vec{a}_k(\phi', 1)) = (\vec{a}_k(\phi'', n''), \vec{a}_k(\phi'', n''+1))$. Since $\vec{V}_k(\phi', n') = \vec{a}_k(\phi', 0)$, then $\vec{V}_k(\phi', n') = \vec{V}_k(\phi'', n'' \cdot (k+1) + k)$.
Case $m \neq 0$**.** Since $g_k(\phi') = g_k(\phi'')$, then there exists $n'' \in \mathbb{N}$ such that $(\vec{a}_k(\phi', m-1), \vec{a}_k(\phi', m)) = (\vec{a}_k(\phi'', n''), \vec{a}_k(\phi'', n''+1))$. Hence, $V_k(\phi', (m-1) \cdot (k+1) + k, i) = V_k(\phi'', n'' \cdot (k+1) + k, i)$ and $V_k(\phi', m \cdot (k+1) + k, i) = V_k(\phi'', (n''+1) \cdot (k+1) + k, i)$ for each $i \in \mathbb{N}$ such that $i \leq k$. Since $n' = m \cdot (k+1) + l$ and $i \leq k - (l+1)$ and $V_k(\phi', m \cdot (k+1) + l, i) = V_k(\phi', (m-1) \cdot (k+1) + k, i + (l+1))$, or $k - l \leq i$ and $V_k(\phi', m \cdot (k+1) + l, i) = V_k(\phi', m \cdot (k+1) + k, i - (k-l))$ for each $i \in \mathbb{N}$ such that $i \leq k$, then $i \leq k - (l+1)$ and $V_k(\phi', n', i) = V_k(\phi'', n'' \cdot (k+1) + k, i + (l+1))$, or $k - l \leq i$ and $V_k(\phi', n', i) = V_k(\phi'', (n''+1) \cdot (k+1) + k, i - (k-l))$ for each

$i \in \mathbb{N}$ such that $i \leq k$. Thus, $V_k(\phi', n', i) = V_k(\phi'', (n''+1) \cdot (k+1) + l, i)$ for each $i \in \mathbb{N}$ such that $i \leq k$. Therefore, $\vec{V}_k(\phi', n') = \vec{V}_k(\phi'', (n''+1) \cdot (k+1) + l)$.
In both cases, $\vec{V}_k(\phi', n') \in f_k(\phi'')$: a contradiction. $\dashv$

Lemma 8 *For all $\phi', \phi'' \in AFF$, if $\phi' \cong_k \phi''$, then $\Diamond^k \top \rightarrow \psi(\phi') \in Alt_1$ iff $\Diamond^k \top \rightarrow \psi(\phi'') \in Alt_1$.*

Proof: By Lemmas 5 and 7. $\dashv$

We shall say that a nonempty set B of pairs of $(k+1)$-tuples of bits is modally definable iff there exists $\phi \in AFF$ such that $B = g_k(\phi)$. For all nonempty sets B of pairs of $(k+1)$-tuples of bits, let $\triangleright_B$ be the domino relation on B. A path in the directed graph $(B, \triangleright_B)$ is said to be weakly Hamiltonian iff it visits each vertex at least once. Let $\vec{1}_{k+1}$ be the $(k+1)$-tuple of 1 and $\vec{0}_{k+1}$ be the $(k+1)$-tuple of 0.

Lemma 9 *For all nonempty sets B of pairs of $(k+1)$-tuples of bits, B is modally definable iff the directed graph $(B, \triangleright_B)$ contains a weakly Hamiltonian path ending with $(\vec{1}_{k+1}, \vec{1}_{k+1})$, or ending with $(\vec{0}_{k+1}, \vec{0}_{k+1})$.*

Proof: Let B be a nonempty set of pairs of $(k+1)$-tuples of bits.
If. Suppose the directed graph $(B, \triangleright_B)$ contains a weakly Hamiltonian path ending with $(\vec{1}_{k+1}, \vec{1}_{k+1})$, or ending with $(\vec{0}_{k+1}, \vec{0}_{k+1})$. Hence, there exists $s \in \mathbb{N}$ and there exists $(b'_0, b''_0), \ldots, (b'_s, b''_s) \in B$ such that $((b'_0, b''_0), \ldots, (b'_s, b''_s))$ is a weakly Hamiltonian path ending with $(\vec{1}_{k+1}, \vec{1}_{k+1})$, or ending with $(\vec{0}_{k+1}, \vec{0}_{k+1})$. Let $(\beta_0, \ldots, \beta_{s \cdot (k+1)+k})$ be the sequence of bits determined by the sequence $(b'_0, , \ldots, b'_s)$ of $(k+1)$-tuples of bits.
Case $(b'_s, b''_s) = (\vec{1}_{k+1}, \vec{1}_{k+1})$**.** Let $\phi = \bigvee\{\Diamond^i \Box \bot \colon i \in \mathbb{N}$ is such that $i < s \cdot (k+1)$ and $\beta_i = 1\} \vee \Diamond^{s \cdot (k+1)} \top$.
Case $(b'_s, b''_s) = (\vec{0}_{k+1}, \vec{0}_{k+1})$**.** Let $\phi = \bigvee\{\Diamond^i \Box \bot \colon i \in \mathbb{N}$ is such that $i < s \cdot (k+1)$ and $\beta_i = 1\}$.
In both cases, the reader may easily verify that for all $n \in \mathbb{N}$, if $n \leq s$, then $V_k(\phi, n \cdot (k+1) + k, i) = \beta_{n \cdot (k+1) + i}$ for each $i \in \mathbb{N}$ such that $i \leq k$. Hence, for all $n \in \mathbb{N}$, if $n \leq s$, then $\vec{V}_k(\phi, n \cdot (k+1) + k) = b'_n$. Thus, for all $n \in \mathbb{N}$, if $n \leq s$, then $(\vec{a}_k(\phi, n), \vec{a}_k(\phi, n+1)) = (b'_n, b''_n)$. Therefore, $B = g_k(\phi)$.
Only if. Suppose B is modally definable. Hence, there exists $\phi \in AFF$ such that $B = g_k(\phi)$. Obviously, there exists $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n_0 \leq n$, then $\vec{a}_k(\phi, n) = \vec{1}_{k+1}$, or for all $n \in \mathbb{N}$, if $n_0 \leq n$, then $\vec{a}_k(\phi, n) = \vec{0}_{k+1}$. Thus, $((\vec{a}_k(\phi, 0), \vec{a}_k(\phi, 1)), \ldots, (\vec{a}_k(\phi, n_0), \vec{a}_k(\phi, n_0+1)))$ is a weakly Hamiltonian path ending with $(\vec{1}_{k+1}, \vec{1}_{k+1})$, or ending with $(\vec{0}_{k+1}, \vec{0}_{k+1})$. $\dashv$

4 Algorithm

We are now in a position to formulate the main result of this paper.

Proposition 1 *The unification problem in Alt_1 is in $PSPACE$.*

Proof: Using the above Lemmas, when k is such that $deg(\psi(x)) \leq k$, the given formula $\psi(x)$ is unifiable iff there exists a modally definable set B of pairs of $(k+1)$-tuples of bits from which, by means of its domino relation, an infinite sequence of bits respecting $\psi(x)$ and ending with 1s, or ending with 0s can be constructed. Hence, it suffices to consider the following procedure:

```
procedure UNI(ψ(x))
begin
k := deg(ψ(x))
guess a tuple (b(0), ..., b(k)) of bits of size k + 1
bool := ⊤
i := 0
while bool ∧ i ≤ k do
    begin
    bool := MC(b(0), ..., b(i), ψ(x))
    i := i + 1
    end
if ¬bool, then reject
while (b(0), ..., b(k)) ≠ 0⃗_{k+1} ∧ (b(0), ..., b(k)) ≠ 1⃗_{k+1} do
    begin
    guess a tuple (b(k + 1), ..., b(2k + 1)) of bits of size k + 1
    bool := ⊤
    i := 0
    while bool ∧ i ≤ k do
        begin
        bool := MC(b(i + 1), ..., b(i + k + 1), ψ(x))
        i := i + 1
        end
    if ¬bool, then reject
    (b(0), ..., b(k)) := (b(k + 1), ..., b(2k + 1))
    end
accept
end
```

The function $MC(\cdot)$ takes as input a tuple $(b(i), \ldots, b(i+j))$ of bits and a formula $\psi(x)$ and returns the Boolean value $MC(b(i), \ldots, b(i+j), \psi(x)) =$ "if $(b(i), \ldots, b(i+j)) \models \psi(x), \text{then } \top, \text{else } \bot$". It can be implemented as a deterministic Turing machine working in polynomial time. The procedure $UNI(\cdot)$ takes as input a formula $\psi(x)$ and accepts it iff, when $k = deg(\psi(x))$, there exists a modally definable set B of pairs of $(k+1)$-tuples of bits from which, by means of its domino relation, an infinite sequence of bits respecting $\psi(x)$ and ending with 1s, or ending with 0s can be constructed. By Lemma 9, the procedure $UNI(\cdot)$ accepts its input $\psi(x)$ iff $\psi(x)$ is unifiable. It can be implemented as a nondeterministic Turing machine working in polynomial space. Hence, the unification problem is in $NPSPACE$. Since $NPSPACE = PSPACE$, the unification problem is in $PSPACE$. $\dashv$

Still, we do not know whether the unification problem in Alt_1 is $PSPACE$-hard.

5 Conclusion

Much remains to be done. For example, one may consider the unification problem when the ordinary modal language is extended by a set AP of parameters (denoted p, q, etc). In this case, the unification problem is to determine, given a formula $\psi(p_1, \ldots, p_\alpha, x_1, \ldots, x_\beta)$, whether there exists formulas $\phi_1, \ldots, \phi_\beta$ such that $\psi(p_1, \ldots, p_\alpha, \phi_1, \ldots, \phi_\beta) \in Alt_1$. For each $k \geq 2$, one may also consider the unification problem in Alt_k, the least normal logic containing the formula $\Diamond(x_1 \wedge \neg x_2 \wedge \ldots \wedge \neg x_{k-1} \wedge \neg x_k) \wedge \ldots \wedge \Diamond(\neg x_1 \wedge \neg x_2 \wedge \ldots \wedge \neg x_{k-1} \wedge x_k) \rightarrow \Box(x_1 \vee x_2 \vee \ldots \vee x_{k-1} \vee x_k)$. In other respects, one may consider the unification problem when the ordinary modal language is replaced by its multimodal variant. Finally, what becomes of these problems when the ordinary modal language is extended by the universal modality?

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