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THE BERRICK-CASACUBERTA PLUS-CONSTRUCTION SPACE IS A WEDGE OF GROPES

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The Berrick-Casacuberta plus-construction space is a wedge of gropes

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Berrick and Casacuberta have recently constructed a space W such that for every space X its plus construction X^+ is homotopically equivalent to $P_W X$, the W-nullification of X. We show that W is the wedge of an infinite family of gropes.

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1 Introduction

Berrick and Casacuberta have recently constructed a space W such that for every space X its plus construction X^+ is homotopically equivalent to $P_W X$, the W-nullification [5] of X. The space W is an Eilenberg-MacLane complex $K(\mathcal{F}, 1)$ where the group $\mathcal{F}$ is defined as follows (cf. [1], Example 5.3, where this group is denoted by $\mathcal{F}'$). For each sequence $\mathbf{n} = (n_1, n_2, \ldots)$ of positive integers and each $r \geq 1$ let $F_{\mathbf{n},r}$ be the free group on $2^r n_1 \ldots n_r$ symbols:

$$\{x_r(\varepsilon_1, \ldots, \varepsilon_r; i_1, \ldots, i_r); \varepsilon_k \in \{0,1\}, 1 \leq i_k \leq n_k\}.$$

For $r = 0$ define $F_{\mathbf{n},0}$ to be infinite cyclic with generator x_0. Define homomorphism $\varphi_r : F_{\mathbf{n},r} \to F_{\mathbf{n},r+1}$ for $r \geq 0$ by

$$x_r(\varepsilon_1, \ldots, \varepsilon_r; i_1, \ldots, i_r) \longmapsto$$

$$\prod_{i_{r+1}=1}^{n_{r+1}} [x_{r+1}(\varepsilon_1, \ldots, \varepsilon_r, 0; i_1, \ldots, i_r, i_{r+1}),\ x_{r+1}(\varepsilon_1, \ldots, \varepsilon_r, 1; i_1, \ldots, i_r, i_{r+1})]\ ,$$

where $[x, y]$ denotes the commutator of x and y.

Let $F_{\mathbf{n}}$ be the direct limit of the direct system $(F_{\mathbf{n},r}, \varphi_r)$ and let $\mathcal{F}$ be the free product of the groups $(F_{\mathbf{n},r}, \varphi_r)$, where $\mathbf{n}$ ranges over all increasing sequences of positive integers.

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2 The construction

Let D be a disk with one handle. It is the identification space of a square with a hole where we identify the edges α and β, respectively, as shown in Fig. 1 below. The space D has an obvious CW decomposition with two 0-cells, four 1-cells and one 2-cell.

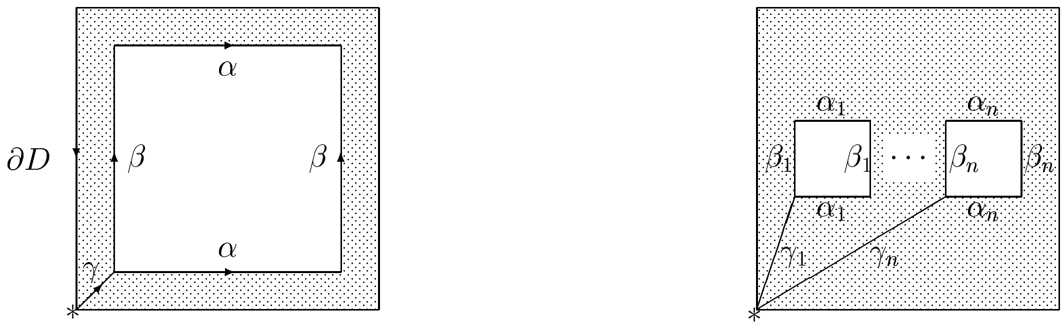

Figure 1: Disk with a handle (left) and disk with n-handles (right).

Using the notation of Fig. 1, using the symbols α, β, γ and ∂D also for the paths given by the natural parametrization of the respective edges and denoting $\alpha^\gamma = \gamma\alpha\gamma^{-1}$, we see the following fact.

Claim: The inclusion

$$i : \partial D = K(\mathbb{Z}, 1) \longrightarrow K(\mathbb{Z} * \mathbb{Z}, 1) = D$$

induces a homomorphism

$$i_\sharp : \pi_1(\partial D) \longrightarrow \pi_1(D)\,, \qquad [\partial D] \longmapsto [\,[\alpha^\gamma], [\beta^\gamma]\,]\,,$$

where ∂D is a free generator of $\pi_1(\partial D)$ and α and β freely generate $\pi_1(D)$. □

Let D_n be a disk with n handles, it is the quotient space of a disk with n holes as shown in Fig. 1.

Let γ_i be paths in D_n from the basepoint $*$ to the initial points of the paths α_i and β_i as in Fig. 1. Taking into account that there is a homotopy equivalence

$$D_n \longrightarrow \bigvee_{i=1}^{i=n} D_1^i$$

from D_n onto the wedge of n disks with one handle, mapping handles to handles, we obtain the following fact.

Claim: The inclusion

$$i : \partial D_n = K(\mathbb{Z}, 1) \longrightarrow K(\underbrace{\mathbb{Z} * \mathbb{Z} * \ldots * \mathbb{Z}}_{n}, 1) = D_n$$

induces a homomorphism

$$\pi_1(\partial D_n) \longrightarrow \pi_1(D_n)\,, \qquad [\partial D_n] \longmapsto [\,[\alpha_1^{\gamma_1}]\,,[\beta_1^{\gamma_1}]\,] \cdot \cdots \cdot [\,[\alpha_n^{\gamma_n}]\,,[\beta_n^{\gamma_n}]\,]\,.$$

where $[\partial D_n]$ freely generates $\pi_1(\partial D_n)$ and $\alpha_i^{\gamma_i}$, $\beta_i^{\gamma_i}$, $1 \le i \le n$, freely generate $\pi_1(D_n)$. $\square$

Let us construct $K(F_{\mathbf{n}}, 1)$ for the increasing sequence $\mathbf{n} = (n_1, n_2, \ldots)$. Let $G_0 = S^1 = K(F_{\mathbf{n},0}, 1)$ and let $G_1 = D_{n_1}$ (which is a $K(F_{\mathbf{n},1}, 1)$ space). Denote a generator of $\pi_1(G_0) = F_{\mathbf{n},0}$ by x_0 and the generators of $\pi_1(G_1) = F_{\mathbf{n},1}$ by $x_1(0, i_k)$ and $x_1(1, i_k)$, where $1 \le k \le n_1$. Then the homomorphism of π_1 induced by the inclusion

$$G_0 = \partial D_{n_1} \longrightarrow D_{n_1}$$

maps

$$x_0 \longmapsto \prod_{i_1=1}^{n_1} [x_1(0, i_1),\, x_1(1, i_1)]\,.$$

Let G_2 be the CW complex obtained by attaching a copy of D_{n_2} onto each loop α_i and β_i of G_1. Then $G_2 = K(F_{\mathbf{n},2}, 1)$ and the inclusion of G_1 into G_2 induces the homomorphism of the fundamental groups mapping the generators of the fundamental group according to the construction of Berrick and Casacuberta.

Assume we have constructed G_{m-1}. The complex G_m is obtained by attaching a copy of D_{n_m} onto each loop α_j and β_j of the disks with handles in $G_{m-1} \setminus G_{m-2}$. Let G be the direct limit of G_m. Using the fact that every compact subset of G is contained in a subcomplex G_m it is not difficult to check that

$$G = K(F_{\mathbf{n}}, 1)\,.$$

Clearly the desired space $W = K(\mathcal{F}, 1)$ is the wedge of gropes $K(F_{\mathbf{n}}, 1)$ as above, over the family $\{\mathbf{n}\}$ of all increasing sequences of positive integers.

Figure 2: A grope.

Every space obtained in terms of the construction like the one above for G, but with the numbers of handles of all the attached disks being arbitrary, is called a grope. Gropes appear to have originated in [6], they have been used mainly in geometric topology [2], and more recently in some aspects of cohomological dimension theory [3], [4].

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