A COHOMOLOGICAL CHARACTERISATION OF YU'S PROPERTY A FOR METRIC SPACES

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ABSTRACT. We introduce the notion of an asymptotically invariant mean as a coarse averaging operator for a metric space and show that the existence of such an operator is equivalent to Yu's property A. As an application we obtain a positive answer to Higson's question concerning the existence of a cohomological characterisation of property A. Specifically we provide coarse analogues of group cohomology and bounded cohomology (controlled cohomology and asymptotically invariant cohomology, respectively) for a metric space X, and provide a cohomological characterisation of property A which generalises the results of Johnson and Ringrose describing amenability in terms of bounded cohomology. These results amplify Guentner's observation that property A should be viewed as coarse amenability for a metric space. We further provide a generalisation of Guentner's result that box spaces of a finitely generated group have property A if and only if the group is amenable. This is used to derive Nowak's theorem that the union of finite cubes of all dimensions does not have property A.

A locally compact group G is said to be *amenable* if it has an invariant mean [**?**], that is, there exists an element $\mu \in \ell^\infty(G)^*$ such that $\mu(1) = 1$ and $g\mu = \mu$ for all $g \in G$.

For a countable discrete group this is equivalent to the Reiter condition [**?**]:

For each $g \in G$ and each $n \in \mathbb{N}$ there is an element $f_n(g) \in \text{Prob}(G)$ of finite support with

(1) $hf_n(g) = f_n(hg)$, and
(2) for all g_0, g_1, $\quad \|f_n(g_1) - f_n(g_0)\|_{\ell^1} \to 0$ as $n \to \infty$.

In [**?**] Yu generalised the notion of amenability to the context of a discrete metric space. This property played a key role in the proof of the Novikov conjecture for hyperbolic groups, and it has become a focus for study in the crossover between non-commutative geometry and geometric group theory. We will use the following definition of property A, which is equivalent to Yu's original definition for spaces of bounded geometry [**?**].

Definition 0.1. *A metric space X is said to have property A if for each $x \in X$ and each $n \in \mathbb{N}$ there is an element $f_n(x) \in$ Prob(X) with*

(1) *a sequence S_n such that* $\text{Supp}(f_n(x)) \subseteq B_{S_n}(x)$ *and*
(2) *for any $R \geq 0$, $\|f_n(x_1) - f_n(x_0)\|_{\ell^1} \to 0$ as $n \to \infty$, uniformly on the set $\{(x_0, x_1) \mid d(x_0, x_1) \leq R\}$.*

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This is an analogue of the Reiter condition for amenability and we refer to the sequence f_n as a *generalised Reiter sequence for* X. In Reiter's condition uniform convergence and the controlled support condition both follow, via equivariance, from pointwise convergence and the finite support condition. It is natural to ask if there is a generalisation of invariant mean which captures property A. In this paper we propose such a generalisation, the notion of an asymptotically invariant mean for a discrete metric space. In the absence of a group action, invariance is measured in terms of bounded variation as controlled by a family of norms. An asymptotically invariant mean is a functional satisfying a summation condition and which is invariant under an appropriate differential. To describe it we need the following background. A bounded function $\phi : X \to \ell^1(X)$ is regarded as a kernel on X, and is said to be controlled if its support lies in a bounded neighbourhood of the diagonal in $X \times X$. We assign it the sup $-\ell^1$ norm, $\|\phi\| = \sup_{x\in X} \sum_{z\in X} \phi(x)(z)$. We let $\mathcal{E}^{0,-1}(X, \ell^1(X))$ denote the Banach space of all such bounded, controlled kernels (the superscripts will be explained in section 2). By continuity the standard differential $D\phi(x_0, x_1) = \phi(x_1) - \phi(x_0)$ and the summation map π defined by $\pi(\phi)(x) = \sum_{z\in X} \phi(x)(z)$ both extend to maps defined on the double dual of $\mathcal{E}^{0,-1}(X, \ell^1(X))$ which we denote $\mathcal{E}^{0,-1}_W(X, \ell^1(X))$. (The target of the differential here is of crucial importance, but we suppress it for the purpose of this overview.) We then define an *asymptotically invariant mean on* X to be an element $\mu \in \mathcal{E}^{0,-1}_W(X, \ell^1(X))$ such that $D\mu = 0$ and $\pi_*\mu = \mathbf{1_W}$, where $\mathbf{1_W}$ denotes the element of $\mathcal{E}^{0,-1}_W(X, \ell^1(X))$ corresponding to the constant function $\mathbf{1}$ on X. This is described in detail in section **??**.

In Theorem **??** we show that a space X admits an asymptotically invariant mean if and only if it satisfies Yu's property A. We give, as an application of this result, an answer to the question of Higson, who asked for a cohomological characterisation of property A. In particular we provide a generalisation of the following classical theorem of Johnson and Ringrose, [**?**], who characterised amenability for a locally compact group in terms of bounded cohomology.

Theorem (Johnson and Ringrose)**.** *Let* G *be a locally compact group. The following are equivalent:*

- G *is amenable;*
- *The bounded cohomology* $H^q_b(G, V^*)$ *is* 0 *for all* $q \geq 1$ *and all Banach* G*-modules* V*;*
- *The class in* $H^1_b(G, (\ell^\infty/\mathbb{C})^*)$ *represented by the cocycle* $\mathcal{J}(g_0, g_1) = \delta_{g_1} - \delta_{g_0}$ *is trivial.*

Here δ_g denotes the Dirac delta function supported at g in $\ell^1(G)$ which is included in $\ell^1_0(G)^{**} \cong (\ell^\infty/\mathbb{C})^*$ in the usual way. The cocycle $\mathcal{J}(g_0, g_1) = \delta_{g_1} - \delta_{g_0}$ will play a crucial role throughout this paper and we have chosen to refer to it and its generalisations as the Johnson class. We note here that in [**?**] we also observed that the characterisation of amenability in terms of the vanishing of the Johnson class also applies in classical (unbounded) group cohomology $H^1(G, (\ell^\infty/\mathbb{C})^*)$.

For a metric space X, and an X-module $\mathcal{V}$ (suitably defined, see section **??**) we will construct two controlled cohomology theories $H^*_Q(X, \mathcal{V}), H^*_W(X, \mathcal{V})$ as coarse analogues of group cohomology. The subscripts Q and W denote suitable completions of a bicomplex from which the cohomology is defined. The cohomology groups $H^1_Q(X, \ell^1_0(X))$ and $H^1_W(X, \ell^1_0(X))$ are of particular interest. Both contain a Johnson class, and while vanishing of the Johnson class in $H^1_Q(X, \ell^1_0(X))$ yields a generalised Reiter sequence for X, its vanishing in $H^1_W(X, \ell^1_0(X))$ yields an asymptotically invariant mean on the space X. We are therefore able to characterise property A in terms of vanishing of the Johnson class in controlled cohomology (Theorem **??**). To obtain a vanishing theorem for the cohomology of spaces with property A we need in addition analogues of bounded cohomology for a group. To play this role, we introduce two asymptotically invariant cohomology theories $H^*_{QA}(X, \mathcal{V}), H^*_{WA}(X, \mathcal{V})$, in which Johnson elements again characterise property A (Theorem **??**), and we obtain a vanishing theorem for both theories by an averaging argument utilising the asymptotically invariant mean, (Theorems **??**, **??**). We can summarise these results as follows.

Theorem. *Let X be a metric space. Then the following are equivalent.*

(1) *The space X has property A;*

(2) *X admits an asymptotically invariant mean;*

(3) *$H^q_\sim(X, \mathcal{V}) = 0$ for all $q \geq 1$ and all X-modules $\mathcal{V}$;*

(4) *$[\mathcal{J}^{0,1}] = 0$ in $H_\sim(X, \ell^1_0(X))$.*

where $\sim$ denotes either of the decorations QA, WA.

As in [**?**], the equivalence of property A and vanishing of the Johnson elements in controlled cohomology is exhibited using a long exact sequence in cohomology, arising from a short exact sequence of coefficients:

$$0 \to \ell^1_0(X) \to \ell^1(X) \to \mathbb{C} \to 0.$$

The Johnson class appears naturally as the image of the constant function **1** under the connecting map in the long exact sequence. The asymptotically invariant mean may be equivalently regarded as a 0-cocycle for either the controlled cohomology $H^*_W(X, \ell^1(X))$, or the asymptotically invariant cohomology $H^*_{WA}(X, \ell^1(X))$.

There are equivariant versions of our cohomology theories when X is a G-space, and $\mathcal{V}$ a G-equivariant X-module, for some group G. For convenience of exposition, we will assume throughout that we have such a G-action, allowing the possibility that $G = \{e\}$ to obtain the non-equivariant case described above. In the case that $X = G$ is a countable discrete group with a proper left-invariant metric we identify the controlled cohomology theories with classical group cohomology for suitable coefficients.

We apply these results in section **??** to generalise Guentner's result that a box space of a residually finite, finitely generated group has property A if and only if the group is amenable. The controlled cohomology theories H_Q and H_W both play key roles in the proof. Our generalisation may be applied to the group $\bigoplus_{i\in\mathbb{N}} \mathbb{Z}_2$ to obtain Nowak's theorem that the union of finite cubes of all dimensions (which we identify as a box space of the group) does not have property A. This result is at first surprising since the group $\bigoplus_{i\in\mathbb{N}} \mathbb{Z}_2$ is clearly amenable, however it is not "metrically amenable" and as we will show, this is the source of Nowak's result. The concept of metric amenability is introduced in section **??** where we discuss asymptotically invariant means.

Several of the results in this paper were announced at the Ascona conference in June 2009, and the first and second authors would like to thank the organisers for their hospitality. This work is related in spirit to the results appearing in [**?**]. In their paper Douglas and Nowak prove that exactness of a finitely generated group G (which by [**?**] is equivalent to property A) implies vanishing of the bounded cohomology $H^q_b(G, V)$, for so-called Hopf G-modules of continuous linear operators with values in $\ell^\infty(G)$. In [**?**] the results of [**?**] are generalised to give a characterisation of topological amenability for a group action in terms of vanishing bounded cohomology.

The paper is organised as follows. In section **??**, by way of motivating what follows, we recall the definition of classical group cohomology and bounded cohomology, and derive the forgetful map from bounded cohomology to classical cohomology from a bicomplex encoding them both; the rows of this bicomplex are acyclic and the forgetful map from bounded cohomology to classical group cohomology arises naturally from the collapse of the bicomplex onto the left hand column obtained by taking the augmentations. This point of view is reflected in section **??** where we construct an algebraic bicomplex which will lead, via completion processes outlined in section **??**, to the definition of our controlled-cohomology theories $H_Q(X, \mathcal{V})$ and $H_W(X, \mathcal{V})$ in section **??**. In section **??** we pause to consider the case of the equivariant cohomology of a discrete group and show that the completions in this case may be carried out at the level of the coefficient module, making a link with group cohomology, and, via our results in [**?**], with the theory of Johnson and Ringrose. In section **??** we introduce the notion of an asymptotically invariant mean and characterise property A in these terms. Asymptotic invariance is a cocycle condition, while we formulate the normalisation condition in terms of a map on coefficients. This leads naturally to consideration of the long exact sequence in cohomology arising from a short exact sequence of coefficient modules. In section **??** we give our first cohomological characterisation of property A in terms of vanishing of the Johnson class in controlled cohomology, which should be thought of as the analogue of classical (unbounded) group cohomology. In section **??** we introduce the asymptotically invariant cohomology which is the analogue of bounded cohomology, and in section **??** we show that vanishing of asymptotically invariant cohomology in dimensions greater than or equal to one is equivalent to property A. In section **??** we consider two of the three known classes of spaces which do not have property A: box spaces of non-amenable residually finite groups, and the union of finite cubes of all dimensions. We use our cohomological approach to give a new unified proof that these classes do not satisfy property A. The remaining known example of a non-A space, that of an expander sequence, is considered from a cohomological point of view in a companion note, [**?**].

1. GROUP COHOMOLOGY

In this section we will motivate the definitions to follow by examining the familiar objects of real valued group cohomology and bounded cohomology in a framework that will generalise to our context. In this section the group G is taken to be a countable discrete group equipped with a proper left-invariant metric.

Recall that for a group G the homogeneous bar resolution with real coefficients is given by the cochain complex $(C^p(G,\mathbb{R}), D)$ where $C^p(G,\mathbb{R})$ is the vector space of all G-invariant real valued functions on G^{p+1}, and D is induced by the differential $(g_0, \dots, g_p) \mapsto \sum_{i=0}^{p}(-1)^p \hat{g}_i$ (where $\hat{g}_i$ denotes $(g_0, \dots, g_p)$ with the i-th term deleted). We regard $\mathbb{R}$ as a G-module with the trivial G-action, so that invariance of a function ϕ is equivalent to equivariance. The cohomology of this complex is the classical group cohomology $H^p(G,\mathbb{R})$.

On the other hand the bounded cohomology $H^p_b(G,\mathbb{R})$ is computed using the subcomplex of $(C^p(G,\mathbb{R}), D)$ consisting of bounded functions. The forgetful map which regards a bounded function as a function gives a map from $H^p_b(G,\mathbb{R})$ to $H^p(G,\mathbb{R})$.

We give an alternative description of this setup using the following bicomplex. Let $\mathcal{E}^{p,q}(G,\mathbb{R})$ consist of those real valued G-equivariant functions defined on $G^{p+1} \times G^{q+1}$, such that for each R the function is bounded over $\{((g_0, \dots, g_p), (h_0, \dots, h_q)) \mid d(g_i, g_j) \le R \text{ for all } i, j\}$. There are natural anti-commuting differentials $D : \mathcal{E}^{p,q}(G,\mathbb{R}) \to \mathcal{E}^{p+1,q}(G,\mathbb{R})$ and $d:\mathcal{E}^{p,q}(G,\mathbb{R}) \to \mathcal{E}^{p+1,q}(G,\mathbb{R})$ induced by the standard differential as above, so we may construct the totalised complex and compute its cohomology.

It is easy to see that the rows of the bicomplex (which correspond to fixing p and varying q) are exact. A splitting is given by setting $s\phi((g_0, \dots, g_p), (h_0, \dots, h_q)) = \phi((g_0, \dots, g_p), (g_0, h_0, \dots, h_q))$. Standard arguments then show that the cohomology of the bicomplex collapses onto the left hand column obtained by augmenting the rows. This means that taking $\mathcal{E}^{p,-1}(G,\mathbb{R})$ to be equal to the kernel of the differential $d : \mathcal{E}^{p,0}(G,\mathbb{R}) \to \mathcal{E}^{p,1}(G,\mathbb{R})$, we obtain a cochain complex $(\mathcal{E}^{p,-1}(G,\mathbb{R}), D)$ and that the cohomology of this complex coincides with the cohomology of the totalised complex.

The cocycles in $(\mathcal{E}^{p,-1}(G,\mathbb{R}), D)$ are equivariant functions $\phi : G^{p+1} \times G \to \mathbb{R}$ which are constant in the final variable. The constraint that these functions should be bounded over the subset $\{((g_0, \dots, g_p), (h_0, \dots, h_q)) \mid d(g_i, g_j) \le R \text{ for all } i, j\}$ is then trivially satisfied by equivariance, since bounded geometry ensures that there are only finitely many orbits of points $(g_0, \dots, g_p)$ for which $d(g_i, g_j) \le R$ for all i, j. Hence the cochain complex $(\mathcal{E}^{p,-1}(G,\mathbb{R}), D)$ is isomorphic to the cochain complex $(C^p(G,\mathbb{R}), D)$ and its cohomology is standard group cohomology.

There is a second augmentation that we may construct by taking the kernels $\mathcal{E}^{-1,q}(G,\mathbb{R})$ of the maps $D : \mathcal{E}^{0,q}(G,\mathbb{R}) \to \mathcal{E}^{1,q}(G,\mathbb{R})$. Since the columns are not acyclic the corresponding cochain complex will not compute the cohomology of the bicomplex. Instead we claim that it computes the bounded cohomology of the group. To see this note that the kernels are now independent of the first variable, and that the boundedness condition simplifies to the condition that the cocycles are bounded over all choices of $(h_0, \dots, h_q)$. Hence the kernels give rise to a cochain complex isomorphic to the complex of bounded equivariant functions on the group, and the cohomology is the bounded cohomology of G.

Finally we consider the map on cohomology induced by the inclusion of $(\mathcal{E}^{-1,q}(G,\mathbb{R}), d)$ into the bicomplex. It is routine to establish that this is the standard forgetful map from bounded to ordinary cohomology.

It should be noted that this discussion applies more generally to group cohomology with coefficients in any Banach G-module.

In what follows we will generalise this construction to give cohomology theories which detect property A for a metric space. The principal ingredients are the definition of a module over a space and the construction of a bicomplex in which the notion of controlled support acts as a proxy for invariance. The cohomology of the completed bicomplex (the *controlled cohomology of* X) is analogous to group cohomology, while an augmentation of the vertical differential will provide an analogue of bounded cohomology, the *asymptotically invariant cohomology of* X. While the controlled cohomology detects property A it does not necessarily vanish for a property A space, any more than group cohomology necessarily vanishes for an amenable group. On the other hand the asymptotically invariant cohomology vanishes for a property A space, just as bounded cohomology vanishes for an amenable group.

2. The algebraic bicomplex

In order to define a suitable cohomology theory we need first to define the notion of a module over a metric space. The definition is motivated by considering the case of the classical Banach G-module of bounded, equivariant, real valued functions on a group G. Each element of the module is equipped with a support in the group, consisting of those elements for which the function is non-zero. The equivariance condition controls the variation of the support of a function. In the absence of a group action we want to capture some notion of controlled supports, but we need to do so in a "coarse" manner:

Definition 2.1. *Let* X *be a topological space. An* X-module *is a triple* $\mathcal{V} = (V, \|\cdot\|, \mathrm{Supp})$ *where the pair* $(V, \|\cdot\|)$ *is a Banach space, and Supp is a function (the support function) from* V *to the set of closed subsets of* X *satisfying the following axioms:*

1. *Supp*$(v) = \emptyset$ *if* $v = 0$,
2. *Supp*$(v + w) \subseteq$ *Supp*$(v) \cup$ *Supp*(w) *for every* $v, w \in V$,
3. *Supp*$(\lambda v) =$ *Supp*(v) *for every* $v \in V$ *and every* $\lambda \neq 0$.
4. *if* v_n *is a sequence converging to* v *then* $\mathrm{Supp}(v) \subseteq \overline{\bigcup_n \mathrm{Supp}(v_n)}$.

Example 2.2. *Let* X *be a topological space and let* $V = \ell^1(X)$ *be equipped with the* ℓ^1*- norm. The standard support structure for this module sets Supp*$(f) = \overline{\{x \in X \mid f(x) \neq 0\}}$ *for each* $f \in \ell^1(X)$.

Note that if $(W, \|\cdot\|)$ is a closed subspace of $(V, \|\cdot\|)$ then any support function on X restricts to a support function $\mathrm{Supp}|_W$ on W so that $(W, \|\cdot\|, \mathrm{Supp}|_W)$ is also an X-module. We will later consider the special case of the subspace $\ell^1_0(X)$ of $\ell^1(X)$ consisting of functions f such that $\sum_{x\in X} f(x) = 0$, by analogy with the Johnson-Ringrose characterisation of amenability.

If X is equipped with a G action for some group G, then we may also consider the notion of a G-equivariant X module. This is an X-module $(V, \|\cdot\|, \mathrm{Supp})$ equipped with a linear isometric action of G such that $g\mathrm{Supp}(v) = \mathrm{Supp}(gv)$ for every $g \in G$ and every $v \in V$.

Let X be a metric space, G be a group acting by isometries on X and $\mathcal{V} = (V, \|\cdot\|_V, \mathrm{Supp})$ be a G-equivariant X module. Associated to this data we will construct an algebraic bicomplex $\mathcal{E}^{p,q}(X, \mathcal{V})$. This bicomplex also depends on the group G, however for concision we will generally omit G from our notation.

For $\mathbf{x} \in X^{p+1}, \mathbf{y} \in X^{q+1}$, we adopt the standard convention that coordinates of $\mathbf{x}, \mathbf{y}$ are written $x_0, \dots, x_p$ and $y_0, \dots, y_q$.

For a positive real number R let Δ^{p+1}_R denote the set $\{\mathbf{x} \in X^{p+1} \mid d(x_i, x_j) \leq R, \forall i, j\}$, and let $\Delta^{p+1,q+1}_R$ denote the set

$$\left\{(\mathbf{x}, \mathbf{y}) \in X^{p+1} \times X^{q+1} \mid d(u, v) \leq R, \forall u, v \in \{x_0, \dots x_p, y_0, \dots, y_q\}\right\}.$$

Identifying $X^{p+1} \times X^{q+1}$ with X^{p+q+2} in the obvious way, $\Delta^{p+1,q+1}_R$ can be identified with Δ^{p+q+2}_R.

Given a function $\phi : X^{p+1} \times X^{q+1} \to V$ we set

$$\|\phi\|_R = \sup_{\mathbf{x}\in\Delta^{p+1}_R, \mathbf{y}\in X^{q+1}} \|\phi(\mathbf{x}, \mathbf{y})\|_V.$$

Definition 2.3.

(i) *We say that a function* ϕ *is* of controlled supports *if for every* $R > 0$ *there exists* $S > 0$ *such that whenever* $(\mathbf{x}, \mathbf{y}) \in \Delta^{p+1,q+1}_R$ *then Supp*$(\phi(\mathbf{x}, \mathbf{y}))$ *is contained in* $B_S(u)$ *for all* $u \in \{x_0, \dots x_p, y_0, \dots, y_q\}$.

(ii) *We denote by* $\mathcal{E}^{p,q}(X, \mathcal{V})$ *the space of all* G*-equivariant maps* $\phi : X^{p+1} \times X^{q+1} \to V$ *which are of controlled supports and such that* $\|\phi\|_R < \infty$ *for all* $R > 0$.

We equip the space $\mathcal{E}^{p,q}(X, \mathcal{V})$ with the topology arising from the semi-norms $\|\cdot\|_R$. While it is natural to allow R to range over all positive values we note that the topology this induces is the same as the topology arising from the countable family of seminorms $\|\cdot\|_R$ for $R \in \mathbb{N}$.

The usual boundary map $\partial : X^{m+1} \mapsto X^m$ induces a pair of anti-commuting coboundary maps D, d which yield the bicomplex

$$
\begin{array}{ccccccccc}
 & & D\uparrow & & D\uparrow & & D\uparrow & & \\
 & & \mathcal{E}^{2,0} & \xrightarrow{d} & \mathcal{E}^{2,1} & \xrightarrow{d} & \mathcal{E}^{2,2} & \xrightarrow{d} & \\
p\uparrow & & D\uparrow & & D\uparrow & & D\uparrow & & \\
 & & \mathcal{E}^{1,0} & \xrightarrow{d} & \mathcal{E}^{1,1} & \xrightarrow{d} & \mathcal{E}^{1,2} & \xrightarrow{d} & \\
 & & D\uparrow & & D\uparrow & & D\uparrow & & \\
 & & \mathcal{E}^{0,0} & \xrightarrow{d} & \mathcal{E}^{0,1} & \xrightarrow{d} & \mathcal{E}^{0,2} & \xrightarrow{d} & \\
 & & & & & \xrightarrow[q]{} & & &
\end{array}
$$

Specifically, $D : \mathcal{E}^{p,q} \to \mathcal{E}^{p+1,q}$ is given by

$$D\phi\,((x_0, \dots, x_{p+1}), \mathbf{y}) = \sum_{i=0}^{p+1} (-1)^i \phi((x_0, \dots, \widehat{x_i}, \dots, x_{p+1}), \mathbf{y})$$

while $d : \mathcal{E}^{p,q} \to \mathcal{E}^{p,q+1}$ is

$$d\phi(\mathbf{x}, (y_0, \dots, y_{q+1})) = \sum_{i=0}^{q+1} (-1)^{i+p} \phi(\mathbf{x}, (y_0, \dots, \widehat{y_i}, \dots, y_{q+1})).$$

In Proposition **??** we will show that the rows of our bicomplex are acyclic, and for this reason it makes sense to consider an augmentation of the rows making them exact at $q = 0$. We note that the definition of $\mathcal{E}^{p,q}$ and the maps $D : \mathcal{E}^{p,q} \to \mathcal{E}^{p+1,q}$, $d : \mathcal{E}^{p,q} \to \mathcal{E}^{p,q+1}$ make sense not just for positive p, q but also when one of p or q is -1. We will be interested in $\mathcal{E}^{p,-1}(X, \mathcal{V})$ which we will identify as the augmentation of row p. Elements of $\mathcal{E}^{p,-1}(X, \mathcal{V})$ are maps $\phi : X^{p+1} \times X^0 \to V$; for convenience of notation, we will suppress the X^0 factor, and write $\phi(\mathbf{x})$ for $\phi(\mathbf{x}, ())$. We note that the augmentation map is the differential $d : \mathcal{E}^{p,-1} \to \mathcal{E}^{p,0}$ defined by $d\phi(\mathbf{x}, (y)) = \phi(\mathbf{x}, (\widehat{y}))$. Suppressing the empty vector we see that $d\phi(\mathbf{x}, (y)) = \phi(\mathbf{x})$, i.e. d is the inclusion of $\mathcal{E}^{p,-1}(X, \mathcal{V})$ into $\mathcal{E}^{p,0}(X, \mathcal{V})$ as functions which are constant in the y variable.

Lemma 2.4. *The maps D and d are well-defined, continuous, anti-commuting differentials.*

Proof. The fact that D and d are anti-commuting differentials on the larger space of *all* equivariant functions from $X^{p+1} \times X^{q+1}$ to V is standard. We must show that D, d preserve finiteness of the semi-norms, and controlled supports. We note that $\|D\phi\|_R \leq (p+2)\|\phi\|_R$ by the triangle inequality, and a corresponding estimate holds for $\|d\phi\|_R$. Hence D, d are continuous, and the semi-norms are finite as required.

For ϕ of controlled supports we now show that $D\phi$ is of controlled supports. Given $R > 0$, take $(\mathbf{x}, \mathbf{y}) \in \Delta_R^{p+2,q+1}$. Since ϕ is of controlled supports, there exists S such that the support $\mathrm{Supp}(\phi((x_0, \dots, \widehat{x_i}, \dots, x_{p+1}), \mathbf{y}))$ is contained in $B_S(x_{i'})$ and $B_S(y_j)$ for all $i' \neq i$, and for all j.

Since for any $i' \neq i$ we have $d(x_i, x_{i'}) \leq R$ we deduce that $\mathrm{Supp}(\phi(\widehat{\mathbf{x}_i}, \mathbf{y}))$ lies in $B_{S+R}(x_{i'})$ for all i'. By the axioms for Supp the support of $D\phi$ is contained in $B_{S+R}(x_{i'})$ and $B_S(y_j)$ for all i' and all j, since this holds for the summands.

The argument for $d\phi$ is identical, exchanging the roles of $\mathbf{x}, \mathbf{y}$. □

We note in passing that continuity of the differentials will enable us to extend them to differentials on completions of the algebraic bicomplex defined in section 3.

Definition 2.5. *Let $H^*_{\mathcal{E}}(X, \mathcal{V})$ denote the cohomology of the totalisation of the bicomplex $\mathcal{E}^{p,q}, p, q \geq 0$, with the differentials D, d.*

Remark. *If X is equipped with two coarsely equivalent G-invariant metrics d, d' then for any module over X the controlled support conditions arising from these metrics are the same. Moreover the family of semi-norms is equivalent in the sense that for each R there is an S such that $\|\cdot\|_{R,d} \leq \|\cdot\|_{S,d'}$ and for each R there is an S such that $\|\cdot\|_{R,d'} \leq \|\cdot\|_{S,d}$. Hence the bicomplexes and the cohomology we obtain from each metric are identical. This applies in particular if $X = G$ is a countable group and the two metrics are both left-invariant proper metrics on G.*

We will now demonstrate exactness of the rows in the augmented complex. This allows the cohomology of the totalisation to be computed in terms of the left-hand column.

Proposition 2.6. *For each p the augmented row $(\mathcal{E}^{p,*}(X, \mathcal{V}), d)$, $p \geq -1$ is exact.*

Specifically, for all $p \geq 0$ there is a continuous splitting $s : \mathcal{E}^{p,q} \to \mathcal{E}^{p,q-1}$ given by

$$s\phi((x_0, \dots, x_p), (y_0, \dots, y_{q-1})) = (-1)^p \phi((x_0, \dots, x_p), (x_0, y_0, \dots, y_{q-1})).$$

We have $(ds + sd)\phi = \phi$ for $\phi \in \mathcal{E}^{p,q}$ with $p \geq 0$, and $sd\phi = \phi$ for ϕ in $\mathcal{E}^{p,-1}$

Proof. The fact that s defines a splitting on the larger space of all equivariant functions from $X^{p+1} \times X^{q+1}$ to V is standard homological algebra. We must verify that if ϕ is of controlled supports then so is $s\phi$, and that if ϕ has bounded R-norms then so does $s\phi$. The latter will follow from continuity of s, which will also allow us to extend the splitting to the completed complexes later on.

Continuity is straightforward. For each $R \geq 0$ we have $\|s\phi\|_R \leq \|\phi\|_R$; this is immediate from the observation that if $(x_0, \dots, x_p)$ in Δ_R^{p+1} then

$$\|\phi((x_0, \dots, x_p), (x_0, y_0, \dots, y_{q-1}))\|_V \leq \|\phi\|_R.$$

It remains to verify that $s\phi$ is of controlled supports. Given $R > 0$, since ϕ is of controlled supports we know there exists S such that if $(\mathbf{x}, \mathbf{y}) \in \Delta_R^{p+1,q+1}$ then $\mathrm{Supp}(\phi(\mathbf{x}, \mathbf{y}))$ is contained in $B_S(x_i)$ and $B_S(y_j)$ for all i, j. If $((x_0, \dots, x_p), (y_0, \dots, y_{q-1})) \in \Delta_R^{p+1,q}$ then the element $\big((x_0, \dots, x_p), (x_0, y_0, \dots, y_{q-1})\big) \in \Delta_R^{p+1,q+1}$, hence $\mathrm{Supp}(s\phi((x_0, \dots, x_p), (y_0, \dots, y_{q-1})))$ is also contained in $B_S(x_i)$ and $B_S(y_j)$ for all i, j.

This completes the proof. □

We remark that the corresponding statement is false for the vertical differential D, since for $\phi \in \mathcal{E}^{p,q}(X,\mathcal{V})$, the function $((x_0,\dots,x_{p-1}),(y_0,\dots,y_q)) \mapsto \phi((y_0,x_0,\dots,x_{p-1}),(y_0,\dots,y_q)$ is only guaranteed to be bounded on sets of the form $\{((x_0,\dots,x_{p-1}),(y_0,\dots,y_q)) \mid d(u,v) \leq R \text{ for all } u,v \in \{x_0,\dots,x_{p-1},y_0\}\}$, and not on $\Delta_R^p \times X^{q+1}$.

Corollary 2.7. *The cohomology* $H^*_{\mathcal{E}}(X,\mathcal{V})$ *is isomorphic to the cohomology of the cochain complex* $(\mathcal{E}^{*,-1}(X,\mathcal{V}), D)$.

Proof. This follows from the exactness of the augmented rows of the bicomplex - the cocycle $\phi \in \mathcal{E}^{p,q}(X,\mathcal{V})$ is cohomologous to the cocycle $(-Ds)^q(\phi)\mathcal{E}^{p+q,0}(X,\mathcal{V})$, whence $H^*_{\mathcal{E}}(X,\mathcal{V})$ is isomorphic to the cohomology of the complex $\ker(d : \mathcal{E}^{p,0} \to \mathcal{E}^{p,1})$ with the differential D. The augmentation map $d : \mathcal{E}^{p,-1} \to \mathcal{E}^{p,0}$ yields an isomorphism from $(\mathcal{E}^{p,-1}, D)$ to the kernel $\ker(d : \mathcal{E}^{p,0} \to \mathcal{E}^{p,1})$, and as D, d anti-commute, the differential D on the kernels is identified with the differential $-D$ on $\mathcal{E}^{p,-1}$. We note however that the change of sign does not affect the cohomology, so $H^*_{\mathcal{E}}(X,\mathcal{V})$ is isomorphic to the cohomology of $(\mathcal{E}^{*,-1}(X,\mathcal{V}), D)$ as claimed. □

To this point our construction is entirely algebraic. Generalised Reiter sequences and asymptotically invariant means will appear in suitable completions of this bicomplex which will be introduced in the following section.

3. Generalised completions

Let $\mathcal{E}$ be a vector space equipped with a countable family of seminorms $\|\cdot\|_i$ which separates points. We will call such a space a pre-Fréchet space. We have in mind that $\mathcal{E} = \mathcal{E}^{p,q}(X,\mathcal{V})$, for some p, q, X, G and $\mathcal{V}$, equipped with the R-norms as R ranges over the natural numbers.

If $\mathcal{E}$ is not complete then one constructs the classical completion $\overline{\mathcal{E}}$ of $\mathcal{E}$ as follows. Let E_{cs} denote the space of Cauchy sequences in $\mathcal{E}$ (i.e. sequences which are Cauchy with respect to each seminorm), and let E_0 denote the space of sequences in $\mathcal{E}$ which converge to 0. Then the completion of $\mathcal{E}$ is precisely the quotient space E_{cs}/E_0. As the topology of $\mathcal{E}$ is given by a countable family of seminorms, this completion is a Fréchet space.

In this section we will define two generalised completions which are somewhat larger than the classical one, and we will demonstrate various properties of the completions, and relations between the two.

Definition 3.1. *The* quotient completion *of* $\mathcal{E}$*, denoted* $\mathcal{E}_Q$ *is the quotient space* E/E_0 *where* E *denotes the space of bounded sequences in* $\mathcal{E}$ *and* E_0 *denotes the space of sequences in* $\mathcal{E}$ *which converge to 0. The family of seminorms on* $\mathcal{E}$ *yields a family of seminorms on* E *and hence on the quotient* $\mathcal{E}_Q$. *The* weak-* completion *of* $\mathcal{E}$*, denoted* $\mathcal{E}_W$ *is the double dual of* $\mathcal{E}$*. The family of seminorms on* $\mathcal{E}$ *gives rise to a family of seminorms on* $\mathcal{E}_W$*.*

We will adopt the convention that a class ψ in the sequential completion $\mathcal{E}_Q$ may be represented by a sequence of functions ψ_n, where each $\psi_n \in \mathcal{E}$.

Let I_Q denote the inclusion of $\mathcal{E}$ in $\mathcal{E}_Q$ as the space of constant sequences and let I_W be the natural inclusion of $\mathcal{E}$ in its double dual $\mathcal{E}_W$. The space $\mathcal{E}$ is not assumed to be complete, but the maps I_Q, I_W extend to an embedding of the classical completion $\overline{\mathcal{E}}$ in $\mathcal{E}_Q, \mathcal{E}_W$ respectively, and indeed $\mathcal{E}_Q, \mathcal{E}_W$ are isomorphic to the quotient and weak-* completions of $\overline{\mathcal{E}}$.

Since the space $\mathcal{E}$ need not be a normed space, we recall some basic theory of duals of Fréchet spaces. For simplicity we assume that the seminorms on $\mathcal{E}$ are monotonic, i.e. $\|\cdot\|_i \leq \|\cdot\|_j$ for $i < j$, this being easy to arrange.

For $\alpha \in \mathcal{E}^*$, we can define $\|\alpha\|^i = \sup\{|\langle \alpha, \phi \rangle| \mid \|\phi\|_i \leq 1\}$. We note that $\|\alpha\|^i$ takes values in $[0, \infty]$, and $\|\cdot\|^i \geq \|\cdot\|^j$ for $i < j$. The condition that α is continuous is the condition that $\|\alpha\|^i$ is finite for some i. For any sequence $r_1, r_2, \ldots$ the set $\{\alpha \in \mathcal{E}^* \mid \|\alpha\|^i < r_i \text{ for some } i\}$ is a neighbourhood of 0, and every neighbourhood of 0 contains such a set. Hence these sets determine the topology on $\mathcal{E}^*$.

Having equipped $\mathcal{E}^*$ with this topology, we can then form the space $\mathcal{E}^{**}$ of continuous linear functionals on $\mathcal{E}^*$. A linear functional η on $\mathcal{E}^*$ is continuous if for all i, setting $\|\eta\|_i = \sup\{|\langle \eta, \alpha \rangle| \mid \|\alpha\|^i \leq 1\}$ we have $\|\eta\|_i < \infty$.

The space $\mathcal{E}_W = \mathcal{E}^{**}$ will be equipped with the weak-* topology. It follows by the Banach-Alaoglu theorem that all bounded subsets of $\mathcal{E}_W$ are relatively compact. In the language of Bourbaki, if $A \subseteq \mathcal{E}_W$ is bounded, i.e. there exists a sequence r_i such that $\|\eta\|_i \leq r_i$ for all i, then A is contained in the polar of $\{\alpha \in \mathcal{E}^* \mid \exists i,\ \|\alpha\|^i \leq 1/r_i\}$, which is compact.

Remark. *From an abstract perspective, the weak-* completion is a natural way to enlarge $\mathcal{E}$. On the other hand, from the point of view of explicitly constructing elements of the space, the quotient completion is more tractable.*

Definition 3.2. *We say that a short exact sequence $0 \to \mathcal{E} \xrightarrow{\iota} \mathcal{E}' \xrightarrow{\pi} \mathcal{E}'' \to 0$ of locally convex topological vector spaces is* topologically exact *if the maps ι, π are open.*

Note that if the spaces are complete then the requirement that ι, π are open is automatic by the open mapping theorem.

Proposition 3.3. *Let $\mathcal{E}, \mathcal{E}'$ be pre-Fréchet spaces. Then a continuous map $T : \mathcal{E} \to \mathcal{E}'$ induces maps $T^Q : \mathcal{E}_Q \to \mathcal{E}'_Q$ and $T^W : \mathcal{E}_W \to \mathcal{E}'_W$. Moreover this process is functorial and exact, i.e., it takes short topologically exact sequences to short exact sequences.*

Proof. For the quotient completion, continuity of the map $T : \mathcal{E} \to \mathcal{E}'$ guarantees that applying T to each term of a bounded sequence ϕ_n in $\mathcal{E}$ we obtain a bounded sequence $T\phi_n$ in $\mathcal{E}'$. If $\phi_n \to 0$ then $T\phi_n \to 0$ by continuity, hence we obtain a map $T^Q : \mathcal{E}_Q \to \mathcal{E}'_Q$. It is clear that this respects compositions.

Now suppose $0 \to \mathcal{E} \xrightarrow{\iota} \mathcal{E}' \xrightarrow{\pi} \mathcal{E}'' \to 0$ is a short exact sequence. If ι^Q vanishes on a class $\phi \in \mathcal{E}_Q$ then evaluating ι on a sequence ϕ_n representing ϕ we see that $\iota\phi_n \to 0$. Since ι is open and injective, $\phi_n \to 0$. Hence $\phi = 0$ and we have shown that ι^Q is injective.

Similarly if $\phi' \in \mathcal{E}'_Q$ with $\pi^Q\phi' = 0$ then $\pi\phi'_n \to 0$. The map π induces a map $\mathcal{E}'/\iota\mathcal{E} \to \mathcal{E}''$ which is an isomorphism of pre-Fréchet spaces, hence the image of ϕ'_n in the quotient $\mathcal{E}'/\iota\mathcal{E}$ tends to 0. That is, there exists a sequence ψ'_n in $\iota\mathcal{E}$ such that $\phi'_n - \psi'_n \to 0$. We have $\psi'_n = \iota\psi_n$ for some $\psi_n \in \mathcal{E}$, and since ψ_n is a bounded sequence by topological injectivity of ι, it represents a class ψ in $\mathcal{E}_Q$. Then $\phi' = \iota^Q\psi$ in $\mathcal{E}'_Q$, hence we deduce that ϕ' is in the image of ι^Q.

Finally, for surjectivity of π^Q we note that if $\phi'' \in \mathcal{E}''_Q$ then there exists a sequence ϕ'_n such that $\phi''_n = \pi\phi'_n$. By openness of π, the sequence ϕ'_n can be chosen to be bounded as required.

In the case of the weak-* completion, the maps T^W, ι^W, π^W are simply the double duals of T, ι, π. The fact that this respects composition is then standard. The hypothesis that ι, π are open ensures that the corresponding sequence of classical completions is exact, whence exactness of the double duals is standard functional analysis. □

We now give a connection between the two completions.

Proposition 3.4. *Let ω be a non-principal ultrafilter on $\mathbb{N}$. Then for any pre-Fréchet space $\mathcal{E}$ there is a linear map $e_\omega : \mathcal{E}_Q \to \mathcal{E}_W$ satisfying $\langle e_\omega(\phi), \alpha\rangle = \lim_\omega \langle \alpha, \phi_n\rangle$ for all α. Moreover $I_W = e_\omega \circ I_Q$ for all ω and for $T : \mathcal{E} \to \mathcal{E}'$ we have $e_\omega \circ T^Q = T^W \circ e_\omega$.*

Proof. Let ϕ denote an element of $\mathcal{E}_Q$ represented by a bounded sequence ϕ_n of elements of $\mathcal{E}$. We regard this sequence as a bounded map $\Phi : \mathbb{N} \to \mathcal{E}$. Regarding $\mathcal{E}$ as a subspace of its double dual $\mathcal{E}_W$, the closure of the range of this map is compact in the weak-* topology on $\mathcal{E}_W$ by the Banach-Alaoglu theorem. By the universal property of the Stone-Čech compactification it follows that Φ extends to a map $\overline{\Phi} : \beta\mathbb{N} \to \mathcal{E}_W$ which is continuous with respect to the weak-* topology on $\mathcal{E}_W$. Note that if $\phi_n \to 0$ then $\overline{\Phi}$ is identically 0 on $\partial\beta\mathbb{N}$, in particular it vanishes at ω. We therefore define $e_\omega(\phi) = \overline{\Phi}(\omega)$ to obtain a well defined map $e_\omega : \mathcal{E}_Q \to \mathcal{E}_W$.

By continuity of $\overline{\Phi}$, for each $\alpha \in \mathcal{E}^*$, we obtain a continuous function on $\beta\mathbb{N}$ defined as $\langle\overline{\Phi}(\cdot), \alpha\rangle$. This is the extension to $\beta\mathbb{N}$ of the bounded function $n \mapsto \langle\alpha, \phi_n\rangle$, hence evaluating at ω we have

$$\langle e_\omega(\phi), \alpha\rangle = \langle\overline{\Phi}(\omega), \alpha\rangle = \lim_\omega \langle\alpha, \phi_n\rangle.$$

The fact that $e_\omega \circ T^Q = T^W \circ e_\omega$ is now easily verified as

$$\langle e_\omega(T^Q\phi), \alpha\rangle = \lim_\omega \langle\alpha, T\phi_n\rangle = \lim_\omega \langle T^*\alpha, \phi_n\rangle = \langle e_\omega(\phi), T^*\alpha\rangle = \langle T^W e_\omega(\phi), \alpha\rangle$$

for all $\alpha \in \mathcal{E}^*$. The compatibility of e_ω with the inclusion maps I_Q, I_W is simply the observation that the extension to $\beta\mathbb{N}$ of a constant sequence is again constant. □

We are now in a position to define our controlled cohomology theories.

4. Controlled cohomology

For $p \geq 0, q \geq -1$, let $\mathcal{E}_Q^{p,q}(X, \mathcal{V})$ denote the quotient completion of $\mathcal{E}^{p,q}(X, \mathcal{V})$, and let $\mathcal{E}_W^{p,q}(X, \mathcal{V})$ denote the weak-* completion of $\mathcal{E}^{p,q}(X, \mathcal{V})$. As (D, d) are continuous anti-commuting differentials, the extensions of these to the completions (which we will also denote by D, d) are again anti-commuting differentials, hence taking $p, q \geq 0$ we have bicomplexes $\left(\mathcal{E}_Q^{p,q}(X, \mathcal{V}), (D, d)\right)$ and $\left(\mathcal{E}_W^{p,q}(X, \mathcal{V}), (D, d)\right)$.

Definition 4.1. *For $\sim = Q$ or W, the $\sim$-controlled cohomology of X with coefficients in $\mathcal{V}$, denoted $H_\sim^*(X, \mathcal{V})$ is the cohomology of the totalisation of the bicomplex $\mathcal{E}_\sim^{p,q}(X, \mathcal{V}), p, q \geq 0$.*

Since the splitting s is continuous it also extends to the completions and we deduce that the augmented rows of the completed bicomplexes are exact. This gives rise to the following.

Corollary 4.2. *The cohomologies $H_Q^*(X, \mathcal{V}), H_W^*(X, \mathcal{V})$ are isomorphic respectively to the cohomologies of the cochain complexes $\left(\mathcal{E}_Q^{*,-1}(X, \mathcal{V}), D\right), \left(\mathcal{E}_W^{*,-1}(X, \mathcal{V}), D\right)$.*

The argument is identical to Corollary **??**.

We note that the extension of s to the completions ensures that taking the kernel of $d : \mathcal{E}^{p,0} \to \mathcal{E}^{p,1}$ and then completing (in either way), yields the same result as first completing and then taking the kernel; one obtains the completion of $\mathcal{E}^{p,-1}$. The corresponding statement for D would be false. The kernel of $D : \mathcal{E}_Q^{0,q} \to \mathcal{E}_Q^{1,q}$ will typically be much larger than the completion of the kernel of $D : \mathcal{E}^{0,q} \to \mathcal{E}^{1,q}$, and similarly for $\mathcal{E}_W$. We will study these kernels in Section **??**, where we will use them to define the asymptotically invariant cohomology theories.

We now make a connection between the three cohomology theories $H_\mathcal{E}, H_Q, H_W$.

Theorem 4.3. *For each non-principal ultrafilter ω on $\mathbb{N}$ the inclusions $I_Q : \mathcal{E}^{p,q}(X, \mathcal{V}) \hookrightarrow \mathcal{E}_Q^{p,q}(X, \mathcal{V})$ and $I_W : \mathcal{E}^{p,q}(X, \mathcal{V}) \hookrightarrow \mathcal{E}_W^{p,q}(X, \mathcal{V})$ together with the map $e_\omega : \mathcal{E}_Q^{p,q}(X, \mathcal{V}) \to \mathcal{E}_W^{p,q}(X, \mathcal{V})$ defined in Proposition **??** induce a commutative diagram at the level of cohomology:*

$$\begin{array}{ccc} H_\mathcal{E}^*(X, \mathcal{V}) & \xrightarrow{\mathbf{I_Q}} & H_Q^*(X, \mathcal{V}) \\ & \searrow^{\mathbf{I_W}} & \downarrow \mathbf{e}_\omega \\ & & H_W^*(X, \mathcal{V}) \end{array}$$

Moreover the kernels $\ker \mathbf{I_Q}$ and $\ker \mathbf{I_W}$ are equal, that is, a cocycle in $\mathcal{E}^{p,q}(X, \mathcal{V})$ is a coboundary in $\mathcal{E}_Q^{p,q}(X, \mathcal{V})$ if and only if it is a coboundary in $\mathcal{E}_W^{p,q}(X, \mathcal{V})$.

Proof. The existence of the maps at the level of cohomology follows from the fact that D, d commute with the inclusion maps and with each of the maps e_ω. The diagram commutes at the level of cochains by Proposition **??**. It is then immediate that $\ker \mathbf{I_Q} \subseteq \ker \mathbf{I_W}$. It remains to prove that if ϕ is a cocycle in $\mathcal{E}^{p,q}(X, \mathcal{V})$ with $I_W\phi$ a coboundary, then $I_Q\phi$ is also a coboundary.

By exactness of the rows, every cocycle in $\mathcal{E}^{p,q}(X, \mathcal{V})$ is cohomologous to an element of $\mathcal{E}^{p+q,0}(X, \mathcal{V})$, hence without loss of generality we may assume that $q = 0$. Moreover any cocycle in $\mathcal{E}^{p,0}(X, \mathcal{V})$ is $d\phi$ for some ϕ in $\mathcal{E}^{p,-1}(X, \mathcal{V})$, and the images of $d\phi$ under I_Q, I_W will be coboundaries if and only if $I_Q\phi, I_W\phi$ are coboundaries in the completions of the complex $(\mathcal{E}^{p,-1}(X, \mathcal{V}), D)$.

Suppose that $I_W\phi$ is a coboundary, that is viewing ϕ as an element of the double dual $\mathcal{E}^{p,-1}_W$, there exists ψ in $\mathcal{E}^{p-1,-1}_W$ such that $D\psi = \phi$. We now appeal to density of $\mathcal{E}^{p-1,-1}$ in $\mathcal{E}^{p-1,-1}_W$ to deduce that there is a bounded net θ_λ in $\mathcal{E}^{p-1,-1}$ converging to ψ in the weak-* topology. By continuity of D we have that $D\theta_\lambda \to D\psi = \phi$. As $D\theta_\lambda$ and ϕ lie in $\mathcal{E}^{p,-1}$, this converges in the weak topology on $\mathcal{E}^{p,-1}$. On any locally convex topological vector space, a convex set is closed in the locally convex topology if and only if it is closed in the associated weak topology. Hence (as the locally convex topology of $\mathcal{E}^{p,-1}$ is metrizable) there is a sequence θ_n of convex combinations of the net θ_λ such that $D\theta_n$ converges to ϕ in the R-semi-norm topology on $\mathcal{E}^{p,-1}$. Thus, letting θ denote the element of $\mathcal{E}_Q$ represented by the sequence θ_n, we have $D\theta = I_Q\phi$ in $\mathcal{E}^{p,-1}_Q$, so $I_Q\phi$ is a coboundary, as required.

□

5. GROUP COHOMOLOGY REVISITED

In this section we consider the case where our space is a discrete group acting on itself by left multiplication and equipped with a proper left invariant metric. We show that in this case the controlled cohomology of G can be identified with the standard group cohomology for suitably completed coefficients.

We say that an X-module $\mathcal{V} = (V, \|-\|, \mathrm{Supp})$ is *non-degenerate* if the subspace V_c of compactly supported elements of V is dense in V. We remark that for every X-module $\mathcal{V}$ the module $\mathcal{V}_c = (\overline{V_c}, \|-\|, \mathrm{Supp})$ is a non-degenerate submodule, and moreover $H^p_\sim(X, \mathcal{V}_c) = H^p_\sim(X, \mathcal{V})$ for $\sim = \mathcal{E}, Q, W$.

We denote by V_Q the Q-completion of V, similarly V_W will mean the weak-* completion of V. Note that since V is a Banach space so too are the completions, and a linear isometric action of G on V extends to a linear isometric action on each of them. We adopt our usual convention that $V_\sim$ denotes either of these completions. Let $\big(C^p(G, V_\sim), D\big)$, denote the homogeneous bar resolution computing the classical group cohomology with coefficients in $V_\sim$.

Theorem 5.1. *Let* G *be a group acting on itself by left translation equipped with a proper left-invariant metric, and let* $\mathcal{V} = (V, \|\cdot\|, \mathrm{Supp})$ *be a non-degenerate* G*-module. Then there is an isomorphism of the cochain complexes* $\big(C^p(G, V_\sim), D\big)$ *and* $\big(\mathcal{E}^{p,-1}_\sim(G, \mathcal{V}), D\big)$ *inducing an isomorphism* $H^p(G, V_\sim) \cong H^p_\sim(G, \mathcal{V})$.

Proof. For $\sim$ denoting Q or W set $\mathcal{V}^\emptyset_\sim = (V_\sim, \|-\|, \mathrm{Supp}^\emptyset)$, where $\mathrm{Supp}^\emptyset$ is the support function that assigns the empty set to every element of $V_\sim$.

We will show that $C^p(G, V_\sim) = \mathcal{E}^{p,-1}(G, \mathcal{V}^\emptyset_\sim) \cong \mathcal{E}^{p,-1}_\sim(G, \mathcal{V})$.

It is immediate that $\mathcal{E}^{p,-1}(G, \mathcal{V}^\emptyset_\sim) \subseteq C^p(G, V_\sim)$. Conversely, for every cochain $\phi \in C^p(G, V_\sim)$, $\|\phi\|_R$ is finite for every R, since the R-neighbourhood of the diagonal contains only finitely many G-orbits, by properness of the metric. The condition of controlled supports is vacuous for $\mathcal{V}^\emptyset_\sim$ and this gives the reverse inclusion.

By definition, $\mathcal{E}^{p,-1}(G, \mathcal{V}^\emptyset_\sim) \cong \varprojlim_R \ell^\infty(\Delta^p_R, V_\sim)^G$. Since G has only finitely many orbits in Δ^p_R, we have that

$$\varprojlim_R \ell^\infty(\Delta^p_R, V_\sim)^G \cong \varprojlim_R \bigoplus_{G\backslash\Delta^p_R} V_\sim \cong \varprojlim_R \bigoplus_{G\backslash\Delta^p_R} (V_c)_\sim \cong \varprojlim_R (\bigoplus_{G\backslash\Delta^p_R} V_c)_\sim$$

where we use that V_c is dense in V, and that completions commute with direct sums. We now use the fact that the structure maps in the inverse system are surjective to conclude that

$$\varprojlim_R (\bigoplus_{G\backslash\Delta^p_R} V_c)_\sim \cong (\varprojlim_R \bigoplus_{G\backslash\Delta^p_R} V_c)_\sim \cong (\varprojlim_R \ell^\infty(\Delta^p_R, V_c)^G)_\sim.$$

Since G has only finitely many orbits on Δ^p_R the controlled support condition in $\mathcal{E}^{p,-1}(G, \mathcal{V})$ is precisely that cochains take values in V_c, hence $(\varprojlim_R \ell^\infty(\Delta^p_R, V_c)^G)_\sim \cong \mathcal{E}^{p,-1}_\sim(G, \mathcal{V})$, as required.

□

In the case of a group equipped with a proper metric the theorem tells us that the completion of the cochain complex can be obtained simply by completing the coefficients. This will not be true when the metric is not proper, nor will it hold in the general case of a metric space unless it is equipped with a cocompact group action.

Now let G be a countable discrete group equipped with a proper left invariant metric and let $\mathcal{J}$ be defined by $\mathcal{J}(g_0, g_1) = \delta_{g_1} - \delta_{g_0}$. In [**?**] we noted that amenability is characterised in classical (as opposed to bounded) cohomology by the vanishing of $[\mathcal{J}]$ as an element of $H^1(G, (\ell^1_0(G))^{**})$, where $\ell^1_0(G)$ denotes the subspace of $\ell^1(G)$ consisting of functions which sum to 0. Applying the above theorem we conclude that amenability is also characterised in terms of $H_{\mathcal{W}}$.

Corollary 5.2. *Let G be a countable discrete group equipped with a proper left invariant metric and let $\mathcal{J} \in \mathcal{E}^{1,-1}(G, \ell^1_0(G))$ be defined by $\mathcal{J}(g_0, g_1) = \delta_{g_1} - \delta_{g_0}$. Then G is amenable if and only if $[I_{\mathcal{W}}\mathcal{J}] = 0$ in $H^1_{\mathcal{W}}(G, \ell^1_0(G))$.*

The analogue of this result in the context of property A will be given in Theorem **??**.

6. Asymptotically invariant means

An invariant mean on a group G is a functional μ on $\ell^\infty(G)$ with total mass 1 which is invariant under the group action. Regarding μ as a 0-cochain for the inhomogeneous bar resolution of G over $(\ell^\infty(G))^*$, the invariance condition is the assertion that μ is in fact a cocycle. Switching to the homogeneous picture and applying Theorem **??** we see that the invariance condition is equivalent to regarding μ as a cocycle in $\mathcal{E}_W^{0,-1}(G,\ell^1(G))$. For our purposes it is then convenient to express the normalisation condition using a map on coefficients, in the spirit of the results in [**?**]. To this end we will consider the short exact sequence of coefficients:

$$0 \to \ell^1_0(X) \xrightarrow{\iota} \ell^1(X) \xrightarrow{\pi} \mathbb{C} \to 0.$$

The first question we need to address is what it means for this to be a short exact sequence of X-modules We begin with the concept of a morphism of coefficient modules. Let X be a metric space, G be a group acting by isometries on X and let $\mathcal{U} = (U, |\cdot|_U, \mathrm{Supp}_U)$ and $\mathcal{V} = (V, |\cdot|_V, \mathrm{Supp}_V)$ be G-equivariant X-modules.

Definition 6.1. *A G-equivariant* X-morphism *from $\mathcal{U}$ to $\mathcal{V}$ is an equivariant bounded linear map $\Psi : U \to V$ for which there exists $S \geq 0$ such that for all $u \in U$, $\mathrm{Supp}_V(\Psi(u)) \subseteq B_S(\mathrm{Supp}_U(u))$. When the group action is clear from the context, in particular when G is trivial, we will simply refer to this as an X-morphism.*

An X-morphism Ψ is said to be a monomorphism *if it is injective and if there exists $T \geq 0$ such that for all $u \in U$, $\mathrm{Supp}_U(u) \subseteq B_T(\mathrm{Supp}_V(\Psi(u)))$.*

An X-morphism Ψ is said to be an epimorphism *if it is surjective and there exists $M \geq 0$ such that for all $R \geq 0$ there exists $S \geq 0$ such that for all $v \in V$ if $\mathrm{Supp}_V(v) \subseteq B_R(x)$ then there exists $u \in \Psi^{-1}(v)$ such that $\|u\|_U \leq M\|v\|_V$ and $\mathrm{Supp}_U(u) \subseteq B_S(x)$.*

An X-morphism Ψ is said to be an isomorphism *if it is both an epimorphism and a monomorphism.*

We adopt the convention that the term morphism refers to an X-morphism when both the space X and the group G are clear from the context.

It is straightforward to show that an X-morphism $\Psi : \mathcal{U} \to \mathcal{V}$ induces a continuous linear map $\Psi_* : \mathcal{E}^{p,q}(X,\mathcal{U}) \to \mathcal{E}^{p,q}(X,\mathcal{V})$ commuting with both differentials. This extends to give maps on both completed bicomplexes, $\mathcal{E}^{p,q}_{\sim}(X,\mathcal{U}) \to \mathcal{E}^{p,q}_{\sim}(X,\mathcal{V})$.

Given a space X, and a group G acting by isometries on X, a short exact sequence of X-modules is a short exact sequence of Banach spaces

$$0 \to U \xrightarrow{\iota} V \xrightarrow{\pi} W \to 0$$

each with the structure of a G-equivariant X-module, and where ι is a monomorphism of X-modules and π is an epimorphism.

Example 6.2. *Consider the following short exact sequence.*

$$0 \to \ell^1_0(X) \xrightarrow{\iota} \ell^1(X) \xrightarrow{\pi} \mathbb{C} \to 0$$

For a G*-space* X *we regard these Banach spaces as* G*-modules in the natural way, where* G *is regarded as acting trivially on* $\mathbb{C}$*. The function spaces are equipped with their usual support functions* $\mathrm{Supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}}$ *and* $\mathbb{C}$ *is equipped with the trivial support function* $\mathrm{Supp}(\lambda) = \emptyset$ *for all* $\lambda \in \mathbb{C}$*. We will show that this is a short exact sequence of* X*-modules.*

The map ι *is the standard "forgetful" inclusion of* $\ell^1_0(X)$ *into* $\ell^1(X)$ *and is easily seen to be a monomorphism. The map* π *is the summation map and this is an epimorphism. To see this we argue as follows: since the support of any* $\lambda \in \mathbb{C}$ *is empty it lies within* R *of any point* $x \in X$*. We choose the scaled Dirac delta function* $\lambda\delta_x \in \ell^1(X)$ *which clearly maps to* λ*, has norm* $|\lambda|$ *and* $\mathrm{Supp}(\lambda\delta_x) = \{x\}$*, so putting* $M = 1$ *and* $S = 0$ *satisfies the conditions.*

Note that the constant function $\mathbf{1} \in \mathcal{E}^{0,-1}(X, \mathbb{C})$ taking the value 1 is a cocycle. Hence (applying I_Q, I_W respectively) it represents a class $[\mathbf{1}_Q] \in H^0_Q(X, \mathbb{C})$, and another class $[\mathbf{1}_W] \in H^0_Q(X, \mathbb{C})$.

As in the previous section, consider the case of a group G equipped with a proper left invariant metric. The map $\pi : \ell^1(G) \to \mathbb{C}$ induces a map $\pi_* : \mathcal{E}^{0,-1}_W(G, \ell^1(G)) \to \mathcal{E}^{0,-1}_W(G, \mathbb{C})$ and, as discussed above, an invariant mean on the group is an element μ in $\mathcal{E}^{0,-1}_W(G, \ell^1(G))$ such that $D\mu = 0$ and $\pi_*(\mu) = \mathbf{1}_W$.

This motivates the following definition.

Definition 6.3. *Let* X *be a metric space equipped with an isometric action of a group* G*. An* equivariant asymptotically invariant mean *for* X *is an element* μ *in* $\mathcal{E}^{0,-1}_W(X, \ell^1(X))$ *such that* $D\mu = 0$ *and* $\pi_*(\mu) = \mathbf{1}_W$*. In the special case when* G *is the trivial group we simply call this an* asymptotically invariant mean*.*

For a group G equipped with a proper left invariant metric an equivariant asymptotically invariant mean is just an invariant mean, however this does not hold in general and we make the following definition.

Definition 6.4. *Let* G *be a group equipped with a left invariant metric d (which we do not assume to be proper). We say that the pair* (G, d) *is* metrically amenable *if it admits an equivariant asymptotically invariant mean.*

We will consider this further in section **??** in our discussion of Guentner's theorem concerning box spaces, [**?**].

We conclude this section by establishing the existence of the long exact sequence in cohomology.

Proposition 6.5. *A short exact sequence of* X*-modules induces a short exact sequence of bicomplexes for* $\mathcal{E}, \mathcal{E}_Q$ *and* $\mathcal{E}_W$*. Hence, by the snake lemma, we obtain long exact sequences in cohomology for* $H^*_\sim(X, -)$*, for each decoration* $\sim = \mathcal{E}, Q$ *or* W*.*

Proof. Suppose we are given a short exact sequence of X-modules

$$0 \to \mathcal{U} \xrightarrow{\iota} \mathcal{V} \xrightarrow{\pi} \mathcal{W} \to 0.$$

We will show that the sequence

$$0 \to \mathcal{E}^{p,q}(X, \mathcal{U}) \xrightarrow{\iota_*} \mathcal{E}^{p,q}(X, \mathcal{V}) \xrightarrow{\pi_*} \mathcal{E}^{p,q}(X, \mathcal{W}) \to 0$$

is topologically exact, i.e. it is exact and the maps are open.

Injectivity and openness of ι_* follows directly from the corresponding properties of ι; as ι has closed range, it is open by the open mapping theorem.

Exactness at the middle term follows from the observation that if $\pi_*(\phi) = 0$ then $\phi = \iota \circ \phi'$ for some function $\phi' : X^{p+1} \times X^{q+1} \to U$, where ϕ' is uniquely defined by injectivity of ι. We need to verify that ϕ' is an element of $\mathcal{E}^{p,q}(X, \mathcal{U})$. Openness of ι yields the required norm estimates, whereas the support condition is satisfied because ι is a monomorphism, hence $\mathrm{Supp}_U(\phi') \subseteq B_T(\mathrm{Supp}_V(\iota \circ \phi')) = B_T(\mathrm{Supp}_V(\phi))$ for some $T \geq 0$.

Surjectivity of π_* follows from the definition of an epimorphism: Given $\phi \in \mathcal{E}^{p,q}(X, \mathcal{W})$, for each $R > 0$ there exists $S > 0$ such that $(\mathbf{x}, \mathbf{y}) \in \Delta_R \leq R$ implies that $\mathrm{Supp}_W(\phi(\mathbf{x}, \mathbf{y})) \subseteq B_S(x_i), B_S(y_j)$ for all i, j. Since π is an epimorphism, there exists $M, T > 0$ such that for each $(\mathbf{x}, \mathbf{y})$ there exists an element of V, which we denote $\phi'(\mathbf{x}, \mathbf{y})$ such that $\|\phi'(\mathbf{x}, \mathbf{y})\|_V \leq M\|\phi(\mathbf{x}, \mathbf{y})\|_W$ and $\mathrm{Supp}_V(\phi'(\mathbf{x}, \mathbf{y})) \subseteq B_T(x_i), B_T(y_j)$ for each i, j, so ϕ' is of controlled supports and has finite R-norms as required. These estimates for the R-norms also ensure that π_* is open.

Proposition **??** allows us to extend these maps to the completions $\mathcal{E}_Q$ and $\mathcal{E}_W$ to obtain short exact sequences for both the $\mathcal{E}_Q$ and $\mathcal{E}_W$ bicomplexes. It is now immediate from the snake lemma that for each decoration $\sim = \mathcal{E}, Q, W$ there is a connecting homomorphism D inducing a long exact sequence in cohomology:

$$0 \to H^0_\sim(X, \mathcal{U}) \to H^0_\sim(X, \mathcal{V}) \to H^0_\sim(X, \mathcal{W}) \xrightarrow{D} H^1_\sim(X, \mathcal{U}) \to H^1_\sim(X, \mathcal{V}) \to H^1_\sim(X, \mathcal{W}) \to \cdots$$

□

7. A COHOMOLOGICAL CHARACTERISATION OF PROPERTY A

As an application of the long exact sequence we give our first cohomological characterisation of Yu's property A in terms of the vanishing of the Johnson class in the controlled cohomology $H^1_\sim(X, \ell^1_0(X))$.

Let X be a metric space (as usual we have in the background a group acting by isometries on X, but our applications in this section will assume that the action is trivial). Recall the short exact sequence of X-modules introduced in Example **??**

$$0 \to \ell^1_0(X) \xrightarrow{\iota} \ell^1(X) \xrightarrow{\pi} \mathbb{C} \to 0.$$

As usual let $\mathbf{1} \in \mathcal{E}^{0,-1}(X, \mathbb{C})$ denote the constant function 1 on X.

Lemma 7.1. *Suppose the action of* G *on* X *is trivial. Then* X *has property* A *if and only if* $\mathcal{E}_Q^{0,-1}(X, \ell^1(X))$ *contains an element* ϕ *such that* $D\phi = 0$ *and* $\pi_*\phi = \mathbf{1}_Q$.

Proof. Any generalised Reiter sequence ϕ_n for X provides (as in Definition **??**) an element of $\mathcal{E}_Q$ with the required properties: the fact that $\phi_n(x)$ is a probability measure ensures that $\pi\phi_n(x) = 1$ for all x, n – that is $\pi_*\phi = I_Q\mathbf{1}$. The other hypotheses of Definition **??** are precisely the assertions that ϕ is of controlled supports and that $D\phi = 0$ in $\mathcal{E}_Q^{1,-1}(X, \ell^1(X))$.

Conversely, given an element $\phi \in \mathcal{E}_Q^{-1}(X, \ell^1(X))$ such that $D\phi = 0$ and $\pi_*\phi = I_Q\mathbf{1}$, represented by a sequence ϕ_n, we set $f_n(x)(z) = \frac{|\phi_n(x)(z)|}{\|\phi_n(x)\|_{\ell^1}}$. Since $\pi\phi_n(x) = 1$ for all x, n we have $\frac{1}{\|\phi_n(x)\|_{\ell^1}} \leq 1$. As an element of $\ell^1(X)$, $f_n(x)$ has the same supports as $\phi_n(x)$, in particular f_n is of controlled supports for all n. The verification that $\|f_n(x_1) - f_n(x_0)\|_{\ell^1}$ tends to 0 uniformly on $\{(x_0, x_1) \mid d(x_0, x_1) \leq R\}$ follows from the fact that $D\phi = 0$ and the estimate $\frac{1}{\|\phi_n(x)\|_{\ell^1}} \leq 1$. □

Now recall the long exact sequences for $H_\sim$, where $\sim = Q$ or W:

$$0 \to H^0_\sim(X, \ell^1_0(X)) \xrightarrow{\iota_*} H^0_\sim(X, \ell^1(X)) \xrightarrow{\pi_*} H^0_\sim(X, \mathbb{C}) \xrightarrow{D} H^1_\sim(X, \ell^1_0(X)) \xrightarrow{\iota_*} H^1_\sim(X, \ell^1(X)) \xrightarrow{\pi_*}$$

The connecting map D yields classes $D[\mathbf{1}_Q]$ in $H^1_Q(X, \ell^1_0(X))$, and $D[\mathbf{1}_W]$ in $H^1_W(X, \ell^1_0(X))$.

Now recall that the classical Johnson class on a group G is defined by the 1-cocycle $\mathcal{J}(g_0, g_1) = \delta_{g_1} - \delta_{g_0}$, where δ_g denotes the Dirac delta function which takes the value 1 at g and 0 elsewhere. By analogy we define the element $\mathcal{J}^{1,0} \in \mathcal{E}^{1,0}(X, \ell^1_0(X))$ by $\mathcal{J}^{1,0}\big((x_0, x_1), y\big) = \delta_{x_1} - \delta_{x_0}$. This element is a cocycle so by applying I_Q, I_W we obtain elements $[\mathcal{J}^{1,0}_Q] \in H^1_Q(X, \ell^1_0(X))$ and $[\mathcal{J}^{1,0}_W] \in H^1_W(X, \ell^1_0(X))$ which we refer to as the Johnson classes for X. We note that the class of $\mathcal{J}^{1,0}$ in $H_{\mathcal{E}}(X, \ell^1_0(X))$ is the image of $[\mathbf{1}]$ under the connecting homomorphism, since we may pull back the function $\mathbf{1}$ to the Dirac element $x \mapsto \delta_x$ and applying the coboundary map to this we obtain $\mathcal{J}^{1,0}$. By applying I_Q and I_W we observe that $[\mathcal{J}^{1,0}_Q] = D[\mathbf{1}_Q]$ and $[\mathcal{J}^{1,0}_W] = D[\mathbf{1}_W]$.

We are now ready to give a cohomological characterisation of the existence of an equivariant asymptotically invariant mean. In the case where the group action is trivial, we characterise property A both in terms of the existence of an asymptotically invariant mean and in cohomological terms.

Theorem 7.2. *Let* X *be a discrete metric space equipped with an isometric action of a group* G. *Then the following are equivalent:*

(1) $[\mathbf{1}_Q] \in \operatorname{Im}\pi_*$ *in* $H^0_Q(X, \mathbb{C})$.

(2) $[\mathcal{J}^{1,0}_Q] = D[\mathbf{1}_Q] = 0$ *in* $H^1_Q(X, \ell^1_0(X))$.

(3) $[\mathcal{J}^{1,0}_W] = D[\mathbf{1}_W] = 0$ *in* $H^1_W(X, \ell^1_0(X))$.

(4) $[\mathbf{1}_W] \in \operatorname{Im} \pi_*$ *in* $H^0_W(X, \mathbb{C})$.

(5) X *admits an equivariant asymptotically invariant mean.*

If the group G *acts trivially on* X *then these conditions are all equivalent to property A for the metric space* X.

Proof. Conditions (**??**) and (**??**) are equivalent by exactness of the long exact sequence in cohomology, while (**??**) is equivalent to (**??**) by Theorem **??**. Conditions (**??**) and (**??**) are equivalent by a further application of the long exact sequence (this time for the weak-* completion). The equivalence of (**??**) and (**??**) is immediate from the definition of asymptotically invariant mean.

To prove the final statement we note that (**??**) is equivalent to property A by Lemma **??**. □

8. Asymptotically invariant cohomology

We pause for a moment to recall the classical definition of bounded cohomology for a group. One first takes the homogeneous bar resolution wherein the k-dimensional cochains consist of all bounded functions from G^{k+1} to $\mathbb{C}$. This cochain complex is exact so has trivial cohomology. This is exhibited by taking a basepoint splitting which is induced by the map $G^k \to G^{k+1}$ given by inserting the basepoint as an additional (first) co-ordinate. Now one takes the G-invariant part of this complex, where G acts diagonally and $\mathbb{C}$ is equipped with the trivial action of G. Since the splitting is not equivariant the corresponding cochain complex is not necessarily exact. When the group G is amenable one can average the splitting over orbits using the invariant mean, and this produces an equivariant splitting which therefore kills the cohomology in dimensions greater than or equal to 1.

As outlined in Section **??** the bounded cochain complex may be regarded as the bottom row of an augmented complex obtained by taking the kernels of the vertical differentials in degree 0. In this section and the following we will carry out an analogous process for property A. In this section we will construct the asymptotically invariant cohomology of a space as the analogue of bounded cohomology. Replacing the classical (split) cochain complex by the first row of the $\mathcal{E}_\sim$ bicomplex, $(\mathcal{E}^{0,q}_\sim, d)$, (which is acyclic since $(\mathcal{E}^{0,q}, d)$ is acyclic by Proposition **??**) we take the kernels under the vertical differential D to produce a new cochain complex, which is the analogue of taking the G-invariant parts in group cohomology.

The splitting s of the horizontal differential d does not restrict to this cochain complex leaving room for interesting cohomology. In the following section we will show that if the space X has property A one can asymptotically average the splitting s to obtain a splitting of the asymptotically invariant complex. Hence we will deduce that if X has property A then the asymptotically invariant cohomology vanishes in all dimensions greater than or equal to 1.

Definition 8.1. *We say that an element* ϕ *of* $\mathcal{E}^{0,q}_Q$ *(respectively* $\mathcal{E}^{0,q}_W$*) is* asymptotically invariant *if* $D\phi = 0$ *in* $\mathcal{E}^{1,q}_Q$ *(respectively* $\mathcal{E}^{1,q}_W$*). Let* $\mathcal{E}^q_{QA}$, $\mathcal{E}^q_{WA}$*, denote the spaces of asymptotically invariant elements in* $\mathcal{E}^{0,q}_Q$ *and* $\mathcal{E}^{0,q}_W$ *respectively. We note as usual that this is defined for* $q \geq -1$.

For notational convenience when considering elements of $\mathcal{E}^{0,q}$ we will write $\phi(x,(y_0,\dots,y_q))$, suppressing the parentheses around the single x variable.

The term asymptotically invariant is motivated by the case of $\mathcal{E}_Q^{0,q}$. An element of $\mathcal{E}_Q^{0,q}$ is asymptotically invariant if it is represented by a sequence $\phi_n : X \times X^{q+1} \to V$ which is asymptotically invariant in the x variable the following sense. For all $R > 0$ the difference $\phi_n(x_1,\mathbf{y}) - \phi_n(x_0,\mathbf{y})$ tends to zero uniformly on $\Delta_R^2 \times X^{q+1}$.

We remark that it is essential that we first complete the complex $\mathcal{E}$ and then take the kernels of D, not the other way around. If we were to take the kernel of $D : \mathcal{E}^{0,q} \to \mathcal{E}^{1,q}$ we would get functions $\phi(x,(y_0,\dots,y_q))$ which are constant in the x variable, that is, we would have invariant rather than asymptotically invariant elements. The kernel of $D : \mathcal{E}_\sim^{0,q} \to \mathcal{E}_\sim^{1,q}$ will typically be much larger than the completion of these x-invariant functions.

We now make the following elementary observation.

Proposition 8.2. *The differential* d *maps* $\mathcal{E}_{QA}^q(X,\mathcal{V})$ *to* $\mathcal{E}_{QA}^{q+1}(X,\mathcal{V})$*, and maps* $\mathcal{E}_{WA}^q(X,\mathcal{V})$ *to* $\mathcal{E}_{WA}^{q+1}(X,\mathcal{V})$*. Hence* $(\mathcal{E}_{QA}^q(X,\mathcal{V}),d)$*,* $(\mathcal{E}_{WA}^q(X,\mathcal{V}),d)$ *are complexes.*

Proof. This is immediate from anti-commutativity of the differentials D, d. □

Recall that there is a splitting $s : \mathcal{E}^{0,q} \to \mathcal{E}^{0,q-1}$ extending to both generalised completions. To see that s does not necessarily map the asymptotically invariant subcomplex into itself consider the following example.

Example 8.3. *For a metric space* X *we define a Johnson element* $\mathcal{J}^{0,1} \in \mathcal{E}^{0,1}(X,\ell^1(X))$ *by* $\mathcal{J}^{0,1}(x,(y_0,y_1)) = \delta_{y_1} - \delta_{y_0}$*. Since* $\mathcal{J}^{0,1}$ *is independent of* x*,* $D\mathcal{J}^{0,1} = 0$*, so* $\mathcal{J}_Q^{0,1} = I_Q\mathcal{J}^{0,1}$ *lies in* $\mathcal{E}_{QA}^1$*, and* $\mathcal{J}_W^{0,1} = I_W\mathcal{J}^{0,1}$ *lies in* $\mathcal{E}_{WA}^1$*. However*

$$Ds\mathcal{J}^{0,1}((x_0,x_1),(y_0)) = s\mathcal{J}^{0,1}(x_1,(y_0)) - s\mathcal{J}^{0,1}(x_0,(y_0)) = (\delta_{y_0}-\delta_{x_1})-(\delta_{y_0}-\delta_{x_0}) = \delta_{x_0}-\delta_{x_1}$$

which has ℓ^1*-norm equal to 2 for all* $x_0 \neq x_1$*. Hence* $Ds\mathcal{J}_Q^{0,1} = I_Q Ds\mathcal{J}^{0,1} \neq 0$ *and* $Ds\mathcal{J}_W^{0,1} = I_W Ds\mathcal{J}^{0,1} \neq 0$*, so neither* $s\mathcal{J}_Q^{0,1}$ *nor* $s\mathcal{J}_W^{0,1}$ *is asymptotically invariant.*

We can now define the asymptotically invariant cohomology.

Definition 8.4. *For* $\sim$ *equal to either of the decorations* Q *or* W*, the* $\sim$*-asymptotically invariant cohomology of* X *with coefficients in the module* $\mathcal{V}$ *is the cohomology of the complex* $(\mathcal{E}_{\sim A}^*(X,\mathcal{V}),d)$*. It is denoted* $H_{\sim A}^*(X,\mathcal{V})$*. Where the completion used is clear from the context we will refer to this as simply* the asymptotically invariant cohomology of X.

Lemma 8.5. *The augmentation maps* $\mathcal{E}_{QA}^q(X,\mathcal{V}) \hookrightarrow \mathcal{E}_Q^{0,q}(X,\mathcal{V})$ *and* $\mathcal{E}_{WA}^q(X,\mathcal{V}) \hookrightarrow \mathcal{E}_W^{0,q}(X,\mathcal{V})$ *induce maps on cohomology* $H_{QA}^q(X,\mathcal{V}) \to H_Q^q(X,\mathcal{V})$ *and* $H_{WA}^q(X,\mathcal{V}) \to H_W^q(X,\mathcal{V})$*, which are isomorphisms for* $q = 0$*, and are injective for* $q = 1$*.*

Proof. Since D vanishes on $\mathcal{E}^q_{QA}(X, \mathcal{V})$, the differential on the asymptotically invariant complex is the restriction of the differential $D+d$ on the totalisation of the bicomplex, so the augmentation map induces a map $H^q_{QA}(X, \mathcal{V}) \to H^q_Q(X, \mathcal{V})$. In degree 0 every cocycle is non-trivial, and if $\phi \in \mathcal{E}^{0,0}_Q$ is a cocycle then $D\phi = 0$ so ϕ is asymptotically invariant whence the map is an isomorphism. In degree 1, if $\phi \in \mathcal{E}^{0,1}_Q(X, \mathcal{V})$ is a coboundary in the totalisation of the bicomplex then there is an element ψ of $\mathcal{E}^{0,0}_Q(X, \mathcal{V})$ such that $(D + d)\psi$ is $(0 \oplus \phi)$ in $\mathcal{E}^{1,0}_Q(X, \mathcal{V}) \oplus \mathcal{E}^{0,1}_Q(X, \mathcal{V})$. That is $D\psi = 0$, so ψ is an element of $\mathcal{E}^0_{QA}(X, \mathcal{V})$, and $d\psi = \phi$. Hence ϕ is also a coboundary in $\mathcal{E}^1_{QA}(X, \mathcal{V})$. Hence the inclusion of $\mathcal{E}^1_{QA}(X, \mathcal{V})$ into $\mathcal{E}^{0,1}_Q(X, \mathcal{V})$ gives an injection of cohomology.

The proof for $\mathcal{E}_W$ is identical. □

Now we restrict to the case where G is trivial, and $\mathcal{V}$ is $\ell^1_0(X)$. The Johnson element $\mathcal{J}^{0,1}(x, (y_0, y_1)) = \delta_{y_1} - \delta_{y_0}$ in $\mathcal{E}^{0,1}(X, \mathcal{V})$ gives classes $[\mathcal{J}^{0,1}_Q] \in H^1_{QA}((X, \mathcal{V}))$ and $[\mathcal{J}^{0,1}_W] \in H^1_{QA}((X, \mathcal{V}))$. Applying the augmentation map we obtain elements of $H^1_Q(X, \mathcal{V})$ and $H^1_W(X, \mathcal{V})$. As noted above $Ds(\mathcal{J}^{0,1})\big((x_0, x_1), y\big) = \delta_{x_0} - \delta_{x_1} = -\mathcal{J}^{1,0}\big((x_0, x_1), y\big)$, so $\mathcal{J}^{1,0}$ is cohomologous to $\mathcal{J}^{0,1}$ in the totalisation of $\mathcal{E}^{*,*}(X, \ell^1_0(X))$. From this it is immediate that we have $[\mathcal{J}^{0,1}_Q] = [\mathcal{J}^{1,0}_Q] = D[\mathbf{1}_Q]$ in $H^1_Q(X, \ell^1_0(X))$ and $[\mathcal{J}^{0,1}_W] = [\mathcal{J}^{1,0}_W] = D[\mathbf{1}_W]$ in $H^1_W(X, \ell^1_0(X))$.

We thus obtain the following theorem.

Theorem 8.6. *Let X be a metric space with trivial G action. Then the following are equivalent:*

(1) X *has property* A.

(2) $[\mathcal{J}^{0,1}_Q] = 0$ *in* $H^1_{QA}(X, \ell^1_0(X))$.

(3) $[\mathcal{J}^{0,1}_W] = 0$ *in* $H^1_{WA}(X, \ell^1_0(X))$.

Proof. By Lemma **??**, for $\sim$ denoting either Q or W, $[\mathcal{J}^{0,1}_\sim]$ is zero in $H^1_{\sim A}(X, \ell^1_0(X))$ if and only if it is zero in $H^1_\sim(X, \ell^1_0(X))$, and we have seen that its image is equal to $[\mathcal{J}^{1,0}_\sim]$. By Theorem **??**, this vanishes if and only if X has property A. □

9. Vanishing theorems

Throughout this section we will consider a metric space X with trivial group action.

We have seen that the map s does not in general split the coboundary map d in the complexes $\mathcal{E}^*_{QA}, \mathcal{E}^*_{WA}$, however if X has property A then we can use the generalised Reiter sequence in the case of the Q-completion, and the asymptotically invariant mean in the case of the W-completion, to asymptotically average $s\phi$. Having done so we will obtain a splitting for the asymptotically invariant complexes, demonstrating the vanishing of the cohomology.

We will make use of the following convolution operator.

Definition 9.1. *For* $f \in \mathcal{E}^{p,-1}(X, \ell^1(X))$ *and* $\theta \in \mathcal{E}^{0,q}(X, V)$*, define* $f * \theta$ *by*

$$(f * \theta)(\mathbf{x}, \mathbf{y}) = \sum_{z} f(\mathbf{x})(z)\theta(z, \mathbf{y}).$$

We remark that as θ lies in the bottom row of the bicomplex, its R-norm does not in fact depend on R, hence we suppress it from the notation. We make the following estimate:

$$\|f * \theta\|_R \leq \sup_{\mathbf{x}\in\Delta_R^{p+1}, \mathbf{y}\in X^{q+1}} \sum_{z\in X} |f(\mathbf{x})(z)| \|\theta(z, \mathbf{y})\|_V \leq \sup_{\mathbf{x}\in\Delta_R^{p+1}} \sum_{z} |f(\mathbf{x}, (z))| \|\theta\| = \|f\|_R \|\theta\|.$$

This estimate shows that for each f the map $\theta \mapsto f * \theta$ is continuous, and for each θ the map $f \mapsto f * \theta$ is continuous.

We note that $D(f * \phi)(\mathbf{x}, \mathbf{y}) = \sum_{z} \sum_{i} (-1)^i f(\widehat{\mathbf{x}_i})(z)\phi(z, \mathbf{y}) = \big((Df) * \phi\big)(\mathbf{x}, \mathbf{y})$, by exchanging the order of summation.

Similarly $d(f * \phi)(\mathbf{x}, \mathbf{y}) = (f * d\phi)(\mathbf{x}, \mathbf{y})$.

The convolution extends in an obvious way to the quotient completion. For $f \in \mathcal{E}_Q^{q,-1}(X, \ell^1(X))$, $\phi \in \mathcal{E}_Q^{0,q}(X, V)$ we define $f * \phi \in \mathcal{E}_Q^{p,q}(X, V)$ by $(f * \phi)_n = f_n * \phi_n$. We note that if either of the sequences f_n, ϕ_n tends to 0 as $n \to \infty$, then $(f * \phi)_n$ tends to 0 by the above norm estimate. Hence the convolution is a well defined map

$$\mathcal{E}_Q^{p,-1}(X, \ell^1(X)) \times \mathcal{E}_Q^{0,q}(X, V) \to \mathcal{E}_Q^{p,q}(X, V),$$

i.e. as an element of $\mathcal{E}_Q^{p,q}(X, V)$, the convolution $f * \phi$ does not depend on the choice of sequences representing f, ϕ.

Since the convolution is defined term-by-term in n, the identities $D(f * \phi) = (Df) * \phi$ and $d(f * \phi) = f * d\phi$ carry over to the quotient completion.

We recall that by Lemma **??** property A is equivalent to the existence of an element f of $\mathcal{E}_Q^{0,-1}(X, \ell^1(X))$ with $Df = 0$ and $\pi_*(f) = \mathbf{1}_Q$. Convolving with such an f allows us to average the splitting $s\phi$ to get an asymptotically invariant element. We use this idea to prove the following theorem.

Theorem 9.2. *If* X *is a metric space satisfying Yu's property A, then the asymptotically invariant cohomology* $H_{QA}^q(X, \mathcal{V})$ *is zero for every* $q \geq 1$ *and every* X*-module* $\mathcal{V}$*.*

Proof. Let $\phi \in \mathcal{E}_{QA}^q(X, \mathcal{V})$ with $q \geq 1$. The element ϕ is represented by a sequence ϕ_n in $\mathcal{E}^q(X, \mathcal{V})$ and $s\phi$ is represented by the sequence

$$s\phi_n(x, (y_0, \ldots, y_{q-1})) = \phi_n(x, (x, y_0, \ldots, y_{q-1})).$$

Since $D\phi = 0$, the sequence $D\phi_n$ tends to zero, that is for all $R > 0$, $\|D\phi_n\|_R \to 0$ as $n \to \infty$. By a diagonal argument, if S_n is a sequence tending to infinity sufficiently slowly, then $\|D\phi_n\|_{S_n} \to 0$ as $n \to \infty$. We choose a sequence S_n with this property.

Take a generalised Reiter sequence f in $\mathcal{E}_Q^{0,-1}(X,\ell^1(X))$ so that $Df=0$ and $\pi_*(f)=\mathbf{1}_Q$, and let f_n be a sequence representing f. If f'_n is a sequence representing f, then $f'_n(x)+(1-\pi(f_n(x)))\delta_x$ also represents f and has sum 1, so without loss of generality we may assume that $\pi(f_n(x))=1$ for all x,n.

By repeating the terms of the sequence f_n we can arrange that $\mathrm{Supp}(f_n(x))\subseteq B_{S_n}(x)$ for all x,n. Note that our choice of f therefore depends on S_n and hence on ϕ.

As a remark in passing, we note that taking such a 'supersequence' of f_n corresponds in some sense to taking a subsequence of ϕ_n. If we were working in the classical completion E_{cs}/E_0, then the subsequence would represent the same element of E_{cs}/E_0, however for $\mathcal{E}_Q$ this need not be true.

For each q' we now define $s_f:\mathcal{E}_Q^{0,q'}(X,V)\to\mathcal{E}_Q^{0,q'-1}(X,V)$ by $s_f\psi=f*s\psi$. We first note that for any ψ the element $s_f\psi$ is asymptotically invariant. This follows from asymptotic invariance of f, since $Ds_f\phi=D(f*s\phi)=(Df)*s\phi=0$. Hence in fact we have a map $s_f:\mathcal{E}_Q^{0,q'}(X,V)\to\mathcal{E}_{QA}^{q'-1}(X,V)$ which restricts to the asymptotically invariant complex.

We claim that for our given ϕ we have $(ds_f+s_fd)\phi=\phi$. We have $ds_f\phi=d(f*s\phi)=f*ds\phi$, while $s_fd\phi=f*sd\phi$ by definition. Hence $(ds_f+s_fd)\phi=f*(ds+sd)\phi=f*\phi$ since $ds+sd=1$. It thus remains to show that $f*\phi=\phi$. Notice that since $\sum\limits_{z\in X}f_n(x)(z)=1$ we have $\phi_n(x,\mathbf{y})=\sum\limits_{z\in X}f_n(x)(z)\phi_n(x,\mathbf{y})$, so we have

$$\begin{aligned}(f_n*\phi_n-\phi_n)(x,\mathbf{y})&=\sum_{z\in X}f_n(x)(z)(\phi_n(z,\mathbf{y})-\phi_n(x,\mathbf{y}))\\&=\sum_{z\in X}f_n(x)(z)D\phi_n((x,z),\mathbf{y}).\end{aligned}$$

Taking norms we have $\|f_n*\phi_n-\phi_n\|\leq\|f_n\|\|D\phi_n\|_{S_n}$, since if $d(x,z)>S_n$ then $f_n(x)(z)$ vanishes. We know that $\|D\phi_n\|_{S_n}\to0$ as $n\to\infty$, hence we conclude that $f*\phi-\phi=0$ in $\mathcal{E}_{QA}^q(X,\mathcal{V})$.

We have shown that for every element $\phi\in\mathcal{E}_{QA}^q(X,\mathcal{V})$ with $q\geq1$, we can construct maps $s_f:\mathcal{E}_{QA}^{q'}(X,\mathcal{V})\to\mathcal{E}_{QA}^{q'-1}(X,\mathcal{V})$ such that $(ds_f+s_fd)\phi=\phi$. (As noted above, f, and hence s_f, depend on the element ϕ.) It follows that if ϕ is a cocycle then $\phi=(ds_f+s_fd)\phi=ds_f\phi$, so every cocycle is a coboundary. Thus we deduce that $H_{QA}^q(X,\mathcal{V})=0$ for $q\geq1$. □

We will now prove a corresponding result for the weak-* completion. The role of the generalised Reiter sequence f_n in the previous argument will be replaced by an asymptotically invariant mean μ in $\mathcal{E}_W^{0,-1}(X,\ell^1(X))$.

We begin by extending the convolutions to the weak-* completions. First we define $f * \phi$ for $f \in \mathcal{E}^{p,-1}(X, \ell^1(X))$ and $\phi \in \mathcal{E}^{0,q}_W(X, V)$. This is defined via its pairing with an element α of $\mathcal{E}^{p,q}(X, V)^*$:

$$\langle f * \phi, \alpha\rangle = \langle \phi, \alpha_f\rangle, \text{ where } \langle \alpha_f, \theta\rangle = \langle \alpha, f * \theta\rangle, \text{ for all } \theta \in \mathcal{E}^{0,q}(X, V).$$

In other words the operator $\phi \mapsto f * \phi$ on $\mathcal{E}^{0,q}_W(X, V)$ is the double dual of the operator $\theta \mapsto f * \theta$ on $\mathcal{E}^{0,q}(X, V)$.

We have $|\langle \alpha_f, \theta\rangle| \leq \|\alpha\|_R \|f * \theta\|_R \leq \|\alpha\|_R \|f\|_R \|\theta\|$ for some R (depending on α). Hence for each α there exists R such that

$$|\langle f * \phi, \alpha\rangle| \leq \|\phi\|_R \|\alpha\|_R \|f\|_R$$

so $f * \phi$ is a continuous linear functional.

We now want to further extend the convolution to define $\eta * \phi$ in $\mathcal{E}^{p,q}_W(X, V)$, for $\eta \in \mathcal{E}^{p,-1}_W(X, \ell^1(X))$ and $\phi \in \mathcal{E}^{0,q}_W(X, V)$. The definition is motivated by the requirement that $(I_W f) * \phi = f * \phi$. Hence for α in $\mathcal{E}^{p,q}(X, V)^*$ we will require

$$\langle (I_W f) * \phi, \alpha\rangle = \langle f * \phi, \alpha\rangle.$$

For $\phi \in \mathcal{E}^{0,q}_W(X, V)$, $\alpha \in \mathcal{E}^{p,q}(X, V)^*$, define $\sigma_{\phi,\alpha} \in \mathcal{E}^{p,-1}(X, \ell^1(X))^*$ by

$$\langle \sigma_{\phi,\alpha}, f\rangle = \langle f * \phi, \alpha\rangle = \langle \phi, \alpha_f\rangle.$$

The above inequalities ensure that $\sigma_{\phi,\alpha}$ is a continuous linear functional.

We observe that $f * \phi$ is determined by the property that $\langle f * \phi, \alpha\rangle = \langle \sigma_{\phi,\alpha}, f\rangle = \langle I_W f, \sigma_{\phi,\alpha}\rangle$. We use this to give the general definition: For $\eta \in \mathcal{E}^{p,-1}_W(X, \ell^1(X))$ and $\phi \in \mathcal{E}^{0,q}_W(X, V)$, we define $\eta * \phi$ in $\mathcal{E}^{p,q}_W(X, V)$ by

$$\langle \eta * \phi, \alpha\rangle = \langle \eta, \sigma_{\phi,\alpha}\rangle$$

for all α in $\mathcal{E}^{p,q}(X, V)^*$.

Lemma 9.3. *For* $\eta \in \mathcal{E}^{p,-1}_W(X, \ell^1(X))$ *and* $\phi \in \mathcal{E}^{0,q}_W(X, \mathcal{V})$ *we have* $D(\eta * \phi) = (D\eta) * \phi$ *and* $d(\eta * \phi) = \eta * d\phi$.

Proof. The elements $D(\eta * \phi), d(\eta * \phi)$ are defined by their pairings with respectively α in $\mathcal{E}^{p+1,q}(X, \mathcal{V})^*$ and β in $\mathcal{E}^{p,q+1}(X, \mathcal{V})^*$. These are given by pairing η with respectively $\sigma_{\phi,D^*\alpha}$ and $\sigma_{\phi,d^*\beta}$.

Since for $f \in \mathcal{E}^{p,-1}(X, \ell^1(X))$ we have $\langle \sigma_{\phi,D^*\alpha}, f\rangle = \langle \phi, (D^*\alpha)_f\rangle$ and $\langle \sigma_{\phi,d^*\beta}, f\rangle = \langle \phi, (d^*\beta)_f\rangle$, we must determine $(D^*\alpha)_f$ and $(d^*\beta)_f$. Pairing these with an element θ in $\mathcal{E}^{0,q}(X, \mathcal{V})$ we have

$$\langle (D^*\alpha)_f, \theta\rangle = \langle \alpha, D(f * \theta)\rangle = \langle \alpha, (Df) * \theta\rangle, \text{ and } \langle (d^*\beta)_f, \theta\rangle = \langle \beta, d(f * \theta)\rangle = \langle \beta, f * d\theta\rangle.$$

Hence $(D^*\alpha)_f = \alpha_{Df}$ and $(d^*\beta)_f = d^*(\beta_f)$, so we have $\sigma_{\phi,D^*\alpha} = D^*\sigma_{\phi,\alpha}$ and $\sigma_{\phi,d^*\beta} = \sigma_{d\phi,\beta}$. It follows that $D(\eta * \phi) = (D\eta) * \phi$ and $d(\eta * \phi) = \eta * d\phi$ as required. ☐

Before proceeding with the proof of the vanishing theorem we first establish the following result.

Lemma 9.4. *If* $\eta \in \mathcal{E}_W^{0,-1}(X, \ell^1(X))$ *is in the image of* $\mathcal{E}_W^{0,-1}(X, \ell_0^1(X))$*, and* $\phi \in \mathcal{E}_W^{0,q}(X, V)$ *with* $D\phi = 0$ *then* $\eta * \phi = 0$.

Proof. The statement that $\eta * \phi = 0$, amounts to the assertion that $\langle \eta, \sigma_{\phi,\alpha} \rangle = 0$ for all α in $\mathcal{E}^{0,q}(X, V)^*$. Since the image of I_W is dense in $\mathcal{E}_W^{0,-1}(X, \ell_0^1(X))$ in the weak-* topology, it suffices to show that $\langle \sigma_{\phi,\alpha}, f \rangle = 0$ for all $f \in \mathcal{E}^{0,-1}(X, \ell_0^1(X))$. We note that

$$\langle \sigma_{\phi,\alpha}, f \rangle = \langle f * \phi, \alpha \rangle = \langle \phi, \alpha_f \rangle.$$

We will show that α_f is a 'boundary,' that is α_f is in the range of the map D^*. As $D\phi = 0$ it will follow that the pairing is trivial.

We define a boundary map $\partial : \ell^1(X \times X) \to \ell_0^1(X)$ by $(\partial H)(z_0) = \sum_{z_1 \in X} H(z, z_0) - H(z_0, z)$. Equivalently, we can write $\partial H = \sum_{z_0, z_1 \in X} H(z_0, z_1)(\delta_{z_1} - \delta_{z_0})$.

We note that ∂ is surjective: For $h \in \ell_0^1(X)$ and x in X, let $H(z_0, z_1) = h(z_1)$ if $z_0 = x, z_1 \neq x$ and let $H(z_0, z_1) = 0$ otherwise. Then $\partial H = h$. We note that $\|H\|_{\ell^1} \leq \|h\|_{\ell^1}$, and $\mathrm{Supp}(H) \subseteq \{x\} \times \mathrm{Supp}(h)$. For each x, let $F(x)$ be the lift of $f(x)$ constructed in this way, so that $\|F(x)\|_{\ell^1} \leq \|f(x)\|_{\ell^1}$ for all x, and as f is of controlled supports there exists R such that if $F(x)(z_0, z_1) \neq 0$ then $z_0, z_1 \in B_R(x)$.

Writing $(\partial F)(x) = \partial(F(x))$, for $\theta \in \mathcal{E}^{0,q}(X, V)$, we have

$$\langle \alpha_f, \theta \rangle = \langle \alpha, f * \theta \rangle = \langle \alpha, (\partial F) * \theta \rangle.$$

Now compute $(\partial F) * \theta$. We have

$$\begin{aligned} ((\partial F) * \theta)(x, \mathbf{y}) &= \sum_z \partial F(x)(z)\theta(z, \mathbf{y}) = \sum_{z, z_0, z_1} F(x)(z_0, z_1)(\delta_{z_1}(z) - \delta_{z_0}(z))\theta(z, \mathbf{y}) \\ &= \sum_{z_0, z_1} F(x)(z_0, z_1) D\theta((z_0, z_1), \mathbf{y}) \end{aligned}$$

We define $T_F : \mathcal{E}^{1,q}(X, V) \to \mathcal{E}^{0,q}(X, V)$ by $(T_F\zeta)(x, \mathbf{y}) = \sum_{z_0, z_1} F(x)(z_0, z_1)\zeta((z_0, z_1), \mathbf{y})$. As $F(x)(z_0, z_1) \neq 0$ implies z_0, z_1 lie in the ball $B_R(x)$, we have the estimate

$$\|T_F\zeta\| \leq \sup_{x \in X, \mathbf{y} \in X^{q+1}} \sum_{z_0, z_1 \in X} |F(x)(z_0, z_1)| \|\zeta((z_0, z_1), \mathbf{y})\|_V \leq \sup_{x \in X} \|F(x)\|_{\ell^1} \|\zeta\|_R \leq \|f\| \|\zeta\|_R.$$

hence T_F is continuous.

We conclude that

$$\langle \alpha_f, \theta \rangle = \langle \alpha, (\partial F) * \theta \rangle = \langle \alpha, T_F D\theta \rangle = \langle D^* T_F^* \alpha, \theta \rangle$$

for all θ, hence $\alpha_f = D^* T_F^* \alpha$, so that

$$\langle \phi, \alpha_f \rangle = \langle \phi, D^* T_F^* \alpha \rangle = \langle D\phi, T_F^* \alpha \rangle = 0.$$

This completes the proof. □

We now prove the vanishing theorem.

Theorem 9.5. *If X is a metric space satisfying Yu's property A, then the asymptotically invariant cohomology $H^q_{WA}(X,\mathcal{V})$ is zero for every $q \geq 1$ and every X-module $\mathcal{V}$.*

Specifically, if μ is an asymptotically invariant mean then $s_\mu\phi = \mu * s\phi$ defines a splitting of the asymptotically invariant complex.

Proof. By Theorem **??**, property A guarantees the existence of an asymptotically invariant mean μ, that is an element μ in $\mathcal{E}^{0,-1}_{WA}$ such that $\pi_*(\mu) = 0$.

We define $s_\mu : \mathcal{E}^{0,q}_{W}(X,\mathcal{V}) \to \mathcal{E}^{0,q-1}_{W}(X,\mathcal{V})$ by $s_\mu\phi = \mu * s\phi$. By Lemma **??** we have $Ds_\mu\phi = D(\mu * s\phi) = (D\mu) * s\phi$. Since μ is asymptotically invariant $D\mu = 0$, so $s_\mu\phi$ is also asymptotically invariant. Hence s_μ restricts to a map $s_\mu : \mathcal{E}^{0,q}_{WA}(X,\mathcal{V}) \to \mathcal{E}^{0,q-1}_{WA}(X,\mathcal{V})$. We must now verify that s_μ is a splitting. By Lemma **??**, and using the fact that $ds + sd = 1$ we have

$$(ds_\mu + s_\mu d)\phi = d(\mu * s\phi) + \mu * sd\phi = \mu * ds\phi + \mu * sd\phi = \mu * \phi.$$

It thus remains to show that $\mu * \phi = \phi$.

Let δ denote the map $X \to \ell^1(X), x \mapsto \delta_x$. We have $\pi_*(I_W\delta) = 1 = \pi_*(\mu)$, so for $\eta = \delta - \mu$ we have $\pi_*(\eta) = 0$. Hence η is in the image of $\mathcal{E}^{0,-1}_{W}(X,\ell^1_0(X))$. As $D\phi = 0$, it follows from Lemma **??** that $\eta * \phi = 0$. Thus $\mu * \phi = (I_W\delta) * \phi = \delta * \phi$. It is easy to see that convolution with δ yields the identity map on $\mathcal{E}^{0,q}(X,\mathcal{V})$, hence its double dual is again the identity map. Thus $\mu * \phi = \delta * \phi = \phi$ as required.

This completes the proof. □

Combining Theorems **??**, **??**, **??** we obtain the following.

Theorem 9.6. *Let X be a discrete metric space with trivial group action. Then the following are equivalent:*

(1) *X has property A.*

(2) *$H^q_{QA}(X,\mathcal{V}) = 0$ for all $q \geq 1$ and all X-modules $\mathcal{V}$.*

(3) *$[\mathcal{J}^{0,1}_Q] = 0$ in $H^1_{QA}(X,\ell^1_0(X))$.*

(4) *$H^q_{WA}(X,\mathcal{V}) = 0$ for all $q \geq 1$ and all X-modules $\mathcal{V}$.*

(5) *$[\mathcal{J}^{0,1}_W] = 0$ in $H^1_{WA}(X,\ell^1_0(X))$.*

10. NON-PROPERTY A SPACES

There are essentially three examples of spaces known not to have property A: expander sequences, box spaces of non-amenable groups and the union of finite cubes of all dimensions. The existing proofs are distinct in character; in this section we unify the latter two examples, while in a companion note [**?**] Ana Khukhro and Nick Wright apply cohomological methods in the case of expanders.

We first consider box spaces of non-amenable groups. This example, which is due to Guenter, was explored by Roe in [**?**].

Let G be a residually finite, finitely generated group and let $\Box G$ be the disjoint union of the finite quotients of G. The quotients are equipped with the word metric arising from the finite generating set of G and this is extended to a metric on the disjoint union, insisting that the distance between components tends to infinity. Then G is amenable if and only if $\Box G$ (the box space of G) has property A [**?**, Proposition 11.39].

Many examples of box spaces fail to embed uniformly in Hilbert space. In [**?**] Nowak gave an example of a space which does admit a uniform embedding in Hilbert space, but still does not possess property A: let X denote the disjoint union of the finite cubes $X_n = \{0,1\}^n$ each equipped with the ℓ^1 metric, and extend this to a proper metric on X insisting that the distance from X_n to its complement tends to ∞ with n. Then X does not have property A.

Here we will show that Nowak's example arises as an application of a new generalisation of Guentner's theorem. We identify the finite cubes as quotients of the group $G = \bigoplus_{i \in \mathbb{N}} \mathbb{Z}_2$, equipped with the word metric on the natural generators. We note that G is amenable since it is an ascending union of finite groups, so we might expect the box space to have property A. As we will see the fact that it does not have property A follows from the observation that G is not metrically amenable.

Lemma 10.1. *Let G be the group $\bigoplus_{i \in \mathbb{N}} \mathbb{Z}_2$ equipped with the word metric induced by its natural generating set. Then G is not metrically amenable.*

Proof. Suppose that $\phi \in \mathcal{E}_Q^{0,-1}(G, \ell^1(G))$ such that $\pi_*(\phi) = \mathbf{1_Q}$. We will show that $D\phi \neq 0$. For each n and for each ϵ there is a finite subset F such that $\|\phi_n(e)|_{F^c}\| < \epsilon$ so that $\|\phi_n(e)|_F\| \geq 1-\epsilon$. Since F is finite may now choose a generator s of G such that $sF \cap F = \emptyset$. Then by equivariance $\phi_n(s) = s\phi_n(e)$ so $\|\phi_n(s)|_{sF^c}\| < \epsilon$ and $\|\phi_n(s)|_{sF}\| \geq 1-\epsilon$. It follows that

$$
\begin{aligned}
&\|\phi_n(s) - \phi_n(e)\| = \sum_{g \in G} |\phi_n(s)(g) - \phi_n(e)(g)| \\
&= \sum_{g \in F} |\phi_n(e)(g) - \phi_n(s)(g)| + \sum_{g \in sF} |\phi_n(e)(g) - \phi_n(s)(g)| + \sum_{g \notin F \cup sF} |\phi_n(e)(g) - \phi_n(s)(g)| \\
&\geq 1 - 2\epsilon + 1 - 2\epsilon + 0 = 2 - 4\epsilon.
\end{aligned}
$$

Hence $\|D\phi_n\|_{R=1} \geq 2$ for all n, and $D\phi \neq 0$. It follows that $[\mathbf{1}_Q] \notin \operatorname{Im} \pi_*$ in $H_Q^0(G, \mathbb{C})$ so by Theorem **??**, G does not admit an equivariant asymptotically invariant mean. $\Box$

Let G be a countable residually finite group equipped with a (not necessarily proper) left invariant metric d valued in $\mathbb{Z}$. Let $X_\lambda = G/N_\lambda, \lambda \in \Lambda$ be a family of finite quotients of G, such that for any $\lambda, \mu \in \Lambda$, there exists $\nu \in \Lambda$ such that $N_\nu \leq N_\mu \cap N_\lambda$. We regard Λ as a directed system by defining $\lambda < \mu$ when $N_\lambda \geq N_\mu$. We equip each of the quotients X_λ with the quotient metric d_λ

and extend this to a metric on $\coprod_{\lambda\in\Lambda} X_\lambda$ in the usual way, requiring that for each positive integer K, the set of all pairs (λ, λ') for which $\lambda \neq \lambda'$ and the distance from X_λ to $X_{\lambda'}$ is less than K, is finite. The disjoint union $\coprod_{\lambda\in\Lambda} X_\lambda$ is said to be a *box space for* G if the intersection of the kernels N_λ is trivial.

Now for any X_λ in the system we denote the natural map $G \to X_\lambda$ by q_λ and the image of an element g under q_λ by g_λ. Similarly if $\lambda < \mu$ there is a natural map $q_{\lambda\mu} : X_\mu \to X_\lambda$ and for any $x \in X_\mu$ we denote $q_{\lambda\mu}(x) = x_\lambda$. Note that if $\lambda < \mu < \nu$ and $x \in X\nu$ then $(x_\mu)_\lambda = x_\lambda$, and similarly, for $g \in G$, $(g_\mu)_\lambda = g_\lambda$.

Theorem 10.2. *Let* G *be a countable residually finite group equipped with a left invariant metric* d*, and let* $X = \coprod_{\lambda\in\Lambda} X_\lambda$ *be a countable box space for* (G, d) *with the following properties:*

(1) *For every* g *in* G *there exists* $\lambda \in \Lambda$ *such that* $d(e, g) = d(e_\lambda, g_\lambda)$.

(2) *For each* $\lambda \in \Lambda$ *and for each* $x \in X_\lambda$ *there are finitely many elements* $g \in G$ *such that* $d(e_\lambda, x) = d(e, g)$ *and* $g_\lambda = x$.

Then G *is metrically amenable if and only if* X *has property A.*

If G is a finitely generated residually finite group then we may equip it with a proper left invariant metric and any box space of G will then satisfy conditions (1) and (2) above so we recover Guentner's theorem. The disjoint union of the finite quotients $X = \coprod_n \bigoplus_{i=1}^{n} \mathbb{Z}_2$ is a box space of the group $G = \bigoplus_{i\in\mathbb{N}} \mathbb{Z}_2$ which also satisfies conditions (1) and (2). (The second condition follows from the fact that while the metric is not proper, given any element $g \in G$ there are only finitely many geodesics in G from e to g.) Since X is a union of finite cubes of all dimensions we recover Nowak's theorem since we have noted that G is not metrically amenable.

Proof of Theorem **??**. Our strategy is to establish a correspondence between asymptotically invariant means for X and equivariant asymptotically invariant means for G by constructing maps $\mathcal{E}^{0,-1}(G, \ell^1(G)) \to \mathcal{E}^{0,-1}(X, \ell^1(X))$ and $\mathcal{E}^{0,-1}(X, \ell^1(X)) \to \mathcal{E}^{0,-1}(G, \ell^1(G))$ which are compatible with the differential D and the summation map π_*.

The first of these is the map $q_* : \mathcal{E}^{0,-1}(G, \ell^1(G)) \to \mathcal{E}^{0,-1}(X, \ell^1(X))$ defined by

$$q_*\phi(x) = x(q_\lambda)_*\phi(e)$$

where $x \in X_\lambda$ and $(q_\lambda)_*$ is the push-forward map from $\ell^1(G)$ to $\ell^1(X_\lambda)$.

For a $\theta \in \mathcal{E}^{1,-1}(X, \ell^1(X))$ define

$$\|\theta\|_R^{\mathrm{loc}} = \sup \|\theta(x_0, x_1)\|$$

where the supremum is taken over all $(x_0, x_1) \in \coprod_\lambda X_\lambda^2$ such that $d(x_0, x_1) \le R$.

Note that for any $\phi \in \mathcal{E}^{0,-1}(G, \ell^1(G))$ and for any pre-images $g_0, g_1 \in G$ of $x_0, x_1 \in X_\lambda$,

$Dq_*\phi(x_0, x_1) = x_1(q_\lambda)_*\phi(e) - x_0(q_\lambda)_*\phi(e) = (q_\lambda)_*\phi(g_1) - (q_\lambda)_*\phi(g_0) = (q_\lambda)_*D\phi(g_0, g_1).$

Since there exist pre-images with $d(g_0, g_1) = d(x_0, x_1)$ we deduce that $\|Dq_*\phi\|_R^{loc} \leq \|D\phi\|_R$. In fact we have equality: for each g_0, g_1 in G, the norm $\|(q_\lambda)_* D\phi(g_0, g_1)\|$ converges to $\|D\phi(g_0, g_1)\|$ (by residual finiteness). By the same argument $\|q_*\phi\| = \|\phi\|$ for all ϕ, hence q_* is injective.

Note also that $\pi_*(q_*\phi)$ is the constant function $\pi_*(\phi)$.

Suppose now that G is metrically amenable. We will work for the moment with the Q-completion. By Theorem **??** there is an element $\phi = (\phi_n) \in \mathcal{E}_Q^{0,-1}(G, \ell^1(G))$ such that $D\phi = 0$ and $\pi_*(\phi) = \mathbf{1_Q}$. But then $\pi_*(q_*\phi) = \mathbf{1_Q}$ and $\|Dq_*\phi_n\|_R^{loc} \to 0$ as $n \to \infty$ for all R. For each $\lambda \in \Lambda$ let $n_\lambda = d(X_\lambda, X_\lambda^c)$, and for each n define

$$\psi_n(x) = \begin{cases} q_*\phi_n(x) & \text{for } x \in X_\lambda, n < n_\lambda \\ \delta_{e_0} & \text{for } x \in X_\lambda, n \geq n_\lambda \end{cases}$$

We wish to show that for each R, $\|D\psi_n\|_R \to 0$ as $n \to \infty$. Choose $n > R$. For $x_0 \in X_{\lambda_0}$ and $x_1 \in X_{\lambda_1}$ consider $\|D\psi(x_0, x_1)\|$. If both of $n_{\lambda_0}, n_{\lambda_1}$ are less than or equal to n then $D\psi_n(x_0, x_1) = 0$, so it suffices to consider the case when at least one of the values $n_{\lambda_0}, n_{\lambda_1}$ is greater than n. Since $d(x_0, x_1) < R < n < n_{\lambda_i}$ and $d(X_i, X_i^c)) > n_{\lambda_i}$ we conclude that $X_{\lambda_0} = X_{\lambda_1}$ and so $\psi_n(x_i) = q_*\phi_n(x_i)$ for each i. It follows that $\|D\psi_n(x_0, x_1)\| = \|Dq_*\phi_n(x_0, x_1)\| \leq \|Dq_*\phi_n(x_0, x_1)\|^{loc}$. Hence $\|D\psi_n\|_R \leq \|Dq_*\phi_n\|_R^{loc}$, which converges to 0 as required. Hence the sequence $\psi = (\psi_n)$ provides an element of $\mathcal{E}_Q^{0,-1}(X, \ell^1(X))$ with $\pi_*\psi = \mathbf{1_Q}$ and $D\psi = 0$, and X has property A as required.

Now suppose that X has property A. We will show how to transfer an asymptotically invariant mean for X to an equivariant asymptotically invariant mean for G. For this part of the argument we will use the weak-* completion. We note that if $\psi \in \mathcal{E}^{0,-1}(X, \ell^1(X))$ is in the image of q_* then:

(i) ψ is equivariant,

(ii) for $x \in X_\lambda$, $\psi(x)$ is supported in X_λ,

(iii) For $\lambda < \mu$ we have $\psi(e_\lambda) = (q_{\lambda\mu})_*\psi(e_\mu)$.

Let $\mathcal{F}$ denote the subspace of $\mathcal{E}^{0,-1}(X, \ell^1(X))$ of cochains with these properties.

We will first prove that the image of q_* is $\mathcal{F}$. Suppose that $\psi \in \mathcal{F}$ and for each $x \in X$, $\psi(x)$ is supported in the ball of radius R about x. For $g \in G$ let

$$\phi_R(e)(g) = \begin{cases} \lim_\mu \psi(e_\mu)(g_\mu), & d(e, g) = R \\ 0, & \text{otherwise.} \end{cases}$$

We extend equivariantly. We will show that the limit exists, that $\phi_R \in \mathcal{E}^{0,-1}(G, \ell^1(G))$, and $(q_*\phi_R)(e_\lambda)(y) = \psi(e_\lambda)(y)$ for $y \in X_\lambda$ with $d(e_\lambda, y) = R$.

To show that the limit exists, choose λ such that $d(e_\lambda, g_\lambda) = R$, which is possible by hypothesis (**??**). By hypothesis (**??**) the set $\mathcal{S}$ of $k \in G$ such that $d(e, k) = R$ and $k_\lambda = g_\lambda$ is finite. If μ is sufficiently large, by residual finiteness of G, q_μ is injective on $\mathcal{S}$. Now for any $\nu > \mu$ we have $\psi(e_\mu) = (q_{\mu\nu})_*(\psi(e_\nu))$ so

$$\psi(e_\mu)(g_\mu) = \sum_{\substack{y \in X_\nu \\ y_\mu = g_\mu}} \psi(e_\nu)(y).$$

As $y_\mu = g_\mu$, $d(e_\nu, y) \geq d(e_\mu, g_\mu) \geq d(e_\lambda, g_\lambda) = R$, hence the only non-zero terms are for $d(e_\nu, y) = R$. We have $y = k_\nu$ for some $k \in G$ with $d(e, k) = R$. Then $k_\lambda = y_\lambda = g_\lambda$ so $k \in \mathcal{S}$ and $k_\mu = y_\mu = g_\mu$, so $k = g$ as q_μ is injective on $\mathcal{S}$. Hence there is only one term in the sum and we have $\psi(e_\mu)(g_\mu) = \psi(e_\nu)(g_\nu)$. The net is therefore ultimately constant, so it converges.

We now show that $\phi_R(e)$ is in $\ell^1(G)$. Let F be a finite subset of G. Then

$$\sum_{g \in F} |\phi_R(e)(g)| = \lim_\mu \sum_{g \in F} |\psi(e_\mu)(g_\mu)|.$$

Choosing μ sufficiently large that q_μ is injective on F we see that $\sum\limits_{g \in F} |\phi_R(e)(g)| \leq \|\psi(e_\mu)\|_{\ell^1} \leq \|\psi\|$. As this holds for all finite F we deduce that $\phi_R(e)$ is in $\ell^1(G)$, and $\|\phi_R(e)\|_{\ell^1} \leq \|\psi\|$. So $\phi_R \in \mathcal{E}^{0,-1}(G, \ell^1(G))$ as required.

For $y \in X_\lambda$ with $d(e_\lambda, y) = R$ we have

$$(q_*\phi_R)(e_\lambda)(y) = \sum_{g \in G, g_\lambda = y} \phi_R(e)(g) = \sum_{g \in \mathcal{T}} \phi_R(e)(g)$$

where $\mathcal{T}$ is the set of all $g \in G$ such that $d(e, g) = R$ and $g_\lambda = y$. As $\mathcal{T}$ is finite by hypothesis (**??**), we can choose μ sufficiently large that $\phi_R(e)(g) = \psi(e_\mu)(g_\mu)$ for all $g \in \mathcal{T}$ and so that q_μ is injective on $\mathcal{T}$.

We note that $q_\mu(\mathcal{T})$ is the set of $z \in X_\mu$ such that $d(e_\mu, z) = R$ and $z_\lambda = y$. Since $\psi(e_\mu)(z)$ vanishes if $d(e_\mu, z) > R$ we have

$$(q_*\phi_R)(e_\lambda)(y) = \sum_{g \in \mathcal{T}} \psi(e_\mu)(g_\mu) = \sum_{\substack{z \in X_\mu \\ z_\lambda = y}} \psi(e_\mu)(z) = \psi(e_\lambda)(y)$$

as required.

It now follows that if $R > 0$ then $\psi - q_*\phi_R$ is supported on the ball of radius $R - 1$, while if $R = 0$ then $\psi = q_*\phi_R$. Hence, by induction on R every $\psi \in \mathcal{F}$ is in the image of q_*.

We now establish the following claim:

Claim. *There exists a retraction r from $\mathcal{E}^{0,-1}(X, \ell^1(X))$ onto $\mathcal{F}$, such that for $\theta \in \mathcal{E}^{0,-1}(X, \ell^1(X))$, if $\pi_*(\theta) = \mathbf{1}$ then $\pi_*(r\theta) = \mathbf{1}$, and $\|Dr\theta\|_R^{\mathrm{loc}} \leq \|D\theta\|_R$.*

To prove this we will need to extract limits from bounded nets indexed by Λ; to do so we choose an ultrafilter ω extending the natural filter on the directed system Λ.

Given $\theta \in \mathcal{E}^{0,-1}(X, \ell^1(X))$ we set $\overline{\theta}(e_\lambda)(y) = \frac{1}{|X_\lambda|}\sum_{z \in X_\lambda} \theta(z)(zy)$, for each $y \in X_\lambda$.

We now set $\psi(e_\lambda)(y) = \lim_{\mu \in \omega} \sum_{\substack{z \in X_\mu \\ z_\lambda = y}} \overline{\theta}(e_\mu)(z)$, for each $y \in X_\lambda$ and set it to 0 for all other $y \in X$.

We extend this equivariantly to a map from X to $\ell^1(X)$. By definition this satisfies conditions (i) and (ii). It also satisfies condition (iii) as

$$\begin{aligned}\sum_{\substack{z \in X_\mu \\ z_\lambda = y}} \psi(e_\mu)(z) &= \sum_{\substack{z \in X_\mu \\ z_\lambda = y}} \lim_{\nu \in \omega} \sum_{\substack{w \in X_\nu \\ w_\mu = z}} \overline{\theta}(e_\nu)(w) \\ &= \lim_{\nu \in \omega} \sum_{\substack{z \in X_\mu \\ z_\lambda = y}} \sum_{\substack{w \in X_\nu \\ w_\mu = z}} \overline{\theta}(e_\nu)(w) \\ &= \lim_{\nu \in \omega} \sum_{\substack{w \in X_\nu \\ w_\lambda = y}} \overline{\theta}(e_\nu)(w) = \psi(e_\lambda)(y).\end{aligned}$$

It is straightforward to verify that ψ is bounded and of controlled supports, so we obtain a map $r : \mathcal{E}^{0,-1}(X, \ell^1(X)) \to \mathcal{F}$ as required. If $\theta \in F$ then $\overline{\theta} = \theta$ by equivariance and $\psi = \overline{\theta}$ by condition (iii) so the map r is a retraction onto $\mathcal{F}$.

We now compute $\pi_*(\psi)$.

$$\begin{aligned}\pi_*(\psi)(e_\lambda) &= \sum_{y \in X_\lambda} \lim_{\mu \in \omega} \sum_{\substack{z \in X_\mu \\ z_\lambda = y}} \overline{\theta}(e_\mu)(z) = \lim_{\mu \in \omega} \sum_{y \in X_\lambda} \sum_{\substack{z \in X_\mu \\ z_\lambda = y}} \overline{\theta}(e_\mu)(z) \\ &= \lim_{\mu \in \omega} \sum_{z \in X_\mu} \overline{\theta}(e_\mu)(z) = \lim_{\mu \in \omega} \sum_{z \in X_\mu} \frac{1}{|X_\mu|} \sum_{w \in X_\mu} \theta(w)(wz) \\ &= \lim_{\mu \in \omega} \frac{1}{|X_\mu|} \sum_{w \in X_\mu} \sum_{z \in X_\mu} \theta(w)(wz) = \lim_{\mu \in \omega} \frac{1}{|X_\mu|} \sum_{w \in X_\mu} \pi_*(\theta)(w).\end{aligned}$$

In particular $\pi_*(\psi)$ is constant, and if $\pi_*(\theta)$ is the constant function $\mathbf{1}$ then $\pi_*(\psi) = \mathbf{1}$ as well.

We now consider the norm $\|D\psi\|_R^{\mathrm{loc}}$.

By equivariance it suffices to consider $\|\psi(x) - \psi(e_\lambda)\|_{\ell^1}$ where $x \in X_\lambda$ and $d(e_\lambda, x) \leq R$. Pick $g \in G$ such that $d(e, g) = d(e_\lambda, x)$ and $g_\lambda = x$.

$$
\begin{aligned}
\|\psi(x)-\psi(e_\lambda)\|_{\ell^1} &= \sum_{y\in X_\lambda} |\psi(x)(y)-\psi(e_\lambda)(y)| \\
&= \lim_{\mu} \sum_{y\in X_\lambda} \Big| \sum_{\substack{z\in X_\mu \\ z_\lambda = y}} \overline{\theta}(g_\mu)(y) - \overline{\theta}(e_\mu)(z) \Big| \\
&\le \lim_{\mu} \sum_{y\in X_\lambda} \sum_{\substack{z\in X_\mu \\ z_\lambda = y}} \Big| \overline{\theta}(g_\mu)(y) - \overline{\theta}(e_\mu)(z) \Big| \\
&= \lim_{\mu} \sum_{z\in X_\mu} \Big| \overline{\theta}(g_\mu)(y) - \overline{\theta}(e_\mu)(z) \Big| \\
&\le \|D\overline{\theta}\|_R^{\mathrm{loc}}.
\end{aligned}
$$

It is easy to see that $\|D\overline{\theta}\|_R^{\mathrm{loc}} \le \|D\theta\|_R$ since taking the average is a norm-decreasing operation. Hence $\|\psi\|_R^{\mathrm{loc}} \le \|D\theta\|_R$ as required.

We complete the proof of Theorem **??** as follows.

Having noted that q_* is an isometric bijection $\mathcal{E}^{0,-1}(G, \ell^1(G)) \to \mathcal{F}$ we obtain a bounded map $q_*^{-1}r : \mathcal{E}^{0,-1}(X, \ell^1(X)) \to \mathcal{E}^{0,-1}(G, \ell^1(G))$. This extends to a map $\mathcal{E}_W^{0,-1}(X, \ell^1(X)) \to \mathcal{E}_W^{0,-1}(G, \ell^1(G))$ and for any invariant mean $\mu \in \mathcal{E}_W^{0,-1}(X, \ell^1(X))$ we obtain an element $\mu' \in \mathcal{E}_W^{0,-1}(G, \ell^1(G))$ which we will show is an equivariant, asymptotically invariant mean for G. This will establish that G is metrically amenable. Since $\pi_*(\mu) = \mathbf{1_W}$, $\pi_*(\mu') = \mathbf{1_W}$, so it only remains to show that $D\mu' = 0$.

To do this we will use $q_*^{-1}r$ to induce a map from $D\mathcal{E}^{0,-1}(X, \ell^1(X))$ to $D\mathcal{E}^{0,-1}(G, \ell^1(G))$, which, abusing notation, we will also denote by $q_*^{-1}r$. Define $q_*^{-1}r(D\theta) = D(q_*^{-1}r(\theta))$, noting that D is injective on $\mathcal{E}^{0,-1}(X, \ell^1(X))$ so this is well defined. It is continuous since for $\theta \in \mathcal{E}^{0,-1}(X, \ell^1(X))$ and $\phi = q_*^{-1}r(\theta)$ we have

$$
\|q_*^{-1}r(D\theta)\|_R = \|D\phi\|_R = \|Dq_*\phi\|_R^{\mathrm{loc}} = \|Dr\theta\|_R^{\mathrm{loc}} \le \|D\theta\|_R.
$$

Now the weak-* completion of $D\mathcal{E}^{0,-1}(X, \ell^1(X))$ injects into the weak-* completion of $\mathcal{E}^{1,-1}(X, \ell^1(X))$ so by continuity $q_*^{-1}r$ extends and in particular $D\mu' = q_*^{-1}rD\mu = 0$. Hence an asymptotically invariant mean for X maps to an equivariant asymptotically invariant mean for G. □

REFERENCES

[1] J. Block and S. Weinberger, Aperiodic tilings, positive scalar curvature and amenability of spaces. J. Amer. Math. Soc. 5 (1992), no. 4, 907–918.

[2] J. Brodzki, G. A. Niblo, P. W. Nowak and N.J. Wright, *Amenable actions, invariant means and bounded cohomology*, arXiv:1004.0295, 2010.

[3] J. Brodzki, G. A. Niblo and N. J. Wright, *Pairings, Duality, Amenability, and Bounded Cohomology*, arXiv:1003.2584, 2010.

[4] R. G. Douglas and P. W. Nowak, *Invariant Expectations and Vanishing of Bounded Cohomology for Exact Groups*, arXiv:1001.0718v3, 2010.
[5] N. Higson and J. Roe, *Amenable group actions and the Novikov conjecture*, J. Reine Angew. Math. 519 (2000), 143–153.
[6] B. E. Johnson, *Cohomology of Banach Algebras*, Memoirs of the AMS Number 127, 1972, AMS, Providence, Rhode Island.
[7] A. Khukhro and N. J. Wright, *A cohomological proof that expanders do not have property A*. In preparation, 2010.
[8] P. W. Nowak, *Coarsely embeddable metric spaces without Property A*, J. Funct. Anal. 252 (2007), no. 1, 126–136.
[9] N. Ozawa, *Amenable actions and exactness for discrete groups*, C. R. Acad. Sci. Paris Sér. I Math. 330 (2000), no. 8, 691–695.
[10] H. Reiter, *Classical harmonic analysis and locally compact groups*. Clarendon Press, Oxford, 1968.
[11] J. Roe, *Lectures on Coarse Geometry*. University Lecture Series Volume 31, American Mathematical Society, 2003.
[12] J. von-Neumann, *Zur allgemeinen Theorie des Maßes*, Fund. Math. 13 (1929), 73–111.
[13] G. Yu, The Coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space, Inventiones Math. **139** (2000), 201–240.

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