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Do coalitions matter in designing institutions? ☆

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Abstract

In this paper, we re-examine the classical questions of implementation theory under complete information in a setting where coalitions are fundamental behavioral units, and the outcomes of their interactions are predicted by applying the solution concept of the core. The planner's exercise includes designing a code of rights that specifies the collection of coalitions having the right to block one outcome by moving to another. A code of individual rights is a code of rights in which only unit coalitions may have blocking powers. We provide the necessary and sufficient conditions for implementation (under core equilibria) by codes of rights, as well as by codes of individual rights. We also show that these two modes of implementation are not equivalent. The results are robust and extend to alternative notions of core, such as an externally stable core. Therefore, coalitions are shown to bring value added to institutional design. The characterization results address the limitations that restrict the relevance of the existing implementation theory.

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1. Introduction

The challenge of implementation lies in designing a mechanism (i.e., game form) in which the equilibrium behavior of agents always coincides with the recommendations given by a *social choice rule* (SCR). If such a mechanism exists, the SCR is said to be implementable.

As such, the key question is how to design an implementing mechanism so that its outcomes are predicted through the application of game theoretic concepts. Most early studies on implementation focused on noncooperative solution concepts, such as the Nash equilibrium and its refinements. However, one of the difficulties with this approach is that canonical mechanisms are typically complex and difficult to explain in natural terms, as they rely on tail-chasing constructions, such as the integer game.

As demonstrated in the seminal paper of Koray and Yildiz (2018) [henceforth KY], an alternative to the noncooperative approach is to allow groups of agents to coordinate their behaviors in a mutually beneficial way. To move away from noncooperative modeling, the details of coalition formation are not modeled. Then, coalitions—not individuals—become basic decision-making units. Here, the role of the solution concept is to explain why, when, and which coalition forms and what it can achieve.

By using the notion of a *rights structure*, introduced by Sertel (2001), KY study implementation problems by rights structures.[1] A rights structure Γ consists of a state space S, an outcome function h that associates every state to an outcome, and a code of rights γ. A code of rights specifies, for each pair of distinct states (s, t), a collection of coalitions $\gamma(s, t)$ effective in moving from s to t.

As a coalitional solution, KY adopt a version of the *core*, referred to as the Γ-*equilibrium*. A state s is an equilibrium state under a given rights structure and agents' preferences if no effective coalition can guarantee each of its members a utility level higher than the one they receive under s. Then, the implementation problem consists of designing a rights structure Γ, with the property that the equilibrium outcomes always coincide with the recommendations of the given SCR. If such a rights structure exists, the SCR is said to be *implementable by a rights structure*.

The implications of KY's approach are interesting. They show that a SCR that is implementable by any rights structure is implementable, in particular, by a rights structure that only uses singleton coalitions. A puzzling consequence of this result is that coalitions appear to have no value added to the implementation problem. For all purposes, it is sufficient to focus on individuals as actors alone. The question, then, is why should institutions be designed based on coalition formation, as is often the case? For example, under a typical democratic constitution, a bill can only be passed by consent from a majority of individuals.

To analyze this question, we focus on the simple and natural restriction on rights structures: the set of states is assumed to coincide with the set outcomes. This mode of implementation was dubbed by KY as *implementation by a code of rights*.[2] Our analysis complements KY, who characterize implementation by codes of rights by assuming universal domain of strict preferences. Assuming an unrestricted domain of preferences, our main results are as follows: we provide an alternative characterization of implementation via a code of rights (Theorem 2), we characterize

[1] McQuillin and Sugden (2011) have more recently proposed a similar notion named *game in transition function form* as a generalization of effectivity functions.

[2] There is a bijection between the effectiveness relationships proposed by Rosenthal (1972) and codes of rights. A code of rights is more flexible than the effectivity function, as it allows the strategic options of coalitions to depend on how the status quo outcome is reached (i.e., on the current state).

implementation by a code of individual rights (Theorem 4), and we show that restricting the code of rights to be individual-based reduces the set of implementable SCR (Corollary 1).[3]

Note that a code of rights captures the allocation of blocking powers in many real-life situations in a natural way, allowing us to formulate implementation problems in an everyday language as feasible options available to agents, or to groups of them, that only depend on the status quo outcome. Conversely, a mechanism that cannot be defined in this way must be conditioned on the history of the play. Many real life schemes lack this feature. For example, what is achievable to the agents or coalitions is often defined on the grounds of property rights or alike concepts. Marriage market is a case in point. Feasible changes in marriage relationships are dictated by who is married to who, and not by who has been married to who in the past. In political decision making, what a party can do depends on its popularity. In the house allocation problem, a coalition of individuals should be allowed to exchange their own houses. And so on.

We demonstrate that implementation by a code of rights is fundamentally dependent on non-singleton coalitions. That is, when the effective coalitions can only depend on the status quo outcome, mechanisms cannot rely on individual behavior alone, i.e., coalitions matter. To show this, we identify two necessary and sufficient conditions for implementability: one is the unanimity condition and the other one is what we call *strong monotonicity*, which is stronger than (Maskin) monotonicity (Maskin, 1999). When we restrict our attention to the full domain of linear orderings, our characterization result is equivalent to that provided by KY.

To understand the way our restriction on the state space affects implementable SCRs, we study implementation by codes of rights under which only singleton coalitions can induce new outcomes. We call this type of codes of rights *codes of individual rights*. Under this setting, we identify the necessary condition for implementability, which we call *singleton strong monotonicity*. This condition is also sufficient when combined with unanimity.

Singleton strong monotonicity implies strong monotonicity but, as we will demonstrate, they are *not* equivalent. Therefore, the key insight for the implementation by rights structures—that coalitions do not matter—does *not* extend to the implementation by codes of rights. The underlying reason for this observation is that implementation by an individual rights structure requires a significant amount of information concerning the preferences of the agent permitted to move at a particular state. When the state space is coarsened, the needed information may no longer be conveyed by the underlying state. Since the preferences of a coalition are less volatile than those of an individual agent, coalitions may no longer be usefully replaceable by individuals. An example is the *majority solution*, which is implementable by a majority coalition-based code of rights but not an individual-based one.

The conclusion that coalitions make a difference is robust and can be extended to the implementation by codes of rights for alternative core definitions. Indeed, we add to the notion of core as the requirement that blocking must be achievable through outcomes that are themselves in the core, meaning we also consider implementation by codes of rights of what is often referred to as an *externally stable core*. We call this type of implementation *externally stable implementation by*

[3] Our approach is different from Peleg and Winter's (2002), who use the notion of effectivity of Moulin and Peleg (1982) to appeal to the notion of implementation, where the game form not only implements an SCR under Nash equilibrium but also induces the same distribution of power as that of the implemented SCR (see also Peleg et al., 2005). Moreover, Andjiga and Moulen's (1988, 1989) analysis is a special case of ours, because there is a simple game in their model, rather than a code of rights that specifies coalitions have the right to block one outcome by moving to another.

codes of rights.[4] KY were the first to study externally stable implementation by rights structures under the assumption of universal domain of linear orderings (KY, 2018; Proposition 2).

The remainder of this paper is divided into four sections. Section 2 sets out the theoretical framework and outlines the basic model. Section 3 provides a novel characterization of the class of SCRs implementable via codes of rights, whereas Section 4 fully characterizes the class of SCRs that are implementable by codes of individual rights. Section 5 concludes the paper. Appendix includes proofs not in the main body.

2. Preliminaries

We consider a finite (nonempty) set of *agents*, denoted by $N = \{1, \cdots, n\}$, and a (nonempty) set of *outcomes*, denoted by Z. For every set A, the power set of A is denoted by $\mathcal{A}$ and $\mathcal{A}_0 \equiv \mathcal{A} - \{\varnothing\}$ is the set of all nonempty subsets of A. Each element K of $\mathcal{N}_0$ is called a *coalition*. A *preference ordering* R_i is a complete and transitive binary relation over Z. Each agent i ($\in N$) has a preference ordering R_i over Z. The *asymmetric* part P_i of R_i is defined by $x P_i y$ if and only if $x R_i y$ and not $y R_i x$, while the *symmetric* part I_i of R_i is defined by $x I_i y$ if and only if $x R_i y$ and $y R_i x$. A *preference profile* is thus an n-tuple of preference orderings $R \equiv (R_i)_{i \in N}$. The *preference domain*, denoted by $\mathcal{R}$, consists of the set of admissible preference profiles.

For R and K, we write $x R_K y$ for $x R_i y$ for all $i \in K$ and $x P_K y$ for $x P_i y$ for all $i \in K$.

The goal of the designer is to implement an SCR F, defined by $F : \mathcal{R} \to \mathcal{Z}_0$. We refer to $x \in F(R)$ as an F-optimal outcome at R. The *range of* F is the set

$$F(\mathcal{R}) \equiv \left\{x \in Z | x \in F(R) \text{ for some } R \in \mathcal{R}\right\}.$$

Following KY, to implement F, the designer designs a *rights structure* Γ, which is a triplet (S, h, γ), where S is the *state space*, $h : S \to Z$ the *outcome function*, and γ a *code of rights*, which is a (possibly empty) correspondence $\gamma : S \times S \twoheadrightarrow \mathcal{N}$. Subsequently, a code of rights specifies, for each pair of distinct states (s, t), a family of coalitions $\gamma(s, t)$ entitled to approve a change from state s to t. A rights structure Γ is said to be an *individual-based rights structure* if, for each pair of distinct states (s, t), $\gamma(s, t)$ contains only unit coalitions if it is nonempty.

For any rights structure Γ and any preference profile R, a state $s \in S$ is an *equilibrium* at R if, for no $t \in S$ and no $K \in \gamma(s, t)$ is $h(t) P_K h(s)$. We express $C(\Gamma, R)$ for the set of Γ-equilibria at R.

Definition 1. A rights structure Γ *implements* F if $F(R) = h \circ C(\Gamma, R)$ for all $R \in \mathcal{R}$. If such a rights structure exists, F is *implementable by a rights structure*.

Definition 2. F is *implementable by an individual-based rights structure* if there exists an individual-based rights structure Γ, so that Γ implements F.

KY show that the following monotonicity condition is necessary and sufficient for implementation by rights structures. We formalize the condition as follows. For any preference ordering

[4] This externally stable implementation is a robust way of implementing outcomes, being more reliable than the implementation of core outcomes, since external stability guarantees no outcome outside the core can be sustained. Moreover, an externally stable core is also more robust than the Von Neumann–Morgenstern (vNM) stable set or its derivatives, since it avoids indirect internal stability problems (i.e., the *Harsanyi critique*; Harsanyi, 1974).

R_i and outcome x, the *lower contour set of* R_i *at* x is defined by $L(x, R_i) = \{x' \in Z | x R_i x'\}$. Therefore,

Definition 3 *(KY).* F satisfies the condition of *image monotonicity* provided that, for all $x \in Z$ and all $R, R' \in \mathcal{R}$, if $x \in F(R)$,

$$L(x, R_i) \bigcap F(\mathcal{R}) \subseteq L(x, R'_i) \text{ for all } i \in N,$$

then $x \in F(R')$.

A linear ordering of agent i, denoted by P_i, is a complete, transitive, and antisymmetric binary relation over Z. We denote by $\mathcal{P}_Z$ the collection of all profiles of linear orderings, that is, the *unrestricted domain of linear orderings*. KY's equivalence result can be stated as follows.

Theorem 1 *(KY, p. 488). Let* $F : \mathcal{P}_Z \to \mathcal{Z}_0$ *be any SCR. Then, the following statements are equivalent:*

(i) F *is* implementable by a rights structure*;*
(ii) F *satisfies the condition of* image monotonicity*;*
(iii) F *is* implementable by an individual-based rights structure.

Korpela et al. (2018) generalize the above theorem by relaxing the preference domain assumption. They show that only SCRs satisfying *(Maskin) monotonicity* and *unanimity*, both being restricted to a superset of the SCR image, are necessary and sufficient conditions for implementation by rights structures.[5] Moreover, what can be implemented by a rights structure can also be implemented by an individual-based rights structure, and vice versa. This generalization is based on a strong notion of blocking. An alternative weaker notion of blocking can be applied to (Γ, R). Given a game (Γ, R), a pair (K, t), where K is a coalition and t is a state, is a *weak objection* to $s \in S$ (or equivalently, K *weakly blocks* s with t) if $K \in \gamma(s, t)$ is entitled to approve a change from state s to t, $h(t) R_K h(s)$ and $h(t) P_i h(s)$ for some agent $i \in K$. This notion of blocking leads to the concept of *strong (core) equilibrium*. For any rights structure Γ and any preference profile R, a state $s \in S$ is a strong equilibrium at R if there does not exist a weak objection to it. One may wonder whether the conclusions of Korpela et al.'s characterization (Korpela et al., 2018, Theorem 2) would change if we adopted this notion of strong equilibrium as our solution concept. The answer is no. One can easily check that unanimity and monotonicity w.r.t. Y are necessary conditions for implementation in strong equilibrium by a rights structure. Moreover, one can check that the individual-based rights structure devised in their proof also implements F in strong equilibrium.

The notion of Γ-equilibrium is a myopic notion of equilibrium. The reason is that a coalition that has the power as well as the incentive to move from one state s to another state t approves

[5] Formally, the two implementing conditions can be stated as follows: F is *(Maskin) monotonic w.r.t.* $Y \subseteq Z$ provided that, for all $x \in Y$ and all $R, R' \in \mathcal{R}$, if $x \in F(R)$ and

$$L(x, R_i) \bigcap W \subseteq L(x, R'_i) \bigcap W \text{ for all } i \in N,$$

then $x \in F(R')$. F satisfies *unanimity w.r.t.* $Y \subseteq Z$ provided that $F(\mathcal{R}) \subseteq Y$ and that, for all $x \in Y$ and all $R \in \mathcal{R}$, if $Y \subseteq L(x, R_i)$ for all $i \in N$, then $x \in F(R)$.

the change regardless of whether t itself is an equilibrium or not. To address this point, for each nonequilibrium state s, KY require that there exists an agent i as well as an equilibrium state t such that agent i has the power and the incentive to move from s to t. Formally:

Definition 4 *(KY, p. 492).* A rights structure Γ *externally stable implements* F if (i) Γ implements F, and (ii) for each $R \in \mathcal{R}$ and each $s \notin C(\Gamma, R)$ such that $h(s) \notin F(R)$, there exist $t \in C(\Gamma, R)$ and $i \in N$ such that $\{i\} \in \gamma(s, t)$ and $h(t) P_i h(s)$. If such a rights structure exists, F is *externally stable implementable by a rights structure*.

KY fully characterize the class of SCRs that are externally stable implementable by a rights structure (KY, p. 492; Proposition 2). They show that externally stable implementation by a rights structure is equivalent to a condition that is a variant of monotonicity, called *winner monotonicity*. This condition can be stated as follows: F is *winner monotonic* provided that, for all $x \in Z$ and all $R, R' \in \mathcal{R}$, if $x \in F(R)$ and $L(x, R_i) \cap F(R') \subseteq L(x, R'_i)$ for all $i \in N$, then $x \in F(R')$. This characterization result is derived by assuming that the preference domain is $\mathcal{P}_Z$. This insight will be useful for our characterization results.

3. Implementation via codes of rights

KY were the first to study implementation via codes of rights by restricting attention to the universal domain of linear orderings. They show that (Maskin) monotonicity and a condition called *binary consistency* fully characterize the class of SCRs that are implementable by a code of rights (KY, Proposition 5, p. 497). Below, we extend this analysis to any preference domain.

Indeed, a natural candidate for the state space of a rights structure Γ is the set of outcomes Z. Arguably, this captures the most natural way of allocating blocking powers in real-life situations (e.g., Ray and Vohra, 2014). Therefore, by assuming that the outcome function h, defined over Z, is the identity map, the implementation exercise is reduced to the design of *code of rights* $\gamma : Z \times Z \twoheadrightarrow \mathcal{N}$. For each pair of outcomes x and y, γ specifies a collection of coalitions $\gamma(x, y)$ that are effective for moving from x to y. If coalition K is an element of $\gamma(x, y)$ and if $y P_K x$, we say x is *blocked* and K is a *blocking coalition*. If there is no such blocking coalition, we say x is *unblocked*. Here, we consider and analyze implementation exercises by codes of rights, which we call *implementation by codes of rights*.

If we choose the set of outcomes Z as the state space of a rights structure Γ and the identity function as the outcome function in the definition of Γ-equilibria, we revert to the familiar notion of core, which can be defined as follows.

Definition 5. For any γ and any R, x is an *equilibrium* at R if for no $y \neq x$ and no $K \in \gamma(x, y)$ it is $y P_K x$.

The blocking notion yields the concept of core at one preference profile R, which is the set of all unblocked outcomes.[6] Note that, in our definition of blocking, we require that every member of the blocking coalition is strictly better off. For any code of rights γ, we write $C(\gamma, R)$ for the set of the equilibria at R.

[6] Ray (1989) shows that the credibility of blocking coalitions is implicit in the definition of the core.

Implementation by codes of rights Let us define implementation by codes of rights.

Definition 6. A code of rights γ *implements* F if and only if $F(R) = C(\gamma, R)$ for all $R \in \mathcal{R}$. If such a code of rights exists, then F is *implementable by a code of rights*.

One can easily verify that monotonicity is a necessary condition for the implementation via codes of rights. However, one also can check that it is not sufficient. We introduce below a new condition, called *strong monotonicity* using the following additional notation. For any coalition $K \in \mathcal{N}_0$, preference profile $R \in \mathcal{R}$, and outcome $x \in Z$, let

$$L(x, R_K) \equiv \bigcup_{i \in K} L(x, R_i),$$

$$F^{-1}(x) \equiv \left\{R \in \mathcal{R} | x \in F(R)\right\},$$

and

$$\Lambda_K^F(x) \equiv \bigcap_{R \in F^{-1}(x)} L(x, R_K).$$

Here, we present strong monotonicity from the viewpoint of necessity. To this end, assume an SCR F is implementable by a code of rights γ. Taking an outcome $x \in F(R)$ for some preference profile $R \in \mathcal{R}$, x must be an equilibrium at R. We fix any coalition K and denote by $\gamma(x, K)$ the set of outcomes for which coalition K is effective, that is, $\gamma(x, K) \equiv \{y \in Z | K \in \gamma(x, y)\}$. Since x is an equilibrium at R, it follows that x is unblocked, that is, for every outcome $y \neq x$, if coalition $K \in \gamma(x, y)$ is effective in moving from x to y, then coalition K cannot be a blocking one. This entails that y must be an element of $L(x, R_K)$ if $y \in \gamma(x, K)$. Since the choice of outcome $y \in \gamma(x, K)$ is arbitrary, set $\gamma(x, K)$ must be contained in $L(x, R_K)$.

For a canonical rights structure, in which the set of states is $S \equiv \{(z, \bar{R}) | z \in F(\bar{R})$ for some $\bar{R} \in \mathcal{R}\}$, the designer can infer the lower contour set of R_i at x for member $i \in K$, that is, to infer the set $L(x, R_K)$.[7] However, in a setting for which the state space coincides with the set of outcomes, the designer cannot obtain information on member i's lower contour sets. Hence, the designer needs to consider any preference profile $\hat{R}$ satisfying $x \in F(\hat{R})$. We choose such a preference profile $\hat{R}$. Then, the preceding argument leads to the conclusion that the set of outcomes $\gamma(x, K)$ for which coalition K is effective must be contained in $L(x, \hat{R}_K)$. This condition should be satisfied for each admissible preference profile $\hat{R}$ satisfying $x \in F(\hat{R})$, that is, the set $\gamma(x, K)$ must be contained in the intersection $\Lambda_K^F(x)$.

Therefore, if at some preference profile R', set $L(x, R'_K)$ contains $\Lambda_K^F(x)$ and thus the set of outcomes $\gamma(x, K)$ for which coalition K is effective, coalition K cannot be a blocking one. If this conclusion holds for all coalitions, x is unblocked, meaning it is an equilibrium at R'. It follows that x must be F-optimal at R', by implementability. More formally, strong monotonicity can be stated as follows.[8]

Definition 7. F is *strongly monotonic* provided that, for all $x \in Z$ and all $R, R' \in \mathcal{R}$, if $x \in F(R)$ and

7 See KY, proof of Proposition 1 (p. 489).

8 Our strong monotonicity condition must not be confused with the strong monotonicity condition of Peleg and Winter (2002). The latter condition is now widely referred to as essential monotonicity (Danilov, 1992).

$$\Lambda_K^F(x) \subseteq L\big(x, R'_K\big) \text{ for all } K \in \mathcal{N}_0,$$

then $x \in F(R')$.

Note that strong monotonicity implies monotonicity. Conversely, for an example of a monotonic SCR that is not strongly monotonic, see the example below.[9]

Example 1 *(Walrasian solution).* Assume three commodities and two agents. Let $\mathcal{U}_i$ be the class of utility functions admissible for agent i, each assumed to be continuous, quasi-concave, and strictly monotonic. Let $\mathcal{U} \equiv \mathcal{U}_1 \times \mathcal{U}_2$ be the class of profiles of admissible utility functions and let $\mathcal{U}^{CD}$ denote the class of profiles of Cobb-Douglas utility functions.[10] Assume that $\mathcal{U}^{CD} \subseteq \mathcal{U}$ and that agent 1's endowment is $e_1 = (1, 2, 0)$ and agent 2's endowment $e_2 = (1, 0, 2)$. Let Z be the set of all feasible allocations, that is, $Z \equiv \{(x_1, x_2)|x_1 + x_2 = e_1 + e_2\}$. The Walrasian solution, denoted by W, can be defined as follows. For each $u \in \mathcal{U}$ and $x \in Z$, $x \in W(u)$ if and only if there is a price vector $p \in \Delta$ such that for all $i \in N$, $p \cdot x_i = p \cdot e_i$ and for all $y_i \in \mathbb{R}^3_+$, if $p \cdot y_i \leq p \cdot x_i$, then $u_i(y_i) \leq u_i(x_i)$, where $\Delta \equiv \{p \in \mathbb{R}^3_+|\sum_{\ell=1}^3 p_\ell = 1\}$.[11] Let us assume that, for all $u \in \mathcal{U}$ and $x \in W(u)$, it holds that $x_{i\ell} > 0$ for all agents i and commodity ℓ.[12] Let $u^0 \in \mathcal{U}^{CD}$ be so that $u_i^0(x) = x_{i1}^2 x_{i2} x_{i3}$ for each agent i. Then, $\mathbf{1} \equiv (1, 1, 1)$ is a Walrasian allocation at u^0 and the Walrasian equilibrium prices are $p^0 \equiv (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. Let $u^1 \in \mathcal{U}^{CD}$ be so that $u_i^1(x) = x_{i1} x_{i2} x_{i3}$ for each agent i. Again, $\mathbf{1}$ is a Walrasian allocation at u^1 generated by $p^1 \equiv (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. It follows that $u^0, u^1 \in W^{-1}(\mathbf{1})$. Finally, let $u^2 \in \mathcal{U}^{CD}$ be so that $u_1^2 = u_1^0$ and $u_2^2 = u_2^1$. W is not strongly monotonic since $\mathbf{1} \in W(u^0)$, $\Lambda_K^W(\mathbf{1}) \subseteq L(\mathbf{1}, u_K^2)$ for all $K \in \mathcal{N}_0$ and yet $\mathbf{1} \notin W(u^2)$. However, W is monotonic.

As we formally show below, strong monotonicity is a necessary condition for implementation via codes of rights, and it is sufficient when combined with another necessary condition known as unanimity. This condition states that, if an outcome is at the top of the preferences of all agents, then that outcome should be selected by the SCR. The condition can be stated as follows.

Definition 8. F satisfies *unanimity* provided that, for all $x \in Z$ and all $R \in \mathcal{R}$, if $Z \subseteq L(x, R_i)$ for all $i \in N$, then $x \in F(R)$.

We show that strong monotonicity and unanimity are necessary and sufficient for implementation by codes of rights.

Theorem 2. *F is* implementable by a code of rights *if and only if F satisfies the conditions of* strong monotonicity *and* unanimity.

[9] It can also be shown that the (constrained) Walrasian solution violates strong monotonicity in an exchange economy with more than three agents. The details are available upon request from the authors.

[10] Although Cobb-Douglas utility functions are not strictly monotonic on the boundary of consumption set $\mathbb{R}^3_+$, only strict monotonicity on the interior of $\mathbb{R}^3_+$ is necessary to obtain our results.

[11] p_ℓ denotes the price of commodity ℓ.

[12] Under this assumption, the Walrasian solution coincides with the constrained Walrasian solution (Hurwicz et al., 1995), which is known to be monotonic.

Proof. "*Only If*": Assume that code of rights γ implements F. Since it is obvious that F satisfies unanimity, we show that F satisfies strong monotonicity. Take any R and x so that $x \in F(R)$. Furthermore, take any R' so that $\Lambda_K^F(x) \subseteq L(x, R'_K)$ for all K. We show that $x \in F(R') = C(\gamma, R')$.

Assume, to the contrary, that $x \notin C(\gamma, R')$. Then, there exist $y \neq x$ and $K \in \gamma(x, y)$ so that $y P'_K x$. It follows that $y \notin L(x, R'_K)$. Take any $\bar{R} \in F^{-1}(x)$. Since $x \in C(\gamma, \bar{R})$ and since $K \in \gamma(x, y)$, it follows that $y \in L(x, \bar{R}_K)$. Since the choice of $\bar{R} \in F^{-1}(x)$ is arbitrary, we have $y \in \Lambda_K^F(x)$. By our initial assumption that $\Lambda_K^F(x) \subseteq L(x, R'_K)$, it follows that $y \in L(x, R'_K)$, which is a contradiction. Therefore, F is strongly monotonic.

"*If*": Assume that F satisfies the conditions of strong monotonicity and unanimity. Let us define a code of rights $\gamma : Z \times Z \twoheadrightarrow \mathcal{N}$ as follows. For all K,

(a) For all $x \in F(\mathcal{R})$ and all $y \in Z$,

$$K \in \gamma(x, y) \iff y \in \Lambda_K^F(x);$$

(b) For all $x \in Z - F(\mathcal{R})$ and all $y \in Z$, $K \in \gamma(x, y)$.

We show that γ implements F. Fix any R.

Let $x \in F(R)$. We show that $x \in C(\gamma, R)$. Since $x \in F(R)$, it follows that $x \in F(\mathcal{R})$. Then, only part (a) of the definition of γ applies. We fix any K and y. Assume that $y \in \Lambda_K^F(x)$. Then, by the definition of γ, it follows that $K \in \gamma(x, y)$. However, since $y \in \Lambda_K^F(x)$, it also follows that $y \in L(x, R_K)$. Moreover, by the definition of γ, it holds that $K \notin \gamma(x, y)$ if $y \notin \Lambda_K^F(x)$. Then, either $K \notin \gamma(x, y)$ if $y \notin \Lambda_K^F(x)$ or $K \in \gamma(x, y)$ and no $y P_K x$ if $y \in \Lambda_K^F(x)$. Since the choices of K and y are arbitrary, we conclude that $x \in C(\gamma, R)$.

Conversely, we take any $x \in C(\gamma, R)$. We proceed according to whether $x \in F(\mathcal{R})$.

Assume that $x \in Z - F(\mathcal{R})$. Then, only part (b) of the definition of γ applies. We fix any i. Since $i \in \gamma(x, y)$ for all y and $x \in C(\gamma, R)$, $x R_i y$ for all y, and so $Z \subseteq L(x, R_i)$. Since the choice of i is arbitrary, it follows that $Z \subseteq L(x, R_i)$ for all i. Since F satisfies the condition of unanimity, we have $x \in F(R)$.

Assume that $x \in F(\mathcal{R})$. Then, $F^{-1}(x)$ is not empty, meaning only part (a) of the definition of γ applies. Assume, to the contrary, that $x \notin F(R)$. Then, strong monotonicity implies that there exist K and $y \in \Lambda_K^F(x)$ so that $y P_K x$. Since $y \in \Lambda_K^F(x)$, $K \in \gamma(x, y)$ by definition of γ. Therefore, there exists $y \neq x$ so that $y P_K x$, for some $K \in \gamma(x, y)$ and so $x \notin C(\gamma, R)$, which is a contradiction. □

Remark 1. As mentioned in Section 2, our notion of equilibrium is based on a strong notion of blocking. However, the notion of weak blocking defined in that section can easily be adapted to the set-up where $S = Z$, which, in turn, leads to the familiar notion of strong (core) equilibrium. Again, one may wonder whether the conclusion of Theorem 2 would change if we adopted this notion of strong equilibrium as our solution concept. The answer is yes. The reason is that strong monotonicity is not a necessary condition for implementation in strong equilibrium by a code of rights. To see it, we provide below an example which shows that there exists an F that is implementable in strong equilibrium by a code of rights but that is not strongly monotonic.

There are three players in $N \equiv \{1, 2, 3\}$, and two profiles R and R' over set $Z \equiv \{w, z, x, y\}$. Preferences are represented in the table below,

R			R'		
1	2	3	*1*	2	3
y, z	x	w	y	w, z, x, y	w, z, x, y
x	w, y, z	x, z	x		
w		y	w, z		

where, as usual, $\frac{a}{b}$ for agent i means that he/she strictly prefers a to b, while a, b means that i is indifferent between a and b.

Let F be so that $F(R) = \{x\}$ and $F(R') = \{y\}$. F is not strongly monotonic, since $x \in F(R)$, $\Lambda_K^F(x) \subseteq L(x, R'_K)$ for all $K \in \mathcal{N}_0$ and yet $x \notin F(R')$.[13] However, F is implementable in strong equilibrium by a code of rights. To see it, let us define the code of rights γ as follows:

$$\gamma(w, x) = \gamma(y, x) = \gamma(z, x) = \{2\},$$
$$\gamma(w, y) = \gamma(z, y) = \{1\} \text{ and } \gamma(x, y) = \{1, 2\},$$

while it is empty in all other cases. One can easily check that at R the pair $(\{2\}, x)$ is a weak objection to w, to z as well as to y. Since there does not exist any weak objection to x, we have that x is the unique strong equilibrium at R. Furthermore, one can check that at R' the pair $(\{1\}, y)$ is a weak objection to w and to z, whereas the pair $(\{1, 2\}, y)$ is a weak objection to x. Since there does not exist any weak objection to y, it follows that y is the unique strong equilibrium outcome at R'. This shows that γ implements F in strong equilibrium.

As already mentioned, KY (pp. 496–498) provide a characterization of the class of SCRs that are implementable via codes of rights. This result is given in terms of monotonicity as well as of a condition called binary consistency. To introduce this condition, we need the following additional notations. Take any $P \in \mathcal{P}_Z$ and any $a, b \in Z$. a is said to be *not Pareto dominated by b at P* if aP_ib for some agent i. Take any $P \in \mathcal{P}_Z$ and any $x, y \in Z$ so that y is not Pareto dominated by x at P. Let P^{xy} be the profile obtained from P, in which x and y are the two most preferred outcomes for all agents and, for all $z \in Z - \{x, y\}$ and $i \in N$, it holds that $xP_i^{xy}z$ and $yP_i^{xy}z$; further, $\{i \in N | yP_i^{xy}x\} = \{i \in N | yP_ix\}$.

Definition 9. F is *binary consistent* provided that, for all $P \in \mathcal{P}_Z$ and $x \in Z$ if for all $y \in Z$ that are not Pareto-dominated by x at P, it holds that $x \in F(P^{xy})$, then $x \in F(P)$.

Theorem 2 is logically equivalent to the characterization provided by KY if we focus on the universal domain of linear orderings, $\mathcal{P}_Z$. Unfortunately, we were unable to find a direct, intuitive bridge between the implementing conditions in these theorems, although the formal arguments of their equivalence are available from the authors upon request.

Externally stable implementation by codes of rights We achieved the result above by focusing on the traditional notion of core, which has been criticized for not being symmetric (Greenberg, 1990). Indeed, from its original definition, an outcome x is not a core point if it is blocked, that is, if there is a coalition that would reject x and move to another outcome y that would be preferred by all its members. However, outcome y itself can, in turn, be blocked. Then, if we require that an

[13] One can easily check that: 1) for each $K \neq \{i\}$, it holds that $\Lambda_K^F(x) = Z \subseteq L(x, R'_K) = Z$; and 2) $\Lambda_{\{1\}}^F(x) = \{w, x\} \subseteq L(x, R'_1) = \{w, z, x\}$, $\Lambda_{\{2\}}^F(x) = Z \subseteq L(x, R'_2) = Z$ and $\Lambda_{\{3\}}^F(x) = \{z, x, y\} \subseteq L(x, R'_3) = Z$.

outcome be immune against blocking, symmetry requires that the same should hold for y. That is, we should require that blocking be done through outcomes that are themselves unblocked. This consistency requirement leads to the concept widely referred to as an externally stable core.

Definition 10. For any code of rights γ and any preference profile R, a set $Z^* \subseteq Z$ of outcomes is *externally stable* at R if, for all $y \in Z - Z^*$, there is $x \in Z^*$ so that $x P_K y$ for some $K \in \gamma(y, x)$.

Externally stable equilibria at R are denoted by $EC(\gamma, R)$. The set $C(\gamma, R)$ is unique when it is nonempty and, hence, $EC(\gamma, R)$ is unique when it is nonempty.

Here, we consider and analyze the implementation of an externally stable core by codes of rights, which we call *externally stable implementation by codes of rights*.

Definition 11. A code of rights γ *externally stable implements* F if $F(R) = EC(\gamma, R)$ for all $R \in \mathcal{R}$. If such a code of rights exists, F is *externally stable implementable by a code of rights*. F is *externally stable implementable by a code of individual rights* if there exists a code of individual rights γ so that γ externally stable implements F.

Externally stable implementation is a robust way of implementing optimal outcomes, particularly being more reliable than implementation, since external stability guarantees that no outcome outside the core can be sustained. An externally stable core is also more robust than the vNM stable set or its derivatives, since it avoids problems with indirect internal stability (i.e., the *Harsanyi critique*; Harsanyi, 1974).

We propose two conditions that are together necessary and sufficient, when combined with unanimity, for an SCR to be externally stable implemented by a code of rights. The first condition is a variant of strong monotonicity, called *strong winner monotonicity*.

Definition 12. F is *strongly winner monotonic* provided that, for all $x \in Z$ and all $R, R' \in \mathcal{R}$, if $x \in F(R)$ and

$$\Lambda_K^F(x) \bigcap F(R') \subseteq L(x, R'_K) \text{ for all } K \in \mathcal{N}_0,$$

then $x \in F(R')$.

In other words, this condition implies that, whenever x is F-optimal at one profile R and for every coalition K, x is a maximal element in $\Lambda_K^F(x) \cap F(R')$ according to the preferences of coalition K at R'_K, and should be F-optimal at R'. The intuition is straightforward. Assume that x is F-optimal at R. Since the externally stable core at this R is a subset of the core at R, we know from Theorem 2 that whenever preferences change from R to R' and $\Lambda_K^F(x) \subseteq L(x, R'_K)$ for each coalition $K \in \mathcal{N}_0$, x must remain F-optimal at the new profile R'. However, by the requirement of external stability, if x were not F-optimal at R', there should exist a coalition K that can, and wants to, reject x and move to an F-optimal outcome at R'. This means that it is not the entire set $\Lambda_K^F(x)$ that matters to remove x as an equilibrium outcome when preferences change from R to R', but it is set $\Lambda_K^F(x) \bigcap F(R')$ that matters.

The condition is stronger than strong monotonicity, since $\Lambda_K^F(x) \cap F(R') \subseteq \Lambda_K^F(x)$ for all $R' \in \mathcal{R}$ and $K \in \mathcal{N}_0$. Conversely, for an example of a strongly monotonic (and monotonic) SCR that is not strongly winner monotonic, see the example below.

Example 2 *(Pareto solution with veto power).* There are three players in $N \equiv \{1, 2, 3\}$, and two profiles of linear orderings P and P' over set $Z \equiv \{v, x, y\}$. Preferences are represented in the table below,

P			P'		
1	2	3	1	2	3
x	v	x	x	v	x
v	x	v	y	x	v
y	y	y	v	y	y

where, as usual, $\frac{a}{b}$ for agent i means that he/she strictly prefers a to b. Let F be so that $F(P) = \{v, x\}$ and $F(P') = \{x\}$. F is not strongly winner monotonic, since $v \in F(P)$, $\Lambda_K^F(v) \cap F(P') \subseteq L(v, P'_K)$ for all $K \in \mathcal{N}_0$ and yet $v \notin F(P')$. However, one can verify that F is strongly monotonic (and thus monotonic).

The second condition can be stated as follows.

Definition 13. F satisfies the *no-simultaneous domination of* F provided that, for all $x \in Z$ and all $R \in \mathcal{R}$, if $x \in Z - F(R)$, then for some $i \in N$, $y P_i x$ for some $y \in F(R)$.

The condition simply states that, if outcome x is not F-optimal at R, it cannot be that this x dominates every outcome in the range of F at R in the sense that x is at least as good as every F-optimal outcome at R for every agent $i \in N$. When preference domain $\mathcal{R}$ is the domain of linear orderings, the condition implies that an outcome x that is not F-optimal at R cannot Pareto dominate every F-optimal outcome at R.[14]

Our next result is that the class of SCRs externally stable implementable by codes of rights coincides with the class of SCRs that satisfy strong winner monotonicity, unanimity, and the no-simultaneous domination of F.

Theorem 3. *F is* externally stable implementable by a code of rights *if and only if F satisfies the conditions of* strong winner monotonicity*,* unanimity *and* no-simultaneous domination of F.

Proof. See Appendix. □

Remark 2. Strong winner monotonicity is not a necessary condition for externally stable implementation in strong (core) equilibria by a code of rights. Indeed, the example in Remark 1 shows that there exists an F that is externally stable implementable in strong equilibria via codes of rights but that is not strongly winner monotonic. To see it, observe that, by construction, $F(R) = \{x\}$, $\Lambda_K^F(x) \cap F(R') \subseteq L(x, R'_K)$ for all $K \in N_0$ and yet $x \notin F(R') = \{y\}$.[15] One can also check that the code of rights devised in Remark 1 externally stable implements F in strong equilibria.

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14 For any profile R, we say that outcome x *Pareto dominates* y if $x P_i y$ for all $i \in N$.

15 One can easily check that: 1) for each $K \neq \{i\}$, it holds that $\Lambda_K^F(x) \cap F(R') = \{y\} \subseteq L(x, R'_K) = Z$; and 2) $\Lambda_{\{1\}}^F(x) \cap F(R') = \varnothing \subseteq L(x, R'_1) = \{w, z, x\}$, $\Lambda_{\{2\}}^F(x) \cap F(R') = \{y\} \subseteq L(x, R'_2) = Z$ and $\Lambda_{\{3\}}^F(x) \cap F(R') = \{y\} \subseteq L(x, R'_3) = Z$.

As already mentioned in Section 2, KY fully characterize the class of SCRs that are externally stable implementable by a rights structure (KY, p. 492; Proposition 2) in terms of winner monotonicity. One can easily check that winner monotonicity, defined at the end of Section 2, is implied by strong winner monotonicity and that they are not equivalent.

Applications Although strong monotonicity and strong winner monotonicity are demanding monotonicity-type conditions, Korpela et al. (2018) show that the (weak) Pareto solution, the Condorcet solution when there are an odd number of voters and the individual rational solution are implementable by codes of rights as well as externally stable implementable by codes of rights.

Before closing this section, let us show that the stable solution is implementable by a code of rights. A *matching problem* is a quadruplet $(M, W, P, \mathcal{M})$ so that:

- M is a finite nonempty set of men with m as a typical element;
- W is a finite nonempty set of women with w as a typical element;
- $P \in \mathcal{P}$ is a profile of linear orderings so that (i) every man $m \in M$'s preference relation is represented by a linear ordering P_m over $W \cup \{m\}$ and (ii) every woman $w \in W$'s preference relation is represented by a linear ordering P_w over $M \cup \{w\}$.
- $\mathcal{M}$ is a collection of all matchings, with μ as a typical element. $\mu : M \cup W \to M \cup W$ is a bijective function, matching every agent $i \in M \cup W$ either to a partner of the opposite sex or with himself/herself. If an agent i is matched with himself/herself, we say that this i is *single* under μ.

We refer to $(M, W, \mathcal{P}, \mathcal{M})$ as a *class of matching problems*, with $(M, W, P, \mathcal{M})$ as a typical matching problem. Note that $Z = \mathcal{M}$ and $M \cup W = N$.

To apply Theorem 2 to matching problems, we extend the linear ordering P_m of a man $m \in M$ to preference ordering R_m on $\mathcal{M}$ as follows: for every $\mu, \mu' \in \mathcal{M}$,

$$\mu R_m(\theta)\mu' \Leftrightarrow \text{either } \mu(m) P_m(\theta)\mu'(m) \text{ or } \mu(m) = \mu'(m).$$

Similarly, this can be done for every woman $w \in W$. Let $\mathcal{R}$ denote the preference domain over $\mathcal{M}$, obtained by a collection $\mathcal{P}$ of profiles of linear orderings.

A matching μ is *blocked by agent i at $R \in \mathcal{R}$* if $i P_i \mu(i)$. A matching μ is *blocked by a pair $(m, w) \in M \times W$ at $R \in \mathcal{R}$* if $m P_w \mu(w)$ and $w P_m \mu(m)$. A matching μ is *stable at $R \in \mathcal{R}$* if it is not blocked by any agent or any pair of a man and a woman at R. Given a matching problem, the *stable solution*, denoted by St, can be defined, for each $R \in \mathcal{R}$, by

$$St(R) \equiv \{\mu \in \mathcal{M} | \mu \text{ is stable at } R\}.$$

Example 3 *(Stable solution).* Let $(M, W, \mathcal{P}, \mathcal{M})$ be any class of matching problems. The stable solution, St, defined over $\mathcal{R}$, is implementable by a code of rights. To show this, we need to show that St satisfies the conditions of unanimity and strong monotonicity. We omit the straightforward proof that St satisfies the unanimity condition. Then, we need to show that this solution satisfies strong monotonicity as well.

Take any two matching problems $(M, W, P, \mathcal{M})$ and $(M, W, P', \mathcal{M})$. Assume that $\mu \in St(R)$ and that $\Lambda_K^{St}(\mu) \subseteq L(\mu, R'_K)$ for all $K \in \mathcal{N}_0$. We show that $\mu \in St(R')$. To obtain a contradiction, we assume $\mu \notin St(R')$.

Suppose that μ is blocked by agent i at R', that is, $i P'_i \mu(i)$. Then, $\hat{\mu} \notin L(\mu, R'_i)$ for all $\hat{\mu} \in \mathcal{M}$ so that $\hat{\mu}(i) = i$. Next, take any $\bar{R} \in St^{-1}(\mu)$, which exists since $\mu \in St(R)$. Since $\mu \in St(\bar{R})$, it follows that μ is not blocked by agent i at $\bar{R}$, that is, either $\mu(i)\bar{P}_i i$ or $\mu(i) = i$. Then, for all $\hat{\mu} \in \mathcal{M}$, if $\hat{\mu}(i) = i$; then, $\hat{\mu} \in L(\mu, \bar{R}_i)$. Since the choice of $\bar{R} \in St^{-1}(\mu)$ is arbitrary, we have $\hat{\mu} \in \Lambda^{St}_{\{i\}}(\mu)$ for all $\hat{\mu} \in M$ so that $\hat{\mu}(i) = i$. As by our initial assumption, it holds that $\Lambda^{St}_{\{i\}}(\mu) \subseteq L(\mu, R'_i)$; it follows that $\mu R'_i \hat{\mu}$ for $\hat{\mu} \in \mathcal{M}$ so that $\hat{\mu}(i) = i$, which is a contradiction.

Assume that μ is blocked by a pair $(m, w) \in M \times W$ at R', that is, $m P'_w \mu(w)$ and $w P'_m \mu(m)$. Then, $\hat{\mu} \notin L(\mu, R'_{\{m,w\}})$ for all $\hat{\mu} \in \mathcal{M}$ so that $\hat{\mu}(w) = m$ and $\hat{\mu}(m) = w$. Next, take any $\bar{R} \in St^{-1}(\mu)$, which exists since $\mu \in St(R)$. Since $\mu \in St(\bar{R})$, it follows that μ is not blocked by (m, w) at $\bar{R}$, that is, not $m \bar{P}_w \mu(w)$ or not $w \bar{P}_m \mu(m)$. For all $\hat{\mu} \in \mathcal{M}$, if $\hat{\mu}(w) = m$ and $\hat{\mu}(m) = w$, $\hat{\mu} \in L(\mu, \bar{R}_{\{m,w\}})$. Since the choice of $\bar{R} \in St^{-1}(\mu)$ is arbitrary, we have $\hat{\mu} \in \Lambda^{St}_{\{m,w\}}(\mu)$ for all $\hat{\mu} \in M$ so that $\hat{\mu}(w) = m$ and $\hat{\mu}(m) = w$. As by our initial assumption, it holds that $\Lambda^{St}_{\{m,w\}}(\mu) \subseteq L(\mu, R'_{\{m,w\}})$; it follows that $\mu R'_w \hat{\mu}$ or $\mu R'_m \hat{\mu}$ for $\hat{\mu} \in \mathcal{M}$ so that $\hat{\mu}(w) = m$ and $\hat{\mu}(m) = w$, which is a contradiction.

4. Implementation via codes of individual rights

Implementation by codes of individual rights A code of rights γ is said to be a *code of individual rights* if, for each pair of distinct outcomes x and y, $\gamma(x, y)$ contains only unit coalitions if it is not empty, that is, it contains only coalitions of size one. Here, we study implementation exercises in which the designer can devise only codes of individual rights, which we call *implementation by codes of individual rights*.

Definition 14. F is *implementable by a code of individual rights* if there exists a code of individual rights γ so that γ implements F.

Although strong monotonicity is still a necessary condition for implementation by codes of individual rights, one can easily verify that it is not sufficient. We introduce below a stronger variant of strong monotonicity, called *singleton strong monotonicity*, which is shown to be necessary and sufficient for implementation by a code of individual rights when combined with the unanimity condition. To introduce this condition, we need an additional notation as follows. For any outcome x and agent i, let

$$\Lambda^F_i(x) \equiv \bigcap_{R \in F^{-1}(x)} L(x, R_i).$$

Therefore,

Definition 15. F is *singleton strongly monotonic* provided that, for all $x \in Z$ and all $R, R' \in \mathcal{R}$, if $x \in F(R)$ and

$$\Lambda^F_i(x) \subseteq L\big(x, R'_i\big) \text{ for all } i \in N,$$

then $x \in F(R')$.

One can easily verify that the condition above is nothing more than the condition of strong monotonicity restricted to unit coalitions. Roughly, the intuitions of the two conditions are the same. The condition above appears in Saijo et al. (1996, p. 955) under the name Condition W^*. The theorem below characterizes the class of SCRs implementable by codes of individual rights.

Theorem 4. *F is* implementable by a code of individual rights *if and only if F satisfies the conditions of* singleton strong monotonicity *and* unanimity.

Proof. Let us define a code of individual rights $\gamma : Z \times Z \twoheadrightarrow \mathcal{N}$ as follows. For all i,

(a) For all $x \in F(\mathcal{R})$ and all $y \in Z$,

$$\{i\} \in \gamma(x, y) \iff y \in \Lambda_i^F(x);$$

(b) For all $x \in Z - F(\mathcal{R})$ and all $y \in Z$, $\{i\} \in \gamma(x, y)$.

We omit the proof, which uses the γ above to prove the "*If*" part of the statement and similar arguments to the proof of Theorem 2. □

The class of SCRs implementable by a code of individual rights is not empty. The reason is that the individually rational solution and no-envy solution are singleton strongly monotonic (Korpela et al., 2018).

Externally stable implementation by codes of individual rights Let us now study externally stable implementation by codes of individual rights. We define below a variant of strong winner monotonicity, which we call *singleton strong winner monotonicity*.

Definition 16. F is *singleton strongly winner monotonic* provided that, for all $x \in Z$ and all $R, R' \in \mathcal{R}$, if $x \in F(R)$ and

$$\Lambda_i^F(x) \bigcap F\left(R'\right) \subseteq L\left(x, R_i'\right) \text{ for all } i \in N,$$

then $x \in F(R')$.

One can easily verify that singleton strong winner monotonicity is necessary for the externally stable implementation by a code of individual rights. Additionally, this condition is sufficient for the externally stable implementation by a code of individual rights when combined with unanimity and the no-simultaneous domination of F. Indeed, we have the following theorem.

Theorem 5. *F is* externally stable implementable by a code of individual rights *if and only if it satisfies the conditions of* singleton strong winner monotonicity*,* unanimity*, and* no-simultaneous domination of F.

Proof. Consider the code of individual rights γ in the proof of Theorem 4. Since the proof readily follows from arguments similar to those used in the proof of Theorem 3, we omit it here. □

Remark 3. In contrast to Remark 1, singleton strong monotonicity and unanimity also fully characterize the class of SCRs that are implementable in strong equilibria via codes of individual rights. Moreover, singleton strong monotonicity, unanimity and no-simultaneous domination of F fully characterize the class of SCRs that externally stable implementable in strong equilibria via codes of individual rights.

Non-equivalence According to Theorem 1 of KY, coalition formation does not bring any value added to the implementation of core equilibria by rights structures; for all purposes, it is sufficient to focus on unit coalitions. We now show that, once one focuses on the allocation of blocking powers via the design of codes of rights, the implementation of core equilibria may require non-singleton coalitions.

By Theorems 2 and 4, it is not evident whether the class of SCRs implementable via codes of rights is equal to that of SCRs implementable by codes of individual rights. We find they are not identical. An example is the Condorcet solution, which can be defined as follows. For all $P \in \mathcal{P}$:

$$CON(P) \equiv \left\{x \in Z \mid \text{for all } y \in Z : \left|\{i \in N \mid x P_i y\}\right| \geq \left|\{i \in N \mid y P_i x\}\right|\right\},$$

where $\mathcal{P}$ is a (nonempty) set of profiles of linear orderings for which the solution is nonempty. The non-equivalence result can be stated as follows.

Theorem 6.

(i) Singleton strong monotonicity *implies* strong monotonicity.
(ii) Strong monotonicity *does not imply* singleton strong monotonicity.

Proof. The proof of part (i) is obvious and thus omitted. Let us now demonstrate part (ii) by assuming that $N \equiv \{1, 2, 3\}$ and $Z \equiv \{x, y\}$ with $x \neq y$.[16] Let $\mathcal{P}_Z$ be the set of all profiles of linear orderings over Z. Moreover, let CON be the Condorcet solution.

One can verify that there are four profiles $\{P^0, P^3, P^2, P^1\} \subseteq \mathcal{P}_Z$ at each x being CON-optimal, since x is preferred to y by every agent at profile P^0 and by everyone except agent $j \in N$ at profile P^j. Similarly, let $\{\hat{P}^0, \hat{P}^3, \hat{P}^2, \hat{P}^1\} \subseteq \mathcal{P}_Z$ be the profiles at which y is CON-optimal, since y is preferred to x by everyone at profile $\hat{P}^0$ and by everyone except agent $j \in N$ at profile $\hat{P}^j$. By Korpela et al. (2018), we already know that CON is strongly monotonic. To complete the proof, we need only to show that CON violates singleton strong monotonicity. One can verify that

$$\Lambda_i^{CON}(x) = \{x\} \text{ and } \Lambda_i^{CON}(y) = \{y\}, \text{ for all } i \in N.$$

Then, by construction, one can verify that, for each $j = 0, 1, 2, 3$ and $i \in N$, it holds that

$$\Lambda_i^{CON}(x) \subseteq L\big(x, P_i^j\big) \text{ and } \Lambda_i^{CON}(x) \subseteq L\big(x, \hat{P}_i^j\big),$$
$$\Lambda_i^{CON}(y) \subseteq L\big(y, P_i^j\big) \text{ and } \Lambda_i^{CON}(y) \subseteq L\big(y, \hat{P}_i^j\big).$$

Now, to determine whether CON violates the condition of singleton strong monotonicity, it suffices to observe that, for any $j = 0, 1, 2, 3$, we have $x \in CON(P^j) - CON(\hat{P}^j)$ but $\Lambda_i^{CON}(x) \subseteq L(x, \hat{P}_i^j)$ for each $i \in N$, in *violation* of singleton strong monotonicity. □

The example constructed to prove part (ii) of Theorem 6 can be used to show that the Pareto solution, Po, is not singleton strongly monotonic.[17] In light of Theorems 2 and 4, the main implication of Theorem 6 can be formally stated as follows.

[16] For simplicity, we prove the claim by assuming $n = 3$. The proof will be similar for $n > 3$.

[17] To this end, we observe that $x \in Po(P^0)$, $\Lambda_i^{Po}(x) = \{x\} \subseteq L(x, \hat{P}_i^0)$ for each agent i but yet $x \notin Po(\hat{P}^0)$.

Corollary 1. Implementation by codes of rights *is not equivalent to* implementation by codes of individual rights.

One may wonder whether the non-equivalence result of Corollary 1 extends to the notion of externally stable implementation by a code of rights. The answer is yes. This readily follows from the facts that singleton strong winner monotonicity implies singleton strong monotonicity, the Condorcet solution is externally stable implementable when the number of voters is odd (see Korpela et al., 2018), and the Condorcet solution is not singleton strongly monotonic from the proof of part (*b*) of Theorem 6.

Corollary 2. Externally stable implementation by codes of rights *is not equivalent to* externally stable implementation by codes of individual rights.

5. Concluding remarks

Since the seminal contribution of Maskin (1999), economists have been interested in understanding how to circumvent the limitations imposed by Maskin monotonicity by exploring the possibilities offered by approximate (as opposed to exact) implementation (Abreu and Matsushima, 1992; Abreu and Sen, 1991), as well as by implementation under the refinements of Nash equilibria (Moore and Repullo, 1988; Abreu and Sen, 1990; Palfrey and Srivastava, 1991; Jackson, 1992; Vartiainen, 2007) and repeated implementation (Kalai and Ledyard, 1998; Lee and Sabourian, 2011; Mezzetti and Renou, 2017).

Coalitional implementation does not quite fit any of these literature strands, as the coalitional approach relies on a certain degree of coordination by agents within a coalition. The theory is thus silent on how this will take place but assumes it will when the conditions for cooperation are appropriate. As demonstrated by KY, a powerful mode of implementation becomes feasible due to this abstraction from the details.

We complement the seminal analysis of KY by providing a full characterization of the class of SCRs implementable by codes of rights, as well as of the class of SCRs implementable by codes of individual rights. In contrast to implementation by rights structures, the specialization of the state space in the set of outcomes has a rather intuitive implication, as we prove that, to implement an SCR by a code of rights, blocking powers need to be allocated to non-singleton coalitions. This is the case if we want to implement, for example, the Pareto or Condorcet solutions. We have also shown that this insight is robust and extends to the implementation by codes of rights for alternative definitions of the core, such as an externally stable core.

A persistent criticism of the theory of implementation is that the mechanisms used in the constructive proofs have unnatural features (Abreu and Matsushima, 1992; Jackson, 1992, 2001). The reason is that the devised mechanisms rely on tail-chasing constructions, such as the integer or modulo games, to ensure that undesired strategy combinations do not lead to an equilibrium. KY have shown that implementation by rights structures, as well as by codes of rights, do not suffer from this criticism. They have achieved this important result by focusing on the unrestricted domain of linear orderings, for which the class of (Maskin) monotonic SCRs is "small" and the implementing mechanisms are relatively simpler (Saijo, 1987; Dasgupta et al., 1979; Muller and Satterthwaite, 1977). However, it remains unclear whether this important result generalizes to other preference domains or hinges upon their domain assumption. In this regard, the characterization results presented in this paper remove any doubt, by showing that the result generalizes to any type of preference domain, even preference domains admitting indifference.

We believe that the implementation framework in this paper is simple and intuitively appealing, and may thus have an important bearing on mechanism design. The developed methodology is thus likely to prove useful in the important task of analyzing environments, where agents have incomplete information, particularly on the preferences of the other agents they are facing, as well as on incorporating elements of farsightedness.[18] Therefore, this is a fruitful area for future research.

Appendix A

Proof of Theorem 3. "*Only If*": Assume that γ externally stable implements F. Since the external stable core satisfies the property of external stability defined in Definition 10, it is clear that F satisfies unanimity as well as the condition of no-simultaneous domination of F. Then, we only show that F satisfies strong winner monotonicity. Take any R and x so that $x \in F(R)$. Take any R' so that $\Lambda_K^F(x) \cap F(R') \subseteq L(x, R'_K)$ for all $K \in \mathcal{N}_0$. We show that $x \in F(R')$. Assume, to the contrary, that $x \notin F(R') = EC(\gamma, R')$. Then, there exist $y \in EC(\gamma, R')$ and $K \in \gamma(x, y)$ so that $y P'_K x$. Considering implementability, $y \in F(R')$. Take any $\bar{R} \in F^{-1}(x)$. Since $x \in EC(\gamma, \bar{R})$ and $K \in \gamma(x, y)$, it follows that $y \in L(x, \bar{R}_K)$; otherwise, $x \notin C(\Gamma, \bar{R})$, which is a contradiction. Since the choice of $\bar{R} \in F^{-1}(x)$ is arbitrary, we have $y \in \Lambda_K^F(x)$. By our initial assumption that $\Lambda_K^F(x) \cap F(R') \subseteq L(x, R'_K)$, it follows that $y \in L(x, R'_K)$, which is a contradiction. Therefore, F is strongly winner monotonic.

"*If*": Assume that F satisfies strong winner monotonicity, unanimity, and no-simultaneous domination of F. Let us define $\gamma : Z \times Z \twoheadrightarrow \mathcal{N}$ as in the proof of Theorem 2. We show that γ externally stable implements F. We fix any R.

Since strong winner monotonicity implies strong monotonicity, Theorem 2 implies that $F(R) = C(\gamma, R)$. We complete the proof by showing that $C(\gamma, R)$ is an externally stable set.

Conversely, assume that $C(\gamma, R)$ is not externally stable. Then, there exists $y \in Z - C(\gamma, R)$ so that, for all $x \in C(\gamma, R)$, it holds that $y R_K x$ or $K \notin \gamma(y, x)$ for all K. We fix one of outcomes y. We now proceed according to whether $y \in F(\mathcal{R})$ or not.

Case 1: $y \in F(\mathcal{R})$

Then, $y \in F(R')$ for some R'. Since $y \notin F(R) = C(\gamma, R)$, strong winner monotonicity implies that there exist K and x so that $x \in \Lambda_K^F(y) \cap F(R)$ and $x P_K y$. Since $x \in \Lambda_K^F(y)$, it follows that $K \in \gamma(y, x)$ from part (a) of the definition of γ. Then, there exists outcome $x \in C(\gamma, R)$ so that $x P_K y$ for some $K \in \gamma(y, x)$, which is a contradiction.

Case 2: $y \notin F(\mathcal{R})$

Then, $y \in Z - F(\mathcal{R})$. Further, from part (b) of the definition of γ, $K \in \gamma(y, x)$ for all $x \in F(R)$ and all K. Since $C(\gamma, R)$ is not externally stable and $y \in Z - C(\gamma, R)$, it follows that $y R_K x$ for all K, meaning y simultaneously dominates the range of F, which is a contradiction. □

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[18] For some progress in this direction, see Korpela and Lombardi (2019).

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