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Abstract

A generalized thermoelastic problem with temperature- dependent modulus of elasticity and thermal conductivity has been considered in an infinite medium with a cylindrical cavity. A fter applying Laplace-transformation the basic equations are presented in the form of a vector-matrix differential equation and then are solved by eigen-value method. Finally, the expressions of radial displacement, temperature and stress distribution are shown graphically for two different cases to compare the situations between the temperature-dependent and temperature-independent material properties in the inverse-Laplace domain.

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Eigenvalue Approach to the Solution of Generalized Thermoelastic Interactions in an Infinite Body with Cylindrical Cavity

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Abstract. A generalized thermoelastic problem with temperature- dependent modulus of elasticity and thermal conductivity has been considered in an infinite medium with a cylindrical cavity. A fter applying Laplace-transformation the basic equations are presented in the form of a vector-matrix differential equation and then are solved by eigen-value method. Finally, the expressions of radial displacement, temperature and stress distribution are shown graphically for two different cases to compare the situations between the temperature-dependent and temperature-independent material properties in the inverse-Laplace domain.

Keywords. Generalized Thermoelasticity, Laplace Transform, Vector-matrix differential equation, Eigenvalue Method.

1. Introduction

Lord and Shulman [1] introduced the theory of generalized thermoelasticity with one relaxation time parameter for the special case of an isotropic body. Dhaliwal and Sherief[2] extended this theory to include the anisotropic case. In this theory, a modified law of heat conduction including both the heat flux and its time derivative replaces the conventional Fourier's law. The heat equation associate with this theory is hyperbolic and hence predicts finite speed of propagation for heat waves. The uniqueness of solution for this theory was proved by Ignaczak[3,4], by Dhaliwal and Sherief[2] and by Sherief[5].

Green and Lindsay[6] obtained the theory of thermoelasticity with two relaxation time parameters. In this theory, the classical Fourier's law of heat conduction is not violated when the body under consideration has a centre of symmetry. The uniqueness of solution of this solution was established by Green[7]. The fundamental solution was obtained by Sherief[8].

Indealing with coupled or generalized thermoelastic problems, the solution procedure is to choose a suitable thermoelastic potential function, but this method has certain limitations as discussed by Bahar and Hetnarski[9]. Here we prefer to adopt the eigenvalue method as in Das et.al.[10] and as such, the physical quantities involved in the boundary and initial conditions are directly solvable from the governing equations.

Previously, most of the investigations of thermoelasticity were done under the assumption of

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temperature-independent material properties, but at high temperature the material characteristics such as modulus of elasticity, Poisson's ratio, coefficient of thermal expansion and the thermal conductivity are no longer constants[11]. In recent years, it has become necessary to take into account the actual behavior of material characteristics.
In this paper, we have considered a infinite medium with a cylindrical cavity where the modulus of elasticity and thermal conductivity are temperature dependent and comparisons are made graphically between the temperature dependent and temperature independent material properties.

NOMENCLATURE

λ, μ= Lamè constants.
u = Displacement component.
t = Time variable.
σ_{ij}=Stress component.
T= Absolute temperature.
T_0 = Reference temperature.
ρ = Mass density.
C_e = Specific heat.
K = Coefficient of thermal conductivity.
κ=Coefficient of thermal diffusivity.
$\gamma = (3\lambda + 2\mu)\alpha_t$.
α^* = Empirical material constant.
τ_0= Thermal relaxation time parameter.
H(t)= Heaviside unit step function.

2. Formulation of the problem

We have taken into account the generalized thermoelasticity with one relaxation time in an isotropic infinite medium which has a cylindrical cavity of radius R.
We use cylindrical co-ordinate system (r, ψ, z) with z-axis lying along the axis of the cylinder. Due to symmetry, all functions are dependent only on r and t.
i.e. if $\overrightarrow{u}$=(u_r, u_ψ, u_z) be the displacement vector, then
u_r=$u(r,t)$; $u_\psi(r,t) = 0 = u_z(r,t)$
In this paper, the modulus of elasticity and the heat conductivity are taken to be temperature dependent as
$\lambda = \lambda_0 f(T)$; $\mu = \mu_0 f(T)$; $K = K_0 f(T)$; $\gamma = \gamma_0 f(T)$;
where $f(T) \approx f(T_0)$=$1 - \alpha^* T_0 = \frac{1}{\alpha_T}$ considering $|\frac{T-T_0}{T_0}| << 1$
The equation of motion and the heat conduction equations are :

$$\rho\alpha_T\ddot{u} = (\alpha_0 + 2\mu_0)\frac{\partial e}{\partial r} - \gamma_0\frac{\partial \theta}{\partial r} \tag{1}$$

and

$$\nabla^2\theta = (\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2})(\frac{\theta}{\kappa} + \frac{\gamma_0 T_0 e}{K_0}) \tag{2}$$

where $e = \frac{1}{r}\frac{\partial(ru)}{\partial r}$; $\theta = T - T_0$; $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$
The stress components are :

$$\alpha_T \sigma_{rr} = 2\mu_0 \frac{\partial u}{\partial r} + \lambda_0 e - \gamma_0 \theta$$
$$\alpha_T \sigma_{\psi\psi} = 2\mu_0 \frac{u}{r} + \lambda_0 e - \gamma_0 \theta$$
$$\alpha_T \sigma_{zz} = \lambda_0 e - \gamma_0 \theta$$
$$\sigma_{zr} = \sigma_{\psi r} = \sigma_{z\psi} = 0 \tag{3}$$

We define the following non-dimensional variables :
$r' = \sqrt{\frac{\lambda_0+2\mu_0}{\rho}}\frac{r}{\kappa}; u' = \sqrt{\frac{\lambda_0+2\mu_0}{\rho}}\frac{u}{\kappa};\ t' = \sqrt{\frac{\lambda_0+2\mu_0}{\rho}}\frac{t}{\kappa};\ \tau_0' = \sqrt{\frac{\lambda_0+2\mu_0}{\rho}}\frac{\tau_0}{\kappa};\ R' = \sqrt{\frac{\lambda_0+2\mu_0}{\rho}}\frac{R}{\kappa};$
$\theta' = \frac{\theta}{T_0}; \sigma' = \frac{\sigma}{\mu_0}$
Using the above non-dimensional quantities in equations (1), (2) and (3), we get(omiting the primes) :

$$\alpha_T \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} - a\frac{\partial \theta}{\partial r} \tag{4}$$

$$\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} - (\dot{\theta} + \tau_0 \ddot{\theta}) = g(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2})(\frac{u}{r} + \frac{\partial u}{\partial r}) \tag{5}$$

$$\alpha_T \sigma_{rr} = \beta^2 \frac{\partial u}{\partial r} + (\beta^2 - 2)\frac{u}{r} - b\theta$$
$$\alpha_T \sigma_{\psi\psi} = (\beta^2 - 2)\frac{\partial u}{\partial r} + \beta^2 \frac{u}{r} - b\theta$$
$$\alpha_T \sigma_{zz} = (\beta^2 - 2)(\frac{\partial u}{\partial r} + \frac{u}{r} - b\theta \tag{6}$$

3. Method of Solution

Consider the definition of Laplace transform :

$$\bar{T}(r,p) = \int_0^\infty T(r,t)exp(-pt)dt \tag{7}$$

Using the above Laplace transform on time t to the equations (4), (5) and (6), we get:

$$\alpha_T p^2 \overline{u} = \frac{d^2\overline{u}}{dr^2} + \frac{1}{r}\frac{d\overline{u}}{dr} - \frac{\overline{u}}{r^2} - a\frac{d\overline{\theta}}{dr} \tag{8}$$

$$\frac{d^2\overline{\theta}}{dr^2} + \frac{1}{r}\frac{d\overline{\theta}}{dr} - (p + \tau_0 p^2)\overline{\theta} = g(p + \tau_0 p^2)(\frac{\overline{u}}{r} + \frac{d\overline{u}}{dr}) \tag{9}$$

$$\alpha_T \overline{\sigma}_{rr} = \beta^2 \frac{d\overline{u}}{dr} + (\beta^2 - 2)\frac{\overline{u}}{r} - b\overline{\theta}$$
$$\alpha_T \overline{\sigma}_{\psi\psi} = (\beta^2 - 2)\frac{d\overline{u}}{r} + \beta^2 \frac{\overline{u}}{r} - b\overline{\theta}$$
$$\alpha_T \overline{\sigma}_{zz} = (\beta^2 - 2)(\frac{d\overline{u}}{dr} + \frac{\overline{u}}{r} - b\overline{\theta} \tag{10}$$

Differentiating (9) with respect to r and using (8) in the resulting equation :

$$\frac{d^3\overline{\theta}}{dr^3} + \frac{1}{r}\frac{d^2\overline{\theta}}{dr^2} - \frac{1}{r^2}\frac{d\overline{\theta}}{dr} = p(1 + \tau_0 p)(1 + ag)\frac{d\overline{\theta}}{dr} + \alpha_T g p^3 (1 + \tau_0 p)\overline{u} \tag{11}$$

Now we write equations (8) and (11) in the form of a vector-matrix differential equation as :

$$L\underline{V} = A\ \underline{V} \tag{12}$$

where $L \equiv \frac{d^2}{dR^2} + \frac{1}{R}\frac{d}{dR} - \frac{1}{R^2}$ is a Bessel operator.

$$\underline{v} = \begin{bmatrix} \overline{u} & \frac{d\overline{\theta}}{dr} \end{bmatrix}^T \ ; \ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$a_{11} = \alpha_T p^2$; $a_{12} = a$; $a_{21} = \alpha_T g p^3(1+\tau_0 p)$; $a_{22} = p(1+\tau_0 p)(1+\varepsilon)$

Let $\lambda_i = \alpha_i^2$, $i = 1, 2$ be the two eigen values of the matrix A which can be determined from the following characteristic equation :-

$\alpha^4 - (a_{11} + a_{22})\alpha^2 + (a_{11}a_{22} - a_{12}a_{21}) = 0$

The eigen vectors corresponding to the eigen values $\lambda_i = \alpha_i^2$, $i = 1, 2$ are respectively :-

$V_i = \begin{bmatrix} -a_{12} \\ a_{11} - \alpha_i^2 \end{bmatrix}$

Hence by Eigenvalue Method [Appendix], the solution of $\underline{v}$ can be written as :-

$$\underline{v} = \begin{bmatrix} \overline{u} \\ \frac{d\overline{\theta}}{dr} \end{bmatrix} = \sum_{i=1}^{2} [A_i V_i K_1(\alpha_i r) + B_i V_i I_1(\alpha_i r)] \tag{13}$$

where K_1 and I_1 are the modified Bessel Functions of second kind of order 1.

The radial displacement and the temperature in the Laplace transformed domain which are both bounded at infinity, are now can be written as :-

$$\overline{u} = -a_{12}[A_1 K_1(\alpha_1 r) + A_2 K_1(\alpha_2 r)] \tag{14}$$

$$\overline{\theta} = -[A_1 \frac{a_{11} - \alpha_1^2}{\alpha_1} K_0(\alpha_1 r) + A_2 \frac{a_{11} - \alpha_2^2}{\alpha_2} K_0(\alpha_2 r)] \tag{15}$$

4. Boundary Conditions

To calculate the unknown constants A_1 and A_2 , we use the following boundary conditions on the internal boundary $r = R$:-

Case I:-

In this case the cavity surface is assumed to be maintained at zero temperature and is subjected to a ramp-type boundary load , i.e.

$$\sigma_{rr}(R,t) = -\sigma_0 H(t) \ , \ \theta(R,t) = 0 \tag{16}$$

Using Laplace Transformation on above :-

$$\overline{\sigma}_{rr}(R,p) = -\frac{\sigma_0}{p} \ , \ \overline{\theta}(R,p) = 0 \tag{17}$$

Case II:-

Considering the thermoelastic interactions when the surface of the cavity is stress-free and kept at a temperature $\theta(R,t)$,then the boundary condition takes the form

$$\sigma_{rr}(R,t) = 0 \ , \ \theta(R,t) = \theta_0 e^{-\omega t} \tag{18}$$

Using Laplace Transformation on above :-

$$\overline{\sigma}_{rr}(R,p) = 0 \ , \ \overline{\theta}(R,p) = \frac{\theta_0}{p+\omega} \tag{19}$$

5. Numerical Results

For our final result we have to find out the Laplace-inversion of radial displacement , temperature and stress distribution which are very complicated in nature.
To evaluate these we have used the Zakian method [12] . For our numerical calculation we have chosen the Copper material . The values of the constants are given by :-
$\lambda_0 = 7.76$; $\mu_0 = 3.86$; $\gamma_0 = (3\lambda_0 + 2\mu_0)\alpha_t = 55.18$; $\alpha_t = 1.78$; $K_0 = 386$; $c_E = 3.831$; $\rho = 8954$; $T_0 = 293K$; $\beta^2 = 4.01$; $b = 0.042$; $a = 0.0105$; $g = 1.61$; $\tau_0 = 0.01$; ε=ag=0.017 ; $R = 1$; $t = 0.3$; $\sigma_0 = 1.2$; $\theta_0 = 1.5$; $\omega = 0.5$

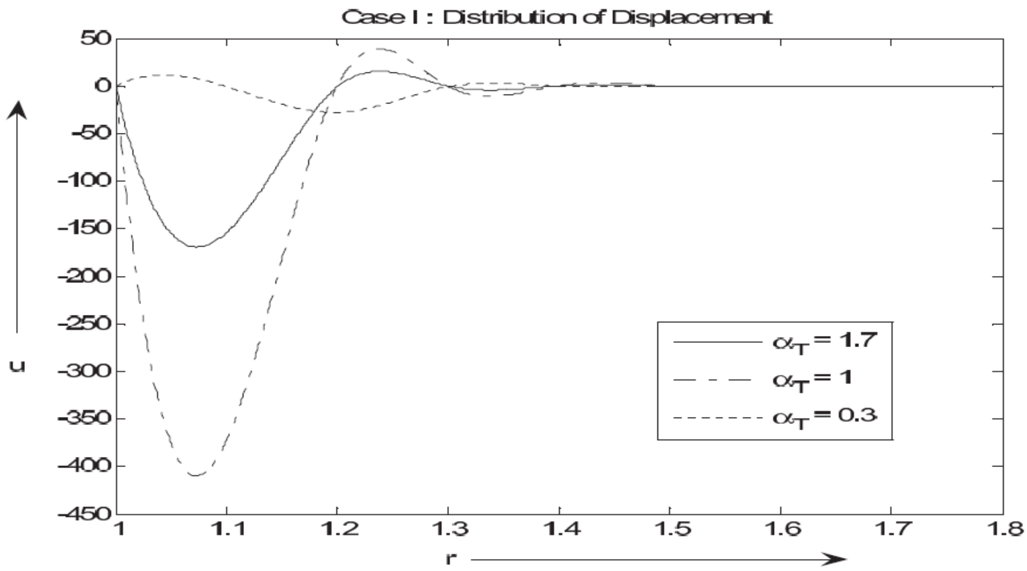
Finally the expressions of radial displacement u , temperature θ and stress distribution σ_{rr} for the above two cases are presented graphically (shown below) where the curves are plotted for different values of α_T (viz. 0.3 , 1 , 1.7). From the following graphs we observe that :-

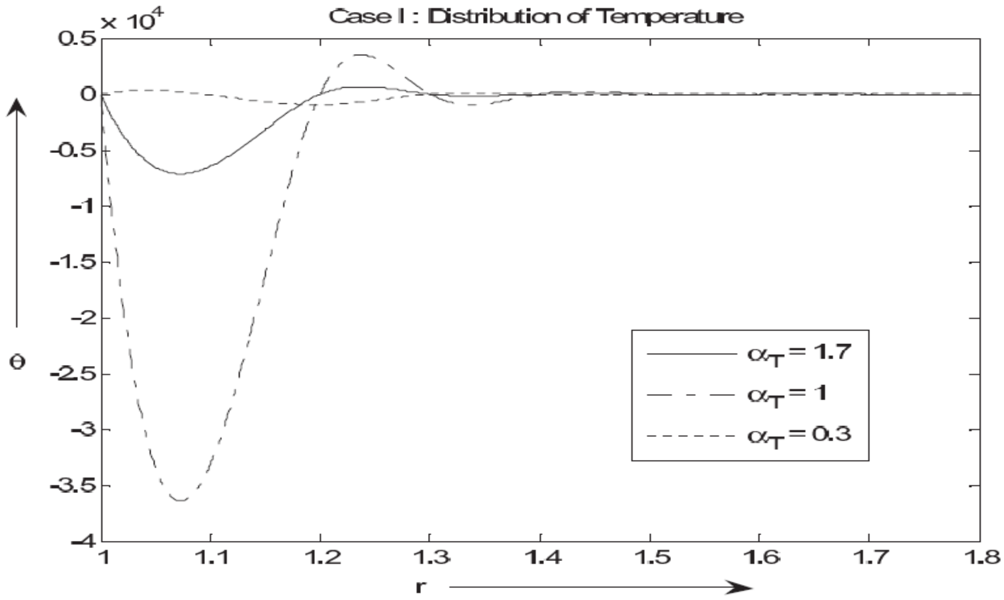
i) we see that the three quantities stated above have their maximum values (absolute value) for temperature independent case i.e. when $\alpha_T = 1$ for both case I and case II.

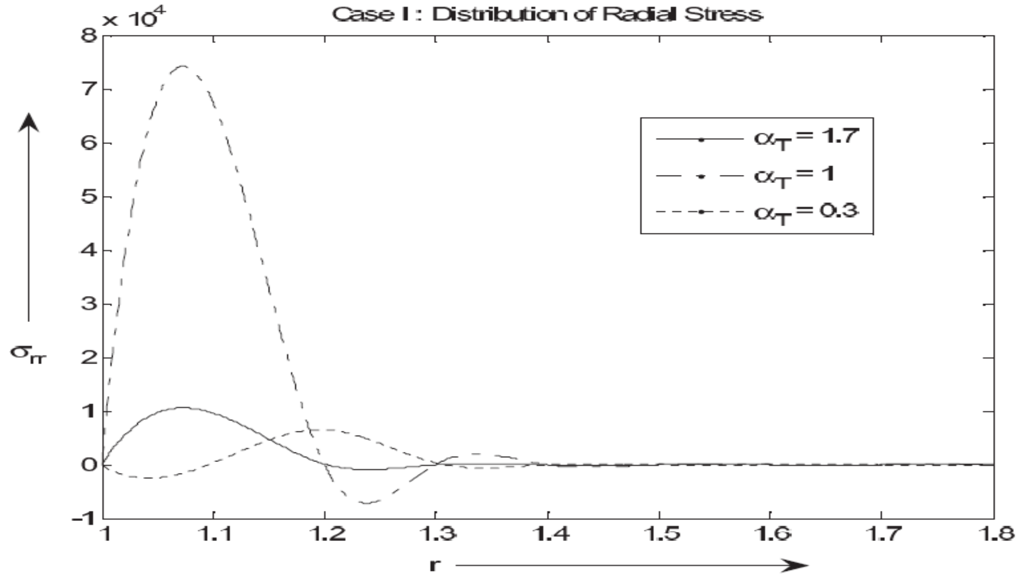
ii) for case I :-
• the distribution of displacement has greater absolute values for $\alpha_T = 1.7$ (i.e. $\alpha_T > 1$) than for $\alpha_T = 0.3$(i.e. $\alpha_T < 1$).
• the temperature distribution has values near about zero for $\alpha_T = 0.3$.
• for stress distribution the absolute values are more or less same for $\alpha_T > 1$ and $\alpha_T < 1$.

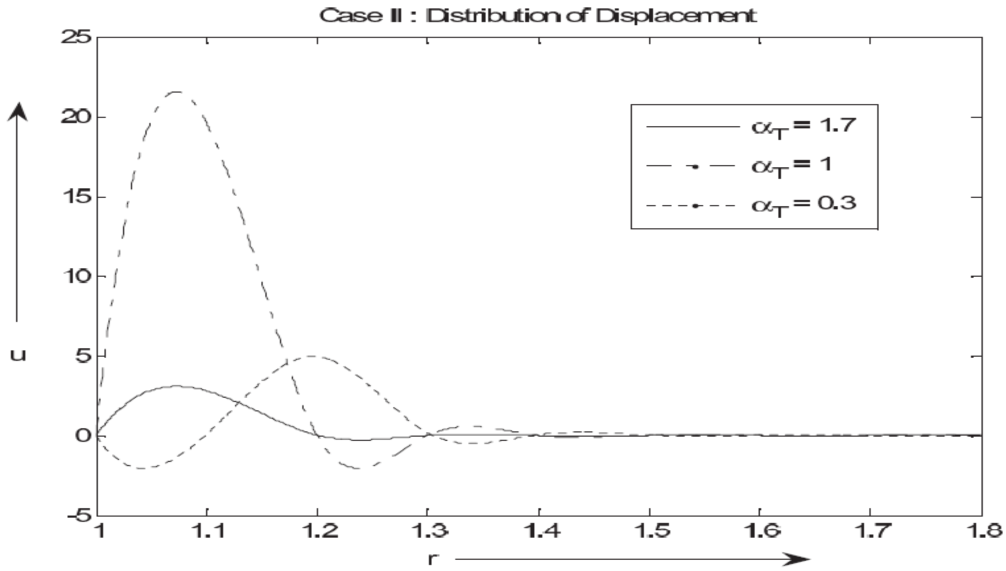
iii) for case II :-
• the displacement distribution has greater absolute values for $\alpha_T < 1$ than for $\alpha_T > 1$ which is the reverse of case I.
• for temperature distribution the absolute values are almost same for both $\alpha_T > 1$ and $\alpha_T < 1$.
• the radial tress distribution has greater absolute values for $\alpha_T < 1$ than for $\alpha_T > 1$.

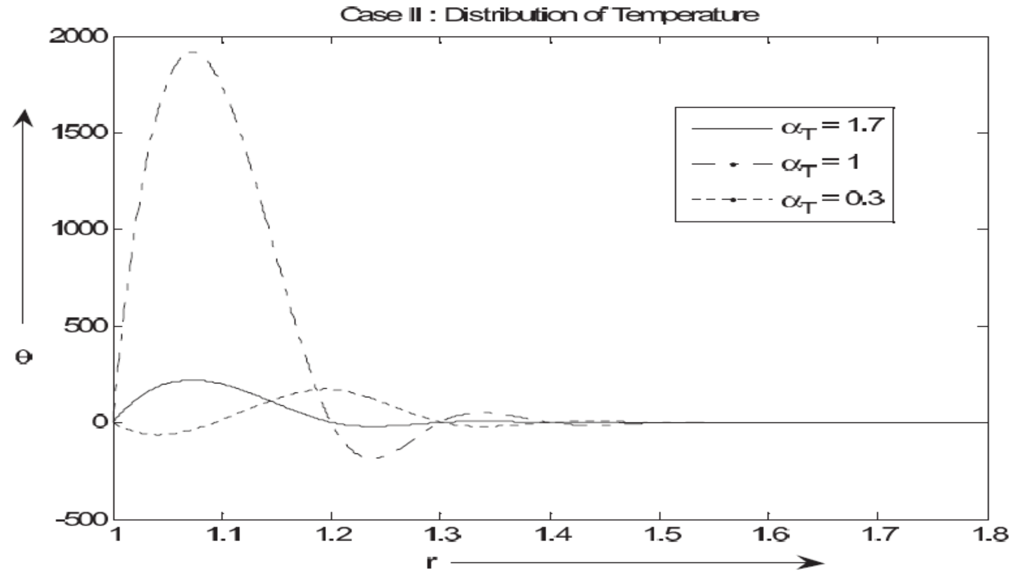
6. Tables, figures and list

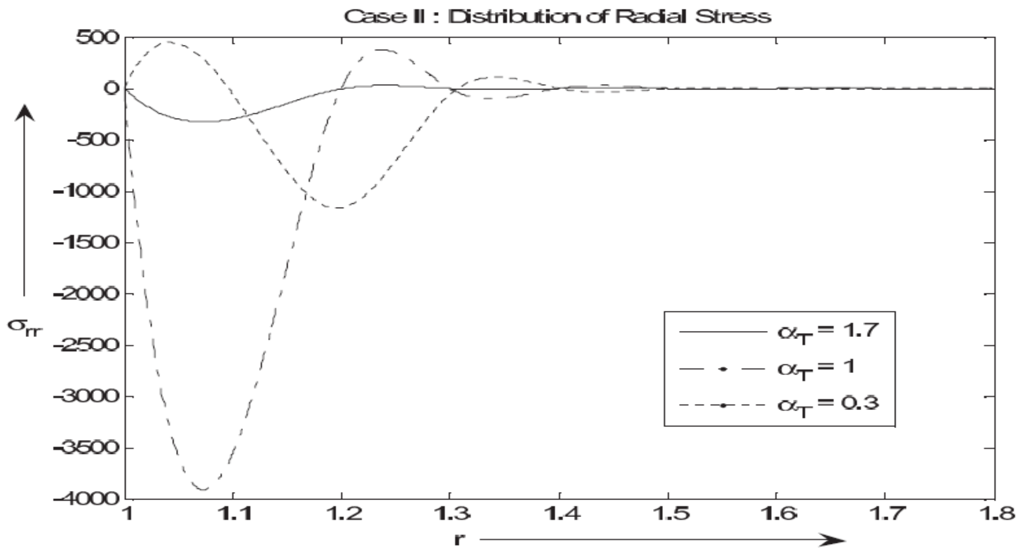





Case I : Distribution of Radial Stress
× 10^4
8
7
6
5
4
3
2
1
0
-1
σrr
α_T = 1.7
α_T = 1
α_T = 0.3
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
r


Case II : Distribution of Displacement
25
20
15
10
5
0
-5
u
α_T = 1.7
α_T = 1
α_T = 0.3
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
r


Case II : Distribution of Temperature
2000
1500
1000
500
0
-500
θ
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
r
$\alpha_T = 1.7$
$\alpha_T = 1$
$\alpha_T = 0.3$


Case II : Distribution of Radial Stress
500
0
-500
-1000
-1500
-2000
-2500
-3000
-3500
-4000
σ_{rr}
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
r
$\alpha_T = 1.7$
$\alpha_T = 1$
$\alpha_T = 0.3$

Appendix

Consider the differential equation in the form :

$$L\underline{V} = A\ \underline{V} \tag{20}$$

where $L \equiv \frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{n^2}{x^2}$ is a Bessel operator.
Let

$$A = V\Lambda V^{-1} \tag{21}$$

where $\Lambda = \begin{bmatrix} \lambda_1 & ... & 0 \\ ... & \lambda_2 & ... \\ ... & ... & ... \\ 0 & ... & \lambda_n \end{bmatrix}$

is a diagonal matrix whose elements λ_1. λ_2. ....,λ_n are the distinct eigenvalues of A. Let $\underline{V}_1$. $\underline{V}_2$. .....,$\underline{V}_n$ be the eigenvectors of A corresponding to λ_1. λ_2. ....,λ_n respectively, and

$$V = [\underline{V}_1\ \underline{V}_2\ ...\ \underline{V}_n] \tag{22}$$

Substituting (21) in (20) and premultiplying by V^{-1}, we get

$$L\underline{y} = \Lambda\underline{y}\ ;\ \underline{y} = V^{-1}\underline{v} \tag{23}$$

as a system of decoupled equations.
A typical rth equation of (23) is

$$Ly_r = \lambda_r y_r$$

$$\frac{d^2y_r}{dx^2} + \frac{1}{x}\frac{dy_r}{dx} - (\lambda_r + \frac{n^2}{x^2})y_r = 0 \tag{24}$$

Case (i)
When $\lambda_r = \alpha_r^2$, the solution of equation (24) can be written as,

$$y_r = A_r K_n(\alpha_r x) + B_r I_n(\alpha_r x) \tag{25}$$

n is integer and A_r, B_r are constants. K_n, I_n are modified Bessel functions of the second kind of order n.

Case (ii)
When $\lambda_r = -\alpha_r^2$, the solution can be written as

$$y_r = A_r J_n(\alpha_r x) + B_r Y_n(\alpha_r x) \tag{26}$$

n is integer and J_n, Y_n are Bessel functions of the first kind of order n.
Hence the complete solution in this case can be written as $\underline{v} = \sum_{r=1}^{n} V_r Y_r$.

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