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Invariant Measures on Hypergroups: An overview

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Abstract

A hypergroup is roughly speaking a locally compact Hausdorff space which has enough structure so that a convolution on the corresponding vector space of Radon measures makes it a Banach algebra. At the center of harmonic analysis is the question of the existence of a Haar measure. The existence of a Haar measures for compact and discrete hypergroup has been done successfully. But for general hypergroup the question remain open. We put together here, work that have been done in attempts to solve this problem. Essentially we present Spector's proof for commutative hypergroups using the more unifying definition known as DJS-definition of a hypergroup.

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Invariant Measures on Hypergroups: An overview

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Abstract. A hypergroup is roughly speaking a locally compact Hausdorff space which has enough structure so that a convolution on the corresponding vector space of Radon measures makes it a Banach algebra. At the center of harmonic analysis is the question of the existence of a Haar measure. The existence of a Haar measures for compact and discrete hypergroup has been done successfully. But for general hypergroup the question remain open. We put together here, work that have been done in attempts to solve this problem. Essentially we present Spector's proof for commutative hypergroups using the more unifying definition known as DJS-definition of a hypergroup.

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I. Introduction

We first recall some standard notations. Let H be a locally compact Hausdorff space :

i.: $C(H)$: the space of complex continuous functions on H,
ii.: $C_b(H)$: the space of bounded elements of $C(H)$
iii.: $C_0(H)$: the space of elements of $C_b(H)$ which tends to 0 at ∞
iv.: $C_c(H)$: the space of elements of $C_0(H)$ with compact support
v.: $C_c^+(H)$: the space of nonnegative elements of $C_c(H)$.
vi.: $M(H)$ denotes the set of finite regular Borel measures.
vii.: $M_1(H)$ denote the set of probability measures.
viii.: If $\mu \in M(H)$ then $Supp(\mu) = \{x \in H : \text{if } V \text{ is any open set containing } x \text{ then } \mu(V) > 0\}$.
ix.: An unspecified topology on $M_+(H)$ is the cone topology.
x.: $B_\infty(H)$ denotes the extended nonnegative real-valued Borel functions.
xi: If A is any subset of H $\bar{A}$ is the closure of A.

The Michael topology is an important topology generally used in the in what we will call the DJS-hypergroup. For completion we give below its definition.

I.1. The Michael topology

Let S be a locally compact space and let $\mathcal{C}(S)$ be the space of all compact subsets of S. For $A, B \subset S$, $\mathcal{C}_A(B) = \{C \in \mathcal{C}(S) : C \cap A \neq \emptyset \text{ and } C \subset B\}$ Then $\mathcal{C}(S)$ can be given the topology

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generated by the sub-basis of all $\mathcal{C}_U(V)$ for which U and V are open subsets of S. This topology which was developed by Michael[Mi55] has the following properties [Je75]

Properties I.1. **i.:** *If S is compact, then $\mathcal{C}(S)$is compact.*
ii.: *$\mathcal{C}(S)$ is a locally compact space.*
iii.: *The mapping $x \mapsto \{x\}$ is a homeomorphism of S onto a closed subset of $\mathcal{C}(S)$.*
iv.: *The collection of nonempty finite subsets of S is a dense subset of $\mathcal{C}(S)$.*
v.: *If Ω is a compact subset of $\mathcal{C}(S)$, then $B = \bigcup\{A : A \in \Omega\}$ is a compact subset of S.*
vi.: *If S is metrizable with metric d, then the Michael topology on $\mathcal{C}(S)$ is stronger than the Hausdorff topology given by the Hausdorff metric ρ which for $A, B \in \mathcal{C}(S)$ is defined by $\rho(A, B) = max\{h(A, B), h(B, A)\}$ where $h(A, B) = sup\{d(x, B) : x \in A\}$*

II. Definitions of a Hypergroups

Several authors define topological hypergroup and most results depends on the author's definition. In the next section we give the most common definitions used in the theory and end with a synthesis of these definition which is called the DJS-hypergroup.

II.1. Dunkl's Definition of a Hypergroup

A locally compact space H is called a hypergroup if there is a map $\lambda : H \times H \to M_1(H)$ with the following properties:

D_1**.:** For each $f \in C_c(H)$ the map

$$(x, y) \longmapsto \int_H f d\lambda(x, y)$$

is in $C_b(H \times H)$ and

$$x \longmapsto \int_H f d\lambda(x, y)$$

is in $C_b(H)$ for each $y \in H$

D_2**.:** The convolution on $M(H)$ defined implicitly by

$$\int_H f d\mu * \nu = \int_H d\mu(x) \int_H d\nu(y) \int_H f d\lambda(x, y)$$

$\mu, \nu \in M(H)$, $f \in C_0(H)$ is associative.

D_3**.:** There is a point (the identity) $e \in H$ such that

$$\lambda(x, e) = \delta_x \qquad (x \in H)$$

D_4**.:** The hypergroup H is said to be commutative if

$$\lambda(x, y) = \lambda(y, x) \qquad \forall x, y \in H$$

Remark II.1. *Dunkl does not require the existence of an involution in his definition, rather he called any hypergroup ,with an involution, which possesses an invariant measure, a $*$-hypergroup. He does not require that the support of the convolution of two point masses be compact and consequently does not use the Michael topology in his definition.*

II.2. Jewett's Definition of a Convos

A pair $(K, *)$ will be called a **semiconvo** if the following five conditions are satisfied:

J_1**.:** K is a nonvoid locally compact Hausdorff space.

J_2**.:** The symbol $*$ denotes a binary operation on $M(K)$ and with this operation $M(K)$ is a complex (associative) algebra.

J_3.: The bilinear mapping $(\mu, \nu) \longmapsto \mu * \nu$ is positive-continuous. (That is, $\mu * \nu \geq 0$ whenever $\mu \geq 0$ and $\nu \geq 0$ and the convolution restricted to $M_+(S) \times M_+(S) \longrightarrow M_+(S)$ is continuous).

J_4.: If $x, y \in K$ then $\delta_x * \delta_y$ is a probability measure with compact support.

J_5.: The mapping $(x, y) \longmapsto Supp(\delta_x * \delta_y)$ from $K \times K$ to $\mathcal{C}(K)$ (with the Michael topology) is continuous.

If in addition we also have,

J_6.: There exists a (necessarily unique) element e of K such that $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$ for all $x \in K$

J_7.: There exists a (necessarily unique) involution $x \longmapsto x^-$ of K such that (for $x, y \in K$) the element e is in the support of $\delta_x * \delta_y$ if and only if $x = y^-$, the semiconvo will be called a **convo**.

Remark II.2. *Dunkl's definition of hypergroup is a commutative semiconvo with identity. Jewett's commutative convo is according to Dunkl's definition, a $*$-hypergroup.*

II.3. Spector's Definition of a Hypergroup

A hypergroup is a locally compact space X, together with a convolution $*$ that makes $M(X)$ a Banach algebra and satisfy the following properties:

S_1.: $M_1(X) * M_1(X) \subset M_1(X)$

S_2.: $*$ is separately continuous from $M_1(X) \times M_1(X)$ to $M_1(X)$ with the weak topology defined by the duality between $M(X)$ and $C_0(X)$ $(\sigma(M(X), C_0))$.

S_3.: The map $(x, y) \longmapsto \delta_x * \delta_y$ is continuous from $X \times X$ onto $M_1(X)$ with the weak topology induced by $\sigma(M(X), C_0)$

S_4.: There is a necessarily unique point e called the "identity element of the hypergroup X", such that δ_e is the identity element of the convolution $*$.

S_5.: There is an involutive homeomorphism of X onto X, denoted by $x \longmapsto x^-$ with natural extension to $M(X)$ satisfying $(\mu * \nu)^- = \nu^- * \mu^-$; in particular $e^- = e$ this homeomorphism will be called the "symmetry of the hypergroup".

S_6.: For every $x, y \in X$, $e \in Supp(\delta_x * \delta_y)$ if and only if $x = y^-$

S_7.: For any compact subset K of X and any neighborhood V of K there exists a neighborhood U of e such that

(1) $Supp(\mu) \subset K$ and $Supp(\nu) \subset U$ imply $Supp(\mu * \nu) \subset V$ and $Supp(\nu * \mu) \subset V$

(2)$Supp(\mu) \subset K$ and $Supp(\nu) \subset U^c$ imply that the support of $\mu * \nu^-$,$\mu^- * \nu$,$\nu * \mu^-$,and $\nu^- * \mu$ are disjoint with U.

Remark II.3. *Spector does not require the support of the convolution of two point masses to be compact, which leads sometimes to some technical complications as he acknowledges himself. Actually he also acknowledges not having any substantial example where this condition fails. Consequently there is no use of the Michael topology in his proofs.*

We now give a general definition of a hypergroup which is now called the DJS-hypergroup. To this end, we start with the definition of a semihypergroup and give simple examples of semihypergroups and hypergroups.

III. The DJS-Hypergroup

A nonempty locally compact Hausdorff space S will be called a **semihypergroup** if the following conditions are satisfied:

(SH_1): $(M_b(S), +, *)$ is a Banach algebra.

(SH_2): For all $x, y \in S$, $\delta_x * \delta_y$ is a probability measure with compact support contained in S.

(SH_3): The mapping $(x, y) \mapsto \delta_x * \delta_y$ of $S \times S$ into $M_1(S)$, where $S \times S$ has the product topology and $M_1(S)$ has the weak topology, is continuous.

(SH_4): The mapping$(x, y) \mapsto Supp(\delta_x * \delta_y)$ of $S \times S$ into $\mathcal{C}(S)$ is continuous, where $\mathcal{C}(S)$ is the space of compact subsets of S endowed with the Michael topology, that is the topology generated by the subbasis of all $\mathcal{C}_U(V) = \{C \in \mathcal{C}(S) : C \cap U \neq \emptyset \text{ and } C \subset V\}$ where U and V are open subsets of S.

Remark If $\delta_x * \delta_y = \delta_y * \delta_x$ for all $x, y \in S$, then we say that $(S, *)$ is a commutative semihypergroup. If, in addition, we also have:

SH_5**:** there exists $e \in S$ such that $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x \ \forall x \in S$, and

SH_6**:** There exists a topological involution (a homeomorphism) from S onto S such that $(x^-)^- = x \ \forall x \in S$, with $(\delta_x * \delta_y)^- = \delta_{y^-} * \delta_{x^-}$ and $e \in Supp(\delta_x * \delta_y)$ if and only if $x = y^-$ where for any Borel set B, $\mu^-(B) = \mu(\{x^- : x \in B\})$,

then $(S, *)$ is called a **hypergroup**

Remark III.1.

(i): If $\delta_x * \delta_y = \delta_y * \delta_x$ for all $x, y \in H$ we say that $(H, *)$ is a commutative hypergroup.

(ii): The convolution $*$ on $M(H)$ is defined by

$$\mu * \nu(f) = \int_H f d\mu * \nu = \int_H \mu(dx) \int_H \nu(dy) \int_H f d\delta_x * \delta_y.$$

for all $f \in C_b(H)$.

Example III.1.

1. If $(G, .)$ is a locally compact Hausdorff group, then with convolution defined by $\delta_x * \delta_y = \delta_{xy}$, $(G, *)$ is a hypergroup. Also if a hypergroup is such that the convolution of two point masses is a point mass then it is a topological group.
2. Consider the segment $[0, 1]$ with convolution defined by

$$\delta_r * \delta_s = \frac{1}{2}\delta_{|r-s|} + \frac{1}{2}\delta_{1-|1-r-s|}$$

for all $r, s \in [0, 1]$,then $([0, 1], *)$ is a hypergroup.
Examples of hypergroups could be found in [BH95],[Je75], [Du73],[Sp78].

Definition III.1.

Let $(H, *)$ be a hypergroup , $f \in C(H)$ and $x, y \in H$. Then we define

$$f(x * y) = f_x(y) = f^y(x) = \int_S f d(\delta_x * \delta_y)$$

if this integral exists, even when it is not finite. f_x is called the left translation of f and f^x is called the right translation of f.

Lemma III.1. [Je75]

Let f be a continuous function on H and let $x \in H$

i.: The mapping $(x, y) \mapsto f(x * y)$ is a continuous function on $H \times H$

ii.: f_x and f^x are continuous functions on H

Lemma III.2. [Je75]

Let $f \in B_\infty(H)$, $\mu, \nu \in M_+(H)$ and $x, y, z \in H$

i.: The mapping $(x, y) \mapsto f(x * y)$ is a Borel function on $H \times H$

ii.: f_x and f^x are Borel functions on H
iii.: $\int_H f_x d\mu = \int_H f d(\delta_x * \mu)$
iv.: $f_x(y * z) = f^z(x * y)$

Remark III.2. *Let H be a locally compact hypergroup. Then $\forall x \in H, \mu \in M(H)$, and $f \in C(H)$*

$$\delta_x * \mu(f) = \int_H f_x d\mu$$

($\equiv \mu(f_x)$, say)

and similarly

$$\mu * \delta_x(f) = \mu(f^x)$$

Definition III.2. *A character χ on a hypergroup H is a continuous complex-valued function on H which is not identically zero and satisfies*

$$\int_H \chi d\delta_x * \delta_y = \chi(x)\chi(y)$$

for all $x, y \in H$. A character χ is said to be Hermitian if and only if $\chi(x^-) = \overline{\chi(x)}$.

Example III.2.

Let $X = [0, +\infty)$ and $\varphi_\lambda(x) = \cos \lambda x$ $\lambda \in [0, +\infty)$ then we have the relation

$$\varphi_\lambda(x)\varphi_\lambda(y) = \frac{1}{2}[\varphi_\lambda(x+y) + \varphi_\lambda(x-y)]$$

for all $\lambda \in [0, +\infty)$ since φ_λ is an even function, this relation is equivalent to

$$\varphi_\lambda(x)\varphi_\lambda(y) = \frac{1}{2}[\varphi_\lambda(x+y) + \varphi_\lambda(|x-y|] =$$

$$\frac{1}{2}[\delta_{x+y}(\varphi_\lambda) + \delta_{|x-y|}(\varphi_\lambda)]$$

Let $\sigma_{x,y} = \frac{1}{2}[\delta_{x+y} + \delta_{|x-y|}]$ then $\{\varphi_\lambda\}$ satisfies the product formula

$$\varphi_\lambda(x)\varphi_\lambda(y) = \int \varphi_\lambda(z)\sigma_{x,y}(dz)$$

Now given two Radon measures μ and ν on X we can define a convolution

$$\mu * \nu(f) = \int\int\int f\sigma_{x,y}\mu(dx)\nu(dy)$$

for all $f \in C_c(X)$. With this convolution, $M(X)$ is a Banach algebra [**?**]. By defining a convolution of point masses by

$$\delta_x * \delta_y = \frac{1}{2}[\delta_{x+y} + \delta_{|x-y|}],$$

so that $(X, *)$ is a hypergroup with identity element 0, the involution is the identity function. Furthermore the characters of $(X, *)$ are the orthogonal system $\{\varphi_\lambda\}$, $\lambda \in [0, +\infty)$.

Definition III.3. *Let H be a locally compact hypergroup. A measure m not necessarily finite, will be called left subinvariant if $\delta_x * m$ is defined and $\delta_x * m \leq m$ for all $x \in H$. If we have $\delta_x * m = m$, m will be called a left invariant measure on H.*

(Right invariant measures are defined in the same way).

Example III.3.

The invariant measure on the hypergroup $(X, *)$ in (III.2) is the Lebesgue measure. In general if a system of orthogonal functions with respect to a measure m, has a product formula which defines a hypergroup H then the measure m is the invariant measure of the hypergroup H.

Remark III.3. i. *Jewett* [Je75] *and Spector* [Sp78] *proved that every locally compact hypergroup H has a left subinvariant measure m and $Supp(m) = H$. For a locally compact group the existence of a subinvariant measure implies that of an invariant measure in fact if G is a group and m is such that $\delta_x * m \leq m$ then given any Borel set A, $\delta_x * m(A) \leq m(A)$ but $m(A) = \delta_e * m(A) = \delta_x * \delta_{x^-} * m(A) \leq \delta_x * m(A)$ (In groups $\delta_x * \delta_{x^-} = \delta_e$ and we have $\delta_{x^-} * m(A) \leq m(A)$) and it follows that $\delta_x * m = m$. This is not the case for hypergroup (see an example of Naimark in* [Je75] *9.5) though it is easy to prove that when a compact hypergroup has a subinvariant measure it is also invariant. Both authors also proved the existence of invariant measures for discrete hypergroups. Spector* [Sp78] *proved that if a hypergroup is commutative it has an invariant measure.*

ii. *Jewett's conjecture* [Je75] *that there exist a left invariant measure on all locally compact hypergroup is yet to be proved.*

iii. *Onipchuk* [On93] *announced the proof of this conjecture but in reading through it we realized that he is using commutativity implicitly in his assumptions.Precisely the enveloping algebra $A \otimes A'$ is not involutive unless the semihypergroup is commutative.*

Definition III.4. *A nonzero measure $m \in M_+(H)$ is called left relatively invariant with associated multiplier $k : H \to \mathbb{R}_+^*$ if*

$$\delta_x * m = k(x)m$$

for all $x \in H$ and $f \in C_c(H)$.

Remark III.4. *The function k is continuous and satisfies $k(x*y) = k(x)k(y)$ for all $x, y \in H$.For if m is relatively invariant then for all $x, y \in H$, $\delta_x * (\delta_y * m) = \delta_x * k(y)m = k(y)\delta_x * m = k(y)k(x)m$ and for any $m-$integrable function f,*

$$(\delta_x * \delta_y) * m(f) = \int_H f d(\delta_x * \delta_y) * m =$$

$$\int_H (\delta_x * \delta_y) * f(z)m(dz) = \int_H m(dz) \int_H f(u * z)(\delta_x * \delta_y)(du) =$$

$$\int_H (\delta_x * \delta_y)(du) \int_H f(u * z)m(dz) = \int_H k(u)(\delta_x * \delta_y)(du) \int_H f(z)m(dz) =$$

$$(\delta_x * \delta_y)(k) \int_H f(z)m(dz) = k(x * y) \int_H f(z)m(dz)$$

that is $(\delta_x * \delta_y) * m = k(x * y)m$ and the result follows.

The left invariant measures are relatively left invariant measure with $k = 1$. Also if m is relatively left invariant with multiplier k then $\frac{m}{k}$ is left invariant [Sp78]. *The following lemmas are proved in* [Je75].

Lemma III.3. *Let f and k be in $C_c^+(H)$. Suppose that $k \neq 0$. Then there exists $\mu \in M_c^+(H)$ such that $f \leq \mu * k$.*

Proof. Choose $a \in H$ such that $k(a) > 0$. One readily sees that , if $x \in H$, then $(\delta_x * \delta_{a^-} * k)(x) > 0$. Thus, μ can be chosen to be a finite linear combination of measures of the form $\delta_x * \delta_{a^-}$.

■

Lemma III.4. *Let $f \in C_c^+(H)$ and let $\epsilon > 0$. Then there exists an open neighborhood W of e with the following property: If $x, y \in H$ and $(\delta_x * \delta_y)(W) > 0$ then $|f(x) - f(y)| < \epsilon$.*

Lemma III.5. *If* $f, k \in C_c^+(H)$ *and* $k \neq 0$ *let* $[f,k] = inf\{\mu(H) : \mu \in M_c^+(H)$ *and* $f \leq \mu * k\}$. *Then for* $f, g, h \in C_c^+(H)$ *we have*

1. $[\mu * f, k] \leq \mu(H)[f,k]$,
2. $[f+g,k] \leq [f,k] + [g,k]$,
3. $[cf,k] = c[f,k]$,
4. $[f,k] \leq [g,k]$ *if* $f \leq g$,
5. $[f,k] \leq [f.g][g,k]$ *if* $g \neq 0]$,
6. $[f,k] > 0$ *if* $f \neq 0$, $[f,k] \geq \frac{\|f\|}{\|k\|}$ *and* $[f,f] = 1$

Proof.

1. If $f \leq \nu * k$ then $\mu * f \leq \mu * \nu * k$ and
$$[\mu * f, k] \leq (\mu * \nu)(H) = \mu(H)\nu(H)$$
and since ν is arbitrary, $[\mu * f, k] \leq \mu(H)[f,k]$.
2. follows from the fact that if $f \leq \mu_1 * k$ and $g \leq \mu_2 * k$ then $f + g \leq \mu_1 * k + mu_2 * k = (\mu_1 + \mu_2) * k$.
3. follows from the fact that if $cf \leq \mu * k$ then $f \leq \frac{1}{c}\mu * k$
4. obvious
5. Follows from the fact that if $g \neq 0$, then if $f \leq \mu * g$ and $g \leq \nu * k$ then $f \leq \mu * \nu * k$
6. follows from tha fact that if $f \leq \mu * k$ then $\|f\|_\infty \leq \|\mu\|\|k\|_\infty$ so if $f \neq 0$,$\|f\|_\infty > 0$

■

Remark III.5. *If* H *is commutative and* $\alpha \in M - 1(H)$ *,we will also have*
$$[\alpha * f, \alpha * g] \leq [f,g]$$
This follows from the fact that if $f \leq \mu * g$ *then if* $\alpha \in M_1(H)$ *then* $\alpha * f \leq \alpha * \mu * g = \mu * \alpha * g$. *This also shows that*
$$[\alpha * f, g] \leq [\alpha * f, \alpha * g] \leq [f,g] \leq [f, \alpha * g]$$

Let F be a fixed nonzero element of $C_c^+(H)$. If $f, k \in C_c^+(H)$ and $k \neq 0$ then set
$$I_k f = \frac{[f,k]}{[F,k]}$$

Lemma III.6. *Let* $\mu \in M_c^+(H)$, $c > 0$ *and* $f, g, k \in C_c^+(H)$. *Suppose that* $k \neq 0$ *then*

i.: $I_k(\mu * f) \leq \mu(H) I_k f$
ii.: $I_k(f+g) \leq I_k f + I_k g$
iii.: $I_k(cf) = cI_k f$
iv.: $I_k f \leq I_k g$ *if* $f \leq g$ *Moreover if* $f \neq 0$ *then*
v.: $\frac{1}{[F,f]} \leq I_k f \leq [f,F]$

Proof.
v. Since
$$I_k f = \frac{[f,k]}{[F,k]} \leq \frac{[f,F][F,k]}{[F,k]} = [f,F]$$
Also, $[F,k] \leq [F,f][f,k]$ so $\frac{1}{[F,f]} \leq \frac{[f,k]}{[F,k]} = I_k f$ so that
$$\frac{1}{[F,f]} \leq I_k f \leq [f,F]$$
The other statements follow from lemma (III.5)

■

Lemma III.7. *Let $f_1, f_2 \in C_c^+(H)$ and let $\epsilon > 0$. Then there exists an open neighborhood W of e with the following property: If $k \in C_c^+(H), k \neq 0$ and $k = 0$ off W, then*

$$I_k f_1 + I_k f_2 < I_k(f_1 + f_2) + \epsilon$$

Proof.
Let $S = Supp(f_1 + f_2)$. Let V be a relatively compact open set containing S and choose $g \in C_c^+(H)$ such that $g = 1$ on V. Take $a > 0$ and put $b = 3a+2a^2$. Now let $h := f_1+f_2+ag \geq a$ on V. Define g_1 and g_2 by letting $g_i = \frac{f_i}{h}$ on V, and 0 elsewhere. Then each g_i is continuous and is zero off S. Also $g_1 + g_2 \leq 1$ on H. By lemma (III.4), there exist open neighborhood W_i of e such that $|g_i(x) - g_i(y)| < a$ when $(\delta_{x^-} * \delta_y)(W_i) > 0$. Let $W = W_1 \cap W_2$.

Suppose that $k \in C_c^+(H)$,$k \neq 0$ and $k = 0$ off W. Using lemma (III.3), choose $\mu \in M_c^+(H)$ such that $h \leq \mu * k$ and such that $\mu(H) \leq (1 + a)[h, k]$. If $x, y \in H$ and $k(x^- * y) > 0$ then $g_i(y) < a + g_i(x)$. Thus if $y \in H$ then

$$f_i(y) = g_i(y)h(y) \leq g_i(y)(\mu * k)(y) = \int_H g_i(y)k(x^- * y)\mu(dx) \leq \int_H (a + g_i(x))k(x^- * y)\mu(dx) =$$

$$\int_H k(x^- * y)[(a + g_i)\mu](dx) = ([(a + g_i)\mu] * k)(y)$$

It follows that $[f_i, k] \leq \int(a + g_i)d\mu$. Combining we have

$$[f_1, k] + [f_2, k] \leq \int_H (2a + g_1 + g_2)d\mu \leq (2a + 1)\mu(H) \leq (2a + 1)(a + 1)[h, k] = (1 + b)[h, k]$$

After dividing by $[F, k]$ we have

$$I_k f_1 + I_k f_2 \leq (1 + b)I_k(f_1 + f_2) + a(1 + b)I_k g < I_k(f_1 + f_2) + \epsilon,$$

if a is sufficiently small.

∎

Theorem III.1. *If H is a locally compact hypergroup then there exist a measure $m \in M_\infty(H)$ such that m is left subinvariant and $Supp(m) = H$.*

Proof. Choose an appropriate net of functions $\{k_l\}$ such that $k_l \neq 0$,$Supp(k_l) \to \{e\}$, and such that the functions I_{k_l} converge pointwise on the set $C_c^+(H)$. Let $J := lim_l I_{k_l}$. Then J is nonnegative and semilinear. Moreover if $f \neq 0$ then $Jf \geq \frac{1}{[F,f]} > 0$. Also, if $\mu \in M_c^+(H)$ and $f \in C_c^+(H)$ then $J(\mu * f) \leq \mu(H)Jf$. By the Riesz representation theorem, there exists a unique $m \in M_\infty(H)$ such that $Jf = \int_H f dm$ for all $f \in C_c^+(H)$ then

$$\int_H f d(\delta_x * m) = \int_H (\delta_{x^-} * f)dm \leq \int_H f dm$$

∎

Lemma III.8. *Let H be a (commutative) hypergroup, $g, h \in C_c^+(H)$ which are not identically zero then*

$$\frac{1}{[h, g]} \leq \frac{m(g)}{m(h)} \leq [g, h]$$

Proof. Note that $I_k g = \frac{[g,k]}{[F,k]}$ and $I_k h = \frac{[h,k]}{[F,k]}$ so that

$$\frac{I_k g}{I_k h} = \frac{\frac{[g,k]}{[F,k]}}{\frac{[h,k]}{[F,k]}} = \frac{I_k g}{I_k h} \leq$$

$$\frac{I_k h I_h g}{I_k h} = I_h g$$

on the other hand,

$$\frac{I_k g}{I_k h} \geq \frac{I_k g}{I_k g I_g h} = \frac{1}{I_g h}$$

so we have

$$\frac{1}{I_g h} \leq \frac{I_k g}{I_k h} \leq I_h g$$

and by the definition of m above, the result follows.

■

So far we have not use commutativity except for remark (III.5). We will now use it.

Lemma III.9. *Let H be a commutative hypergroup, and $f_0 \in C_c^+(H)$ which is not identically zero be fixed. Then for all $g \in C_c^+(H)$, set $a(g) = \frac{1}{[f_0,g]}, b(g) = [g, f_0]$ if $g \neq 0$ (then $0 < a(g) \leq b(g)$, and $a(g) = b(g) = 0$ if $g = 0$). Let A be the set of all left invariant measures on H normalized by f_0 (that is $m(f_0) = 1$) and satisfying for all $g \in C_c^+(H)$*

$$a(g) \leq \frac{1}{[\alpha * f_0, \alpha * g]} \leq \frac{m(\alpha * g)}{m(\alpha * f_0)} \leq$$

$$[\alpha * g, \alpha * f_0] \leq b(g)$$

then A is a nonempty convex, compact set not containing the null measure 0.

Lemma III.10. *Let H be a commutative hypergroup and A be defined as above then the mapping $T : A \to A$ which associate to every element $m \in A$ the element $T_\alpha(m)$ defined at any $g|inC_c^+(H)$ by*

$$T_\alpha(m)g = \frac{m(\alpha * g)}{m(\alpha * f_0)}$$

is a continuous map for every $\alpha \in M_1(H)$.

Theorem III.2. *Let A be a nonempty compact convex subspace of a topological locally convex space. Let $\mathcal{C}$ be a commutative family of continuous function from A to A with the following properties:*

a. *$\mathcal{C}$ is stable under composition;*
b. *$\mathcal{C}$ is convex, that is: for every $x \in A$, the set $\{T(x) : T \in \mathcal{C}\}$ is a convex subset of A. Then there exists in A a fixe point for any $T \in \mathcal{C}$*

This result is a generalization of two classical fixed point theorem of Markov-Kakutani and of Schauder-Tychonoff.

Proof. From The Schauder Tychonoff theorem ,every element of $\mathcal{C}$ has a fix point in A so we need to show that if $F(T)$ is the set of all fix point of T, then $\bigcap_{T\in\mathcal{C}} F(T) \neq \emptyset$. Since A is compact we just need to show that any finite intersection of element of $\mathcal{C}$ is finite.
Suppose $\bigcap_{i=1}^{n-1} F(T_i) \neq \emptyset$. Let $x \in \bigcap_{i=1}^{n} F(T_i)$ and $B = \{T(x) : T \in \mathcal{C}\}$, then since $\mathcal{C}$ is convex, B is convex and is a subset of $\bigcap_{i=1}^{n} F(T_i)$ since $\mathcal{C}$ is commutative; more over $T_n(B) \subset B$ since $\mathcal{C}$ is closed under composition. Now let $C = \bar{B}$ in A, then C is a convex compact subset of A and $T_n(C) \subset C$ by the continuity of T_n so the Schauder Tychonoff theorem applied to the restriction of T_n to C. Hence there is a fixed point of T_n in $C \subset \bigcap_{i=1}^{n-1} F(T_i)$ so $\bigcap_{i=1}^{n} F(T_i) \neq \emptyset$. Therefore there exists a common fix point to all $T \in \mathcal{C}$

■

Theorem III.3. *Every commutative hypergroup has a left invariant measure.*

Proof. The conditions of the theorem are satisfied. To show that $\mathcal{C}$ is convex, Let $m \in A$, and $\alpha, \beta \in M_1(H), 0 \leq t \leq 1$; we need to show that there exists $\gamma \in M_1(H)$ such that

$$tT_\alpha(m) + (1-t)T_\beta(m) = T_\gamma(m)$$

for all $g \in C_c^+(H)$

$$\{tT_\alpha(m) + (1-t)T_\beta(m)\}(g) = t\frac{m(\alpha * g)}{m(\alpha * f_0)} + (1-t)\frac{m(\beta * g)}{m(\beta * f_0)}$$

and this is of the form $T_\gamma(m)(g) = \frac{m(\gamma * g)}{m(\gamma * f_0)}$ if γ is defined by $\gamma = u\alpha + (1-u)\beta$ with

$$u = \frac{\frac{t}{m(\alpha * f_0}}{\frac{t}{m(\alpha * f_0} + \frac{1-t}{m(\beta * f_0}}$$

So there exist $m \in A$ such that $T_\alpha(m) = m$ for all $\alpha \in M_1(H)$ and $g \in C_c^+(H)$, that is

$$T_\alpha(m)(g) = \frac{m(\alpha * g)}{m(\alpha * f_0)} = m(g)$$

for all $\alpha \in M_1(H)$ which implies that $m(\alpha * g) = m(\alpha * f_0)m(g)$ so m is relatively left invariant which implies there exist a left invariant measure in H.

■

References

[BH95] Bloom W.R and Heyer H.,*The Harmonic Analysis of Probability Measures on Hypergroups*, de Gruyter Stud. Math.,vol.20, de Gruyter, Berlin and Hawthorne, New York, 1995.

[DE] Delsartes, J., *Hypergroupes et opérateurs de permutation et de transmutation. (French) 1956 La théorie des équations aux dérivées partielles.* Colloques Internationaux du Centre National de la Recherche Scientifique, LXXI Centre National de la Recherche Scientifique, Paris (Reviewer: J. B. Diaz) 35.00 Nancy, 9-15 avril 1956 pp. 29–45.

[Du73] Dunkl C.F.,*The measure Algebra of a Locally Compact Hypergroup*, Trans. Amer. Math. Soc. 179(1973), 331-348.

[Ge50] Gelfand I.M. *Spherical functions in symmetric Riemann spaces(Russian).*, Dok. Akad. Nauk. SSSR(NS), Vol.70,1950,p.5-8.

[HK75] Hewitt E. and Karl S., *Real and Abstract Analysis*, Third printing Graduate Text in math. 25 Springer Verlag, New York- Heidelberg (1975).

[HR70] Hewitt E. and Ross A. K., *Abstract Harmonic Analysis*, Vol 1,2 Springer-Verlag, Berlin and New York, 1970.MR41 7378.

[Je75] Jewett R.I.,*Spaces with an abstract convolution of measures*, Advances in Math. 18(1975),1-101.

[Mi55] Michael, E.A. *Topologies on the spaces of subsets.* Trans. Amer. Math. Soc. 71(1955), 152-182.

[On93] Onipchuk S. V., *Regular semihypergroups*, Russian Acad. Sci. Sb. Math. Vol. 76 No 1 (1993) 155-164.

[Sp78] Spector R.,*Mésures invariantes sur les hypergroupes*, Trans. Amer. Soc. 239(1978)147-165.

[WA] Walls, H. S. *Hypergroups*Amer. J. Math, 59 (1937)

[Ze89] Zeuner H. *One dimensional hypergroups* Adv. math., 76:1-18.

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