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On n-Fold Positive Implicative Artinian and Noetherian BCI-Algebras

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Abstract

We give a characterization of n-fold Positive Implicative Artinian and Positive Implicative Noetherian BCI-Algebras and study the normalization of n-fold fuzzy positive implicative BCI-ideals. Using the n-fold Positive Implicative and n-fold commutative ideals, we obtain two radical properties: PI^n- and C^n-radicals.

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Abstract. We give a characterization of n-fold Positive Implicative Artinian and Positive Implicative Noetherian BCI-Algebras and study the normalization of n-fold fuzzy positive implicative BCI-ideals. Using the n-fold Positive Implicative and n-fold commutative ideals, we obtain two radical properties: PI^n- and C^n-radicals.

Keywords. BCI-algebras, fuzzy set, PI^n-Noetherian and PI^n-Artinian BCI-algebras, n-fold fuzzy positive implicative BCI-ideal, PI^n- and C^n-radicals.

I. Introduction

The notion of BCK-algebras was initiated by Imai and Iseki in 1966 as a generalization of both classical and non-classical propositional calculus. The research on BCI/ BCK/ MV-algebras have burgeoned in the last decade. Y. B. Jun and K. H. Kim in [5] introduced the notion of n-fold fuzzy positive implicative ideals in BCK-algebras. The authors defined the notion of PI^n-Noetherian BCK-algebras, and give its characterization and the normalization of an n-fold fuzzy positive implicative ideal. Very recently using the intuitionistic fuzzy set, B. Satyanarayana et al. in [6] introduced the notion of PI^n-Noetherian and PI^n-Artinian BCK-algebras and study some of its properties. C. Lele et al. [1, 2, 3] introduced the notion of n-fold positive implicative ideals, n-fold fuzzy positive implicative-ideal and n-fold commutative ideals in BCI-algebras. In this note; we define the notion of PI^n-Noetherian and PI^n-Artinian BCI-algebras, and give its characterization. Furthermore, we study the normalization of an n-fold fuzzy positive

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implicative BCI-ideal. In the last section, using the notions of n-fold Positive Implicative and n-fold commutative ideals, we discuss two radical properties: PI^n- and C^n-radicals. This paper generalizes the corresponding results in BCK-algebras ([5], [6], [7]).

II. Preliminaries

We recall in this section the definitions and results that will be used throughout the paper, most of the time without any further notice.

Definition II.1. [4] *An algebra* $X = \langle X, \star, 0\rangle$ *of type* $< 2, 0 >$, *is said to be a BCI-algebra if it satisfies the following conditions for all* $x, y, z \in X$:

- *BCI1-* $((x \star y) \star (x \star z)) \star (z \star y) = 0$;
- *BCI2-* $x \star 0 = x$;
- *BCI3-* $x \star y = 0$ *and* $y \star x = 0$ *imply* $x = y$.

If a BCI-algebra X satisfies the condition $0 \star x = 0$ for all $x \in X$; then X is called a BCK-algebra. Hence, BCK-algebras form a subclass of BCI-algebras.

Let n be a positive integer. Throughout this paper we appoint that $X := \langle X, \star, 0\rangle$ denotes a BCI-algebra; $x \star y^n := (...((x \star y) \star y) \star ...) \star y$, in which y occurs n times; $x \star y^0 := x$ and $x \star \prod_{i=1}^{n} y_i$ denotes $(...((x \star y_1) \star y_2) \star ...) \star y_n$ where $x, y, y_i \in X$.

For the rest of the paper X shall denote a general BCI-algebra

Definition II.2. [1]

Let X be a BCI-algebra.

(1) We recall that a fuzzy set of a set X is a function $\mu : X \longrightarrow [0; 1]$.

(2) A fuzzy set in X is said to be a fuzzy ideal of X if:

(i) $\mu(0) \geq \mu(x)$ *for all* $x \in X$;

(ii) $\mu(x) \geq Min\{\mu(x); \mu(y)\}$; *for all* $x; y \in X$.

(3) A fuzzy set μ *of a BCI-algebra X is called a fuzzy n-fold positive implicative ideal of X if it satisfies the following conditions:*

(i) $\mu(0) \geq \mu(x)$ *for all* $x \in X$;

(iii) $\mu(x \star y^n) \geq min\{\mu((x \star y^{n+1}) \star (z \star y)), \mu(z)\}$; *for all* $x, y, z \in X$.

Definition II.3. [5] *A BCI-algebra X is said to satisfy the* PI^n*-ascending (resp.,* PI^n*-descending) chain condition (briefly,* PI^n*-ACC (resp.,* PI^n*-DCC)) if for every ascending (resp., descending) sequence* $A_1 \subset A_2 \subset ...$ *(resp.,* $A_1 \supset A_2 \supset ...$*) of n-fold positive implicative ideals of X there* *exists a natural number r such that* $A_r = A_k$ *for all* $r \geq k$. *If X satisfies the* PI^n*-ACC (resp.,* PI^n*-DCC) we say that X is a* PI^n*-Noetherian BCI-algebra (resp.* PI^n*–Artinian BCI-algebra).*

Theorem II.4. *Let* $\{Ak|k \in \mathbb{N}\}$ *be a family of n-fold positive implicative ideals of BCI-algebra X which is chain, that is,* $A_0 \supsetneq A_1 \supsetneq A_2 \supsetneq ...$ *Let* μ *be a fuzzy set in X defined by:*

$$\begin{cases} \mu(x) = 1 \quad \text{if } x \in \bigcap_{k=0}^{\infty} A_k \\ \mu(x) = \frac{k}{1+k} \quad \text{if } x \in A_k \diagdown A_{k+1}; k = 0, 1, 2, ... \end{cases}$$

for all $x \in X$, *where* A_0 *stands for X. Then* μ *is an n-fold fuzzy positive implicative ideal of X.*

Proof. Clearly $\mu(0) \geq \mu(x)$ for all $x \in X$. Let $x, y, z \in X$.

Suppose that $(x \star y^{n+1}) \star (z \star y) \in A_k \diagdown A_{k+1}$ and $z \in A_r \diagdown A_{r+1}$ for $k = 0, 1, 2, ...$, $r = 0, 1, 2, ...$ Without loss of generality, we may assume that $k \leq r$. Then obviously $z \in A_k$. Since A_k is an n-fold positive implicative ideal, it follows that $x \star y^n \in A_k$ so that

$\mu(x \star y^n) \geq \frac{k}{k+1} = \min\{\mu(x \star y^{n+1}) \star (z \star y); \mu(z)\}$

If $(x \star y^{n+1}) \star (z \star y) \in \bigcap_{k=0}^{\infty} A_k$ and $z \in \bigcap_{k=0}^{\infty} A_k$, then $x \star y^n \in \bigcap_{k=0}^{\infty} A_k$. Hence $\mu(x \star y^n) = 1 = \min\{\mu(x \star y^{n+1}) \star (z \star y); \mu(z)\}$.

If $(x \star y^{n+1}) \star (z \star y) \notin \bigcap_{k=0}^{\infty} A_k$ and $z \in \bigcap_{k=0}^{\infty} A_k$, then there exists $i \in \mathbb{N}$ such that $(x \star y^{n+1}) \star (z \star y) \in A_i \diagdown A_{i+1}$. It follows that $x \star y^n \in A_i$ so that

$\mu(x \star y^n) \geq \frac{i}{i+1} = \min\{\mu(x \star y^{n+1}) \star (z \star y); \mu(z)\}$.

Finally, assume that $(x \star y^{n+1}) \star (z \star y) \in \bigcap_{k=0}^{\infty} A_k$ and $z \notin \bigcap_{k=0}^{\infty} A_k$. Then $z \in A_j \diagdown A_{j+1}$ for some $j \in \mathbb{N}$. Hence $x \star y^n \in A_j$, and thus

$\mu(x \star y^n) \geq \frac{j}{j+1} = \min\{\mu(x \star y^{n+1}) \star (z \star y); \mu(z)\}$

Consequently, μ is an n-fold fuzzy positive implicative ideal of X.

■

Theorem II.5. *If every n-fold fuzzy positive implicative ideal of X has a finite number of values, then X is a* PI^n*-Artinian BCI-algebra.*

Proof. Suppose that X is not a PI^n-Artinian BCI-algebra

Then there exists a strictly descending chain $A_0 \supsetneq A_1 \supsetneq A_2 \supsetneq ...$ of n-fold positive implicative ideals of X which does not terminates at finite steps. Hence the fuzzy set in X define by:

$$\begin{cases} \mu(x) = 1 \quad \text{if } x \in \bigcap_{k=0}^{\infty} A_k \\ \mu(x) = \frac{k}{1+k} \quad \text{if } x \in A_k \diagdown A_{k+1}; k = 0, 1, 2, ... \end{cases}$$

is an n-fold fuzzy positive implicative ideal of X (See Theorem II.4) and μ has an infinite number of different values. This is a contradiction and hence X is a PI^n-Artinian.

■

Now we consider the converse of Theorem II.4

Theorem II.6. *Let X be a PI^n-Artinian BCI-algebra and let μ be an n-fold fuzzy positive implicative ideal of X. If a sequence of elements of $Im(\mu)$ is strictly increasing, then μ has a finite number of values.*

Proof.

Let $\{t_k\}$ be a strictly increasing sequence of elements of $Im(\mu)$. Hence $0 \leq t_1 \leq t_2 \leq ... \leq 1$. Then by [[1], theorem 3.12] $\mu_{t_r} := \{x \in X|\mu(x) \geq t_r\}$ is an n-fold positive implicative ideal of X for all $r = 2, 3, ...$ Let $x \in \mu_{t_r}$. Then $\mu(x) \geq t_r \geq t_{r-1}$, and so $x \in \mu_{t_{r-1}}$. Hence $\mu_{t_r} \subset \mu_{t_{r-1}}$. Since $t_{r-1} \in Im(\mu)$, there exists $x_{r-1} \in X$ such that $\mu(x_{r-1}) = t_{r-1}$. It follows that $x_{r-1} \in \mu_{t_{r-1}}$, but $x_{r-1} \notin \mu_{t_r}$. Thus $\mu_{t_r} \subsetneq \mu_{t_{r-1}}$, and so we obtain a strictly descending sequence $\mu_{t_1} \supsetneq \mu_{t_2} \supsetneq \mu_{t_3} \supsetneq ...$ of n-fold positive implicative ideals of X which is not terminating. This contradicts the assumption that X satisfies the PI^n-DCC. Consequently, μ has a finite number of values.

■

Corollary II.7. *Let X be a BCI-algebra such that for any n-fold fuzzy positive implicative ideal μ of X, a sequence of $Im(\mu)$ is strictly increasing.*

Then X be a PI^n-Artinian BCI-algebra if and only if μ has a finite number of values.

We note that a set is well ordered if and only if it does not contain any infinite descending sequence.

Theorem II.8. *The following are equivalent.*

(a) X is a PI^n-Noetherian BCI-algebra.

(b) The set of values of any n-fold fuzzy positive implicative ideal of X is a well ordered subset of $[0, 1]$.

Proof.

$(a) \Longrightarrow (b)$. Let μ be an n-fold fuzzy positive implicative ideal of X. Assume that the set of values of is not a well-ordered subset of $[0, 1]$. Then there exists a strictly decreasing sequence $\{t_k\}$ such that $\mu(x_k) = t_k$. It follows that $\mu_{t_1} \subsetneq \mu_{t_2} \subsetneq \mu_{t_3} \subsetneq ...$

is a strictly ascending chain of n-fold positive implicative ideals of X, where $\mu_{t_r} := \{x \in X|\mu(x) \geq t_r\}$ for every $r = 1, 2, ...$. This contradicts the assumption that X is PI^n-Noetherian.

$(b) \Longrightarrow (a)$. Assume that condition (a) is satisfied and X is not PI^n-Noetherian. Then there exists a strictly ascending chain $A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq ...$ • • • • • • • • (♣)

of n-fold positive implicative ideals of X. Let $A = \bigcup_{k\in\mathbb{N}} A_k$. Then by [[2], theorem 5.15] A is an n-fold positive implicative ideal of X. Define a fuzzy set ν in X by:

$$\begin{cases} \nu(x) = 0 \text{ if } x \notin A_k \\ \nu(x) = \frac{1}{r} \text{ if } r = \min\{k \in \mathbb{N}/x \in A_k\} \end{cases}$$

We claim that ν is an n-fold fuzzy positive implicative ideal of X. Since $0 \in A_k$ for all $k = 1, 2, ...$, we have $\nu(0) = 1 \geq \nu(x)$ for all $x \in X$. Let $x, y, z \in X$. $(x\star y^{n+1})\star(z\star y) \in A_k\diagdown A_{k+1}$ and $z \in A_k\diagdown A_{k+1}$ for $k = 0, 1, 2, ...$, then $x \star y^n \in A_k$. It follows that

$\nu(x \star y^n) \geq \frac{1}{k} = min\{\nu((x \star y^{n+1}) \star (z \star y)), \nu(z)\}$.

Suppose that $(x \star y^{n+1}) \star (z \star y) \in A_k\diagdown A_{k+1}$ and $z \in A_k\diagdown A_r$ for all $r < k$. Since A_k is an n-fold positive implicative ideal, it follows that $x \star y^n \in A_k$. Hence

$\nu(x \star y^n) \geq \frac{1}{k} \geq \dfrac{1}{r+1} \geq \nu(z)$ and,

$\nu(x \star y^n) \geq min\{\nu((x \star y^{n+1}) \star (z \star y)), \nu(z)\}$. Similarly for the case

$(x \star y^{n+1}) \star (z \star y) \in A_k\diagdown A_r$ and $z \in A_k$. we have $\nu(x \star y^n) \geq min\{\nu((x \star y^{n+1}) \star (z \star y)), \nu(z)\}$.

Thus ν is an n-fold fuzzy positive implicative ideal of X. Since the chain (♣) is not terminating, ν has a strictly descending sequence of values. This contradicts the assumption that the value set of any n-fold fuzzy positive implicative ideal is well ordered. Therefore X is PI^n-Noetherian. This completes the proof.

■

Theorem II.9. *Let $S = \{t_k|k = 1, 2, ...\} \cup \{0\}$ where $\{t_k\}$ is a strictly descending sequence in (0,1). Then a BCI-algebra X is PI^n-Noetherian if and only if for each n-fold fuzzy positive implicative ideal μ of X, $Im(\mu) \subset S$ implies that there exists a natural number k such that $Im(\mu) \subset \{t_1, t_2, ..., t_k\} \cup \{0\}$.*

Proof. Assume that X is a PI^n-Noetherian BCI-algebra and let μ be an n-fold fuzzy positive implicative ideal of X. Then by Theorem II.8 we know that $Im(\mu)$ is a well-ordered subset of $[0, 1]$ and so the condition is necessary.

Conversely, suppose that the condition is satisfied. Assume that X is not PI^n- Noetherian. Then there exists a strictly ascending chain of n-fold positive implicative ideals

$A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq ...$

Define a fuzzy set μ in X by

$$\begin{cases} \mu(x) = t_1 & \text{if } x \in A_1 \\ \mu(x) = t_k & \text{if } x \in A_k \diagdown A_{k-1}; k = 2, 3, ... \\ \mu(x) = 0 & \text{if } x \in X \diagdown \bigcup_{k=1}^{\infty} A_k \end{cases}$$

Since $0 \in A_1$, we have $\mu(0) = t_1 \geq \mu(x)$ for all $x \in X$. If either $(x \star y^{n+1}) \star (z \star y)$ or z belongs to $X \diagdown \bigcup_{k=1}^{\infty} A_k$, then either $\mu((x \star y^{n+1}) \star (z \star y))$ or $\mu(z)$ is equal to 0 and hence

$\mu(x \star y^n) \geq 0 = min\{\mu((x \star y^{n+1}) \star (z \star y)), \mu(z)\}$

If $(x \star y^{n+1}) \star (z \star y) \in A_1$ and $z \in A_1$, then $x \star y^n \in A1$ and thus

$\mu(x \star y^n) = t_1 = min\{\mu((x \star y^{n+1}) \star (z \star y)), \mu(z)\}$.

If $((x \star y^{n+1}) \star (z \star y) \in A_k \diagdown A_{k-1}$ and $z \in A_k \diagdown A_{k-1}$, then $x \star y^n \in A_k$. Hence

$\mu(x \star y^n) \geq t_k = min\{\mu((x \star y^{n+1}) \star (z \star y)), \mu(z)\}$

Assume that $(x \star y^{n+1}) \star (z \star y) \in A_1$ and $z \in A_k \diagdown A_{k-1}$ for $k = 2, 3, ...$ Then $x \star y^n \in A_k$ and therefore

$\mu(x \star y^n) \geq t_k = min\{\mu((x \star y^{n+1}) \star (z \star y)), \mu(z)\}$

Similarly for $(x \star y^{n+1}) \star (z \star y) \in A_k \diagdown A_{k-1}$ and $z \in A_1$, $k = 2, 3, ...$, we obtain

$\mu(x \star y^n) \geq t_k = min\{\mu((x \star y^{n+1}) \star (z \star y)), \mu(z)\}$

Consequently, μ is an n-fold fuzzy positive implicative ideal of X. This contradicts our assumption.

■

III. Normalizations of n-fold fuzzy positive implicative BCI-ideals

Definition III.1. *A fuzzy subset μ of a set X is said to normal if $sup_{x \in X} \mu(x) = 1$. In other words, there exists $x \in X$ such that $\mu(x) = 1$.*

Example III.2. *Let $X = \{0, 1, 2, 3\}$ with the operation $\star$ defined by*

$\star$	0	1	2	3
0	0	0	3	2
1	1	0	3	2
2	2	2	0	3
3	3	3	2	0

Then $X = \langle X, \star, 0 \rangle$ is a BCI-algebra.

Then the fuzzy set $\mu : X \longrightarrow [0, 1]$ defined by

$\mu(0) = \mu(1) = 1$; $\mu(2) = \mu(3) = 0.5$ *is a normal 2-fold fuzzy positive implicative BCI-ideal of X.*

Remark III.3. *Note that if μ is a normal n-fold fuzzy positive implicative ideal of X, then clearly $\mu(0) = 1$, and hence μ is normal if and only if $\mu(0) = 1$.*

Lemma III.4. *Given an n-fold fuzzy positive implicative ideal μ of X let μ^+ be a fuzzy set in X defined by $\mu^+(x) = \mu(x) + 1 - \mu(0)$ for all $x \in X$. Then μ^+ is a normal n-fold fuzzy positive implicative ideal of X which contains μ.*

Proof. The proof is straightforward

■

Using [[1], Proposition 3.10], we know that for any n-fold positive implicative ideal I of X, the characteristic function μ_I of I is a normal n-fold fuzzy positive implicative ideal of X. It is clear that μ is a normal n-fold fuzzy positive implicative ideal of X if and only if $\mu^+ = \mu$.

Corollary III.5 ([5], proposition 5.5). *If μ is an n-fold fuzzy positive implicative ideal of X, then $(\mu^+)^+ = \mu^+$.*

Proposition III.6 ([5], proposition 5.7). *Let μ and ν be n-fold fuzzy positive implicative ideals of X. If $\mu \subset \nu$ and $\mu(0) = \nu(0)$, then $X_\mu \subset X_\nu$*

Given an n-fold fuzzy positive implicative ideal, we construct a new normal n-fold fuzzy positive implicative ideal.

Theorem III.7. *Let μ be an n-fold fuzzy positive implicative ideal of X and*

let $f : [0, \mu(0)] \longrightarrow [0, 1]$ be an increasing function. Let $\mu_f : X \longrightarrow [0, 1]$ be a fuzzy set in X defined by $\mu_f(x) = f(\mu(x))$ for all $x \in X$. Then μ_f is an n-fold fuzzy positive implicative ideal of X. In particular, if $f(\mu_f(0)) = 1$ then μ_f is normal; and if $f(t) \geq t$ for all $t \in [0, \mu(0)]$, then $\mu \subset \mu_f$.

Proof. Since $\mu(0) \geq \mu(x)$ for all $x \in X$ and since f is increasing, we have $\mu_f(0) = f(\mu(0)) = f(\mu(x)) = \mu_f(x)$ for all $x \in X$. For any $x, y, z \in X$ we get

$$min\{\mu_f(x \star y^{n+1}) \star (z \star y), \mu_f(z)\} = min\{f(\mu(x \star y^{n+1}) \star (z \star y)), f(\mu(z))\} =$$
$$f(min\{\mu(x \star y^{n+1}) \star (z \star y), (\mu(z))\}) \leq f(\mu(x \star y^n) = \mu_f(x \star y^n)$$

Hence μ_f is an n-fold fuzzy positive implicative ideal of X. If $f(\mu(0)) = 1$, then clearly μ_f is normal. Assume that $f(t) \geq t$ for all $t \in [0, \mu(0)]$. Then $\mu_f(x) = f(\mu(x)) \geq \mu(x)$ for all $x \in X$, which proves $\mu \subset \mu_f$.

■

Corollary III.8. *Let μ be a fuzzy n-fold positive implicative ideal of X, $\mu(0) \neq 0$ and let $\widetilde{\mu}$ be the fuzzy set of X defined by $\widetilde{\mu}(x) = \dfrac{\mu(x)}{\mu(0)}$ for all $x \in X$. Then $\widetilde{\mu}$ is a normal fuzzy n-fold positive implicative ideal of X and $\mu \subset \widetilde{\mu}$.*

Let $\mathcal{N}(X)$ denote the set of all normal n-fold fuzzy positive implicative ideals of X.

Lemma III.9. *Let $\mu \in \mathcal{N}(X)$ be nonconstant such that it is a maximal element of the poset $(\mathcal{N}(X), \subset)$. Then μ takes only the values 0 and 1.*

Proof.

Since μ is normal, we have $\mu(0) = 1$. Let $x \in X$ be such that $\mu(x) \neq 1$. It is sufficient to show that $\mu(x) = 0$. If not, then there exists $a \in X$ such that $0 < \mu(a) < 1$.

Define a fuzzy set in X by $\nu(x) = \frac{1}{2}(\mu(x)+\mu(a))$ for all $x \in X$. Clearly, ν is well defined, and we get

$\nu(0) = \frac{1}{2}[\mu(0) + \mu(a)] = \frac{1}{2}[1 + \mu(a)] \geq \frac{1}{2}[\mu(x) + \mu(a)] = \nu(x)$; for all $x \in X$.

Let $x, y, z \in X$. Then

$\nu(x \star y^n) = \frac{1}{2}[\mu(x \star y^n) + \mu(a)] \geq \frac{1}{2}\min\{\mu(x \star y^{n+1}) \star (z \star y), \mu(z)\} + \frac{1}{2}\mu(a)$.

$= \min\{\frac{1}{2}[\mu(x \star y^{n+1}) \star (z \star y) + \mu(a)]; \frac{1}{2}[\mu(z) + \mu(a)]\}$

$= \min\{\nu((x \star y^{n+1}) \star (z \star y)); \nu(z)\}$

Hence ν is an n-fold fuzzy positive implicative ideal of X. By Lemma III.4, ν^+ is a maximal n-fold fuzzy positive implicative ideal of X, where ν^+ is defined by $\nu^+(x) = \nu(x) + 1 - \nu(0)$ for all $x \in X$. Note that

$\nu^+(a) = \nu(a) + 1 - \nu(0) = \frac{1}{2}[\mu(a) + \mu(a)] + 1 - \frac{1}{2}[\mu(0) + \mu(a)] = \frac{1}{2}[\mu(a) + 1] > \mu(a)$.

Hence $\nu^+(a) > \mu(a)$ and $\nu^+(a) < 1 = \nu^+(0)$. It follows that ν^+ is nonconstant, and μ is not a maximal element of $(\mathcal{N}(X), \subset)$. This is a contradiction.

■

Definition III.10. *An n-fold fuzzy positive implicative ideal μ of X is said to be fuzzy maximal if μ is nonconstant and μ^+ is a maximal element of the poset $(\mathcal{N}(X), \subset)$.*

For any positive implicative ideal I of X let μ_I be a fuzzy set in X defined by:

$$\begin{cases} \mu_I(x) = 1 & \text{if } x \in I \\ \mu_I(x) = 0 & \text{if } x \notin I \end{cases}$$

Theorem III.11. *Let μ be an n-fold fuzzy positive implicative ideal of X. If μ is fuzzy maximal, then*

(a) μ is normal,

(b) μ takes only the values 0 and 1,

(c) $\mu_{X_\mu} = \mu$,

(d) X_μ is a maximal n-fold positive implicative ideal of X.

Proof.

Let μ be an n-fold fuzzy positive implicative ideal of X which is fuzzy maximal. Then μ^+ is a nonconstant maximal element of the poset $(\mathcal{N}(X), \subset)$. Since $\mu \subset \mu^+$ it follows from Lemma III.9 that $\mu = \mu^+$ takes only the values 0 and 1. This proves (a) and (b).

(c) Obviously $\mu_{X_\mu} \subset \mu$, and μ_{X_μ}, takes only the values 0 and 1. Let $x \in X$. If $\mu(x) = 0$, then $\mu_{X_\mu} \supset \mu$. If $\mu(x) = 1$, then $x \in X_\mu$ and so $\mu_{X_\mu}(x) = 1$. This shows that $\mu_{X_\mu} \supset \mu$.

(d) Since μ is nonconstant, by [[1], theorem 3.13] X_μ is a proper n-fold positive implicative ideal of X. Let I be an n-fold positive implicative ideal of X containing X_μ. Then $\mu = \mu_{X_\mu} \subset \mu_I$. Since μ and μ_I are normal n-fold fuzzy positive implicative ideals of X and since $\mu = \mu^+$ is a maximal element of $(\mathcal{N}(X), \subset)$, we have that either $\mu = \mu_I$ or $\mu_I = 1$ where $1 : X \longrightarrow [0, 1]$ is a fuzzy set defined by 1(x) = 1 for all $x \in X$. The later case implies that I = X. If $\mu = \mu_I$, then $X_\mu = X_{\mu_I} = I$. This shows that X_μ is a maximal n-fold positive implicative ideal of X. This completes the proof.

■

Definition III.12. [1] *A fuzzy subset μ of X has a sup property if for any nonempty subset A of X, there exists $a_0 \in A$ such that $\mu(a_0) = Sup\{\mu(a)/a \in A\}$. Using this fact, we can prove the following result.*

Corollary III.13. *Let μ be an n-fold fuzzy positive implicative ideal of X. If μ is fuzzy maximal, then μ has a sup property*

Definition III.14. [1] *Let X, Y be two BCI-algebras. A map $f : X \longrightarrow Y$ is called a BCI-homomorphism if: $f(x \star y) = f(x) \star f(y)$ for all $x, y \in X$.*

Let μ a fuzzy subset of X, ν a fuzzy subset of Y and $f : X \longrightarrow Y$ a BCI-homomorphism. The image of μ under f denoted by $f(\mu)$ is a fuzzy set of Y defined by:

For all $y \in Y$, $f(\mu)(y) = Sup\{\mu(x)/x \in f^{-1}(y)\}$ if $f^{-1}(y) \neq \emptyset$ and $f(\mu)(y) = 0$ if $f^{-1}(y) = \emptyset$.

The preimage of ν under f denoted by $f^{-1}(\nu)$ is a fuzzy set of X defined by:

For all $x \in X$, $f^{-1}(\nu)(x) = \nu(f(x))$.

Corollary III.15. *Let ν be a normal n-fold fuzzy positive implicative ideal of X. The preimage of ν under f is a normal n-fold fuzzy positive implicative ideal*

Proof. Using [[1], Definition 3.16] and Remark III.3.

■

Theorem III.16. *Let $f : X \longrightarrow Y$ be an onto BCI-homomorphism, the image $f(\mu)$ a normal fuzzy n-fold positive implicative ideal μ with a sup property is also a normal fuzzy n-fold positive implicative ideal.*

Proof. By [1, Proposition 3.18], $f(\mu)$ is fuzzy n-fold positive implicative ideal. Using Definition III.1, it easy to prove that $f(\mu)$ is normal.

■

Theorem III.17. *Let $f : X \longrightarrow Y$ be an onto BCI-homomorphism. If X is a PI^n-noetherian (resp. PI^n-artinian), then so is Y.*

Proof. Assume that Y is not PI^n-noetherian. By Theorem II.8, the set of values of μ is not a well-ordered subset of [0; 1], for some n-fold fuzzy positive implicative ideal μ. Hence there exists a strictly decreasing sequence $\{t_k\}$ such that $\mu(x_k) = t_k$. It follows that $\mu_{t_1} \subsetneq \mu_{t_2} \subsetneq \mu_{t_3} \subsetneq ...$

is a strictly ascending chain of n-fold positive implicative ideals of Y, for every $r = 1, 2, ...$

.

By first homomorphism theorem [[4], corollary 1.6.7] $X/kerf \cong Y$. By ([4], theorem 1.5.13) every ideal of $X/kerf$ is the form $I/kerf$; where I is an ideal of X with $kerf \subset I$. Take any ascending chain of n-fold positive implicative ideal in $Y \cong X/kerf$ as follows:

$\mu_{t_1}/kerf \subsetneq \mu_{t_2}/kerf \subsetneq \mu_{t_3}/kerf \subsetneq ...$

Then $kerf \subset \mu_{t_1} \subsetneq \mu_{t_2} \subsetneq \mu_{t_3} \subsetneq ...$ is a strictly ascending chain of n-fold positive implicative ideal in X, for every r = 1, 2,... This contradicts the assumption that X is PI^n-Noetherian.

Similar arguments can be applied to artinian case.

■

IV. PI^n-and C^n-radicals

Let us discuss two radical properties in BCI-algebras. For the concepts of radical theory in BCI-algebras, please refer to ([4], [7]). By ([4], theorem 1.6.1 p. 69) we know that a homomorphic image of a BCK/BCI-algebra might not be a BCK/BCI-algebra and so it is necessary in discussion of the section to restrict that the homomorphic images of a BCI-algebra are still BCI-algebra.

Definition IV.1. *Let X be a nonzero BCI-algebra. If every proper ideal of X is not an n-fold positive implicative (resp. n-fold commutative) ideal of X then X is called a BCI-algebra with the property PI^n- (resp. C^n-) or a PI^n-(C^n-) algebra for short. If $X = \{0\}$, we appoint that X itself is a PI^n-(C^n-) algebra.*

Definition IV.2. *Let X be a nonzero BCI-algebra and I an ideal of X. If I itself regarded as a BCI-algebra is a PI^n-(C^n-)algebra then I is called a PI^n-(C^n-) ideal of X.*

Theorem IV.3. *Let X be a nonzero BCI-algebra and Y a homomorphic image of X. If X is a PI^n-(C^n-) algebra then so is Y.*

Proof. Let X be a PI^n-algebra. If Y not a PI^n-algebra then there is an n-fold positive implicative proper ideal J of Y. Suppose that f is surjective homomorphism from X onto Y and $I = f^{-1}(J)$ then I is a proper ideal of X. If $(x \star y^{n+1}) \star (0 \star y) \in I$ then $f[(x \star y^{n+1}) \star (0 \star y)] = (f(x) \star f^{n+1}(y)) \star (0 \star f(y)) \in J$ and so $f(x \star y^n) = f(x) \star f^n(y) \in J$ by J an n-fold positive implicative ideal of Y and hence $x \star y^n \in I$. This proves that I is an n-fold positive implicative ideal of X but $X \neq I$, a contradiction with X a PI^n-algebra.

Similarly we can prove the case that X is an C^n-algebra.

■

Lemma IV.4. *(*[4, Exercises 1.4.14]*) Let S be a nonempty subset of a BCI-algebra. The least closed ideal of X containing S is $< S \cup \{0 \star x / x \in S\} >$ where $< S \cup \{0 \star x / x \in S\} >$ is the generated ideal of X by $S \cup \{0 \star x / x \in S\}$*

Theorem IV.5. *Let X be a nonzero BCI-algebra. If every nonzero homomorphic image X' of X contains at least a nonzero PI^n-(C^n-) ideal of X' then X is a PI^n-(C^n-) algebra.*

Proof. if X is not a PI^n-algebra then there is an n-fold positive implicative proper ideal I of X. By ([2], theorem 5.15) and Lemma IV.4, $J =< I \cup \{0 \star x / x \in I\} >$ is an n-fold positive implicative proper ideal of X. On the other hand X/J (which is a nonzero homomorphic image of X) is an n-fold positive implicative algebra (see [2], theorem 5.18). By ([2], corollary 5.20), the zero ideal of X/J is n-fold positive implicative ideal. Hence every nonzero ideal of X/J is not PI^n-algebra of X/J, a contradiction.

Using ([3]; Theorem 4.10, Corollary 4.12 and Proposition 4.13), we can similarly prove the other case.

■

According to Theorems IV.3 and IV.5 in BCI-algebra, the property PI^n (C^n) is a radical property. In fact, every homomorphic image of a PI^n-algebra is a PI^n-algebra (See [4]).

Definition IV.6. *Let X be a BCI-algebra. Then the greatest PI^n closed ideal, denoted by $PI^n(X)$ of X is called the PI^n-radical of X. If $PI^n(X) = X$, X is called a PI^n-radical algebra. If $PI^n(X) = \{0\}$, X is called a PI^n-semisimple algebra. For the properties C^n, we can define those corresponding concepts. It obvious that the concepts of PI^n-and C^n-algebras are the same as that of PI^n- and C^n-radical algebras respectively.*

Example IV.7. *(1) Every n-fold positive implicative (n-fold commutative) BCI-algebra X is PI^n-(C^n)-semisimple since the zero ideal alone is the PI^n- (C^n) ideal of X.*

(2) Consider the BCI-algebra X whose Cayley's table is given by:

$\star$	0	1	a	b	c
0	0	0	a	a	a
1	1	0	a	a	a
a	a	a	0	0	0
b	b	a	1	0	0
c	c	a	1	1	0

Then $I = \{0, 1\}$ is C^1-ideal of X. Hence since $PI^n(X) = I$, X is not C^1-radical algebra.

Lemma IV.8. *Let X be a BCI-algebra. If I, J are two $PI^n - (C^n-)$ closed ideals of X then so is $C =< I \cup J >$.*

Proof. Suppose that I, J are two PI^n-closed ideals of X. Without loss generality we suppose $C =< I \cup J > \neq \{0\}$. Let D be a nonzero homomorphic image of C and K the kernel of the corresponding homomorphism then $D \cong C/K$. As $C/K \neq \{0\}$. we have $I \nsubseteq K$ or $J \nsubseteq K$. Assume that $I \nsubseteq K$ thus $I \cap K \neq I$ therefore according to the second isomorphism theorem, we have

$IK/k \cong I/I \cap K$ where $IK = \bigcup_{x \in I} K_x$ see[[4], theorem 1.6.8].

By [[4], theorem 1.7.5 (1)] $IK =< I \cup K >$ and we have

$< I \cup K > /K \cong I/I \cap K \neq \{0\}$ ••••••♣♣

Since $I/I \cap K$ is a homomorphism image of I and I is a PI^n-ideal of X, by Theorem IV.5 $I/I \cap K$ is a PI^n-algebra. Note that $< I \cup K > /K$ is an ideal of $C/K \cong D$, by (♣♣), we get D contains at least a nonzero PI^n-ideal. Now by Theorem IV.3; $C =< I \cup J >$ is a PI^n-ideal of X.

Similarly we can prove the case that I, J are C^n-ideals of X

■

Theorem IV.9. *Let X be a BCI-algebra and $\{A_i\}_{i\in I}$ all $PI^n - (C^n-)$ closed ideal family of X. Then so is $PI^n(X) = \bigcup_{i\in I} A_i$ $(C^n(X) = \bigcup_{i\in I} A_i)$*

Proof. We denote $W = \bigcup_{i\in I} A_i$. Clearly $0 \in W$. If $x \star y, y \in W$ then there exist $i, j \in I$ such that $x \star y \in A_i$ and $y \in A_j$. So $x \star y, y \in C =< A_i \cup A_j >$ and C is a PI^n-ideal of X by Lemma IV.8 hence $x \in C \subset W$ and we already proved that W is an ideal of X. Now if W is not a PI^n-ideal of X then there is an n-fold positive implicative proper ideal I of W thus there is some $A_i \in \{A_i\}_{i\in I}$ such that $A_i \nsubseteq I$, namely $A_i \cap I \neq A_i$. Let $x, y \in A_i$ such that $(x \star y^{n+1}) \star (0 \star y) \in A_i \cap I$. Then $x \star y^n \in I$, since A_i is closed ideal containing x and y, we have $x \star y^n \in A_i$, that is $x \star y^n \in A_i \cap I$. Hence $A_i \cap I$ is an n-fold positive implicative ideal of A_i, a contradiction with A_i a PI^n-close ideal of X. Because W is the union of all PI^n-closed ideal, W is the greatest PI^n-closed ideal of X, namely $PI^n(X) = \bigcup_{i\in I} A_i$

■

Theorem IV.10. *The $PI^n - (C^n-)$ radical in BCI-algebras is a lower radical determined by the algebraic class Ω consisting of all $PI^n - (C^n-)$ algebras.*

Proof. We prove alone the first case, yet. Let

$\overline{\Omega} = H(\Omega)$ where $X \in H(\Omega)$ iff X is a homomorphic image of some member of Ω. According to Theorem IV.3, $\overline{\Omega} = \Omega$ so X is a $L(\Omega)$-algebra if and only if X is a PI^n-algebra where X called a $L(\Omega)$-algebra means that for any proper ideal I of X there is a nonzero ideal J of X/I such that $J \in \overline{\Omega}$. Hence the PI^n- radical is a lower radical determined by Ω.

■

Theorem IV.11. *The $PI^n - (C^n-)$ radical in BCI-algebras is a upper radical determined by the algebraic class M consisting of all n-fold positive implicative (n-fold commutative) BCI-algebras.*

Proof. We also prove the first case. Let $\overline{M} = H(M)$.

Clearly $\overline{M} = M$. We can easily prove that X is an UM-algebra if and only if X is a PI^n-algebra where X called an UM-algebra means that every nonzero homomorphic image of X is not in $\overline{M} = M$. Hence the PI^n-radical is an upper radical determined by M.

■

Theorem IV.12. *The PI^n-radical in BCI-algebras are not the hereditary radicals.*

Proof. suppose that $I = \{0, a\}$ is a BCI-algebra of order 2 in which $a \star 0 = 0 \star a = a$ and $0 \star 0 = a \star a = 0$. Consider the Li's union of A and $\mathbb{N}$(with $A \cap \mathbb{N} = \{0\}$) $X = A \cup_L \mathbb{N}$. Define

$$\begin{cases} x \star_L y = max(0, x - y) & \text{if } x, y \in \mathbb{N} \\ x \star_L y = x \star y & \text{if } x, y \in A \\ x \star_L y = a & \text{if } x \in \mathbb{N} \\ x \star_L y = x & \text{if } y \in \mathbb{N}; x \in A \end{cases}$$

We can check that $X = \langle X, \star_L, 0\rangle$ is a BCI-algebra (see [4], example 1.3.4) and there are altogether three ideals of X: $\{0\}$ A and X. For any $n \in \mathbb{N}$, take $x = n + 1$ et $y = 1$ then $(x \star_L y^{n+1}) \star (0 \star y) = 0 \in A$ but $x \star_L y^n = 1 \notin A$ so A is not an n-fold positive implicative ideal of X, hence the zero ideal of X is neither n-fold positive implicative. This prove that X is a PI^n-radical algebra (for any $n \in \mathbb{N}$). However A itself regarded as a BCI-algebra is an n-fold positive implicative BCI-algebra, thus $PI^n(A) = \{0\} \neq A$, that is, A is PI^n-radical algebras. Therefore the PI^n-radical is not a hereditary radical.

■

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