IMHOTEP

AFRICAN JOURNAL OF PURE AND APPLIED MATHEMATICS

Imhotep Mathematical Journal
Volume 3, Numéro 1, (2018),
pp. 5 – 20.

Robust observation intervals and stabilization of a wastewater treatment model

Mustapha Serhani
mustapha.serhani@gmail.com

TSI team, Faculty of SJES,
University Moulay Ismail, B.P.
21301, Toulal, Meknes, Morocco.

Hamid Boutanfit
hboutanfit@gmail.com

ITSI team, Faculty of Science,
University Moulay Ismail, B.P.
11201, Zitoune, Meknes, Morocco.

Abstract

In this work, we intend to provide a robust regulation of an aerobic wastewater process model. The model contains two on-line measured states, substrate biomass and dissolved oxygen and, due to exogenous disturbances and functional uncertainties of the kinetics, two unmeasured states, bacteria biomass and recycled bacteria biomass. The first aim consist to find an observation intervals of the unmeasured state variables. The second object aims to build a pair of feedback controls, using the dilution and aeration rates as functions of observed states, to stabilize the substrate biomass and the dissolved oxygen around suitable values.

IMHOTEP - Afr. J. Pure Appl. Math. **3** (2018), 5–20
1608-9324/010005–4, DOI 13.1007/s00009-003-0000

IMHOTEP African Journal
of Pure and Applied Mathematics

Robust observation intervals and stabilization of a wastewater treatment model

Mustapha Serhani and Hamid Boutanfit

Abstract. In this work, we intend to provide a robust regulation of an aerobic wastewater process model. The model contains two on-line measured states, substrate biomass and dissolved oxygen and, due to exogenous disturbances and functional uncertainties of the kinetics, two unmeasured states, bacteria biomass and recycled bacteria biomass. The first aim consist to find an observation intervals of the unmeasured state variables. The second object aims to build a pair of feedback controls, using the dilution and aeration rates as functions of observed states, to stabilize the substrate biomass and the dissolved oxygen around suitable values.

Mathematics Subject Classification (2010). 37N35, 93D15, 93D05, 92D40.

Keywords. Observation intervals, Feedback stabilization, Dynamical system, Wastewater treatment.

1. Introduction

The activated sludge station is mainly constituted by two sequential tanks, an aerator and a settler. The process can be summarized as follows: the wastewater is discharged in an aerator with a flow Q_{in} and a concentration S_{in} in the feed stream. The phase of biological oxidation of the polluted water (substrate) S, by a blend of one bacterial population x in an aerobic reaction consuming the oxygen D_o, occurs in the aerator tank.The working principle can be found in literature, see for example [2, 5, 16, 17, 22, 23]. The settler is a gravitational sedimentation tank, where the sludge and the clean residual water are separated. Part of the settled active sludge and hence bacteria x_r is re-circulated from the settler into the aeration reactor, so that the content of micro-organisms in the aerobic reactor could be maintained at an appropriate level, in order to continue the wastewater treatment process. The excess of active sludge is discharged from the biological wastewater treatment system. The control problem consists of feeding a continuous stirred reactor with wastewater loaded with a high pollutant concentration.The objective is that the pollutant concentration in the output reaches a prescribed value, and is then regulated about it (stabilization stage).
The main obejct here is to use robust control to stabilize the process. Note that the concept of robust command addresses behavior of a controlled dynamical system on presence of interne perturbations as unknown parameters modeling the system, not well known interactions between compartments of the model and extern disturbances as environmental conditions, perturbations in initial conditions, ... The aim is to study the persistence of the system under these sorts of perturbations. The main feature that can be achieved is to be able to derive the system with an

adequate control law to a desired situation. Our case is an ideal case to use robust control since reactions inside the bioreactor are often highly nonlinear, not very well known and subject to external disturbances, such as the unknown presence of pollutants. Indeed, due to metabolic variations and the influence of many physical-chemical factors (pH, Temperature, oxygen,...), it's very hard to have an accurate idea of kinetic growth function of the bacteria μ. On the other hand, substrate concentration S_{in} in the feed stream fluctuates and is badly known, by the way, biomasses of bacteria and recycle bacteria are unmeasured and hence we can't use them to build the feedback control. In [1, 12, 15, 18] the authors investigate the partial stabilization of a model of wastewater treatment, the stabilization concerns a single output, that is the substrate biomass. In this work, we analyze two dimension stabilization of a nonlinear control model with imperfect information stemming from the activated sludge process. We consider uncertainties, both of the kinetic and the exogenous inputs, requiring that the kinetic function is functionally bounded by two known kinetic functions and the disturbances or unknown inputs are bounded with known bounds. The first aim consist to build a robust set-valued observers, in terms of intervals for the unmeasured variables. The second aim is to build a feedback control in term of the dilution and aeration rates, involving online measured states and observed states instead the single-valued unknown states, to stabilize the substrate biomass and the dissolved oxygen around desired fixed levels.

The paper is organized as follows: In the second section we present the model equations, describe variables and parameters and formulate assumptions. Section (3) is devoted to build the observer intervals. In section (4) we construct a pair of feedback controls and prove that with this law control we can stabilize the output variables around a suitable value. Finaly, in section (5) we give some numerical simulations to illustrate our results.

2. Mathematical model

The model is inspired from [13], see also [22, 23], it is based on four fundamental mass balance equations describing the biological degradation of the pollutant by aerobic bacteria in the aerator. The aerobic microorganisms x consume the substrate S (organic carbon) to fuel their metabolism, the result is the conversion of organic pollutants into inorganic compounds and new microbial cells. The following diagram gives flows in the station.

The mass conservation law in the aerator stipulate that the input flows are equal to the output flows. The input flows in the aerator are the influent flow Q_{in} with which the wastewater is coming into the aerator, with substrate concentration S_{in} and dissolved oxygen concentration $D_{o_{in}}$ in the feed stream and the recycle flow flow Q_r with which a part of waste containing bacteria biomass is recycled from the settler into the aerator to stimulate the biological reactions. The output flow is the same $Q_{in} + Q_r$ at the exit of the aerator. This output flow is divided in the settler tank at three flows: Q_r recycle flow, the waste flow Q_w with which the waste is discharged out of the station and the effluent flow Q_e with wich the treated water is spilled into the river as illustrated in figure (1).

For the substrate biomass S the input is S_{in} with flow Q_{in} since there is no recycle substrate S then no input flow Q_r. The output is S with flow $Q_{in} + Q_r$ as illustrated in figure (1). So, in the aerator the mass balance of substrate biomass $S(t)$ gives the following equation describing their evolution

$$V_a \frac{dS(t)}{dt} = -V_a \frac{\mu(t)}{Y_f} x(t) - (Q_{in} + Q_r)S(t) + Q_{in}S_{in}.$$

By dividing with V_a we obtain the dimensionless equation

$$\frac{dS(t)}{dt} = -\frac{\mu(t)}{Y_f} x(t) - D(t)(1 + r)S(t) + D(t)S_{in}.$$

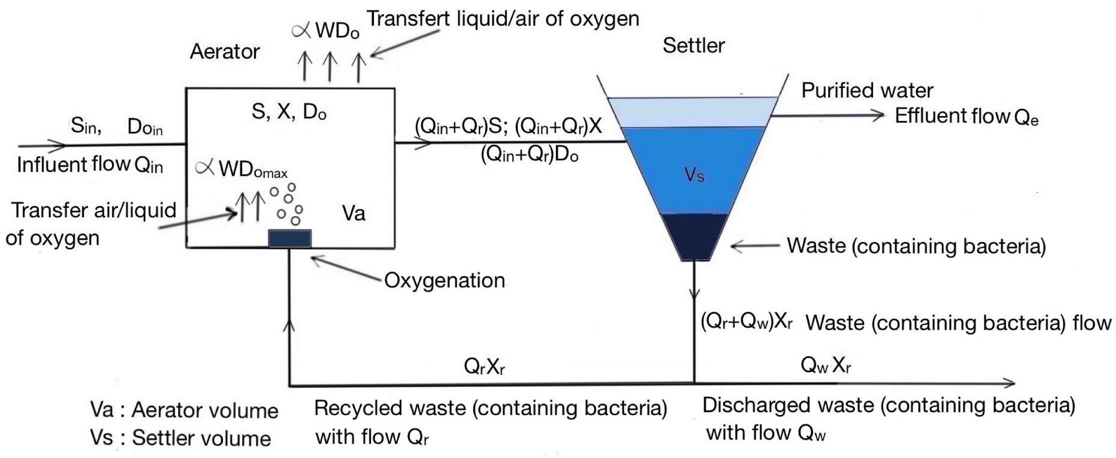

FIGURE 1. Flow diagram of activated sludge station.

where $D(t) = \dfrac{Q_{in}}{V_a}$ is the dilution rate, it represents the flow of medium per time, $Q_{in}(t)$, over the volume V_a of culture in the bioreactor and $r = \dfrac{Q_r}{Q_{in}}$ is the recycle rate as described in the table (2).

For the bacteria biomass x the input is x_r with flow Q_r and the output is x with flow $Q_{in} + Q_r$ as shown in figure (1). So in the aeration tank the mass balance of bacteria biomass $x(t)$, taking account of the recycled biomass, is subject to the equation:

$$V_a \frac{dx(t)}{dt} = V_a \mu(t) x(t) - (Q_{in} + Q_r)x(t) + Q_r x_r(t).$$

By dividing with V_a we obtain the dimensionless equation

$$\frac{dx(t)}{dt} = \mu(t)x(t) - D(t)(1 + r)x(t) + rD(t)x_r(t).$$

Dissolved oxygen $D_o(t)$ is added in the aerator to enhance the oxygenation process by providing oxygen to aerobic microorganisms, so they can successfully turn organic wastes into inorganic byproducts.

Inputs are $D_{o_{in}}$ with flow Q_{in} and $D_{o_{max}}$ with rate of air/liquid transfer $\alpha W(t)$ and outputs are $D_o(t)$ with flow $Q_{in} + Q_r$ and $D_o(t)$ with rate of liquid/air transfer $\alpha W(t)$, see figure (1). So in the aerator tank the mass balance of oxygen concentration $D_o(t)$ in the mixture gives

$$V_a \frac{dD_o(t)}{dt} = -V_a K_0 \frac{\mu(t)}{Y_f} x(t) - (Q_{in} + Q_r)D_o(t) + Q_{in} D_{o_{in}} + V_a \alpha W(t)[D_{o_{max}} - D_o(t)].$$

Similarly by dividing with V_a we obtain the dimensionless equation

$$\frac{dD_o(t)}{dt} = -K_0 \frac{\mu(t)}{Y_f} x(t) - D(t)(1+r)D_o(t) + D(t)D_{o_{in}} + \alpha W(t)[D_{o_{max}} - D_o(t)].$$

The flocks formed in the aerated tank, is removed in a secondary tank (Settler) by gravity settling. A part of this sludge is recycled back to the reactor. Hence the input of recycled bacteria biomass x_r is x with rate $(Q_{in} + Q_r)$ and the ouput is x_r with flow $Q_r + Q_w$ as illustrated in figure (1). So the mass balance of recycled bacteria biomass $x_r(t)$ is expressed by

$$V_a \frac{dx_r(t)}{dt} = (Q_{in} + Q_r)x(t) - (Q_r + Q_w)x_r(t).$$

and by the same way we divide this equation by V_a which leads to

$$\frac{dx_r(t)}{dt} = D(t)(1+r)x(t) - D(t)(\beta + r)x_r(t).$$

where $\beta = \dfrac{Q_w}{Q_{in}}$ is the rate of the waste and hence the recycled biomass which is discharged out of the station.
The list with description and units of variables used in the model as summarized in the table below.

Variables	Description	Unit
$S(t)$	Substrate biomass	$\mid S \mid = mg/L$: milligram by liter
$x(t)$	Bacteria biomass	$\mid x \mid = numbre/mL$: numbre by milliliter
$D_o(t)$	Dissolved oxygen concentration	$\mid D_o \mid = mgO_2/L$: milligram of oxygen by liter
$x_r(t)$	Recycled bacteria biomass	$\mid x \mid$

Table 1: States variables description and units

and the list with descriptions and units of parameters used in the model as summarized as follows.

Parameters	Description	Unit
$D_{o_{max}}$	Maximum amount of dissolved oxygen	$\mid D_o \mid$
$D(t)$	The dilution rate: is the ratio between the influent flow Q_{in} and the aerator volume V_a	Day^{-1}
S_{in}	Substrate concentration in the feed stream	$\mid S \mid$
$D_{o_{in}}$	Dissolved oxygen concentration in the feed stream	$\mid D_o \mid$
Y_f	Scaling factor	$\mid x \mid . \mid S^{-1} \mid$
α	Air/Liquid Oxygen transfer	Day^{-1}
W	Aeration rate	Dimensionless
K_0	Scaling factor	$\mid D_o \mid . \mid S^{-1} \mid$
r	The recycle rate representing the rate of bacteria biomass recycled from the Settler to the Aerator	Dimensionless
β	Represents the rate of waste and recycled bacteria that is discharged out of the station	Dimensionless

Table 2: Parameters description and units

As discussed in the introduction, kinetic growth rate μ is not well known but its upper and lower bounds μ^+ and μ^- are known. We suppose that those bounds obey to a modified

Monod law and depend on concentration of substrate together with that of dissolved oxygen, as introduced in [5, 6, 11, 14, 16, 22, 23]

$$\mu^*(S(t), D_o(t)) = \mu^*_{max} \frac{S(t)}{K_s + S(t)} \frac{D_o(t)}{K_{D_o} + D_o(t)}. \tag{1}$$

With $^* = ^{+,-}$.
In section "Microbial Growth on Multiple Substrates", of [10], the author introduces and discusses the use of the same kinetic.

We assume also that $S_{in}(t)$, $D_{o_{in}}(t)$ and $D_{o_{max}}(t)$ are not well known but their bounds are known.

Hypotheses:
Along this paper, we assume that
H_1. Parameters r, Y_f, K_0, α and β are given constants and D, $W : [0, +\infty[\longrightarrow \mathbb{R}^+$ are known nonnegative functions.

H_2. There exists functions $S^*_{in}(t)$, $D^*_{o_{in}}(t)$ and $D^*_{o_{max}}(t)$, with $^* = ^{+,-}$, such that

$$\underline{S}_{in} \le S^-_{in}(t) \le S_{in}(t) \le S^+_{in}(t) \le \overline{S}_{in}, \ \forall t \ge 0,$$
$$\underline{D}_{o_{in}} \le D^-_{o_{in}}(t) \le D_{o_{in}}(t) \le D^+_{o_{in}}(t) \le \overline{D}_{o_{in}}, \ \forall t \ge 0,$$
$$\underline{D}_{o_{max}} \le D^-_{o_{max}}(t) \le D_{o_{max}}(t) \le D^+_{o_{max}}(t) \le \overline{D}_{o_{max}}, \ \forall t \ge 0,$$

where $\underline{S}_{in}$, $\overline{S}_{in}$, $\underline{D_{o}}_{in}$, $\overline{D}_{oin}$, $\underline{D_o}_{max}$ and $\overline{D}_{o_{max}}$ are given constants.

H_3. There exists μ^- and μ^+ such that

$$\underline{\mu} \le \mu^-(\xi) \le \mu(\xi) \le \mu^+(\xi) \le \overline{\mu}, \ \forall \xi \in \mathbb{R}^2_+,$$

where $\mu^-(\xi)$ and $\mu^+(\xi)$ obey to law (1), while $\underline{\mu}$ and $\overline{\mu}$ are given constants.
Unless otherwise stated, we will adopt the notation $\mu(t) := \mu(S(t), D_o(t))$.

3. Observation Intervals

The dynamical system modeling the wastewater problem is summarized as:

$$\begin{cases} \dfrac{dx_r(t)}{dt} = D(t)(1+r)x(t) - D(t)(\beta + r)x_r(t) \\ \dfrac{dx(t)}{dt} = \mu(t)x(t) - D(t)(1+r)x(t) + rD(t)x_r(t) \\ \dfrac{dS(t)}{dt} = -\dfrac{\mu(t)}{Y_f}x(t) - D(t)(1+r)S(t) + D(t)S_{in} \\ \dfrac{dD_o(t)}{dt} = -K_0\dfrac{\mu(t)}{Y_f}x(t) - D(t)(1+r)D_o(t) + D(t)D_{o_{in}} + \alpha W(t)[D_{o_{max}} - D_o(t)] \end{cases} \tag{2}$$

Model (2) can be rewritten as

$$\dot{X} = CR(t)X(t) + A(t)X(t) + B(t), \tag{3}$$

with

$$X = \begin{pmatrix} x_r & x & S & D_o \end{pmatrix}^T, \quad C = \begin{pmatrix} 0 & 1 & -\frac{1}{Y_f} & \frac{-K_0}{Y_f} \end{pmatrix}^T,$$

$$R(t) = \begin{pmatrix} 0 & \mu(t) & 0 & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 & 0 & D(t)S_{in} & D(t)D_{o_{in}} + \alpha W(t)D_{o_{max}} \end{pmatrix}^T,$$

and

$$A(t) = \begin{pmatrix} -(\beta+r)D(t) & (1+r)D(t) & 0 & 0 \\ rD(t) & -(1+r)D(t) & 0 & 0 \\ 0 & 0 & -(1+r)D(t) & 0 \\ 0 & 0 & 0 & \alpha W(t) - (1+r)D(t) \end{pmatrix}.$$

Model (3) can also be reformulated as:

$$\begin{cases} \dot{X}_1 = C_1 R_1(t) X_1(t) + A_1(t) X_1(t) \\ \dot{X}_2 = C_2 R_1(t) X_1(t) + A_2(t) X_2(t) + B_2(t) \\ Y = X_2(t) \end{cases}, \tag{4}$$

with

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad X_1 = \begin{pmatrix} x_r \\ x \end{pmatrix}, \quad X_2 = \begin{pmatrix} S \\ D_o \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} -\frac{1}{Y_f} \\ -\frac{K_0}{Y_f} \end{pmatrix},$$

$$B_2(t) = \begin{pmatrix} D(t)S_{in} \\ D(t)D_{o_{in}} + \alpha W(t)D_{o_{max}} \end{pmatrix},$$

$$A_1(t) = \begin{pmatrix} -(\beta+r)D(t) & (1+r)D(t) \\ rD(t) & -(1+r)D(t) \end{pmatrix},$$

$$A_2(t) = \begin{pmatrix} -(1+r)D(t) & 0 \\ 0 & \alpha W(t) - (1+r)D(t) \end{pmatrix},$$

$$R_1(t) = \begin{pmatrix} 0 & \mu(t) \end{pmatrix},$$

The equation

$$Y(t) = X_2(t), \tag{5}$$

represents available variables. Since it's practically possible to install captors in the aerator tank to measure substrate and dissolved oxygen concentrations, see [7, 22], we assume that state variable $Y = X_2$, regrouping S and D_o, can be on-line measured and X_1, regrouping x and x_r, is not available.

As mentioned in introduction, author of [15] considers a single output observation equation and use a feedback control involving this single observation to stabilize the system. Our case, is more complicated, since there are two observed variables, each of them is used to build two combined feedback control stabilizing the system.

The first aim is to build an observation intervals of state X_1 by invoking only the known data. Consider

$$B_2^+(t) = \begin{pmatrix} D(t)S_{in}^+ \\ D(t)D_{o_{in}}^+ + \alpha W(t)D_{o_{max}}^+ \end{pmatrix},$$

and

$$B_2^-(t) = \begin{pmatrix} D(t)S_{in}^- \\ D(t)D_{o_{in}}^- + \alpha W(t)D_{o_{max}}^- \end{pmatrix}.$$

It is possible to make a variable change to reformulate system (2) free of kinetic function. Define a new state variable Z as

$$Z(t) = NX(t), \tag{6}$$

where $X = (X_1\ X_2)^T$ and $N \in \mathcal{M}_{2,4}(\mathbb{R})$ is a matrix to be determined. The new system will be kinetic independent if and only if it satisfies

$$NC = 0. \tag{7}$$

Equation (7) is a matrix equation whose unknown is N. It suffices to find at least one solution of the above equation.
From (7) we have

$$N_1C_1 + N_2C_2 = 0,$$

where $N = (N_1\ N_2)$, such that $N_1,\ N_2 \in \mathcal{M}_{2,2}(\mathbb{R})$. Hence,

$$N_2C_2 = -N_1C_1,$$

then

$$C_2^T N_2^T = -C_1^T N_1^T.$$

Since $rank(C_2) = 1$ then $(C_2^T)^{-1}$, the right pseudo inverse of C_2^T, exists, (see [20])

$$C_2^T = -\begin{pmatrix} \frac{Y_f}{1+k_0^2} \\ \\ \frac{K_0 Y_f}{1+k_0^2} \end{pmatrix},$$

we can arbitrary choose N_1^T to compute

$$N_2^T = -(C_2^T)^{-1} C_1^T N_1^T.$$

So, if we choose $N_1 = I_2$, then

$$N_2 = \frac{Y_f}{1+k_0^2} \quad \begin{pmatrix} 0 & 0 \\ 1 & k_0 \end{pmatrix},$$

which prove the existence of the matrix N.
By variable change (6) the system (4) can be reformulated, free of kinetic function, with Z and X_2.

$$\dot{Z} = \beta_1(t)Z(t) + \beta_2(t)X_2(t) + NB(t), \tag{8}$$

with

$$\begin{aligned} \beta_1(t) &= A_1(t) \\ \beta_2(t) &= N_2A_2(t) - A_1(t)N_2 \end{aligned}$$

Proposition 3.1. *Suppose that there exists $\alpha^+,\ \alpha^- \in \mathbb{R}^4$ such that*

$$\alpha^- \leq X(0) \leq \alpha^+, \tag{9}$$

then a robust observation intervals $[X_1^-\ ,\ X_1^+]$ for X_1 can be built such that

$$X_1^-(t) \leq X_1(t) \leq X_1^+(t),\ \ \forall\, t \geq 0, \tag{10}$$

where

$$\textit{for the upper bound}: \begin{cases} X_1^+ = Z^+ - N_2X_2 \\ \dot{Z}^+ = \beta_1 Z^+(t) + \beta_2 X_2(t) + N_2 B_2^+ \\ Z^+(0) = N\alpha^+(0) \end{cases} \tag{11}$$

$$\textit{for the lower bound}: \begin{cases} X_1^- = Z^- - N_2 X_2 \\ \dot{Z^-} = \beta_1 Z^-(t) + \beta_2 X_2(t) + N_2 B_2^- \\ Z^-(0) = N\alpha^-(0) \end{cases} \tag{12}$$

Inequalities (9) and (10) must be understand coordinate by coordinate.

Proof. To show this proposition we need to recall the following result about cooperative systems.
Consider a dynamical system.

$$\dot{y} = f(t, y) \tag{13}$$

on open convex domain $\mathbb{U} \subset \mathbb{R}^n$.

Definition 3.2. *([3, 4, 19]) System (13) is said to be cooperative if Jacobian matrix $\frac{\partial f}{\partial x}(t, a)$ is Metzler for all $a \in \mathbb{U}$, i.e.*

$$\forall i \neq j,\ \forall t \geq 0,\ \frac{\partial f_i}{\partial x_j}(t, x) \geq 0.$$

One of the most interesting result on cooperative systems, is the fact that they ensure monotony properties, that is, two trajectories initialized at $x_0 \leq y_0$ retain this inequality during forwarding time, see [3, 4, 19].

Theorem 3.3. *([19]) Consider cooperative system (13) and two initial conditions x_0 and y_0 such that*

$$x_0 \leq y_0.$$

Then

$$y(t, x_0) \leq y(t, y_0),\ \ \forall t \geq 0.$$

where $y(t, \xi)$ is the trajectory of (13) starting at ξ $\square$

Consider a system of the form

$$\dot{Q}(t) = V(t)Q(t) + \gamma(t), \tag{14}$$

with $Q : [0\ ,\ T] \to \mathbb{R}^n$, $\gamma : \mathbb{R}^+ \longrightarrow \mathbb{R}^n_+$ and $V(t)$ is a (n, n) matrix.
If $V(t)$ is a Metzler matrix then system (14) is cooperative and we can easy prove that trajectories verify

$$Q(0) \geq 0 \Longrightarrow Q(t) \geq 0,\ \ \forall t \geq 0. \tag{15}$$

Hence, our hope is to formulate errors $e^+ = X_1^+ - X_1$ and $e^- = X_1 - X_1^-$ under the form (14) and apply cooperative results.
Error e^+ verify equation (14)

$$\dot{e}^+ = \beta_1(t)e^+ + (N_2 B_2^+(t) - N_2 B_2(t)), \tag{16}$$

with $Q = e^+$, $V = \beta_1$ and $\gamma(t) = N_2 B_2^+(t) - N_2 B_2(t)$. Indeed,
Firstly, the matrix $\beta_1(t) = \begin{pmatrix} -(\beta + r)D(t) & (1 + r)D(t) \\ rD(t) & -(1 + r)D(t) \end{pmatrix}$ has non negative off-diagonal entries and then β_1 is a Metzler.
Secondly,

$$\gamma(t) = N_2 B_2^+(t) - N_2 B_2(t) \geq 0.$$

Finally according to assumption of proposition (3.1) we have

$$e^+(0) = X^+(0) - X_1(0) \geq 0.$$

We conclude that the system (16) is cooperative and hence (15) implies that

$$e^+(t) \geq 0,\ \ \forall t \geq 0,$$

in other word,

$$X^+(t) \geq X_1(t), \quad \forall t \geq 0.$$

The same proof for $X^-(t) \leq X_1(t)$ can be derived by using $e^- = X_1 - X^-$ $\square$

4. Robust output stabilization

The aim of this section is to build a feedback control, using dilution $D(t)$ and aeration $W(t)$ rates as controls, to stabilize output variable $Y(t) = (S(t), D_o(t))$ around a suitable value $Y^d = (S^d, D_o^d)$.
To be able to define controls, we need to introduce some tools and considerations:
If we put

$$Y(t) = (y_1(t), y_2(t)), \quad y_1^{in}(t) = \frac{S_{in}(t)}{1+r}, \quad y_2^{in}(t) = \frac{D_{o_{in}}(t)}{1+r},$$

$$y_2^{max}(t) = D_{o_{max}}(t), \quad D_1(t) = D(t)(1+r), \quad and \quad D_2(t) = \alpha W(t).$$

The output equation in (4), becomes

$$\dot{y_1} = -R_1(y_1, y_2, x) + D_1(y_1^{in} - y_1) \tag{17}$$
$$\dot{y_2} = -R_2(y_1, y_2, x) + D_2(y_2^{max} - y_2), \tag{18}$$

since $X_1 = \begin{pmatrix} x_r \\ x \end{pmatrix}$ and

$$R_1(y_1, y_2, x) = \frac{\mu(y_1, y_2)x}{Y_f}, \quad R_2(y_1, y_2, x) = K_0 \frac{\mu(y_1, y_2)x}{Y_f} - D_1(y_2^{in} - y_2).$$

Using observation intervals $X_1^-(t) \leq X_1(t) \leq X_1^+(t)$ developed in the previous section we provide observers of x and x_r

$$X_1^- = \begin{pmatrix} x_r^- \\ x^- \end{pmatrix} \leq \begin{pmatrix} x_r \\ x \end{pmatrix} \leq X_1^+ = \begin{pmatrix} x_r^+ \\ x^+ \end{pmatrix}$$

Define now, the upper and lower functions of R_i, for $i = 1, 2$, as functions of y_1, y_2 and observations x^-, x^+ instead unknown state x.

$$R_1^-(y_1, y_2, x^-, x^+) = \frac{\mu^-(y_1, y_2)x^-}{Y_f},$$

$$R_1^+(y_1, y_2, x^-, x^+) = \frac{\mu^+(y_1, y_2)x^+}{Y_f},$$

$$R_2^-(y_1, y_2, x^-, x^+) = K_0 \frac{\mu^-(y_1, y_2)x^-}{Y_f} - D_1(y_2^{in+} - y_2),$$

$$R_2^+(y_1, y_2, x^-, x^+) = K_0 \frac{\mu^+(y_1, y_2)x^+}{Y_f} - D_1(y_2^{in-} - y_2),$$

where $\mu^*(y_1, y_2)$, $(^* =^{+,\,-})$ is already defined in (1) and

$$y_2^{in-}(t) = \frac{D_{o_{in}}^-(t)}{1+r}, \quad y_2^{in+}(t) = \frac{D_{o_{in}}^+(t)}{1+r}.$$

Let

$$y_1^{in-}(t) = \frac{\underline{S}_{in}(t)}{1+r}, \quad y_1^{in+}(t) = \frac{\overline{S}_{in}(t)}{1+r}, \quad y_2^{max-}(t) = D_{o_{max}}^-(t) \text{ and } y_2^{max+}(t) = D_{o_{max}}^+(t)$$

and consider, for $i = 1, 2$, functions

$$g_i(y_i) = \frac{1}{2}(y_i - y_i^d)^2,$$

and

$$G_i(y_i) = \begin{cases} \dfrac{1 - \exp^{-g_i(y_i)}}{y_i - y_i^d} & if \quad y_i \neq y_i^d \\ 0 & if \quad y_i = y_i^d. \end{cases}$$

Remark that G_i is a continuous function on $\mathbb{R}$.
Finally, note that the level of dissolved oxygen concentration in the aerobic reactors has a significant influence, it has to be neither too low for stability of the process, or too high to economize energy consumption. For this reason, the dissolved oxygen concentration is supposed to be lower than the maximum level $D^-_{o_{max}}$, see [2, 9, 22]. In our context this means that

$$y_2 < y_2^{max-}. \tag{19}$$

On its part, the substrate provides abundant food to sustain growth of microorganisms. This growth corresponds to the synthesis of new microorganism cells. At this phase substrate concentration is considerably depleted (exponentially). The level of $S(t)$ is very low compared with S^-_{in}, $(S(t) << S^-_{in})$, there is not enough food left to sustain growth of microorganisms. The cells start consuming their fellow as food, and hence the concentration of substrate decreases to amount of non biodegradable matter. So, through regulation of the recycling pump, we can choose the recycle rate as follows

$$0 < r < \frac{S^-_{in} - S_0}{S_0},$$

where $S(0) = S_0$ is the substrate biomass at the time $t = 0$.
The function $S \mapsto \frac{S^-_{in} - S}{S}$ is decreasing, so when S decreases, this function increases and by the way

$$r < \frac{S^-_{in} - S}{S}, \ \forall S \leq S_0,$$

which implies that

$$S < \frac{S^-_{in}}{1 + r} = y_1^{in-},$$

and hence that

$$y_1 < y_1^{in-}. \tag{20}$$

Note that this consideration is used in the most industrial effluents, see for example [1, 15] . In our case we will consider the same assumptions 19 and 20.
Define now feedback controls

$$D_1(y_1, y_2, x^-, x^+) := \begin{cases} D_1^+ = \dfrac{R_1^+ - \lambda_1^+ G_1(y_1)}{y_1^{in-} - y_1} & if \quad y_1 < y_1^d \\ D_1^- = \dfrac{R_1^- - \lambda_1^- G_1(y_1)}{y_1^{in+} - y_1} & if \quad y_1 \geq y_1^d. \end{cases} \tag{21}$$

$$D_2(y_1, y_2, x^-, x^+) := \begin{cases} D_2^+ = \dfrac{R_2^+ - \lambda_2^+ G_2(y_2)}{y_2^{max-} - y_2} & if \quad y_2 < y_2^d \\ D_2^- = \dfrac{R_2^- - \lambda_2^- G_2(y_2)}{y_2^{max+} - y_2} & if \quad y_2 \geq y_2^d. \end{cases} \tag{22}$$

The parameters λ_i^*, with $i = 1,\ 2$ and $* =^{+,\ -}$ are positive constants, chosen to adjust the rate of the exponential stability convergence as we will show in the proof of the Theorem (4.1).

With this control law we prove that the stabilization of output $Y(t) = (y_1(t), y_2(t))$ is possible, the argument is based on a suitable choice of a Lyapunov function, (see [8, 21] for Lyapunov theory).

Theorem 4.1. *Feedback control laws* (D_1, D_2) *exponentially stabilizes output* $Y(t) = (y_1(t), y_2(t))$ *around desired point* $Y^d = (y_1^d, y_2^d)$.

Proof. Consider the following Lyapunov function

$$V(y_1, y_2) = V_1(y_1) + V_2(y_2),$$

with

$$V_i(y_i) = \exp(g_i(y_i)) - 1, \quad \forall\, i = 1, 2.$$

The time derivative of V along a trajectory $y(t) = (y_1(t), y_2(t))$ is

$$\frac{dV(y(t))}{dt} = \dot{V}_1(y_1(t)) + \dot{V}_2(y_2(t)),$$

where

$$\dot{V}_i(y_i(t)) = (y_i(t) - y_i^d)\exp(g_i(y_i(t)))\dot{y}_i(t). \tag{23}$$

Firstly, it is easy to show that

$$(y_i - y_i^d)(y_i^{in} - y_i) \leq (y_i - y_i^d)(y_i^{in*} - y_i), \tag{24}$$

with $* = -$ if $y_i - y_i^d < 0$ and $* = +$ if $y_i - y_i^d > 0$
and that

$$(y_i - y_i^d)(-R_i + R_i^*) \leq 0, \tag{25}$$

where $* = +$ if $y_i - y_i^d < 0$ and $* = -$ if $y_i - y_i^d > 0$.
Indeed, let us prove (25) for $i = 2$. If $y_2 - y_2^d > 0$ then we choose $* = -$.
Hence

$$-R_2 + R_2^- = D_1(y_2^{in} - y_2^{in+}) + \frac{K_0}{Y}(\mu^- x^- - \mu x) < 0,$$

as required.
Proof of (25) for $i = 1$ is similar.
The hope now, is to prove that

$$\dot{V}_1(y_1(t)) < -\lambda_1^* V_1(y_1(t)), \quad \text{for all } y_1 \neq y_1^d.$$

where $* = -$ if $y_1 - y_1^d > 0$ and $* = +$ if $y_1 - y_1^d < 0$

- Suppose that $y_1 - y_1^d > 0$,
  using equations (23), (24), (25), (17) and control law (21), one gets

$$\begin{aligned}
\dot{V}_1(y_1(t)) &= (y_1 - y_1^d)\exp(g_1(y_1))(-R_1 + D_1(y_1^{in} - y_1)) \\
&= \exp(g_1(y_1))[-R_1(y_1 - y_1^d) + D_1(y_1 - y_1^d)(y_1^{in} - y_1)] \\
&< \exp(g_1(y_1))[-R_1(y_1 - y_1^d) + D_1(y_1 - y_1^d)(y_1^{in+} - y_1)] \\
&< (y_1 - y_1^d)\exp(g_1(y_1))[-R_1 + (\frac{R_1^- - \lambda_1^- G_1(y_1)}{y_1^{in+} - y_1})(y_1^{in+} - y_1)] \\
&< (y_1 - y_1^d)\exp(g_1(y_1))[-R_1 + R_1^- - \lambda_1^- G_1(y_1)] \\
&< \exp(g_1(y_1))[(y_1 - y_1^d)(-R_1 + R_1^-) - \lambda_1^- G_1(y_1)(y_1 - y_1^d)] \\
&< -\lambda_1^- \exp(g_1(y_1))G_1(y_1)(y_1 - y_1^d) \\
&< -\lambda_1^- \exp(g_1(y_1))(\frac{1 - \exp^{-g_1(y_1)}}{y_1 - y_1^d})(y_1 - y_1^d) \\
&< -\lambda_1^- V_1(y_1(t))
\end{aligned}$$

- If $y_1 - y_1^d < 0$ then

$$\begin{aligned}
\dot{V}_1(y_1(t)) &= (y_1 - y_1^d)\exp(g_1(y_1))(-R_1 + D_1(y_1^{in} - y_1)) \\
&= \exp(g_1(y_1))[-R_1(y_1 - y_1^d) + D_1(y_1 - y_1^d)(y_1^{in} - y_1)] \\
&< \exp(g_1(y_1))[-R_1(y_1 - y_1^d) + D_1(y_1 - y_1^d)(y_1^{in+} - Y_1)] \\
&< (y_1 - y_1^d)\exp(g_1(y_1))[-R_1 + (\frac{R_1^- - \lambda_1^+ G_1(y_1)}{y_1^{in+} - y_1})(y_1^{in+} - y_1)] \\
&< (y_1 - y_1^d)\exp(g_1(y_1))[-R_1 + R_1^- - \lambda_1^+ G_1(y_1)] \\
&< \exp(g_1(y_1))[(y_1 - y_1^d)(-R_1 + R_1^-) - \lambda_1^+ G_1(y_1)(y_1 - y_1^d)] \\
&< -\lambda_1^+ \exp(g_1(y_1))G_1(y_1)(y_1 - y_1^d) \\
&< -\lambda_1^+ \exp(g_1(y_1))(\frac{1 - \exp(-g_1(y_1))}{y_1 - y_1^d})(y_1 - y_1^d) \\
&< -\lambda_1^+ V_1(y_1(t)).
\end{aligned}$$

It follows that

$$\dot{V}_1(y_1(t)) < -\sigma V_1(y_1(t)), \quad \text{for all } y_1 \neq y_1^d.$$

where $\sigma_1 = \min\{\lambda_1^-,\ \lambda_1^+\}$.
The same arguments can be used to derive case $i = 2$, we obtain then

$$\dot{V}_2(y_2(t)) < -\lambda_2^* V_1(y_2(t)), \quad \text{for all } y_2 \neq y_2^d.$$

where $* = -$ if $y_2 - y_2^d > 0$ and $* = +$ if $y_2 - y_2^d < 0$.
It follows that

$$\dot{V}_2(y_2(t)) < -\sigma_2 V_2(y_2(t)), \quad \text{for all } y_2 \neq y_2^d.$$

where $\sigma_2 = \min\{\lambda_2^-,\ \lambda_2^+\}$.
We conclude that

$$\dot{V}(y(t)) < -\min(\sigma_1,\ \sigma_2)V(y(t))$$

as required □

Remark 1. *To assure continuity of controls D_1 and D_2 we can use the modified controls*

$$\widetilde{D}_{i,\varepsilon} := \begin{cases} D_i^- \quad if \quad y_i - y_i^d \geq \varepsilon \\ \dfrac{(y_i - y_i^d + \varepsilon)D_i^- + (y_i^d - y_i + \varepsilon)D_i^+}{2\varepsilon} \quad if \quad -\varepsilon \leq y_i - y_i^d \leq \varepsilon \\ D_i^+ \quad if \quad y_i - y_i^d < -\varepsilon \end{cases} \tag{26}$$

Then feedback control laws $(\widetilde{D}_{1,\varepsilon}, \widetilde{D}_{2,\varepsilon})$ exponentially stabilizes the output $Y(t)$ around desired point $Y^d(t)$.
Indeed, the case what we would analyze is $-\varepsilon \leq y_i - y_i^d \leq \varepsilon$:

- Case $0 < y_i - y_i^d \leq \varepsilon$,
  we take ε very smaller, so that we can identify

  $$y_i - y_i^d \approx \varepsilon$$

  So,

  $$\frac{(y_i - y_i^d + \varepsilon)}{2\varepsilon} \approx 1 \text{ and } \frac{(y_i^d - y_i + \varepsilon)}{2\varepsilon} \approx 0$$

  Hence we can identify $\widetilde{D}_{i,\varepsilon}$ with D_i^- .

- Conversely if $-\varepsilon < y_1 - y_1^d < 0$ then

$$\frac{(y_i - y_i^d + \varepsilon)}{2\varepsilon} \approx 0 \text{ and } \frac{(y_i^d - y_i + \varepsilon)}{2\varepsilon} \approx 1$$

  Hence, we can identify $\widetilde{D}_{i,\varepsilon}$ with D_i^+ .

With those considerations we find the case treated in the proof of the above theorem.

5. Numerical simulations

In this section we give some numerical simulations to illustrate our results.
Consider the kinetic growth functions:

$$\mu(S, D_o) = \mu_{max}\mu_1(S).\mu_2(D_o) + 10\sqrt{S.D_o}|sin(S.D_o)|,$$
$$\mu^+(S, D_o) = \mu^+_{max}\mu_1(S).\mu_2(D_o) + 10\sqrt{S.D_o},$$
$$\mu^-(S, D_o) = \mu^-_{max}\mu_1(S).\mu_2(D_o),$$

with

$$\mu_1(S) = \frac{S}{K_S + S} \quad and \quad \mu_2(D_o) = \frac{D_o}{K_{D_o} + D_o},$$

Note that we choose in the kinetic function the square and absolute value of the function sinus to produce non smooth variations and disturbances, in order to show that in some degenerated cases we can stabilize the system.
The model parameters and initial conditions are given as follows :
$D = 50$, $\beta = 0.2$, $r = 14$, $K_S = 0.5$, $K_{D_o} = 0.002$

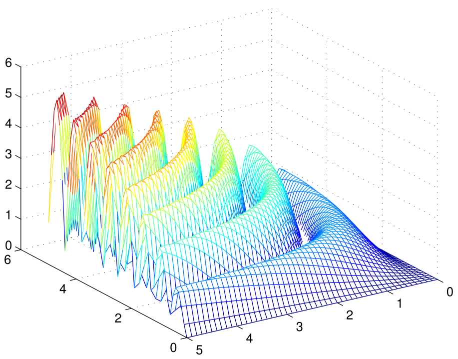

FIGURE 2. Kinetic growth function μ with envelops μ^+ and μ^-.

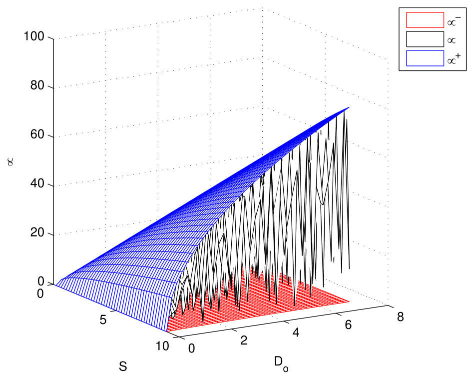

FIGURE 3. Kinetic growth function μ.

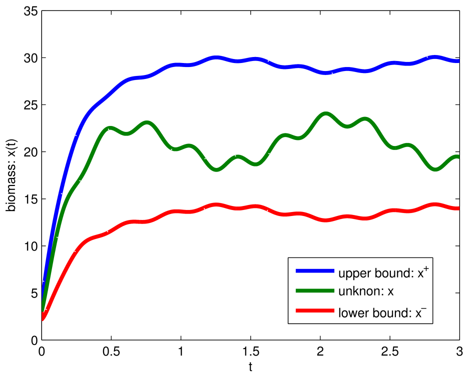

FIGURE 4. Observer interval of the recycled bacteria biomass $x_r(10^9.number/mL)$ with respect t(day).

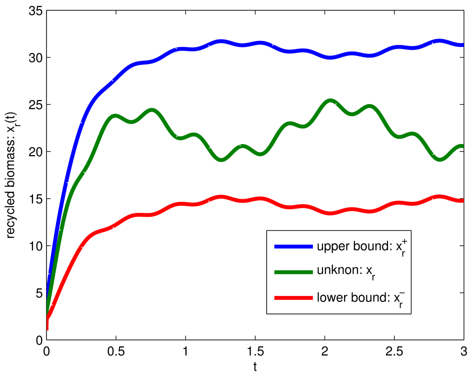

FIGURE 5. Observer Interval of the bacteria biomass $x(10^9.number/mL)$ with respect t(day).

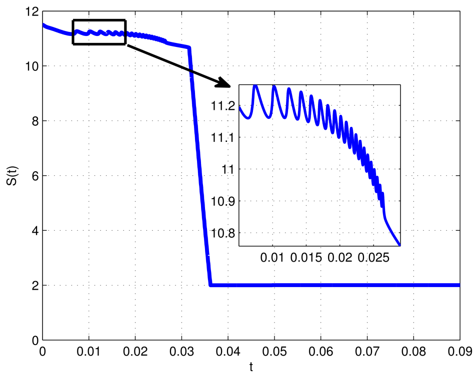

FIGURE 6. Stabilization of substrat biomass $S(mg/L)$ around $S^d(mg/L)$ with respect t(day).

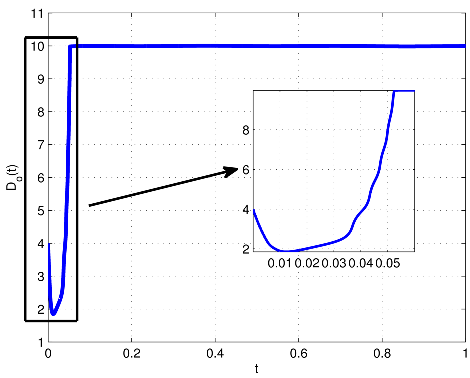

FIGURE 7. Stabilization of dissolved oxygen concentration $D_o(mgO_2/L)$ around $D_o^d(mgO_2/L)$ with respect t(day).

Interpretation

Figure (2) gives an example of a not well known kinetic growth function of bacteria μ, (with variations and non smoothness) and figure (3) shows μ with its upper and lower bounds μ^+ and μ^-.

Figures (4) and (5) illustrate the first aim of this work, that is, the observation intervals for the biomass bacteria x and recycle biomass bacteria x_r. We can observe that $x^-(t) \leq x(t) \leq x^+(t)$ and $x_r^-(t) \leq x_r(t) \leq x_r^+(t)$ for all $t \geq 0$.

Figures (6) and (7) illustrate the second aim of this work, that is, the stabilization around (S^d, D_o^d). We show that after a period of fluctuations (small squares in figures (6) and (7)), due to fluctuations of μ, trajectories of S and D_o are stabilized around the desired levels thanks to feedback controls D_1 and D_2.

6. Conclusion

In this work we have investigated two dimension stabilization of an aerobic activated sludge model. By means of observation intervals of unmeasured states, that is, bacteria and recycle bacteria biomasses, we built feedback controls involving dilution and aeration rates to stabilize the output states, that is, bacteria and recycle bacteria biomasses around a desired fixed level, we gave some numerical simulations to illustrate our results. Our hope in the next works is to investigate the global stability of all states and optimal control by introducing a criterion to minimize energy consuming of recycling and aeration.

References

1. V. Alcaraz-Gonzalez, F.A. Fregoso-Sanchez, J.L. Seyer and H.O. Méndez-Acosta, *Exponential regulation of alkalinity and VFA in continuous anaerobic digestion processes under uncertain operational conditions*, J. Wseas transactions on systems and control **10** (2015) 453–460.
2. M. Barbu and S. Caraman, *Qft robust control of wastewater treatment processes*, Robust Control, Theory and Application, Eds. Andrzej Bartoszewicz, Tech Open Sciences (2011) 577–602.
3. V.S. Bokharaie, *Stability analysis of positive systems with applications to epidemiology* PhD, Hamilton Institute, National University of Ireland, Maynooth, 2012.
4. V.S. Bokharaie, O. Masona and F. Wirth, *Stability and positivity of equilibria for subhomogeneous cooperative systems*, J. Nonlinear Analysis: Theory, Methods & Applications **74** (2011) 6416–6426.
5. H. Boutanfit, M. Serhani and A. Boutoulout, *Stability and bifurcation synthesis in a nonlinear chemostat model*, Global Journal of Pure and Applied Mathematics **12**(4) (2016) 2877–2899.
6. B.M. Dolgonosov, *Modeling of aerobic biodegradation and oxygen consumption in benthic sediments*, J. Wat. Re. Res. **36** (1)(2000) 297–308.
7. K. Gernaey, H. Bogaert, P.D. Vanrolleghem, L.D. Van Vooren and W. Verstraete, *Sensors for nitrogen removal monitoring in wastewater treatment*, IN: Environmental biomonitoring - The biotechnology ecotoxicology interface, Lynch JM, Wiseman A. editors, Environmental Biomonitoring Conference EB '97 Proceedings 6-7 June 1996; Surrey, Guildford, UK: Cambridge University Press, (1998) 180–206.
8. W.M. Haddad and V. Chellaboina, *Nonlinear dynamical systems and Control: A Lyapunov-based approach*, Oxford: Princeton University Press, 2008.
9. B. Holenda, E. Domokos, A. Redey and J. Fazakas, *Dissolved oxygen control of the activated sludge wastewater treatment process using model predictive control*, J. Computers and Chemical Engineering **32**(6) (2008) 1270–1278.
10. D.S. Kompala, *Bioprocess engineering: fundamentals and applications*, CRC Press, 2013.
11. C.F. Lindberg, *Control and estimation strategies applied to the activated sludge process*, Automatic Control: Uppsala University, 1997.
12. L. Mailleret, J.L. Gouzé and O. Bernard, *Global stabilization of a class of partially known nonnegative systems*, Automatica **44** (2008) 2128–2134.
13. F. Nejjari, A. Benhammou, B. Dahhou and G. Roux, *Non-linear multivariable adaptive control of an activated sludge wastewater treatment process*, Int. J. Adapt. Control Signal Process **13** (1999) 347–365.
14. G. Olsson, *State of the art in sewage treatment control*, Engineers Symp Series, American Inst. of Chemical **159**(72) (1997) 52–76.
15. A. Rapaport and J. Harmand, *Robust regulation of a class of partially observed nonlinear continuous bioreactors*, J. of Process Control **12** (2) (2002) 291–302.
16. M. Serhani, H. Boutanfit and A. Boutoulout, *Sensitivity and strong controllability of a nonlinear chemosta model*, ESAIM: Proceedings and Surveys **49** (2015) 115–129.
17. M. Serhani, J.L. Gouzé and , N. Raissi, *Dynamical study and robustness for a nonlinear wastewater treatment model*, J. Nonlinear Analysis: Real World Applications **12**(1) (2011) 487–500.

18. M. Serhani, N. Raissi and P. Cartigny, *Robust feedback control design for a nonlinear wastewater treatment model*, J. Mathematical Modelling of Natural Phenomena **4**(5) (2009) 128–143.
19. H.L. Smith, *Monotone dynamical systems. An introduction to the theory of competitive and cooperative systems*, AMS Mathematical Surveys and Monographs, (1995) 31–53.
20. D. Stanica, *Une nouvelle méthode pour déterminer la pseudo inverse d'une application linéaire*, J. de la Société des Sciences Mathématiques de Roumanie **42**(1) (1995) 65–74.
21. G. Teschl, *Ordinary differential equations and dynamical systems*, **140**, Graduate Studies in Mathematics: Amer. Math. Soc., Providence, 2012.
22. G. Vlad, M. Sbarciog, M. Barbu, S. Caraman and A.V. Wouwer, *Indirect control of substrate concentration for a wastewater treatment process by dissolved oxygen tracking*, J. Control engineering and applied informatics **14**(1) (2012) 37–47.
23. G. Vlad, D.N. Robescu and C.R. Mocanu, *Mathematical modelling of a biological wastewater treatment process. case study: the wastewater treatment station of Romanofir S.A.*, Trading CO-Talmaciu. U.P.B. Sci. Bull **73**(2) (2011) 299–310.

Mustapha Serhani
e-mail: `mustapha.serhani@gmail.com`
TSI team, Faculty of SJES, University Moulay Ismail, B.P. 21301, Toulal, Meknes, Morocco

Hamid Boutanfit
e-mail: `hboutanfit@gmail.com`
ITSI team, Faculty of Science, University Moulay Ismail, B.P. 11201, Zitoune, Meknes, Morocco

Submitted: 20 Mars 2018

Revised: 23 July 2018

Accepted: 6 October 2018