IMHOTEP

AFRICAN JOURNAL OF PURE AND APPLIED MATHEMATICS

Imhotep Mathematical Journal
Volume 2, Numéro 1, (2017),
pp. 1 – 6.

The relations between φ-amenability and some special ideals

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Abstract

In this paper we determine φ-amenability of a Banach algebra A with some certain ideals in its second dual A''. We show that for an idempotent $m \in A''$, the set $\mathbb{C} \cdot m = \{\lambda \cdot m : \lambda \in \mathbb{C}\}$ is a left ideal in A'' if and only if m is a φ-mean. We also give some results on the relationships between weak amenability and φ-amenability of Banach algebras.

IMHOTEP - Afr. J. Pure Appl. Math. **2** (2017), 1–6
1608-9324/010001-0, DOI 13.1007/s00009-003-0000

IMHOTEP African Journal of Pure and Applied Mathematics

The relations between φ-amenability and some special ideals

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Mathematics Subject Classification (2010). 46H25 (Primary), 43A07 (Secondary).

Keywords. Banach algebra, phi-amenability, Ideal, φ-mean, φ-amenability, extreme point.

1. Introduction

Let A be a Banach algebra, and let X be a Banach A-bimodule. Then a linear map $D : A \to X$ is a derivation if

$$D(ab) = a \cdot D(b) + D(a) \cdot b \qquad (a, b \in A).$$

For example, let $x \in X$ and define $\delta_x : A \to X$ by $\delta_x a = a \cdot x - x \cdot a$ then δ_x is a derivation which is called an inner derivation.

The set of all bounded derivations from A into X is denoted by $Z^1(A, X)$, and $N^1(A, X)$ is the set of all inner derivations from A into X. Also, $H^1(A, X) = Z^1(A, X)/N^1(A, X)$ is the first cohomology group with coefficients in X.

The dual space of a Banach A-bimodule with the following multiplicatios can be made into a Banach A-bimodule

$$a \cdot x'(x) = x'(x \cdot a) \qquad , \qquad x' \cdot a(x) = x'(a \cdot x)$$

for each $x \in X, x' \in X', a \in A$.

Let A be a Banach algebra and A'' be its second dual, for each $a, b \in A$, $f \in A'$ and $F, G \in A''$ we define $f \cdot a$, $a \cdot f$ and $F \cdot f, f \cdot F \in A'$ by

$$f \cdot a(b) = f(a \cdot b) \qquad , \qquad a \cdot f(b) = f(b \cdot a)$$
$$F \cdot f(a) = F(f \cdot a) \qquad , \qquad f \cdot F(a) = F(a \cdot f).$$

Now we define $F \cdot G, F \times G \in A''$ as follows

$$F \cdot G(f) = F(G \cdot f), \qquad F \times G(f) = G(f \cdot F).$$

Then A'' is a Banach algebra with respect to either of the products $\cdot$ and $\times$, these products are called respectively, the first and the second Arens products on A''. The original definitions of the two Arens products were given by Arens in 1951, see [1] and [2]. A is called Arens regular if $F \cdot G = F \times G$, for all $F, G \in A''$.

A Banach algebra A is called amenable if $H^1(A, X') = Z^1(A, X')/N^1(A, X') = \{0\}$ for each Banach A-bimodule X. This concept was introduced by B. E. Johnson in [8].

The notion of weak amenability was introduced by W. G. Bade, P. C. Curtis and H. G. Dales in [3] for commutative Banach algebras. Later Johnson defined weak amenability for arbitrary Banach algebras in [7], in fact a Banach algebra A is weakly amenable if $H^1(A, A) = 0$.

In [11] Lau introduced a large class of Banach algebras which are called F-algebras (or Lau algebras). Then E. Kaniuth, A. T. Lau and J. Pym in [9] and [10] investigated φ-amenability which is more general than left amenability for F-algebras.

Let A be a Banach algebra and φ a homomorphism from A onto $\mathbb{C}$, then A is called **φ-amenable** if there exists $m \in A''$ satisfying $m(\varphi) = 1$ and $m(f \cdot a) = \varphi(a)m(f)$ for all $a \in A$ and $f \in A'$ and m is called a **φ-mean**.

In [12], Monfared introduced and investigated the notion of right character amenability. A Banach algebra A is right character amenable if it has a bounded right approximate identity and is φ-amenable for each $\varphi \in \Delta(A)$, where $\Delta(A)$ is the space of all homomorphisms from A onto $\mathbb{C}$. For a locally compact group G, the Fourier algebra $A(G)$ is right character amenable if and only if G is amenable whereas, $A(G)$ is φ-amenable for each $\varphi \in \Delta(A)$, see [10] and [12]. Also, if for some $\varphi \in \Delta(A)$, $\ker \varphi$ has a bounded right approximate identity A is φ-amenable. Since every closed ideal of a C^*-algebra has a bounded approximate identity then every C^*-algebra is φ-amenable for every φ. For more details see [9] and [10]].

In this paper the second dual A'' of a Banach algebra A will always be considered with the first Arens product.
Now, we recall some theorems which are used in this paper.

Theorem 1.1. *Let A be a Banach algebra and $\varphi \in \Delta(A)$. Then A is φ-amenable if and only if $H^1(A, X') = \{0\}$, for each Banach A-bimodule X with $a \cdot x = \varphi(a)x$ for all $x \in X$ and $a \in A$.*

Proof. See Theorem 1.1 of [9].

■

Let A be a Banach algebra then $m \in A''$ is a two-sided φ-mean if for each $f \in A'$ and $a \in A$,

$$m(f \cdot a) = m(a \cdot f) = \varphi(a)m(f).$$

Theorem 1.2. *$m \in A''$ is a two-sided φ-mean for a Banach algebra A, if and only if there exists a bounded net $(u_\alpha)_\alpha$ in A such that $m = w^* - \lim_\alpha \hat{u}_\alpha$ and*

$$\varphi(u_\alpha) = 1, \quad \|a \cdot u_\alpha - \varphi(a) \cdot u_\alpha\| \to 0, \quad \|u_\alpha \cdot a - \varphi(a) \cdot u_\alpha\| \to 0.$$

Proof. See Theorem 1.4 of [10].

■

2. Determination of φ-amenability with special ideals

Let A be a Banach algebra and $m \in A''$ an idempotent. Then the subset

$$J_m = \mathbb{C} \cdot m = \{\lambda m : \ \lambda \in \mathbb{C}\}$$

is a closed one-dimensional subalgebra of A''.

Now consider subsets L_m and R_m in A'' as follows

$$\begin{aligned} L_m &= \{n \in A'' : \ n \cdot m = \lambda m \text{ for some } \lambda \in \mathbb{C}\} \\ R_m &= \{n \in A'' : \ m \cdot n = \lambda m \text{ for some } \lambda \in \mathbb{C}\}. \end{aligned}$$

Then L_m and R_m are closed subalgebras in A''. L_m is always w^*-closed, and R_m is w^*-closed when A is Arens regular.

It is easy to see that J_m is a left (right) ideal in A'' if and only if $L_m = A''$ ($R_m = A''$). These subalgebras are examples of "residual quotients" of ideals, analagous to the concept of a "primitive ideal".

Theorem 2.1. *Let A be a Banach algebra, $m \in A''$ be an idempotent and φ be a multiplicative functional. Then*

(1) *J_m is weakly amenable;*
(2) *$Z^1(J_m, X) = \{0\}$, for each symmetric Banach J_m-bimodule;*
(3) *For each Banach algebra B with $xy = \varphi(x)y$ $(x, y \in B)$, $J_m \hat{\otimes} B^\#$ is weakly amenable where $B^\#$ is the unitization of B;*
(4) *If m is a φ-mean, then $H^1(A, J'_m) = \{0\}$.*

Proof. (1) Let $D : J_m \to J'_m$ be a bounded derivation. Since m is an idempotent and J_m is commutative, then $Dm = D(m \cdot m) = 2m \cdot Dm = 4m \cdot Dm$. So $Dm = 0$.
(2) Use 2.8.63 of [5].
(3) Let $\mathcal{U} = J_m \hat{\otimes} B^\#$ and $D : \mathcal{U} \to \mathcal{U}'$ be a bounded derivation and let $b_0 \in B$ with $\varphi(b_0) = 1$. Then $\lambda \in \mathcal{U}'$ exists such that

$$\lambda(n \otimes b) = D(m \otimes b)(n \otimes b_0) \qquad (n \in J_m, b \in B^\#).$$

For each $n \in J_m$ and $b, c \in B$ we have:

$$\begin{aligned} \delta_\lambda(m \otimes b)(n \otimes c) &= ((m \otimes b) \cdot \lambda - \lambda \cdot (m \otimes b))\,(n \otimes c) \\ &= \lambda(n \otimes c \cdot b - n \otimes bc) \\ &= (\varphi(c) \cdot D(m \otimes b) - \varphi(b) \cdot D(m \otimes c))\,(n \otimes b_0) \\ &= ((m \otimes c) \cdot D(m \otimes b))\,(n \otimes b_0) \\ &= (D(m \otimes b))\,(n \otimes b). \end{aligned}$$

So for each $b \in B$, $D(m \otimes b) = \delta_\lambda(m \otimes b)$. Now we show that $D(m \otimes e_B) = \delta_\lambda(m \otimes e_B) = 0$. To this end we consider $D_1 : J_m \to \mathcal{U}'$ with the definition $D_1(n) = D(n \otimes e_B)$, $(n \in J_m)$. Then D_1 is a bounded derivation. It is easy to see that under the following multiplications, $\mathcal{U}'$ is a symmetric Banach J_m-bimodule

$$n \cdot f = (n \otimes e_B) \cdot f \quad , \quad f \cdot n = f \cdot (n \otimes e_B) \quad (f \in \mathcal{U}',\ n \in J_m).$$

Now by (2) we have $D_1 = 0$, i. e. $D(m \otimes e_B) = 0$ and we have

$$\begin{aligned} D(m \otimes b) &= D\,((m \otimes e_B) \cdot (m \otimes b)) = (m \otimes e_B) \cdot D(m \otimes b) + D(m \otimes e_B) \cdot (m \otimes b) \\ &= (m \otimes e_B) \cdot D(m \otimes b) = (m \otimes e_B) \cdot \delta_\lambda(m \otimes b) = \delta_\lambda(m \otimes b). \end{aligned}$$

So D is an inner derivation and $J_m \hat{\otimes} B^\#$ is weakly amenable.
Use Theorem 1.1 to prove (4).

■

Lemma 2.2. *Let A be a Banach algebra and $m \in A''$ be a non-zero idempotent. If $\hat{A} \subseteq L_m$, then there exists $\varphi \in \Delta(A)$ such that A is φ-amenable.*

Proof. Let $\hat{A} \subseteq L_m$ then $m \neq 0$ and for each $a \in A$ there exists $\lambda_a \in \mathbb{C}$ such that $a \cdot m = \lambda_a m$ and λ_a is unique. Now we define $\varphi : A \to \mathbb{C}$ with $\varphi(a) = \lambda_a$ $(a \in A)$ then $\varphi \in \Delta(A)$. For each $f \in A'$ and $a \in A$ we have $m(f \cdot a) = am(f) = \varphi(a)m(f)$. Moreover $m(\varphi) = m(\varphi)(m\varphi)$ so $m(\varphi) = 1$ and m is a φ-mean.

■

We denote the commutator of A in A'' with A^c which is defined as follows

$$A^c = \{F \in A'' : \ a \cdot F = F \cdot a, \ \forall a \in A\}.$$

Lemma 2.3. *Let A be a Banach algebra and $m \in A''$ be a non-zero idempotent. Then in each of the following cases, there exists $\varphi \in \Delta(A)$ such that m is a two-sided φ-mean:*

(1) $\hat{A} \subseteq L_m$ *and* $m \in A^c$;

(2) $\hat{A} \subseteq L_m \cap R_m$.

Proof. (1) Use Lemma 2.2. For (2) Let $\hat{A} \subseteq L_m \cap R_m$ then by Lemma 2.2 there exists $\varphi \in \Delta(A)$ such that m is a φ-mean. $\hat{A} \subseteq R_m$ implies that for each $a \in A$ there exists $\alpha_a \in \mathbb{C}$ such that $m \cdot a = \alpha_a m$ and α_a is unique. So there exists $\psi \in \Delta(A)$ such that $m \cdot a = \psi(a)m \quad$ and $\quad m(\psi) = 1$.
If $\varphi \neq \psi$, since $\varphi, \psi \in \Delta(A)$ then $\ker\varphi$ and $\ker\psi$ are two different maximal ideals in A. So $a_0 \in \ker\psi$ exists for which $a_0 \notin \ker\varphi$ and we can assume $\varphi(a_0) = 1$. Then $\psi \cdot a_0(b) = \psi(a_0)\psi_b = 0$, $(b \in A)$. Also, m is a φ-mean and $m(\psi) = m(\psi)\varphi(a_0) = m(\psi \cdot a_0) = 0$, which is a contradiction. Therefore $\varphi = \psi$ and m is a two-sided φ-mean.

■

For a Banach algebra A, $Z(A'', \cdot) = \{m \in A'' : \ m \cdot n = m \times n, \text{ for all } n \in A''\}$ is called the center of A''.

Theorem 2.4. *Let A be a Banach algebra. Then:*

(1) *$m \in A''$ is a φ-mean if and only if m is a non-zero idempotent and J_m is a left ideal in A'';*

(2) *$m \in Z(A'', \cdot)$ is a two-sided φ-mean if and only if m is a non-zero idempotent and J_m is a two-sided ideal in A''.*

Proof. (1) Let m be a φ-mean, then for each $f \in A'$ and $(x_\alpha)_\alpha \subseteq A$ with $m = w^* - \lim_\alpha \hat{x}_\alpha$ we have

$$m(f)m = m(m \cdot f) = \lim_\alpha m(f \cdot x_\alpha) = \lim_\alpha mf\varphi(x_\alpha) = m(f)(m\varphi) = m(f).$$

So m is a non-zero idempotent. For $n \in A''$ and for each $f \in A'$ we have $n \cdot m(f) = mf \cdot n(\varphi)$. So $n \cdot m = n(\varphi)m \in J_m$. For the converse let J_m be a left ideal in A'' then $\hat{A} \subseteq L_m$ and by Lemma 2.2, $\varphi \in \Delta(A)$ exists such that m is a φ-mean.
(2) Let m be a two-sided φ-mean then J_m is a left ideal in A'' and m is a non-zero idempotent. For each $f \in A'$ and $n \in A''$ with $n = w^* - \lim_\beta \hat{b}_\beta$ we have

$$\begin{aligned} mn(f) &= m \times n(f) = n(f \cdot m) = \lim_\beta f \cdot m(b_\beta) = \lim_\beta m \cdot b_\beta(f) \\ &= \lim_\beta \varphi(b_\beta)(mf) = n\varphi \cdot mf. \end{aligned}$$

So $m \cdot n = n(\varphi)m \in J_m$
For the converse let J_m be a two-sided ideal in A''. Then by Lemma 2.3, $\varphi \in \Delta(A)$ exists such that m is a two-sided φ-mean. Also for $n \in A''$ with $n = w^* - \lim_\beta \hat{b}_\beta$ and each $f \in A'$ we have

$$\begin{aligned} m \times n(f) &= n(f \cdot m) = \lim_\beta m \cdot b_\beta(f) = \lim_\beta b_\beta \cdot m(f) = n \cdot m(f) \\ &= n(\varphi)m(f) = m \cdot n(f) \end{aligned}$$

so $m \in Z((A'', \cdot))$ and

$$\begin{aligned} m \cdot n(f) &= m \times n(f) = n(f \cdot m) = \lim_\beta f \cdot m(b_\beta) = \lim_\beta m \cdot b_\beta(f) \\ &= \lim_\beta \varphi(b_\beta)m(f) = n(\varphi)m(f). \end{aligned}$$

■

Corollary 2.5. *Let A be a Banach algebra and $m \in A''$ be a non-zero idempotent. Then the following statements are equivalent:*

(1) *m is a two-sided φ-mean;*
(2) $\hat{A} \subseteq L_m \cap R_m$*;*
(3) *$m \in A^c$ and $\hat{A} \subseteq L_m$.*

Proof. Use Lemma 2.3 and Theorem 2.4.

■

Let I be a closed ideal of a Banach algebra A, then I has the **trace extension property** if for each $\lambda \in I'$ with $a \cdot \lambda = \lambda \cdot a$ and each $a \in A$ there exists $f \in A'$ such that $f\big|_I = \lambda$ and $a \cdot f = f \cdot a$ $(a \in A)$.

Corollary 2.6. *Let A be a Banach algebra and $m \in Z(A'', \cdot)$ be a two-sided φ-mean. Then:*

(1) *J_m has the trace extension property;*
(2) *A'' is weakly amenable if and only if A''/J_m is weakly amenable.*

Proof. For (1), use Theorem 2.4. For (2), since J_m has the trace extension property then A''/J_m is weakly amenable, see [6]. For the converse by Theorem 2.1 J_m is weakly amenable and then A'' is weakly amenable.

■

Corollary 2.7. *Let A be a Banach algebra and m be a two-sided φ-mean. If $H^1(A, \ker m) = \{0\}$, then A is weakly amenable.*

Proof. Let $D : A \to A'$ be a bounded derivation, by Theorem1.2 there exists a bounded net $(u_\alpha)_{\alpha\in I}$ in A such that $\varphi(u_\alpha) = 1$ and $\|u_\alpha \cdot a - \varphi(a) \cdot u_\alpha\| \to 0$.
For each $a \in A$ and each α we have

$$\begin{aligned} m \circ D(u_\alpha \cdot a) &= m(Du_\alpha) \cdot \varphi(a) + m(D(a)) \cdot \varphi(u_\alpha) \\ &= m \circ D(u_\alpha)\varphi(a) + m \circ D(a). \end{aligned}$$

Also,

$$\begin{aligned} \|m \circ D(a)\| &= \|(m \circ D)(u_\alpha \cdot a - \varphi(a)u_\alpha)\| \\ &\leq \|m\| \, \|D\| \left(\limsup_\alpha \|u_\alpha \cdot a - \varphi(a)u_\alpha\|\right) = 0. \end{aligned}$$

So, for each $a \in A$ we have $m(Da) = 0$ and $\operatorname{Im} D \subseteq \ker m$.

■

Remark 1. Let A be a Banach algebra. For $\varphi \in \Delta(A)$, consider $S_\varphi = \{m \in A'' : m \text{ is a } \varphi\text{−mean}\}$. If A is an Arens regular commutative Banach algebra then S_φ has at most one element. In the case A admits a φ-mean of norm 1, by using the Krein-Milman Theorem (see [13]) $S_\varphi \cap A''_{[1]}$ has an extreme point which also is the convex hull of its extreme points.

Remark 2. Let φ be the augmentation character(see [4]) on $l^1(G)$. If $l^1(G)$ is φ-amenable then G is amenable. Since there exists $m \in (l^1(G))''$ such that $m(f \cdot a) = \varphi(a)m(f)$ $(f \in (l^1(G))'$ and $a \in l^1(G))$. For $f \in l^\infty(G)$, similar the method used in chapter 43 of [4] define $f' \in (l^1(G))'$ with

$$f'(a) = \sum_{x\in G} a(x).f(x) \qquad (a \in l^1(G))$$

then $(T_{x^{-1}}f)' = f'(\delta_x)$ where $\delta_x(x) = 1$ and $\delta_x(a) = 0$ $(a \neq x)$ and $T_x f(a) = f(x^{-1}a)$ for each $a, x \in G$. Now, consider $\psi \in (l^\infty(G))'$ by $\psi(T_x f) = \psi(f)$, for each $f \in l^\infty(G)$ and $x \in G$. Let $l^\infty_+(G) = \{g \in l^\infty(G) : g \geq 0\}$ and for each $g \in l^\infty_+(G)$ take $\theta(g) = sup\{Re(\psi(f)) : f \in l^\infty(G) : |f| \leq g\}$. Then the extension of $(\theta(1))^{-1}\theta$ to $l^\infty(G)$ is an invariant mean which means G is amenable.

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Submitted: 9 Juin 2016
Revised: 26 November 2016
Accepted: 12 January 2017