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Consensus Stability in the Hegselmann–Krause Model with Coopetition and Cooperosity

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Abstract: Heterogeneous Hegselmann–Krause (HK) models have been used to represent opinion dynamics in social networks. In this framework, the concepts of *coopetition* and *cooperosity* have been recently introduced by the authors in order to characterize different connectivity thresholds for the agents. Inspired by this application, in this paper a sufficient condition for the asymptotic stability of the origin in piecewise linear systems is proved. The result is based on continuous Lyapunov functions which are piecewise differentiable in time. By considering a piecewise quadratic Lyapunov function, the stability result is applied for the consensus in heterogeneous HK models. Examples of heterogeneous HK models with different number of agents show the effectiveness of the proposed approach.

Keywords: Social networks, Multi-agent systems, Stability of hybrid systems, Lyapunov methods.

1. INTRODUCTION

In Hegselmann and Krause [2002] (HK) the authors proposed a model for the representation of opinion dynamics for interacting agents in social networks. The continuous-time version of this model describes the dynamics of each agent by a scalar differential equation whose discontinuous right hand side depends on the differences between the agent state with the others, see Meng et al. [2016] and the references therein. In this paper we refer to the interpretation of the agent state, usually called *opinion*, as a measure of the intensity of his *attitude* toward a particular purpose or action, see Friedkin [2015]. In the classical HK model, the attitudes difference of each pair of agents is weighted by the so-called *influence function* which is zero if the absolute value of such difference is larger than a given *connectivity threshold*, see Motsch and Tadmor [2014], Yang et al. [2014]. Some specific influence functions are used to represent the *coopetition*, an interaction between agents that compete and cooperate at the same time. In Hu and Zheng [2014], Valcher and Misra [2014] coopetitive networks have been modeled as signed graphs where the positive and negative edges represent the cooperation and competition, respectively. A variation of the coopetitive model with sign invariant agents attitudes has been proposed in Ceragioli et al. [2016], while in Ceragioli and Frasca [2015] the effects of the quantization in the information exchanged by the agents is analyzed. In our HK model, by considering each pair of cooperating agents, we say that the agent i is *coopetitive* vs j (and j is *cooperose* vs i) if their cooperation contributes to increase the attitude of i (and to decrease the attitude of j). In this sense the term *cooperosity* introduced in Tangredi et al. [2016] indicates the combination of cooperation and generosity.

We consider the stability problem of the HK model which includes *coopetition* and *cooperosity*. The consensus stability for the classical HK model has been widely considered in the literature, e.g., see Blondel et al. [2010]. If the connectivity thresholds of the agents are different, i.e., the network is heterogeneous, clusters or consensus are more sensitive to the agents initial attitudes, also for the case of few agents so as pointed out in Liang et al. [2013], Scafuti et al. [2015]. A sufficient condition for the consensus depending on the connectivity over the network is proposed in Yang et al. [2014]. The use of continuous Lyapunov functions which are piecewise differentiable is the alternative technique considered herein. Piecewise quadratic (PWQ) functions, see Johansson [2003], and the copositivity approach, see Bundfuss and Dür [2008] and Iervolino et al. [2015], are exploited in order to formulate the stability problem in terms of a set of linear matrix inequalities (LMIs), whose solution provides a PWQ Lyapunov function for the consensus.

We represent the *coopetition* and the *cooperosity* behaviors over a heterogeneous HK dynamics with a piecewise linear (PWL) model. A formal proof for the stability statement in Tangredi et al. [2016] is provided in this paper. To this aim, as a preliminary result, we also provide an asymptotic stability condition for a quite general class of PWL systems. These conditions are expressed in terms of LMIs by using a PWQ Lyapunov function. An HK example with five agents shows the effectiveness of the theorem in a nontrivial heterogeneous scenario. Moreover, numerical results show the positive role of the *cooperosity* for achieving larger consensus values and the effects of varying the connectivity thresholds based on the agent fitness. The rest of the paper is organized as follows. In Section 2 we recall the heterogeneous HK model and its PWL form. A sufficient stability condition for a quite general class of PWL systems is proposed in Section 3. Then in Section 4 this result is applied to the HK model by providing conditions for the consensus in term of LMIs. The numerical results are analyzed in Section 5 and Section 6 concludes the paper.

2. HETEROGENEOUS HK MODEL

In this section the *coopetition* and *cooperosity* concepts are recalled and used to present the HK model in a PWL form.

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2.1 Influence functions

The classical HK model consists of a set of N autonomous agents, whose attitudes are state variables $\xi_i \in [0,1]$, with corresponding dynamics described by

$$\dot{\xi}_i = \sum_{j=1}^{N} \phi_{ij}(\xi_i,\xi_j)(\xi_j - \xi_i) \quad (1)$$

for $i = 1,\dots,N$, where for simplicity we omit the time dependence of the variables ξ_i. The *influence function* $\phi_{ij}(\xi_i,\xi_j) : [0,1]^2 \to \{0,1\}$ is equal to 1 when ξ_j influences the attitude evolution of the agent i, and 0 otherwise. For all agents, we propose an influence function that depends on the difference $\xi_j - \xi_i$ as follows

$$\phi_{ij}(\xi_i,\xi_j) = \begin{cases} 1, & \text{if } -d_{ij}^G \le \xi_j - \xi_i \le d_{ij}^C \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

where the constant $d_{ij}^G \in [0,1]$ ($d_{ij}^C \in [0,1]$) is the connectivity threshold bounding the *generosity* (*competition*) of the agent i versus the agent j. Without loss of generality we set $\phi_{ii} = 0$.

The *cooperosity* and *coopetition* behaviours are described in Fig. 1, see Tangredi et al. [2016] for details. If $\phi_{ij} = 1$ and $\xi_j < \xi_i$, from (1) it follows that the term $\xi_j - \xi_i$ contributes negatively to the derivative of ξ_i, i.e., it decreases the attitude of i. This identifies *generosity* of i versus j if $\phi_{ji} = 0$, i.e. j is not influenced by i (see the region G in Fig. 1), or *cooperosity* if $\phi_{ji} = 1$ (see the region C_r in Fig. 1). Analogously, the *coopetitive* (region C_t in Fig. 1) and *competitive* (region C in Fig. 1) behaviours occur when $\xi_j > \xi_i$. In the particular case $d_{ij}^G = 0$ for all j, then the agent i is a *pure selfish*, while for $d_{ij}^C = 0$ for all j the agent i is a *pure altruist*.

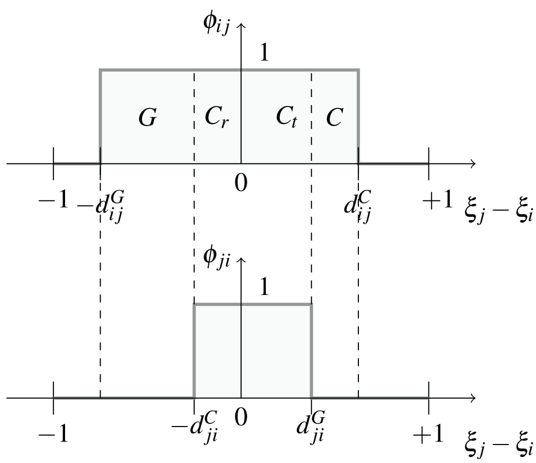

Fig. 1. An example of influence functions ϕ_{ij} and ϕ_{ji}, both vs. $\xi_j - \xi_i$. The letters G, C_r, C_t and C indicate the *generosity*, *cooperosity*, *coopetitive* and *competitive* behaviours of i vs. j, respectively.

2.2 Benefits, costs and fitness

Each agent is characterized by a fitness which depends on benefits and costs due to the relations with the other agents over the network. The fitness is defined by taking inspiration from the repeated Prisoner's Dilemma, see Nowak [2006]. In particular, we define the fitness of the agent i as the average of the fitness pairs f_{ij} evaluated over the number of the agents connected to i:

$$f_i = \frac{\sum_{j=1}^{N} \phi_{ij} f_{ij}}{\sum_{j=1}^{N} \phi_{ij}}, \quad (3)$$

where for simplicity the dependence on the agents attitudes has been omitted. In the homogeneous case, i.e., $d_{ij}^C = d_{ji}^G$ for all i, j, it is easy to verify that (3) can be rewritten as

$$f_i = \beta_i^{C_t} \mu_i - \sigma_i^{C_r}(1 - \mu_i) \quad (4)$$

where

$$\mu_i = \frac{\sum_{j=1}^{N} \phi_{ij} \mathrm{step}(\xi_j - \xi_i)}{\sum_{j=1}^{N} \phi_{ij}} \quad (5)$$

is the fraction of agents connected to the agent i who have a larger attitude, being $\mathrm{step}(\xi) : [-1,1] \to \{0,1\}$ a function equal to 1 for $\xi \ge 0$ and 0 otherwise. Since in the homogeneous HK model there are no intersections among the agents attitudes time evolutions, the number of non zero elements multiplied by ϕ_{ij} in the numerator of (5) depends only on the initial conditions.

Different fitness functions f_{ij} can be considered. In Tangredi et al. [2016] a piecewise constant function has been proposed, while in this paper we consider the dependence on the distance between the agents attitudes with a piecewise linear function, see Fig. 2.

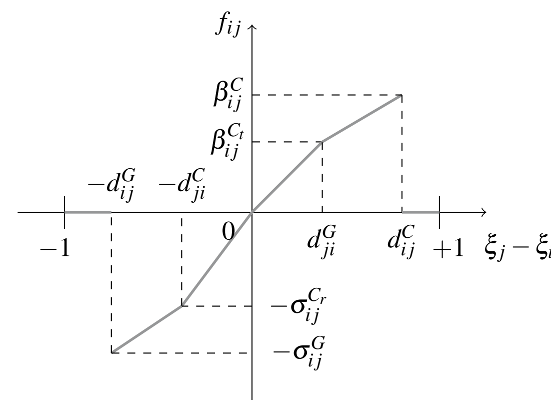

Fig. 2. A piecewise linear fitness f_{ij} characterizing the relation of the agent i with the agent j.

In the numerical analysis the effects of connectivity thresholds depending on the fitness will be considered.

2.3 PWL form of the HK model

For each combination of the influence functions values, the model (1) is linear time invariant and can be rewritten in the matrix form

$$\dot{\xi} = -L_s \xi \quad (6)$$

with

$$-L_s = \begin{pmatrix} -\sum_{j=1}^{N} \phi_{1j} & \phi_{12} & \dots & \phi_{1N} \\ \phi_{21} & -\sum_{j=1}^{N} \phi_{2j} & \dots & \phi_{2N} \\ \vdots & \vdots & \dots & \vdots \\ \phi_{N1} & \phi_{N2} & \dots & -\sum_{j=1}^{N} \phi_{Nj} \end{pmatrix} \quad (7)$$

for $s = 1,\dots,S$, and S is the total number of state space polyhedral regions corresponding to all the feasible combinations of the influence functions values. In the general asymmetric case, the number of functions ϕ_{ij} is $N_\phi = N(N-1)$.

In the case of a static graph, i.e. ϕ_{ij} do not depend on the agents attitudes, the influence functions ϕ_{ij} are constant and equal to 1 if the corresponding agents are connected or 0 otherwise. In this case $S = 1$ and the matrix L_1 is the classical Laplacian matrix. Note that in the case of our interest, the influence functions depend on the agents relative attitudes which change in time. Therefore the connections and the corresponding active mode s will change accordingly.

Usually, in the heterogeneous HK model it is assumed that the connectivity thresholds are not dependent on the direction of the connection, see Hegselmann and Krause [2002], i.e.,

$$d_{ij}^C = d_{ji}^G, \quad d_{ij}^G = d_{ji}^C, \tag{8}$$

which in our vision can be interpreted as i being competitive (generous) versus j so as j is generous (competitive) versus i, see Fig. 1 for an interpretation. Under the assumptions (8), the conditions $\phi_{ij} = \phi_{ji}$ hold for all i, j, and the matrices $\{L_s\}_{s=1}^S$ are symmetric. Therefore, from (1) it follows that the sum of the states time derivatives is identically zero and the agents attitudes preserve their average for any time instant. We consider the general case where (8) do not hold.

Since we are still interested in the convergence analysis to a consensus, it is useful to introduce a state transformation which has the origin as an equilibrium point. Let us introduce the attitudes differences

$$x_i = \xi_i - \xi_N, \quad i = 1,\ldots,N-1. \tag{9}$$

Any difference of two attitudes can be written as a linear combination of the variables (9). Indeed:

$$\xi_j - \xi_i = (\xi_j - \xi_N) - (\xi_i - \xi_N) = x_j - x_i \tag{10a}$$

$$\xi_j - \xi_N = x_j \tag{10b}$$

for any $i = 1,\ldots,N-1$, $j = 1,\ldots,N-1$. By combining (9) and (10) together with (1), for each $i = 1,\ldots,N-1$ with simple algebraic manipulations, see Tangredi et al. [2016] for details, one can write

$$\dot{x}_i = -\left(\sum_{j=1}^{N} \phi_{ij} + \phi_{Ni}\right) x_i + \sum_{j=1, j\neq i}^{N-1} (\phi_{ij} - \phi_{Nj}) x_j. \tag{11}$$

By collecting all (11) we obtain the HK model on relative attitudes in the following PWL form

$$\dot{x} = A_s x, \quad x \in X_s \tag{12}$$

with $s = 1,\ldots,S$, $\{X_s\}_{s=1}^S$ provides a polyhedral partition of the state space $\mathbb{D} = [-1,1]^{N-1}$ and

$$A_s = \begin{pmatrix} -\sum_{j=1}^{N} \phi_{1j} - \phi_{N1} & \phi_{12} - \phi_{N2} & \ldots & \phi_{1N} - \phi_{N,N-1} \\ \phi_{21} - \phi_{N1} & -\sum_{j=1}^{N} \phi_{2j} - \phi_{N2} & \ldots & \phi_{2N} - \phi_{N,N-1} \\ \vdots & \vdots & \ldots & \vdots \\ \phi_{N-1,1} - \phi_{N1} & \phi_{N-1,2} - \phi_{N2} & \ldots & -\sum_{j=1}^{N} \phi_{N-1,j} - \phi_{N,N-1} \end{pmatrix}. \tag{13}$$

Note that when all agents are connected the dynamics (12) is characterized by the Hurwitz matrix $-(N+1)I_{N-1}$. Each index s corresponds to a polyhedral region X_s of the state space and it can be represented by means of inequalities which depend on ϕ_{ij}. Therefore in order to complete the model (12) the influence functions must be expressed in terms of the state $x \in [-1,1]^{N-1}$. The formulation (2) induces a partition of the state space into polyhedral regions which can be defined in terms of their $\mathscr{H}$-representations

$$X_s = \{x \in \mathbb{R}^{N-1} \mid H_s x + g_s \leq 0\} \tag{14}$$

$s = 1,\ldots,S$, where $H_s \in \mathbb{R}^{(N_s+2N-2)\times(N-1)}$ and $g_s \in \mathbb{R}^{(N_s+2N-2)}$ can be obtained by collecting the N_s independent inequalities resulting from (2) and the transformation (9)–(10), together with the $2(N-1)$ inequalities corresponding to the state boundaries $-1 \leq x_i \leq 1$, $i = 1,\ldots,N-1$.

From (9) it is clear that the origin of (12) corresponds to the consensus, i.e. all agents having the same attitude value. Given a set of initial conditions $\{\xi_i(0)\}_{i=1}^N$, with $0 \leq \xi_i(0) \leq 1$, $i = 1,\ldots,N$, a solution of (1) can be determined from a solution of (12). Indeed, the corresponding set $\{x_i(0)\}_{i=1}^{N-1}$ is uniquely defined by (9) and a solution $x(t)$ of (12) can be derived. Then one can obtain a solution $\xi_N(t)$ of the scalar differential equation

$$\dot{\xi}_N = \sum_{j=1}^{N-1} \phi_{Nj}(\xi_N, x_j(t) + \xi_N) x_j(t) \tag{15}$$

and, by computing $\xi_i(t) = x_i(t) + \xi_N(t)$, $i = 1,\ldots,N-1$, it follows a solution $\{\xi_i(t)\}_{i=1}^N$ of (1).

3. A STABILITY CONDITION FOR PWL SYSTEMS

We now provide a sufficient condition for the asymptotic stability of the origin of a PWL system in the form (12) with Lyapunov functions which are continuous and piecewise differentiable along the system solutions.

Let us first present the solution concept adopted. We assume that there are no left accumulation of switches. Given an initial state $x(0) = x_0$, we say that a function $x(t) : [0,\infty) \to \mathbb{R}^{N-1}$ is a solution of (12) in the sense of Caratheodory, if it is absolutely continuous on each compact subinterval of $[0,\infty)$ and satisfies (12) for almost all $t \in [0,\infty)$.

We are now ready to formulate the stability result, which is obtained as a generalization to piecewise differentiable functions of the reasoning adopted for the classical Lyapunov theory in Khalil [2002],

Lemma 1. Consider the PWL system (12) where $X_s \subseteq \mathbb{D}$ is a polyhedron, $\mathbb{D} \subset \mathbb{R}^{N-1}$ is a compact domain, $\{X_s\}_{s=1}^S$ is a polyhedral partition of $\mathbb{D}$ and $0 \in \text{int}(\mathbb{D})$. Assume that (12) has an absolutely continuous solution for each initial condition $x(0)$ and that $\mathbb{D}$ is an invariant set for (12). Let

$$V(x) = V_s(x), \quad x \in X_s, \; s = 1\ldots,S \tag{16}$$

be a continuous function, with $V(0) = 0$ and $V_s : X_s \to \mathbb{R}$ for each s a continuous and differentiable function for all $x \in X_s$. Define the upper derivative of $V(x)$ along the solutions of (12) as

$$\dot{V}^*(x) = \max\{\dot{V}_s(x)\}_{s\in\Sigma(x)}, \quad x \in \bigcap_{s\in\Sigma(x)} X_s \tag{17}$$

where $\Sigma(x)$ is the set of all indices $s \in \{1,\ldots,S\}$ such that $x \in X_s$ and $\dot{V}_s(x)$ is the derivative of $V_s(x)$ along the solutions of (12), $s = 1,\ldots,S$. Note that if x belongs to the interior of a polyhedron $X_{\bar{s}}$, i.e., $x \in \text{int}(X_{\bar{s}})$, then $\Sigma(x) = \bar{s}$ and $\dot{V}^*(x) = \dot{V}_{\bar{s}}(x)$. If

$$V_s(x) > 0, \quad x \in X_s - \{0\}, \quad s = 1,\ldots,S \tag{18a}$$

$$\dot{V}_s(x) < 0, \quad x \in X_s - \{0\}, \quad s = 1,\ldots,S \tag{18b}$$

then $x = 0$ is an asymptotically stable equilibrium point in $\mathbb{D}$.

Proof. Clearly, the origin $x = 0$ is an equilibrium point for (12). Any trajectory starting in the set $\mathbb{D}$ remains in $\mathbb{D}$ for all $t \geq 0$ because it is an invariant set.

Assume that $x(0) \neq 0$. For any $t > 0$ denote by $\{t_i\}_{i=1}^n$ the strictly increasing sequence of time instants in $(0,t)$ such that $x(t_i)$ lies on the boundary of X_s for some s, for all i. With some abuse of notation let $t_0 = 0$ and $t_{n+1} = t$. From (17), (18) and the continuity of $V(x(t))$ one can write

$$\begin{aligned} V(x(t)) &= V(x(0)) + \sum_{i=0}^{n} \int_{t_i}^{t_{i+1}} \dot{V}^*(x(\tau))d\tau \\ &\leq V(x(0)) - \sum_{i=0}^{n} \gamma_i (t_{i+1} - t_i) \\ &\leq V(x(0)) - \gamma t, \end{aligned} \tag{19}$$

where $-\gamma_i = \max_{t\in[t_i,t_{i+1}]} \dot{V}^*(x(t)) \leq 0$ and $-\gamma = \max\{-\gamma_i\}_{i=0}^n \leq 0$. Then it follows

$$V(x(t)) \leq V(x(0)), \ \forall t \geq 0. \tag{20}$$

To prove asymptotic stability we need to show that $\|x(t)\| \to 0$ as $t \to \infty$. Being $V(0) = 0$, by continuity it is sufficient to show that $V(x(t)) \to 0$ as $t \to \infty$. Since $V(x(t))$ is positive, decreasing and bounded from below by zero, it is

$$V(x(t)) \to c \geq 0 \quad t \to \infty. \tag{21}$$

Suppose that c is positive and let

$$\Omega_c = \{x \in \mathbb{D} \mid V(x) \leq c\}. \tag{22}$$

Due to the continuity of $V(x)$, there exists a sufficiently small $d > 0$ such that $B_d \subset \Omega_c$, with $B_d = \{x \in \mathbb{D} \mid \|x\| \leq d\}$. Since $V(x(t)) \to c$, the trajectory $x(t)$ lies outside the set B_d for all $t \geq 0$. Let $-\gamma_d < 0$ the maximum of $\dot{V}^*(x)$ over the compact set $\mathbb{D} - \text{int}(B_d)$ which exists because $\dot{V}^*(x)$ is piecewise continuous. Then

$$\begin{aligned} V(x(t)) &= V(x(0)) + \int_0^t \dot{V}^*(x(\tau))d\tau \\ &\leq V(x(0)) - \gamma_d t. \end{aligned} \tag{23}$$

Since the right-hand side of (23) will eventually become negative, this contradicts the assumption that $c > 0$.

4. CONSENSUS WITH PWQ FUNCTIONS

The result in Lemma 1 will be applied for the analysis of the consensus in the heterogeneous HK model by using a PWQ Lyapunov function. Each polyhedron $X_s \subset \mathbb{D} \subseteq [-1,1]^{N-1}$ in (14) can be equivalently represented by means of its $\mathscr{V}$-representation

$$X_s = conv\{v_{s,\ell}\}_{\ell=1}^{\lambda_s} \tag{24}$$

with $s = 1,\ldots,S$ and *conv* indicates the convex hull. The vertices $\{v_{s,\ell}\}_{\ell=1}^{\lambda_s}$ of the polyhedron X_s can be obtained from the $\mathscr{H}$-representation (14) by using numerical tools, e.g., the tool cddmex in Matlab, see Fukuda [2016]. With reference to the partition $\{X_s\}_{s=1}^S$, say Σ_0 the subset of indices s such that $0 \in X_s$ and Σ_1 its complement, i.e., $\Sigma_0 \cup \Sigma_1 = \{1,\ldots,S\}$. For each polyhedron X_s, if $s \in \Sigma_0$ let us define the ray matrix $R_s \in \mathbb{R}^{(N-1)\times\lambda_s}$ as follows

$$R_s = \begin{pmatrix} v_{s,1} & \cdots & v_{s,\lambda_s} \end{pmatrix}, \tag{25}$$

while if $s \in \Sigma_1$ the matrix $\hat{R}_s \in \mathbb{R}^{N\times\lambda_s}$ given by

$$\hat{R}_s = \begin{pmatrix} v_{s,1} & \cdots & v_{s,\lambda_s} \\ 1 & \cdots & 1 \end{pmatrix}. \tag{26}$$

The stability analysis below is based on a piecewise function $V : \mathbb{R}^{N-1} \to \mathbb{R}$ in the form (16) where, for each $s = 1,\ldots,S$ the function $V_s : X_s \to \mathbb{R}$ is given by

$$V_s(x) = x^\top P_s x + 2q_s^\top x + r_s, \tag{27}$$

where $P_s \in \mathbb{R}^{(N-1)\times(N-1)}$ is a symmetric matrix, $q_s \in \mathbb{R}^{N-1}$ is a vector, r_s is a real scalar with $r_s = 0$ for $s \in \Sigma_0$. Let us define the matrix

$$\hat{P}_s = \begin{pmatrix} P_s & q_s \\ q_s^\top & r_s \end{pmatrix}. \tag{28}$$

An important aspect in order to apply Lemma 1 for our stability analysis is the continuity of (16).

Lemma 2. Consider the matrices $\{\hat{P}_s\}_{s=1}^S$ with $\hat{P}_s$ given by (28). Say X_h and X_k two elements of $\{X_s\}_{s=1}^S$ such that $X_h \cap X_k \neq \emptyset$ and $\Gamma_{hk} \in \mathbb{R}^{N\times m_{hk}}$, $m_{hk} < N$ the matrix of the common colums of the ray matrices corresponding to X_h and X_k. If the following conditions hold

$$\Gamma_{hk}^\top (\hat{P}_h - \hat{P}_k)\Gamma_{hk} = 0 \tag{29}$$

for all $h,k \in \{1,\ldots,S\}$, such that $X_h \cap X_k \neq \emptyset$, then (16) is continuous on the common boundary between X_h and X_k.

Proof. The proof follows from Lemma 8 in Iervolino et al. [2016].

By using Lemma 1 and Lemma 2, we now derive LMI conditions for the asymptotic stability to the consensus through a PWQ Lyapunov function. The following result was only stated in Tangredi et al. [2016] without proof.

Theorem 3. Consider the PWL system (12), (14) with the polyhedra $\{X_s\}_{s=1}^S$ expressed as (24) and the PWQ function (16) with (27) as a candidate Lyapunov function. Assume that (12) has an absolutely continuous solution for each initial condition $x(0)$. Consider the matrices $\{R_s\}_{s\in\Sigma_0}$ with $R_s \in \mathbb{R}^{(N-1)\times\lambda_s}$ given by (25) and the matrices $\{\hat{R}_s\}_{s\in\Sigma_1}$ with $\hat{R}_s \in \mathbb{R}^{N\times\lambda_s}$ given by (26). Define the matrices

$$\hat{A}_s = \begin{pmatrix} A_s & 0_{N-1} \\ 0_{N-1}^\top & 0 \end{pmatrix} \tag{30}$$

with $s \in \Sigma_1$. Consider the set of LMIs

$$R_s^\top P_s R_s - N_s \succcurlyeq 0 \tag{31a}$$

$$-R_s^\top (A_s^\top P_s + P_s A_s) R_s - M_s \succcurlyeq 0 \tag{31b}$$

for all $s \in \Sigma_0$, and

$$\hat{R}_s^\top \hat{P}_s \hat{R}_s - N_s \succcurlyeq 0 \tag{32a}$$

$$-\hat{R}_s^\top (\hat{A}_s^\top \hat{P}_s + \hat{P}_s \hat{A}_s)\hat{R}_s - M_s \succcurlyeq 0 \tag{32b}$$

for all $s \in \Sigma_1$, and the set of inequalities

$$2q_s^\top R_s e_h \geq 0, \quad 2q_s^\top A_s R_s e_h \geq 0 \tag{33}$$

for $h = 1,\ldots,\lambda_s$, $s \in \Sigma_0$. If there exist symmetric matrices $\{P_s\}_{s=1}^S$, $\{q_s\}_{s=1}^S$, $\{r_s\}_{s\in\Sigma_1}$, symmetric (entrywise) positive matrices $\{N_s\}_{s=1}^S$ and $\{M_s\}_{s=1}^S$, such that the set of linear matrix inequalities (31), (32) subject to the equality constraints (29) and to the inequality constraints (33) has a solution, then the origin is asymptotically stable for any initial condition in the partition $\mathbb{D} = \cup_{s=1}^S X_s$, provided it is an invariant set.

Proof. Choose the PWQ function (16), (27) as a candidate Lyapunov function. From (29) and Lemma 2 this function is continuous. If (31a), (32a) and the first of (33) hold, from Corollary 7 in Iervolino et al. [2016] it follows that (18a) hold. Analogously, from (31b), (32b) and the second of (33), the inequalities (18b) are also satisfied. Then the candidate PWQ Lyapunov function is continuous, strictly positive and strictly decreasing along any solution of (12). Since the polyhedral partition $\mathbb{D}$ is an invariant set, from Lemma 1 every trajectory that starts in $\mathbb{D}$ tends to zero asymptotically, i.e. the consensus is asymptotically stable for any initial condition in $\mathbb{D}$.

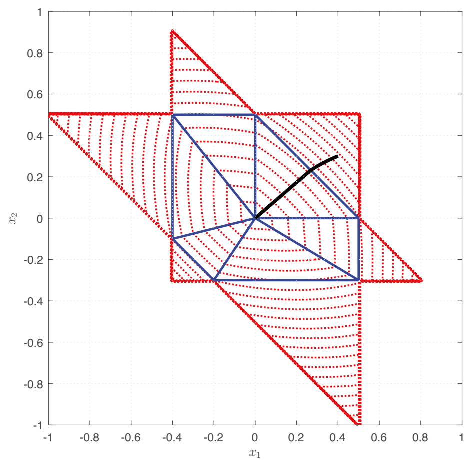

Fig. 3. State space for the model (12)–(14) with $N = 3$ and $d_{ij}^C = d_{ji}^C = 0.5$, $d_{ij}^G = d_{ji}^G$ $i = 1,2,3$, $j = 1,2,3$ with $d_{12}^G = 0.4$, $d_{13}^G = 0.5$, $d_{23}^G = 0.3$: polyhedral partition (14); a state trajectory (black line) and some level curves of the PWQ Lyapunov function (dotted red lines).

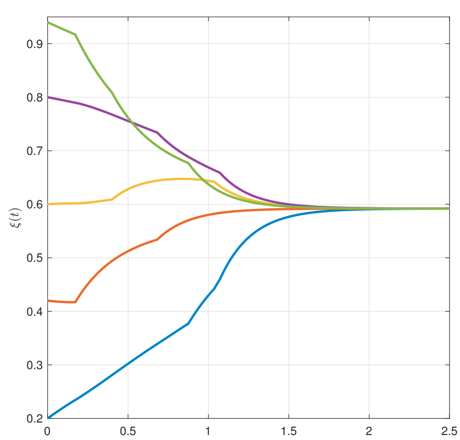

Fig. 4. Time evolutions of the attitude dynamics model (1) with $N = 5$, $d_{12}^{C,G} = d_{21}^{G,C} = 0.25$, $d_{15}^{C,G} = d_{51}^{G,C} = 0.3$, $d_{25}^{C,G} = d_{52}^{G,C} = 0.5$, $d_{45}^{C,G} = d_{54}^{G,C} = 0.15$, and $d_{ij}^{C,G} = d_{ji}^{C,G} = 0.2$ for all the other pairs i, j.

5. SIMULATION RESULTS

Theorem 3 has been applied for the stability analysis of (12)–(14) with $N = 3$ and different thresholds. By solving the inequalities (31)–(33) subject to the equality constraints (29) with `Matlab` and `CVX`, see Grant and Boyd [2014], a PWQ Lyapunov function has been obtained for the star-shape region contained in the feasibility domain shown in Fig. 3. By virtue of Proposition 2.1 in Motsch and Tadmor [2014] and by testing the positive invariance with the approach in Johansson [2003], the star-shape region in Fig. 3 is an invariant set, which allows to conclude the local asymptotic stability of the consensus.

Theorem 3 has been applied also for a heterogeneous HK model with $N = 5$ and initial conditions such that the assumption for the consensus in Theorem 3.2 in Yang et al. [2014] do not holds. We found a solution of (31)–(33) together with (29) in the polyhedral set obtained by excluding from the state space partition the regions with isolated agents. Fig. 4 shows the time evolutions of the agents attitudes.

The effects of a variation of the generosity thresholds are analyzed through a heterogeneous model with 150 agents, starting from a homogeneous one which presents clustering, see Fig. 5. The large number of polyhedra in this case makes prohibitive from a computational point of view the application of our PWQ approach. However, some insights can be argued from the numerical results. The attitudes time evolutions in Fig. 6 show that by introducing a random amount of *generosity*, the agents eventually reach the consensus.

A variation of the thresholds as a function of the agent global fitness f_i has been also considered. In particular, by keeping the sum $d_{ij}^G + d_{ij}^C$ constant, the generosity threshold has been varied according to $d_{ij}^G = \bar{d}_{ij}^G + K_p f_i$ with K_p a positive gain and $\bar{d}_{ij}^G$ the nominal value, i.e., the generosity increases with the fitness. The cost and benefit parameters, for the fitness pairs f_{ij}, have been chosen as $\sigma_i^G = 1$, $\sigma_i^{C_r} = 0.7$, $\beta_i^{C_t} = 0.3$, $\beta_i^C = 0.6$. Fig. 7 shows the histograms of the consensus values with $N = 625$ agents, for $K_p \in [0, 0.5]$. The occurrences have

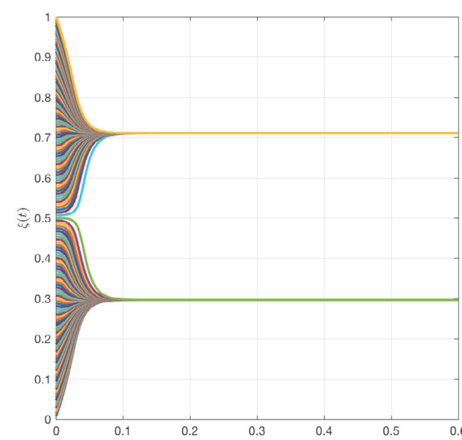

Fig. 5. Clustering without *generosity* highlighted by the time evolutions of (1) with $N = 150$, initial conditions uniformly distributed in $[0,1]$ and $d_{ij}^{C,G} = d_{ji}^{C,G} = 0.25$.

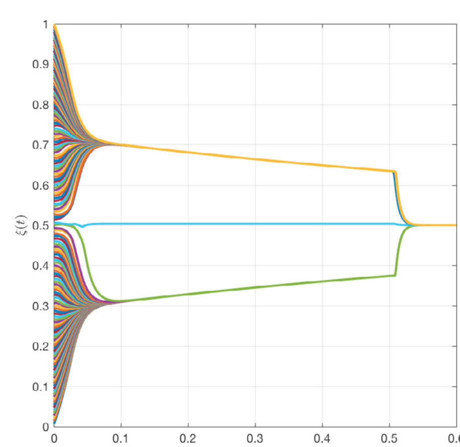

Fig. 6. Consensus with *generosity* for the same scenario as in Fig. 5 but with $d_{ij}^C = d_{ji}^C = 0.25$ and $d_{ij}^G = d_{ji}^G \in [0.24, 0.26]$ randomly chosen with uniform distribution.

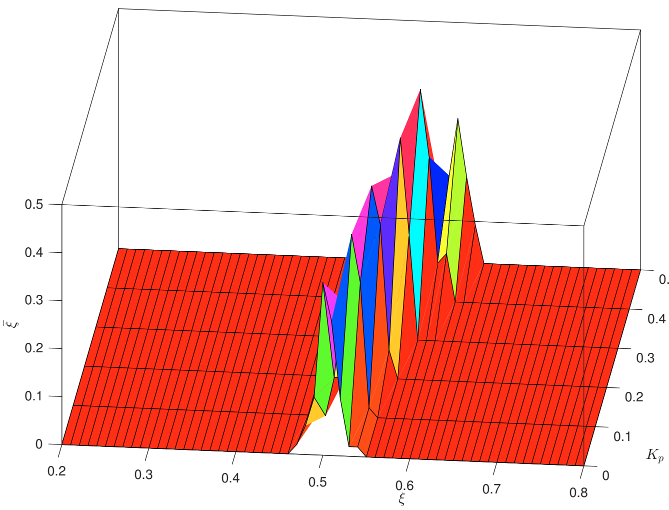

Fig. 7. Normalized number of runs (z-axis) with a given consensus value (x-axis) by varying the gain K_p (y-axis), with $d^G_{ij} = 0.35 + K_p f_i$, and $d^G_{ij} + d^C_{ij} = 0.7$ for all i and j.

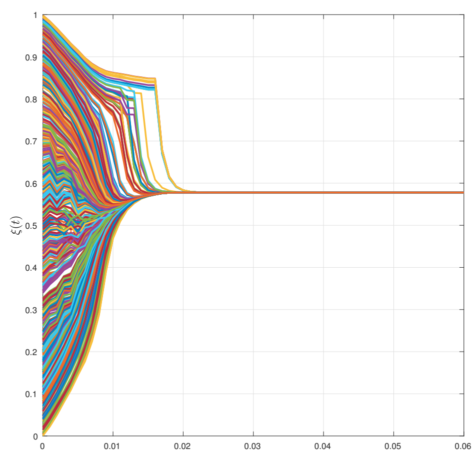

Fig. 8. Time evolutions of the agents attitudes for $K_p = 0.4$.

been computed on 50 runs with random initial conditions. Note that by increasing the value of K_p the average of the consensus increases. Fig. 8 shows the time evolutions for one specific run.

6. CONCLUSIONS

We have analyzed the consensus in a Hegselmann–Krause model where the concepts of *coopetition* and *cooperosity* have been expressed through the influence thresholds of cooperating agents. A sufficient condition for the consensus in the heterogeneous opinion dynamics has been proved by using a Lyapunov technique. Numerical results have shown the effectiveness of the approach and how *coopetition* and *cooperosity* affect the consensus. An investigation about the thresholds dependence on the agent fitness has been also reported; the interesting preliminary results will lead to future work in this direction.

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