Fractional differential equations of multiphase hereditary materials and exact mechanical models

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Abstract: Creep and relaxation tests, performed on various materials like polymers, rubbers and so on are well-fitted by power-laws with exponent $\beta \in [0, 1]$ (Nutting (1921), Di Paola et al. (2011)). The consequence of this observation is that the stress-strain relation of hereditary materials is ruled by fractional operators (Scott Blair (1947), Slonimsky (1961)). A large amount of researches have been performed in the second part of the last century with the aim to connect constitutive fractional relations with some mechanical models by means of fractance trees and ladders (see Podlubny (1999)). Recently, Di Paola and Zingales (2012) proposed a mechanical model that corresponds to fractional stress-strain relation with any real exponent and they have proposed a description of above model (Di Paola et al. (2012)). In this study the authors aim to extend the study to cases with more fractional phases and to fractional Kelvin-Voigt model of hereditariness.

Keywords: Mechanical systems, Power-law description, Fractional hereditary materials, Discretized models, Modal transformation.

1. INTRODUCTION

In recent years fractional differential equations (FDE) have been used more and more often in several fields of physics as well as in applied mechanics context. FDE, in linear or non-linear formulations, have been introduced to describe several complex phenomena occurring at multiple temporal as well as spatial observation scales (Tarasov (2010), Podlubny (1999), Metzeler et al. (2000), Di Paola and Zingales (2008)). Indeed, since the beginning of the twentieth century, the time evolution of polymer stress relaxation as well as of current intensity relaxation in dielectric capacitors proved to be well described, analytically, by time-varying power-laws functions. This observation leads several authors to derive the governing equation of physical and dynamical linear systems with the aid of Boltzmann superposition principle, yielding fractional-order differential equations (Mainardi (2010), Schiessel et al. (1995), Caputo (1969), Hilfer (2000), Chechkin et al. (2008)).

Beside the efficiency of fractional-order operators to describe, phenomenologically, the time-varying evolution of systems properties like stress relaxation or creep, fractional-order differential equations do not correspond to a physical picture of the system and, in this regard, several attempts have been proposed in relevant scientific literature to yield a the mechanical correspondence with fractional-order differential equations (Schiessel et al. (1995), Metzeler et al. (2000), Bagley and Torvik (1983), Bagley and Torvik (1985), Liebst and Torvik (1996), Podlubny (1999), Lazopoulos (2006), Povstenko (2008), Sherief et al. (2010)). Recently the authors introduced exact mechanical model corresponding to spatial-order fractional operators in the context of non-local elasticity (see Di Paola and Zingales (2008), Di Paola et al. (2009), Di Paola et al. (2011)) as well as in the context of non-local heat conduction (see Borino, Di Paola and Zingales (2011), Mongiovì and Zingales (2012)). In more details, the use of spatial fractional order operators corresponds to the presence of spatial multiple scale model with power-law decay of spring stiffness in the context of non-local elasticity or of thermal conductivity in case of non-local thermodynamics.

A different scenario is involved in presence of fractional derivative in time since, in this case the non-local effects correspond to a long-term memory of the considered dynamical system. Fractional-order time derivatives of system' state variables are involved, for example, in the context of polymer viscoelasticity (Heymans and Bauwens (1994)) in voltage-current relations of non-ideal capacitors (see Carlson and Halijak (1964), Westerlund and Ekstam (1994)) as well as in the context of second-sound effects observed in thermal wave propagation in complex materi-

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als (Sherief et al. (2010), Youssef and Al-Lehaibi (2010)). Recently, the authors proposed a multiscale mechanical model that corresponds, exactly to a fractional-order time derivative to represent the rheologic behavior of fractional-order materials (Di Paola and Zingales (2012); Di Paola et al. (2012)). In more details in Di Paola and Zingales (2012) a mechanical model corresponding, exactly, to a fractional-order rheologic stress-strain relation with any real exponent $0 \leq \beta \leq 1$ has been proposed, whereas a numerical assessment have been reported in Di Paola et al. (2012). The mechanical model is represented by a massless plate resting on massless Newtonian fluid that is restrained by means of independent dashpots. The authors termed this rheological model as Elasto-Viscous (EV), as far as $0 \leq \beta \leq 1/2$, and the corresponding material as Elasto-Viscous one since the elastic phase prevail over the viscous one at the beginning of the load history. As instead $1/2 \leq \beta \leq 1$ the mechanical model is represented by a massless shear-type indefinite column resting on a bed of independent dashpots. In this case the authors termed Visco-Elastic (VE) this kind of materials since the viscous phase prevail on the elastic one.

In this paper the authors aim to provide mechanical analogues of fractional-order differential equations in the context of time-varying state variables. Indeed, the presence of multiple order fractional differential operators will be related to a proper mechanical model that corresponds, exactly, to the original fractional differential equation. Some numerical applications will be reported to assess the validity of the model.

2. MULTIPHASE FHM

The time dependent behavior of a viscoelastic material may be introduced starting from so-called relaxation function $G(t)$ that is the stress history $\sigma(t)$ for an assigned strain $\gamma(t) = U(t)$ being $U(t)$ the unit step function. Alternatively the viscoelastic material may be characterized by the creep function $J(t)$ that is the strain history for the assigned stress history $\sigma(t) = U(t)$. In virtue of the Boltzmann superposition principle the stress-strain relations is expressed as

$$\sigma(t) = \int_0^t G(t-\tau) d\gamma(\tau) = \int_0^t G(t-\tau)\dot{\gamma}(\tau) d\tau \qquad (1)$$

Eq. (1) is valid if $\gamma(0) = 0$. Let us now suppose that from experimental relaxation test $G(t)$ is well fitted from

$$G(t) = \sum_{j=1}^{m} \frac{C(\beta_j)}{\Gamma(1-\beta_j)} t^{-\beta_j}; \qquad 0 \leq \beta_j \leq 1 \qquad (2)$$

where $\Gamma(\cdot)$ is the Euler Gamma Function and $C(\beta_j)$, β_j are parameters depending of the material at hands. Introducing Eq. (1) we get

$$\sigma(t) = \sum_{j=1}^{m} C(\beta_j) \left({}^C D_{0^+}^{\beta_j} \gamma \right)(t) \qquad (3)$$

where $\left({}^C D_{0^+}^{\beta_j} \gamma \right)(t)$ is the Caputo's fractional derivative of order β_j. Unless the case $m = 1$ in Eq. (2) or (3) for which the inverse relationship to Eq. (3) is readily found as

$$\gamma(t) = \frac{1}{C(\beta_1)} \left(D_{0^+}^{-\beta_1} \sigma \right)(t) \qquad (4)$$

where $\left(D_{0^+}^{-\beta_1} \sigma \right)(t)$ is the Riemann-Liouville fractional integral, in all other case the inverse of Eq. (3) is not so simple (Suarez and Shokooh (1995), Rossikhin and Shitnikova (2001)). To illustrate difficulties emerging in the evaluation of the inverse relationship (3) the simpler case of $m = 2$ and $\beta_1 = 0$ ($C(\beta_1) = \mathrm{E}$) is taken into consideration. In this case we get

$$\gamma(t) = -\frac{1}{\mathrm{E}} \sum_{k=1}^{\infty} \left(-\frac{\mathrm{E}}{C(\beta_2)} \right)^k \left(D_{0^+}^{-k\beta_2} \sigma \right)(t). \qquad (5)$$

Such an equation is found by using Laplace transform, by manipulating the algebraic equation in terms of Mittag-Leffler functions and using inverse Laplace transform of the so obtained equation. In fact the creep function of the considered system may be obtained starting from the following fractional differential equation

$$\mathrm{E} J(t) + C(\beta_2) \left({}^C D_{0^+}^{\beta_2} J(t) \right) = U(t). \qquad (6)$$

The Eq. (6) is the equilibrium equation of the fractional Kelvin-Voigt model forced by $\sigma(t) = U(t)$. Starting from that equilibrium equation and by performing the Laplace transform we obtain the following algebraic equation

$$\mathrm{E}\hat{J}(s) + C(\beta_2) s^{\beta_2} \hat{J}(s) = \frac{1}{s} \qquad (7)$$

where $\hat{J}(s)$ is the Laplace transform of the creep function $J(t)$. The solution of Eq. (7) is

$$\hat{J}(s) = \frac{1}{\mathrm{E}s + C(\beta_2) s^{1+\beta_2}}. \qquad (8)$$

It should be noted that the Eq. (8) could be obtained starting from the relationship between the creep function and relaxation function in the Laplace domain (Flügge (1967)), according to which the following relationship

$$\hat{G}(s)\hat{J}(s) = \frac{1}{s^2} \qquad (9)$$

holds true.

By performing the Laplace transform of Eq. (8) we obtain a solution which involves the one-parameter Mittag-Leffler function $E_{\beta_2}(\cdot)$ (Podlubny (1999), Mainardi (2010)), in particular for quiescent system at $t = 0$ we obtain

$$\begin{aligned} J(t) &= \frac{1}{\mathrm{E}} \left[1 - E_{\beta_2} \left(-\frac{\mathrm{E}}{C(\beta_2)} t^{\beta_2} \right) \right] \\ &= \frac{1}{\mathrm{E}} \left[1 - \sum_{k=0}^{\infty} \frac{\left(-\mathrm{E}/C(\beta_2) t^{\beta_2} \right)^k}{\Gamma(\beta_2 k + 1)} \right] \\ &= -\frac{1}{\mathrm{E}} \sum_{k=1}^{\infty} \frac{\left(-\mathrm{E}/C(\beta_2) t^{\beta_2} \right)^k}{\Gamma(\beta_2 k + 1)}. \end{aligned} \qquad (10)$$

In addition to Eq. (1) the Boltzmann superposition principle in the stress-strain relations may be expressed as

$$\gamma(t) = \int_0^t J(t-\tau) d\sigma(\tau) = \int_0^t J(t-\tau)\dot{\sigma}(\tau) d\tau \qquad (11)$$

and by replacing in the Eq. (11) the expression of $J(t)$, reported in Eq. (10), we get the deformation history reported in Eq. (5). Moreover, Eq. (5) with simple mathematical manipulations can be rewritten as

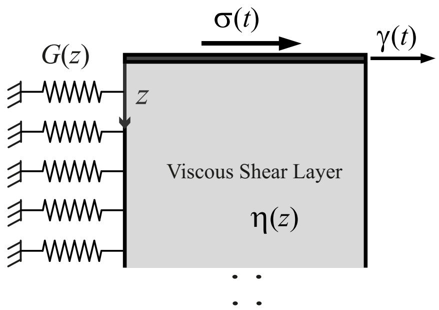
$$\gamma(t) = \frac{1}{\mathrm{E}}\left\{\sigma(t) + \sum_{k=0}^{\infty}\left[\left(\frac{\mathrm{E}}{C(\beta_2)}\right)^{2k+1}\left(D_{0^+}^{-(2k+1)\beta_2}\sigma\right)(t) - \left(\frac{\mathrm{E}}{C(\beta_2)}\right)^{2k}\left(D_{0^+}^{-2k\beta_2}\sigma\right)(t)\right]\right\}. \tag{12}$$

If $\beta_2 = 1$ and $C(\beta_2) = \mu$ the model becomes a classical Kelvin-Voigt model with creep function of the exponential type.

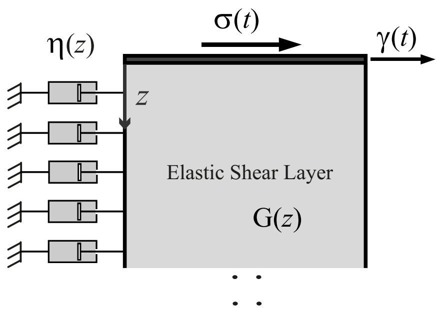
In order to overcome the difficulty encountered for the case of multiphase hereditary kernel expressed in Eq. (2) we will take full advantage of the exact mechanical model already found in Di Paola and Zingales (2012). This issue will be addressed in the next section.

3. EXACT MECHANICAL MODEL OF FHM

In a previous paper (Di Paola, Zingales (2012)) it has been shown, that exact mechanical models of viscoelasticity may be found for the two intervals of $\beta : 0 \le \beta \le 1/2$, $\beta : 1/2 \le \beta \le 1$. In the former case the material was labeled as Elasto-Viscous (EV) and the mechanical model is depicted in Figure 1(a). While in the latter the material was labeled as Visco-Elastic (VE) and is depicted in Figure 1(b).



(a) Elasto-Viscous model ($0 \le \beta \le 1/2$)



(b) Visco-Elastic model ($1/2 \le \beta \le 1$)

Figure 1. Continuous fractional models.

The Elasto-Viscous case ($0 \le \beta \le 1/2$) is a massless indefinite viscous shear layer with a viscosity coefficient $c_E(z)$ resting on a bed of independent springs characterized by an elastic coefficient $k_E(z)$. By contrast the Visco-Elastic case ($1/2 \le \beta \le 1$) is a massless indefinite elastic shear layer characterized by a shear modulus $k_V(z)$ resting on a bed of independent viscous dashpots characterized by the viscosity coefficient $c_V(z)$. The subscripts E and V in $k(z)$ and $c(z)$ are introduced in order to distinguish the predominant behavior (E stands for Elasto-Viscous, while V stands for Visco-Elastic). Moreover we define G_0 and η_0 the reference values of the shear modulus and viscosity coefficient.

As soon as we assume:

$$k_E(z) = \frac{G_0}{\Gamma(1+\alpha)} z^{-\alpha}; \quad c_E(z) = \frac{\eta_0}{\Gamma(1-\alpha)} z^{-\alpha} \tag{13}$$

with $0 \le \alpha \le 1$ and $\beta = (1-\alpha)/2$, and

$$k_V(z) = \frac{G_0}{\Gamma(1-\alpha)} z^{-\alpha}; \quad c_V(z) = \frac{\eta_0}{\Gamma(1+\alpha)} z^{-\alpha} \tag{14}$$

with $\beta = (1+\alpha)/2$, the stress $\sigma(t)$ at the upper lamina and $\gamma(t)$ the corresponding normalized displacement (that is the corresponding strain) reverts to a fractional law expressed in Eq. (3) for $m = 1$, $\beta_1 = \beta$.

The governing equation for $0 \le \beta \le 1/2$ of the mechanical model depicted in Figure 1(a) is

$$\frac{\partial}{\partial z}\left[c_E(z)\frac{\partial \dot{\gamma}(z, t)}{\partial z}\right] = k_E(z)\gamma(z, t) \tag{15}$$

the constitutive law obtained for $\gamma(0, t) = \gamma(t)$ is that obtained in Eq. (3) for $m = 1$, $\beta_1 = \beta$ provided the coefficient $C(\beta) = C_E(\beta)$ in the stress-strain relation is given as

$$C_E(\beta) = \frac{G_0\Gamma(\beta)2^{2\beta-1}}{\Gamma(2-2\beta)\Gamma(1-\beta)}; \quad 0 \le \beta \le 1/2 \tag{16}$$

with $\tau_E(\alpha) = -\eta_0\Gamma(\alpha)/\Gamma(-\alpha)G_0$ and $\beta = (1-\alpha)/2$.

The equilibrium equation of the continuos model depicted in Figure 1(b) is written as:

$$\frac{\partial}{\partial z}\left[k_V(z)\frac{\partial \gamma(z, t)}{\partial z}\right] = c_V(z)\dot{\gamma}(z, t) \tag{17}$$

the solution of such differential equation for $z \to 0$ shows that the stress $\sigma(t)$ at the top is related to the normalized displacement $\gamma(t)$ by means of a fractional derivate of order $\beta = (1+\alpha)/2$. The coefficient $C(\beta) = C_V(\beta)$ in the stress-strain relation reads

$$C_V(\beta) = \frac{G_0\,\Gamma(1-\beta)\,2^{1-2\beta}}{\Gamma(2-2\beta)\,\Gamma(\beta)}(\tau_V(\alpha))^{\beta}; \quad 1/2 \le \beta \le 1 \tag{18}$$

with $\tau_V(\alpha) = -\eta_0\Gamma(-\alpha)/\Gamma(\alpha)G_0$ and $\beta = (1+\alpha)/2$.

4. MECHANICAL MODEL OF MULTIPHASE FHM

With the previous results we now may find the mechanical model whose constitutive law is expressed in Eq. (3). We order β_j in such way that

$$0 \le \beta_1 < \beta_2 < \dots \beta_r \le 1/2 \le \beta_{r+1} < \dots < \beta_m \le 1 \tag{19}$$

For such a material characterized by coefficient β_j we have that the massless lamina is sustained by r columns of massless Newtonian fluid resting on a bed of independent springs and $m - r$ shear type elastic columns resting on a bed of independent dashpots how it is described in Figure 2.

At this stage we have r EV liquid columns sustained by external independent springs and $m-r$ shear type columns sustained by external dashpots. All these elements share a common displacements $\gamma(t)$ at the top of each column and then the load at the each lamina on the top has to

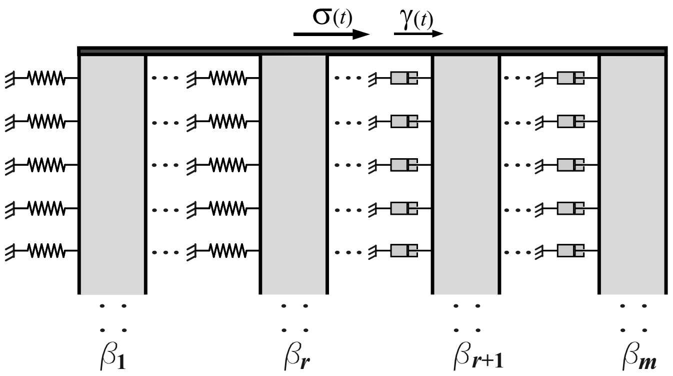


Figure 2. Fractional multiphase hereditary material.

be calibrated in such a way that the displacement on the top is equal for each column (compatibility condition) and $\sigma^{(j)}(t)$, the stress at each column, has to be such that $\sum_{j=1}^{m}\sigma^{(j)}(t)=\sigma(t)$ (equilibrium of the top lamina).

Now the inverse relationship for the general case presented in Eq. (3) may be solved with the proper tools of dynamical systems. This issue will be addressed later on with the aid of discretized model.

5. DISCRETIZATION OF MULTIPHASE FHM

The mechanical representation of fractional order operators discussed in previous section may be used to introduce a discretization scheme that corresponds to evaluate fractional derivative. How it has been shown in the previous section the multiphase FHM has a mechanical equivalent of r EV columns ($\beta\in[0,\,1/2]$) and $m-r$ VE columns ($\beta\in[1/2,\,1]$).

5.1 The discretized model of EV column

By introducing a discretization of the z-axis as $z_j=j\triangle z$ into to the governing equation of the EV material in Eq. (15) yields a finite difference equation of the form:

$$\frac{\triangle}{\triangle z}\left[c_E^{(i)}(z_j)\frac{\triangle\dot{\gamma}^{(i)}(z_j,\,t)}{\triangle z}\right]=k_E^{(i)}(z_j)\gamma^{(i)}(z_j,\,t);\qquad i=1,2,\dots,r \tag{20}$$

so that, denoting $k_{Ej}^{(i)}=k_E^{(i)}(z_j)\triangle z$ and $c_{Ej}^{(i)}=c_E^{(i)}(z_j)/\triangle z$ the continuous model is discretized into a dynamical model constituted by massless shear layers, with horizontal degrees of freedom $\gamma^{(i)}(z_j,t)=\gamma_j^{(i)}(t)$, that are mutually interconnected by linear dashpots with viscosity coefficients $c_{Ej}^{(i)}$ and resting on a bed of independent linear springs $k_{Ej}^{(i)}$.

The stiffness coefficient $k_{Ej}^{(i)}$ and the viscosity coefficient $c_{Ej}^{(i)}$ of the i-th column reads:

$$k_{Ej}^{(i)}=\frac{G_0^{(i)}}{\Gamma(1+\alpha_i)}z_j^{-\alpha_i}\triangle z;\quad c_{Ej}^{(i)}=\frac{\eta_0^{(i)}}{\Gamma(1-\alpha_i)}\frac{z_j^{-\alpha_i}}{\triangle z} \tag{21}$$

with $\alpha_i=1-2\beta_i$.

The equilibrium equations of the generic shear-layer of the i-th model read:

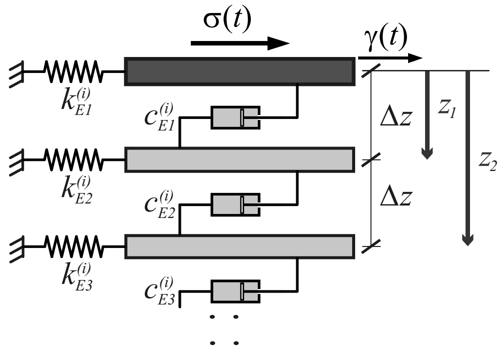


Figure 3. Discretized counterpart of the continuous model Figure 1(a): EV column.

$$\begin{cases} k_{E1}^{(i)}\gamma_1^{(i)}(t)-c_{E1}^{(i)}\triangle\dot{\gamma}_1^{(i)}(t)=\sigma^{(i)}(t),\\ k_{Ej}^{(i)}\gamma_j^{(i)}(t)+c_{Ej-1}^{(i)}\triangle\dot{\gamma}_{j-1}(t)-c_{Ej}\triangle\dot{\gamma}_j^{(i)}(t)=0,\\ \qquad j=1,\,2,\dots,\,\infty,\quad i=1,\,2,\dots,r \end{cases} \tag{22}$$

where $\gamma_1^{(i)}(t)=\gamma(t)$ and $\triangle\dot{\gamma}_j^{(i)}(t)=\dot{\gamma}_{j+1}^{(i)}(t)-\dot{\gamma}_j^{(i)}(t)$. By inserting Eqs. (21) in Eqs. (22), at the limit as $\triangle z\to 0$, the discrete model reverts to Eq. (15). That is the discretized model presented in Figure 3 represents a proper discretization of the continuous EV counterpart. As soon as z increase $\gamma^{(i)}(z,\,t)$ decay and $\lim_{z\to\infty}\gamma^{(i)}(z,t)=0$ it follows that only a certain number, say n, of equilibrium equation may be accounted for the analysis. It follows that the system in Eqs. (22) may be rewritten in the following compact form:

$$p_E^{(i)}\mathbf{A}^{(i)}\dot{\boldsymbol{\gamma}}^{(i)}+q_E^{(i)}\mathbf{B}^{(i)}\boldsymbol{\gamma}^{(i)}=\mathbf{v}\sigma^{(i)}(t) \tag{23}$$

where:

$$p_E^{(i)}=\frac{\eta_0^{(i)}}{\Gamma(1-\alpha_i)}\triangle z^{-(1+\alpha_i)};\qquad q_E^{(i)}=\frac{G_0^{(i)}}{\Gamma(1+\alpha_i)}\triangle z^{1-\alpha_i}. \tag{24}$$

In Eq. (23):

$$\boldsymbol{\gamma}^{(i)^T}=\left[\,\gamma_1^{(i)}(t)\ \gamma_2^{(i)}(t)\ \dots\ \gamma_n^{(i)}(t)\,\right];\quad \mathbf{v}^T=[1\ 0\ 0\ \dots\ 0] \tag{25}$$

where the apex T means transpose. The matrices $\mathbf{A}^{(i)}$ and $\mathbf{B}^{(i)}$ are given as

$$\mathbf{A}^{(i)}=\begin{bmatrix} 1^{-\alpha_i} & -1^{-\alpha_i} & \dots & 0\\ -1^{-\alpha_i} & 1^{-\alpha_i}+2^{-\alpha_i} & \dots & 0\\ 0 & -2^{-\alpha_i} & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & (n-1)^{-\alpha_i}+n^{-\alpha_i} \end{bmatrix} \tag{26}$$

$$\mathbf{B}^{(i)}=\begin{bmatrix} 1^{-\alpha_i} & 0 & 0 & \dots & 0\\ 0 & 2^{-\alpha_i} & 0 & \dots & 0\\ 0 & 0 & 3^{-\alpha_i} & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & n^{-\alpha_i} \end{bmatrix}. \tag{27}$$

The matrices $\mathbf{A}^{(i)}$ and $\mathbf{B}^{(i)}$ are symmetric and positive definite (in particular $\mathbf{B}^{(i)}$ is diagonal) and they may be easily constructed for an assigned value of α_i (depending of the derivative order β_i) and for a fixed truncation order n. Moreover Eq. (23) may now be easily integrated by using standard tools of dynamic analysis how it will be shown later on.

5.2 The discretized model of VE column

As the fractional order derivative of the s-th column ($r < s \leq n$) is $\beta^{(s)} = \beta_V^{(s)} \in [1/2,\, 1]$ the mechanical description of the material is the represented by the continuos model depicted in Figure 4 and ruled by Eq (17).

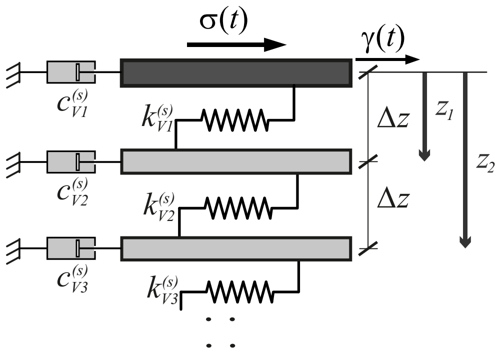


Figure 4. Discretized counterpart of the continuous model Figure 1(b): VE column.

By introducing a discretization of the z-axis in intervals $\triangle z$ in governing equation of the VE materials in Eq. (17) yields a finite difference equation of the form:

$$\frac{\triangle}{\triangle z}\left[k_V^{(s)}(z_j)\frac{\triangle\gamma^{(s)}(z_j,\, t)}{\triangle z}\right] = c_V^{(s)}(z_j)\dot{\gamma}^{(s)}(z_j,\, t) \qquad (28)$$

that corresponds to a discretized mechanical representation of fractional derivatives. The mechanical model is represented by a set of massless shear layers with state variables $\gamma^{(s)}(z_j,\, t) = \gamma_j^{(s)}(t)$ that are mutually interconnected by linear springs with stiffness $k_{Vj}^{(s)} = k_V^{(s)}(z_j,\, t)/\triangle z$ resting on a bed of independent linear dashpots with viscosity coefficient $c_{Vj}^{(s)} = c_V^{(s)}(z_j,\, t)\triangle z$. Springs and dashpots are given as:

$$k_{Vj}^{(s)} = \frac{G_0^{(s)}}{\Gamma(1-\alpha_s)}\frac{z_j^{-\alpha_s}}{\triangle z}; \quad c_{Vj}^{(s)} = \frac{\eta_0^{(s)}}{\Gamma(1+\alpha_s)} z_j^{-\alpha_s}\triangle z \qquad (29)$$

with $\alpha_s = 2\beta_s - 1$.
The set of equilibrium equations reads:

$$\begin{cases} c_{V1}^{(s)}\dot{\gamma}_1^{(s)} - k_{V1}^{(s)}\triangle\gamma_1^{(s)} = \sigma^{(s)}(t), \\ c_{Vj}^{(s)}\dot{\gamma}_j^{(s)} + k_{Vj-1}^{(s)}\triangle\gamma_{j-1}^{(s)} - k_{Vj}^{(s)}\triangle\gamma_j = 0, \\ \qquad j = 1,\, 2, \ldots,\, \infty, \quad s = r+1,\, 2, \ldots,\, n. \end{cases} \qquad (30)$$

So that, accounting for the contribution of the first n shear layers the differential equation system may be written as:

$$p_V^{(s)}\mathbf{B}^{(s)}\dot{\boldsymbol{\gamma}}^{(s)} + q_V^{(s)}\mathbf{A}^{(s)}\boldsymbol{\gamma}^{(s)} = \mathbf{v}\sigma^{(s)}(t) \qquad (31)$$

where:

$$p_V^{(s)} = \frac{\eta_0^{(s)}}{\Gamma(1+\alpha_s)}\triangle z^{1-\alpha_s}; \quad q_V^{(s)} = \frac{G_0^{(s)}}{\Gamma(1-\alpha_s)}\triangle z^{-(1+\alpha_s)} \qquad (32)$$

while $\boldsymbol{\gamma}^{(s)}$, $\mathbf{v}$ and the matrices $\mathbf{A}^{(s)}$ and $\mathbf{B}^{(s)}$ have already been defined in sect. 5.1. Compatibility condition of the upper lamina reads $\gamma^{(s)}(0, t) = \gamma(t)$, $\forall s : 0 \leq s \leq n$; while equilibrium equation reads $\sum_{s=1}^{n}\sigma^{(s)} = \sigma(t)$ being $\sigma(t)$ the external stress of the upper lamina.

6. EXAMPLES

The observations reported in previous section lead to conclude that, whatever class of FHM material is considered, the time-evolution of the material system may be obtained by the introduction of a proper set of inner state variables, collected in the vector $\boldsymbol{\gamma}(t)$ and ruled by a set first-order linear differential equations. In this perspective the mechanical response of the FHM may be obtained in terms of the vector $\boldsymbol{\gamma}(t)$ by means eigenvectors of the differential equations system reported in Eq. (22) for EV materials or in Eq. (30) for VE materials.

In this section three examples are reported, one is a fractional Kelvin-Voigt model with EV Spring-Pot ($\beta = \beta_E = 0.4$), the second case consist in a Kelvin-Voigt model with VE Spring-pot ($\beta = \beta_V = 0.6$), then the last case is fractional Kelvin-Voigt model with critical value of $\beta = 0.5$. The stiffness of the elastic spring is denoted with $\tilde{K}$. The following numerical examples are obtained for $\sigma(t) = U(t)$.

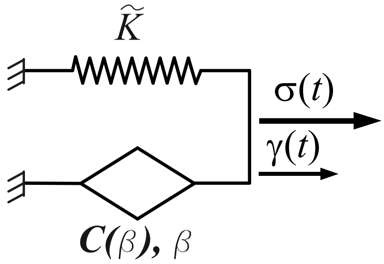


Figure 5. Fractional Kelvin-Voigt model.

6.1 The case of Kelvin-Voigt with EV Spring-Pot

In this section the fractional Kelvin-Voigt model depicted in Figure 5 is investigated. In particular the Spring-Pot connected in parallel with the spring is of Elasto-Viscous type, for which the coefficient $\beta \in [0, 1/2]$. The discretized model is reported in Figure 6. The governing equation is

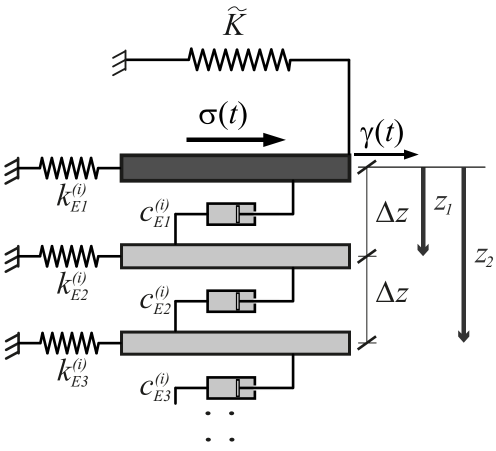


Figure 6. Discretized fractional EV Kelvin-Voigt model.

reported in Eq. (23). However, to take into account for the presence of elastic spring connected in parallel with the EV Spring-Pot the matrix $\mathbf{B} = \tilde{\mathbf{B}}$ becomes:

$$
\tilde{\mathbf{B}} = \begin{bmatrix} 1^{-\alpha} + \dfrac{\tilde{K}}{q_E} & 0 & 0 & \dots & 0 \\ 0 & 2^{-\alpha} & 0 & \dots & 0 \\ 0 & 0 & 3^{-\alpha} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & n^{-\alpha} \end{bmatrix}. \tag{33}
$$

Therefore the differential equations system which governs the equilibrium is rewritten in the following compact form

$$
p_E \mathbf{A} \dot{\boldsymbol{\gamma}} + q_E \tilde{\mathbf{B}} \boldsymbol{\gamma} = \mathbf{v}\sigma(t). \tag{34}
$$

As customary we first solve the homogeneous case, that is as $\sigma(t) = 0$. We introduce a coordinate transformation in the form:

$$
\tilde{\mathbf{B}}^{1/2}\boldsymbol{\gamma} = \mathbf{x} \tag{35}
$$

and premultiplying by $\tilde{\mathbf{B}}^{-1/2}$ a differential equation for the unknown vector $\mathbf{x}$ is obtained as:

$$
p_E \tilde{\mathbf{D}}\,\dot{\mathbf{x}} + q_E \mathbf{x} = \tilde{\mathbf{v}}\sigma(t) \tag{36}
$$

where $\tilde{\mathbf{v}} = \tilde{\mathbf{B}}^{-1/2}\mathbf{v}$ and $\tilde{\mathbf{D}}$ is the dynamical matrix $\tilde{\mathbf{D}} = \tilde{\mathbf{B}}^{-1/2}\mathbf{A}\,\tilde{\mathbf{B}}^{-1/2}$ given as

$$
\tilde{\mathbf{D}} = \begin{bmatrix} \dfrac{q_E}{q_E + \tilde{K}} & -\sqrt{\dfrac{q_E 2^{\alpha}}{\tilde{K} + q_E}} & \dots & 0 \\ -\sqrt{\dfrac{q_E 2^{\alpha}}{\tilde{K} + q_E}} & 1 + \left(\dfrac{2}{1}\right)^{\alpha} & \dots & 0 \\ 0 & -\left(\dfrac{3}{2}\right)^{\frac{\alpha}{2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 + \left(\dfrac{n}{n-1}\right)^{\alpha} \end{bmatrix} \tag{37}
$$

that is $\tilde{\mathbf{D}}$ is symmetric and positive definite and it may be obtained straightforwardly once n and α are fixed. Let $\tilde{\boldsymbol{\Phi}}$ be the modal matrix whose columns are the orthonormal eigenvectors of $\tilde{\mathbf{D}}$ that is

$$
\tilde{\boldsymbol{\Phi}}^T \tilde{\mathbf{D}}\, \tilde{\boldsymbol{\Phi}} = \tilde{\boldsymbol{\Lambda}}; \qquad \tilde{\boldsymbol{\Phi}}^T \tilde{\boldsymbol{\Phi}} = \mathbf{I} \tag{38}
$$

where $\mathbf{I}$ is the identity matrix and $\tilde{\boldsymbol{\Lambda}}$ is the diagonal matrix collecting the eigenvalues $\tilde{\lambda}_j > 0$ of $\tilde{\mathbf{D}}$.

In the following we order $\tilde{\lambda}_j$ in such a way that $\tilde{\lambda}_1 < \tilde{\lambda}_2 < \dots < \tilde{\lambda}_n$. As we indicate $\mathbf{y}(t)$ the modal coordinate vector, defined as

$$
\mathbf{x}(t) = \tilde{\boldsymbol{\Phi}}\,\mathbf{y}(t); \qquad \mathbf{y}(t) = \tilde{\boldsymbol{\Phi}}^T\,\mathbf{x}(t) \tag{39}
$$

and we substitute in Eq. (36) a decoupled set of differential equation is obtained in the form:

$$
p_E \tilde{\boldsymbol{\Lambda}}\,\dot{\mathbf{y}} + q_E \mathbf{y} = \bar{\mathbf{v}}\sigma(t) \tag{40}
$$

where $\bar{\mathbf{v}} = \tilde{\boldsymbol{\Phi}}^T\tilde{\mathbf{v}} = \tilde{\boldsymbol{\Phi}}^T\tilde{\mathbf{B}}^{-1/2}\mathbf{v} = \tilde{\boldsymbol{\Phi}}^T\mathbf{v}$. The j^{th}-equation of Eq. (40) reads:

$$
\dot{y}_j + \rho_j\, y_j = \frac{\tilde{\phi}_{1,j}}{p_E \tilde{\lambda}_j}\sigma(t); \qquad j = 1, 2, 3, \dots n \tag{41}
$$

where $\rho_j = q_E / p_E \tilde{\lambda}_j > 0$ and $\phi_{1,j}$ is the j^{th} element of the first row of the matrix $\tilde{\boldsymbol{\Phi}}$. The solution of Eq. (41) is provided in the form:

$$
y_j(t) = y_j(0)\, e^{-\rho_j t} + \frac{\tilde{\phi}_{1,j}}{p_E \tilde{\lambda}_j} \int_0^t e^{-\rho_j (t-\tau)} \sigma(\tau)\, d\tau \tag{42}
$$

where $y_j(0)$ is the j^{th} component of the vector $\mathbf{y}(0)$ related to the vector of initial conditions $\boldsymbol{\gamma}(0)$ as:

$$
\mathbf{y}(0) = \tilde{\boldsymbol{\Phi}}^T \tilde{\mathbf{B}}^{1/2} \boldsymbol{\gamma}(0). \tag{43}
$$

Solution of the differential equation system in Eq. (34) may be obtained as the modal vector $\mathbf{y}(t)$ has been evaluated by solving Eq. (42) with the aid of Eqs. (35) and (39) as

$$
\boldsymbol{\gamma}(t) = \tilde{\mathbf{B}}^{-1/2} \tilde{\boldsymbol{\Phi}} \mathbf{y}(t). \tag{44}
$$

As we are interested to a relation among the shear stress and the normalized transverse displacement of the upper lamina we must evaluate the first element of vector $\boldsymbol{\gamma}(t)$ that is obtained as:

$$
\gamma(t) = \mathbf{v}^T \boldsymbol{\gamma}(t). \tag{45}
$$

For quiescent system at $t = 0$ and forcing the model with $\sigma(t) = U(t)$ the solution $\gamma(t)$ obtained from Eq. (45) becomes:

$$
\gamma(t) = \sum_{j=1}^{n} \frac{\tilde{\phi}_{1,j}^2}{\tilde{K} + q_E} \left[1 - e^{-\rho_j t}\right]. \tag{46}
$$

6.2 The case of Kelvin-Voigt with VE Spring-Pot

In this section the fractional Kelvin-Voigt is characterized for the presence of VE Spring-Pot connected in parallel with elastic stiffness. The discretized model is depicted in Figure 7. Modal analysis of the differential equations

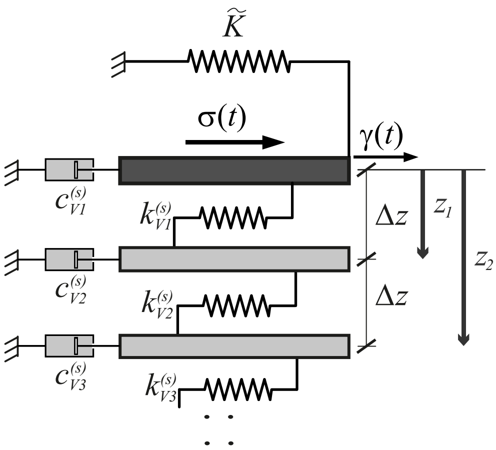


Figure 7. Discretized fractional VE Kelvin-Voigt model.

system representing the behavior of this fractional model is quite similar to previous section. In this case the equilibrium equations system in compact form becomes

$$
p_V \mathbf{B} \dot{\boldsymbol{\gamma}} + q_V \hat{\mathbf{A}} \boldsymbol{\gamma} = \mathbf{v}\sigma(t). \tag{47}
$$

where the matrix $\mathbf{B}$ is defined in Eq. (27), while the matrix $\hat{\mathbf{A}}$ becomes:

$$
\hat{\mathbf{A}} = \begin{bmatrix} 1^{-\alpha_i} + \dfrac{\tilde{K}}{q_V} & -1^{-\alpha_i} & \dots & 0 \\ -1^{-\alpha_i} & 1^{-\alpha_i} + 2^{-\alpha_i} & \dots & 0 \\ 0 & -2^{-\alpha_i} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (n-1)^{-\alpha_i} + n^{-\alpha_i} \end{bmatrix}. \tag{48}
$$

In this case we substitute $\boldsymbol{\gamma} = \mathbf{B}^{-1/2}\mathbf{x}$ in Eq. (47) and we perform premultiplication by $\mathbf{B}^{-1/2}$:

$$p_V \dot{\mathbf{x}} + q_V \hat{\mathbf{D}}\,\mathbf{x} = \tilde{\mathbf{v}}\sigma(t) \tag{49}$$

where $\hat{\mathbf{D}} = \mathbf{B}^{-1/2}\hat{\mathbf{A}}\mathbf{B}^{-1/2}$ is the dynamical matrix defined as:

$$\hat{\mathbf{D}} = \begin{bmatrix} \dfrac{q_V + \tilde{K}}{q_V} & -\left(\dfrac{2}{1}\right)^{\frac{\alpha}{2}} & \dots & 0 \\ -\left(\dfrac{2}{1}\right)^{\frac{\alpha}{2}} & 1 + \left(\dfrac{2}{1}\right)^{\alpha} & \dots & 0 \\ 0 & -\left(\dfrac{3}{2}\right)^{\frac{\alpha}{2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 + \left(\dfrac{n}{n-1}\right)^{\alpha} \end{bmatrix}. \tag{50}$$

The dynamical equilibrium equation in modal coordinate reads:

$$p_V \dot{\mathbf{y}} + q_V \hat{\boldsymbol{\Lambda}}\,\mathbf{y} = \bar{\mathbf{v}}\sigma(t) \tag{51}$$

so that equilibrium of j^{th} Kelvin-Voigt represented by Eq. (51) is given as:

$$\delta_j\,\dot{y}_j + y_j = \frac{\hat{\phi}_{1,j}}{q_V \hat{\lambda}_j}\sigma(t); \qquad j = 1, 2, 3, \dots n \tag{52}$$

where $\delta_j = p_v / q_V \hat{\lambda}_j > 0$.

The solution in terms of modal coordinates are obtained in integral form as:

$$y_j(t) = y_j(0)\, e^{-t/\delta_j} + \frac{\hat{\phi}_{1,j}}{\delta_j\, q_V \hat{\lambda}_j} \int_0^t e^{-(t-\tau)/\delta_j} \sigma(\tau)\, d\tau. \tag{53}$$

The stress-strain relations between shear stress $\sigma(t)$ and normalized displacement $\gamma(t)$ may be obtained as in previous section (see Eqs. (42) and (45)).

For quiescent system at $t = 0$ and forcing the model with $\sigma(t) = U(t)$ the solution $\gamma(t)$ becomes:

$$\gamma(t) = \sum_{j=1}^{n} \frac{\hat{\phi}^2_{1,j}}{q_V \hat{\lambda}_j}\left[1 - e^{-\frac{t}{\delta_j}}\right]. \tag{54}$$

The particular case in which fractional Kelvin-Voigt model consists by perfect spring and Spring-pot with $\beta = 0.5$ may be studied either starting from Eq. (46) or from Eq. (54).

In Figure 8 the results for $\sigma(t) = U(t)$, $G_0 = \eta_0 = 2$, $\tilde{K} = 20\,G_0$, $n = 750$, $\Delta z = 0.05$ and different values of $\beta = 0.4$, 0.5, 0.6, are contrasted with solution reported in Eq. (10) evaluated by *Mathematica*.

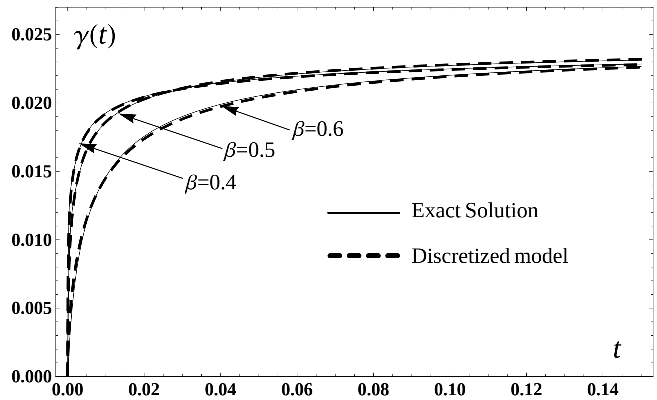


Figure 8. Creep test of EV and VE Kelvin-Voigt model: comparison between the exact and approximate solution.

7. CONCLUSIONS

Inverse relationship of simpler constitutive law $\sigma(t) = C(\beta)\left({}^C D^{\beta}_{0^+}\gamma\right)(t)$ is readily found in the form $\gamma(t) = C(\beta)^{-1}\left(D^{-\beta}_{0^+}\sigma\right)(t)$. Finding the inverse relationship of $\sigma(t) = \sum_{j=1}^{m} C(\beta_j)\left({}^C D^{\beta_j}_{0^+}\gamma\right)(t)$ is a very hard task unless the case $m = 2$ that involves two parameters Mittag-Leffler functions. For $m > 2$ no exact solutions may be found. Recently it has been shown that exact mechanical models of $\sigma(t) = C(\beta)\left({}^C D^{\beta}_{0^+}\gamma\right)(t)$ is represented by indefinite mechanical models constituted by massless fluid resting on a bed of springs ($0 \le \beta \le 1/2$) or by massless shear type column resting on a bed of dashpots ($1/2 \le \beta \le 1$). Springs and dashpots decrease with power-law related to the fractional order derivative β. With the aid of these mechanical models the problem of finding the strain history for an assigned stress history may be faced by using standard tools of dynamic analysis of mechanical systems. It is shown that in the general case $\sigma(t) = \sum_{j=1}^{m} C(\beta_j)\left({}^C D^{\beta_j}_{0^+}\gamma\right)(t)$ the mechanical model is massless plate interconnecting m columns r of them are fluids sustained by independent springs. While the remaining $m - r$ are shear type columns resting on a bed of independent dashpots. Discretization of the two kind of columns and eigenanalysis of each column leads to a simple problem that may be implemented in computer codes.

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