FRACTIONAL CAPUTO ANALYSIS OF DISCRETE SYSTEMS

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Abstract

In this paper we investigate the motion of discrete dynamical systems involving Caputo fractional derivatives using the fractional calculus. The fractional Hamilton's equations and the explicit solutions of Euler-Lagrange equations are calculated by using the canonical transformations. The interesting point in this work is that the classical results are obtained when fractional derivatives are replaced with the integer order derivatives. Two examples are analyzed in detail.

Keywords: Fractional calculus, Caputo fractional derivatives, Lagrangian and Hamiltonian formulation

Introduction

Fractional calculus is one of the generalizations of the differential calculus and it has been used successfully as an alternative tool to solve several problems in various fields of science and engineering (Oldham and Spanier, 1974; Miller and Ross 1993; Samko et al., 1993; Podlubny, 1999; Hilfer, 2000; Kumar and Kumar, 2013

Fractional calculus of variations unifies the calculus of variations and the fractional calculus, by inserting fractional derivatives into the variational integrals. This occurs naturally in many problems of physics or mechanics, in order to provide more accurate models of physical phenomena (Atanackovi and Stankovi, 2007; Baleanu, 2009).

The fractional calculus is nowadays a subject under strong research. Different definitions for fractional derivatives and integrals are used, depending on the purpose under study. Investigations cover problems depending on Riemann-Liouville fractional derivatives, the Caputo fractional derivatives (Agrawal, 2007, 2011; Baleanu and Muslih, 2005; Baleanu and Agrawal, 2006) and others.

However, the differential equations defined in terms of Riemann-Liouville derivatives require fractional initial conditions whereas the differential equations defined in terms of Caputo derivatives require regular boundary conditions. For this reason, Caputo fractional derivatives are popular among scientists and engineers.

The formulation of the fractional variational principles still needs to be more developed in the future and it has an important role for elaboration of a consistent fractional quantization method for both discrete and continuous systems. The first attempt to find the fractional Lagrangian and Hamiltonian for a given dissipative system is due to Riewe (1996, 1997). Riewe formulated the problem of the calculus of variations with fractional derivatives and obtained the respective Euler-Lagrange equations, combining both conservative and nonconservative cases. These Euler-Lagrange equations are then used to investigate problems with Lagrangians that are linear in the velocities (Baleanu and Avkar, 2004). Furthermore, many authors (Baleanu and Muslih, 2005; Muslih and Baleanu, 2005; Baleanu and Agrawal, 2006; Agrawal, 2006, 2007; Rabei et al., 2007; Fahd and

Baleanu, 2007) have introduced a huge amount of mathematical knowledge and important contributions in the field of fractional integrals and derivatives.

The study of fractional problems of the calculus of variations and respective Euler-Lagrange equations is a fairly recent issue and include only the continuous case. The discrete analogues of differential equations can be very useful in applications.

The current work is aimed to apply the fractional calculus to solve differential equation involving discrete classical systems with Caputo Derivatives.

This paper is organized as follows: In Section 2, some basic formulas of the fractional calculus are briefly reviewed. Section 3 contains a briefly review of the fractional Lagrangian and Hamiltonian analysis of discrete systems. In Section 4, two examples of discrete classical systems are presented. Section 5 is dedicated to our conclusions.

Basic Tools

In this section we briefly present some basic definitions of the fractional calculus. The fractional Caputo derivatives are defined as follows:

The left-sided Caputo fractional derivative of order α is defined by

$$ {}_{a}^{C}D_{x}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{x} (x-\tau)^{n-\alpha-1} \left(\frac{d}{d\tau} \right)^{n} f(\tau) d\tau \quad (1) $$

and the right-sided Caputo fractional derivative of order α is defined by

$$ {}_{x}^{C}D_{b}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_{x}^{b} (\tau-x)^{n-\alpha-1} \left(-\frac{d}{d\tau} \right)^{n} f(\tau) d\tau , \quad (2) $$

where the Gamma function has the form

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx \,. \tag{3}$$

Here α is the order of the derivative such that $n-1 \leq \alpha < n$ and is not equal to zero. By definition, the Caputo fractional derivative of a constant is zero. If α is an integer, we recovered the usual definitions, namely,

$$\begin{aligned} {}_a^C D_x^\alpha f(x) &= \left(\frac{d}{dx}\right)^\alpha f(x), \\ {}_x^C D_b^\alpha f(x) &= \left(-\frac{d}{dx}\right)^\alpha f(x). \end{aligned} \tag{4}$$

The fractional Euler-Lagrange equations are obtained in Ref. (Agrawal, 2002), and we present briefly the main results obtained as follows: Let $S(q)$ be a functional of the form

$$S(q) = \int_a^b L(q, {}_a^C D_t^\alpha q, {}_t^C D_b^\beta q, t) dt \,, \qquad 0 < \alpha, \beta < 1 \tag{5}$$

and defined on the set of continuous functions $q(t)$ which have continuous left fractional Caputo derivative of order α and right fractional Caputo derivative of order β in the interval $[a, b]$. Then a necessary condition for $S(q)$ to have an extremum for a given function $q(t)$ is that $q(t)$ satisfies the generalized fractional Euler-Lagrange equation given by

$$\frac{\partial L}{\partial q} + {}_t^C D_b^\alpha \frac{\partial L}{\partial \, {}_a^C D_t^\alpha q} + {}_a^C D_t^\beta \frac{\partial L}{\partial \, {}_t^C D_b^\beta q} = 0 \,. \tag{6}$$

Note that for fractional calculus of variation problems the resulting Euler-Lagrange equation contains both the left Caputo fractional derivative and the right Caputo fractional derivative. This is expected since the optimum function must satisfy both terminal conditions. Further, for $\alpha = \beta = 1$, we have

$$
\begin{aligned}
{}_a^C D_t^\alpha &= \frac{d}{dt}, \\
{}_t^C D_b^\alpha &= -\frac{d}{dt},
\end{aligned}
\tag{7}
$$

and Eq.(6) reduces to the standard Euler-Lagrange equation for classical calculus of variations

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 . \tag{8}$$

Fractional Lagrangian and Hamiltonian Analysis of Discrete Systems

A Hamiltonian dynamics in terms of Caputo derivatives was developed (see e.g. (Baleanu and Agrawal, 2006; Agrawal, 2006, 2007; Rabei et al., 2007). For a given fractional Lagrangian

$$L = L(q, {}_a^C D_t^\alpha q, {}_t^C D_b^\beta q, t) , \tag{9}$$

the fractional canonical momenta are defined as

$$
\begin{aligned}
p_\alpha &= \frac{\partial L}{\partial\, {}_a^C D_t^\alpha q}, \\
p_\beta &= \frac{\partial L}{\partial\, {}_t^C D_b^\beta q}.
\end{aligned}
\tag{10}
$$

Therefore, we construct the corresponding fractional Hamiltonian as follows:

$$H = p_\alpha\ {}_a^C D_t^\alpha q + p_\beta\ {}_t^C D_b^\beta q - L . \tag{11}$$

The total differential of this Hamiltonian can be obtained as [18]

$$
\begin{aligned}
dH = {} & p_\alpha\, d\, {}_a^C D_t^\alpha q + dp_\alpha\ {}_a^C D_t^\alpha q + p_\beta\, d\, {}_t^C D_b^\beta q + dp_\beta\ {}_t^C D_b^\beta q - \frac{\partial L}{\partial q} dq \\
& - \frac{\partial L}{\partial\, {}_a^C D_t^\alpha q} d\, {}_a^C D_t^\alpha q - \frac{\partial L}{\partial\, {}_t^C D_b^\beta q} d\, {}_t^C D_b^\beta q - \frac{\partial L}{\partial t} dt.
\end{aligned}
\tag{12}
$$

We may combine (10) and (12) to obtain

$$dH = dp_\alpha\ {}_a^C D_t^\alpha q + dp_\beta\ {}_t^C D_b^\beta q - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial t} dt . \tag{13}$$

Making use of the Euler-Lagrange equation (6), we get

$$dH = dp_{\alpha}\ {}_{a}^{C}D_{t}^{\alpha}q + dp_{\beta}\ {}_{t}^{C}D_{b}^{\beta}q + \left({}_{a}^{C}D_{t}^{\beta}\ p_{\beta} + {}_{t}^{C}D_{b}^{\alpha}\ p_{\alpha}\right)dq - \frac{\partial L}{\partial t}dt \quad (14)$$

Since the Hamiltonian is required to be a function of the generalized coordinates and momenta:

$$H \equiv H(q, p_{\alpha}, p_{\beta}, t), \quad (15)$$

it follows that

$$dH = \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial p_{\alpha}}dp_{\alpha} + \frac{\partial H}{\partial p_{\beta}}dp_{\beta} + \frac{\partial H}{\partial t}dt\,. \quad (16)$$

Comparing (14) and (16), one obtains the following fractional Hamilton's equations of motion

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t},$$

$$\frac{\partial H}{\partial p_{\alpha}} = {}_{a}^{C}D_{t}^{\alpha}q,$$

$$\frac{\partial H}{\partial p_{\beta}} = {}_{t}^{C}D_{b}^{\beta}q, \quad (17)$$

$$\frac{\partial H}{\partial q} = {}_{t}^{C}D_{b}^{\alpha}\ p_{\alpha} + {}_{a}^{C}D_{t}^{\beta}\ p_{\beta}\,.$$

It is worth to mention that for $\alpha \to 1$, equations (17) reduce to classical Hamilton equations. In addition, the fractional Hamiltonian isn't a constant of motion even though the Lagrangian doesn't depend on the time explicitly, this can be observed by making use of

$$\frac{dH}{dt} = \frac{\partial H}{\partial q}\frac{dq}{dt} + \frac{\partial H}{\partial p_{\alpha}}\frac{dp_{\alpha}}{dt} + \frac{\partial H}{\partial p_{\beta}}\frac{dp_{\beta}}{dt},$$

and by substituting the values of partial derivatives of Hamiltonian from (17). As a result we obtain that

$$\frac{dH}{dt} = \left({}_{t}^{C}D_{b}^{\alpha}\ p_{\alpha} + {}_{a}^{C}D_{t}^{\beta}\ p_{\beta}\right)\frac{dq}{dt} + \left({}_{a}^{C}D_{t}^{\alpha}q\right)\frac{dp_{\alpha}}{dt} + \left({}_{t}^{C}D_{b}^{\beta}q\right)\frac{dp_{\beta}}{dt} \neq 0 \quad (18)$$

Examples

As a first example of discrete systems consider the fractional Lagrangian:

$$L = \frac{1}{2}\left({}_a^C D_t^\alpha q\right)^2 + \frac{1}{2}\left({}_t^C D_b^\beta q\right)^2 \tag{19}$$

which is the analog of free motion in one-dimensional space.

Considering $a = 0, \quad b = 1, \quad 0 < \alpha, \beta < 1$, and by using (6), the fractional Euler-Lagrange equation is given by

$$ {}_t^C D_1^\alpha \left({}_0^C D_t^\alpha q\right) + {}_0^C D_t^\beta \left({}_t^C D_1^\beta q\right) = 0 . \tag{20}$$

The expressions of the generalized fractional momenta read

$$\begin{aligned} p_\alpha &= {}_0^C D_t^\alpha q, \\ p_\beta &= {}_t^C D_1^\beta q. \end{aligned} \tag{21}$$

The fractional Hamiltonian corresponding to (19) is given by

$$H = \frac{1}{2}\left(p_\alpha^2 + p_\beta^2\right). \tag{22}$$

Thus, the fractional equations of motion can be obtained as:

$$\begin{aligned} p_\alpha &= {}_0^C D_t^\alpha q, \\ p_\beta &= {}_t^C D_1^\beta q, \end{aligned} \tag{23}$$

in addition to

$$ {}_t^C D_1^\alpha \left({}_0^C D_t^\alpha q\right) + {}_0^C D_t^\beta \left({}_t^C D_1^\beta q\right) = 0 . \tag{24}$$

In fact, this result shows that the fractional Hamiltonian equations are equivalent to the fractional Euler-Lagrange equation.

As a second example Let us consider the following fractional Lagrangian:

$$L = \frac{1}{2}\left({}_0^C D_t^\alpha q\right)^2 + \frac{1}{2}\left({}_t^C D_1^\beta q\right)^2 + q\, {}_0^C D_t^\alpha q + q\, {}_t^C D_1^\beta q + \frac{q^2}{2} . \tag{25}$$

The fractional Euler-Lagrange equation is given by

$$ {}_t^C D_1^\alpha \left({}_0^C D_t^\alpha q + q\right) + {}_0^C D_t^\beta \left({}_t^C D_1^\beta q + q\right) + q = 0 . \tag{26}$$

Now, the generalized fractional momenta read as

$$\begin{aligned} p_\alpha &= {}_0^C D_t^\alpha q + q, \\ p_\beta &= {}_t^C D_1^\beta q + q, \end{aligned} \tag{27}$$

and therefore the fractional Hamiltonian has the form

$$H = \frac{1}{2}\left(p_\alpha - q\right)^2 + \frac{1}{2}\left(p_\beta - q\right)^2 - \frac{q^2}{2}, \tag{28}$$

Accordingly, the fractional equations of motion can be obtained as:

$$\begin{aligned} p_\alpha - q &= {}_0^C D_t^\alpha q, \\ p_\beta - q &= {}_t^C D_1^\beta q. \end{aligned} \tag{29}$$

$${}_t^C D_1^\alpha \left({}_0^C D_t^\alpha q + q\right) + {}_0^C D_t^\beta \left({}_t^C D_1^\beta q + q\right) + q = 0 \,. \tag{30}$$

Again, these results are in exact agreement with (26) and (27).

Conclusion

The canonical fractional Hamiltonian and the corresponding fractional Hamilton's equations of motion for discrete systems are constructed within the fractional Caputo derivatives. For the given two discrete examples we observed that both fractional Euler-Lagrange equations and fractional Hamiltonian equations give the same results.

The approach presented here and the resulting equations of motion are very similar to those for variational problems containing integral order derivatives.

As it is expected the dynamics of the fractional calculus systems is different from the classical one but the classical dynamics is recovered as a particular case when the derivatives are of integral order only, the results of fractional calculus of variations reduce to those obtained from the ordinary calculus of variations.

We consider two specific examples in this paper. For the first example we consider one of the possible fractional generalization of the free one dimensional particle Lagrangian. Both fractional Euler-Lagrange and Hamilton equations are obtained for this example and it was proved that they are equivalent in the fractional case. The second example deals with the same fractional generalization of the free one dimensional particle except by adding a term of $q\ {}_0^C D_t^\alpha q + q\ {}_t^C D_1^\beta q + \frac{q^2}{2}$.

The advantage of using the method presented in this paper, is that we can easily obtain the action function, which is the essential part for obtaining the WKB quantization for any mechanical fractional system.

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