Affine Processes and Pseudo-Differential Operators with Unbounded Coefficients

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Introduction

In the 1960s, Courrège and von Waldenfels proved that the generator of a Feller semigroup is represented by an integro-differential operator under some reasonable assumptions. By Fourier inversion, we immediately derive that the generator of a Feller process is a pseudo-differential operator

$$Au(x) = -q(x, D)u(x) = -(2\pi)^{\frac{d}{2}} \int_{\mathbb{R}^d} \mathrm{e}^{\mathbf{i}x^\top \xi} q(x, \xi) \hat{u}(\xi) \,\mathrm{d}\xi, \qquad \forall u \in C_c^\infty(\mathbb{R}^d),$$

with symbol $q(x, \xi)$. Since every Markov process is associated with a semigroup, this result establishes a connection between Feller processes and symbols of pseudo-differential operators.

For a Lévy process, this relation is quite natural. It is well-known that a Lévy process $(L_t)_{t\geq 0}$ is characterized by the characteristic exponent ψ_L,

$$\mathbb{E}^x(\mathrm{e}^{\mathbf{i}(L_t - x)^\top \xi}) = \mathrm{e}^{-t\psi_L(\xi)},$$

and that the generator of a Lévy process is given by

$$Au(x) = -(2\pi)^{\frac{d}{2}} \int_{\mathbb{R}^d} \mathrm{e}^{\mathbf{i}x^\top \xi} \psi_L(\xi) \hat{u}(\xi) \,\mathrm{d}\xi.$$

Hence, the characteristic exponent and the symbol of the pseudo-differential operator coincide.

Many results for Lévy processes are based on the characteristic exponent. In other words, it is possible to deduce properties of a Lévy process from its symbol. In recent years, this approach has generated new insights to Feller processes, e.g. path properties, through their symbol. However, most applications require the assumption of bounded coefficients,

$$\sup_{x\in\mathbb{R}^d} |q(x, \xi)| \leq c(1 + |\xi|^2) \quad \text{for all } \xi \in \mathbb{R}^d.$$

Hence, this thesis is devoted to the study of Feller processes with unbounded coefficients through their symbol. We address this task by focusing on affine processes. This wide class of processes has a symbol with coefficients which are affine dependent on x and hence unbounded. We develop new techniques and tools to handle the affine case and then expand our results to Feller processes with unbounded coefficients.

This analytic approach to stochastic processes requires elementary harmonic analysis. Therefore, we introduce the notion and relevant results of positive and negative definite functions in Chapter 1. Making the thesis self-contained, we provide a brief exposition of Markov process, semigroups and pseudo-differential operators.

The second chapter contains several results for Feller processes whose symbols have unbounded coefficients. We start with probability estimates, the maximal inequality and an extension of the upper bound of the tail probability. With the aid of these important tools, we are able to prove a law of iterated logarithm as well as path properties by the Blumenthal-Getoor-Pruitt indices. The analysis of pseudo-differential operators with unbounded coefficients requires weighted norms, which compensate the growth of the coefficients. Section 2.3 provides essential properties of weighted norms, allowing us to characterize the domain of a pseudo-differential operator.

Since affine processes are a major example in this thesis, we thoroughly examine them on the so-called canonical state space $D = \mathbb{R}_+^m \times \mathbb{R}^n$ in Chapter 3. An affine process is defined as a Markov process whose characteristic function is exponential-affine dependent on x. Basic examples show that in general, we have no explicit representation of this definition. However, an affine process is uniquely characterized by its admissible parameters. After establishing some basic properties and identifying the affine semigroup as a Feller semigroup, we use harmonic analysis to verify the admissibility conditions. The parameters correspond to negative definite functions which form the symbol of the pseudo-differential operator. Based on the results of Chapter 2, we give accessible proofs of the domain and cores of an affine process. In Section 3.6, we deduce several path properties through the symbol.

Chapter 4 deals with affine processes on the space of symmetric positive semidefinite matrices. The program is similar to the canonical case. However, the matrix-valued state space causes slight differences. Nevertheless, we see that the study of affine processes through their symbol is independent of the state space to a certain extent.

In Chapter 5, we look at Ornstein-Uhlenbeck processes from a potential theoretical point of view. We consider the corresponding semigroups on the L^2 space. We will see that the Ornstein-Uhlenbeck process generates an L^2 sub-Markov semigroup, and hence a Dirichlet operator. Furthermore, we are interested in the invariance and symmetry of the operator. By a criterion, based on the symbol, we calculate the invariant measure for an Ornstein-Uhlenbeck process. The symmetry of the generator of an Ornstein-Uhlenbeck process implies a functional equation. The solution of the latter shows that the symmetry requires that the process has no jumps. These results also carry over to pertubed Ornstein-Uhlenbeck processes.

The Markov chain approximation is a simulation scheme based on the symbol of the generator. In Chapter 6, we generalize this method to Feller processes with symbols whose drift and diffusion coefficient satisfy a linear growth condition. For affine processes we exploit the special structure of the state space to expand the linear growth condition to all coefficients of the symbol. Furthermore, our results carry over to more general state spaces as positive semidefinite matrices since we use an approach based on the symbol.

Index of Notation

This list is intended to aid cross-referencing, so notation that is specific to a single section is generally not listed. Some symbols are used locally, without ambiguity, in senses other than those given below.

General notation: Analysis

$a \vee b$, $a \wedge b$	$\max(a,b)$, $\min(a,b)$
a^+, a^-	$\max(a,0)$, $-\min(a,0)$
$\mathbf{i}$	imaginary unit
$\lvert x\rvert$	Euclidean vector and matrix norm
$\langle x,y\rangle = \sum_{k=1}^d x_k y_k$	for $x,y \in \mathbb{C}^d$
$e_\xi(x)$	$e^{\xi^\top x}$, $x,\xi \in \mathbb{C}^d$
$\operatorname{supp} f$	support, $\overline{\{f \neq 0\}}$
∇	gradient $\left(\frac{\partial}{\partial x_1}, \dots \frac{\partial}{\partial x_d}\right)^\top$
∇^α	$\frac{\partial^{\alpha_1+\dots+\alpha_d}}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}}$
$\mathcal{F}u = \hat{u}$	Fouier transform of a function u
$(T_t)_{t\geq 0}$	semigroup of operators
$(A, \mathcal{D}(A))$	generator
$q(x,D)$	pseudo-differential operator
$q(x,\xi)$	continuous negative definite symbol
$p(x,\xi)$	probabilistic symbol
S_d^+	space of $d \times d$ dimensional symmetric positive semidefinite matrices
S_d	space of $d \times d$ dimensional symmetric matrices
$\langle x,y\rangle = \operatorname{Tr}(xy)$	for $x,y \in S_d$

General notation: Probability

$(\Omega, \mathcal{F}, \mathbb{P})$	probability space
$\sim$	'is distributed as'
a.s.	almost surely
$(X_t, \mathcal{F}_t)_{t\geq 0}$	adapted process
$B = (B_t)_{t\geq 0}$	Brownian motion
$L = (L_t)_{t\geq 0}$	Lévy process
ψ_L	characteristic exponent of a Lévy process $(L_t)_{t\geq 0}$
(l, Q, ν)	Lévy triplet
χ	truncation function
τ_r^x	$\inf\{t > 0;\ X_t \in \overline{B}^c(x,r)\}$

Sets and σ-algebras

A^c	complement of the set A
A°	open interior of the set A
$\overline{A}$	closure of the set A
$B(x,r)$	open ball, centre x, radius r
$\overline{B}(x,r)$	closed ball, centre x, radius r
$\mathcal{B}(D)$	Borel sets of D
$\mathcal{F}_t^X$	$\sigma(X_s : s \leq t)$

Spaces of measures and functions

$B(D)$	Borel functions on D
$B_b(D)$	— —, bounded
$C(D)$	continuous functions on D
$C_b(D)$	— —, bounded
$C_\infty(D)$	— —, $\lim_{\lvert x\rvert\to\infty} u(x) = 0$
$C_c(D)$	— —, compact support
$C^k(D)$	k times continuously diff'ble functions on D
$C_\infty^k(D)$	— —, 0 at infinity (with their derivatives)
$C_c^k(D)$	— —, compact support

$\|u\|_\infty = \sup_x |u(x)|$ supremum norm

$\|u\|_{(k)}$ $\sum_{|\alpha|\le k} \|u\|_\infty$

$C^p_{\rho,\infty}$ weighted function space

$\|u\|_{(p),\rho} = \sum_{|\alpha|\le p} \|\rho D^\alpha u\|_\infty$ weighted norm

$L^p(D,\mu), L^p(\mu), L^p(D)$ L^p space w.r.t. the measure space $(D, \mathcal{F}, \mu)$

$\|f\|_{L^p(\mu)}$ $\left(\int |f|^p \,\mathrm{d}\mu\right)^{1/p}$

$\mathcal{S}(\mathbb{R}^d)$ Schwartz space of rapidly decreasing smooth functions

Affine processes

$\mathbb{R}^m_+$ $\{x \in \mathbb{R}^m;\ x_i \ge 0\ \forall i = 1,\dots,m\}$

$\mathbb{C}^m_-$ $\{x \in \mathbb{C}^m;\ Re(x_i) \le 0\ \forall i = 1,\dots,m\}$

$D = \mathbb{R}^m_+ \times \mathbb{R}^n$ "canonical" state space

$d = m + n$

$I = \{1,\dots,m\}$ index set

$II = \{m+1,\dots,d\}$ index set

$x_I = (x_1,\dots,x_m)^\top$ projection on I coordinates for $x \in \mathbb{R}^m_+ \times \mathbb{R}^n$

$x_{II} = (x_{m+1},\dots,x_d)^\top$ projection on II coordinates for $x \in \mathbb{R}^m_+ \times \mathbb{R}^n$

F, R functional characteristics

Further abbreviations

CIR Cox-Ingersoll-Ross process

GOU generalized Ornstein-Uhlenbeck process

SDE stochastic differential equation

Chapter 1

Preliminaries

This chapter presents some preliminaries including a brief introduction to Markov processes. From there, we continue with semigroups, in particular Feller semigroups. Finally, we summarize relevant results on generators and pseudo-differential operators. Since continuous negative definite functions appear in all these parts, we start by briefly introducing them in the next section.

1.1 Continuous Negative Definite Functions

Let us recall several known facts on positive and negative definite functions. Usually, these functions are considered on the spaces $\mathbb{R}^n$ as well as $\mathbb{R}^m_+$. For our purpose, we require a more general setting. Therefore, we examine positive and negative definite functions on abelian semigroups[1] with an involution[2] $* : S \to S$, cf. Berg, Christensen and Ressel [8] and Ressel [43]. Naturally, $\mathbb{R}^n$ with $x^* = -x$ and $\mathbb{R}^m_+$ with $x^* = x$ are abelian semigroups with involutions. Further examples are the product space $D = \mathbb{C}^m_- \times \mathbf{i}\mathbb{R}^n$ including the special cases[3] $m = 0$ or $n = 0$ and the space of matrices[4] $D = S_d^+ \times \mathbf{i}S_d$, both equipped with the complex conjugate as the involution $x^* = \overline{x}$. We are mainly interested in the latter examples. Therefore, we choose the general approach of positive and negative definite functions on semigroups unless it yields unreasonable technicalities.

We use the definitions of positive and negative definiteness according to Berg et al. [8, Definition 4.1.5, Definition 4.1.8].

Definition 1.1. A function $\phi : S \to \mathbb{C}$ is called positive definite if

$$\sum_{j,k=1}^{n} c_j \overline{c}_k \phi(s_j^* + s_k) \geq 0$$

[1] An abelian semigroup is a set S with a binary operation $+ : S \times S \to S$ which is commutative and associative.

[2] A mapping $* : S \to S$ is called involution, if $(s+t)^* = s^* + t^*$ and $(s^*)^* = s$ for all $s, t \in S$.

[3] The case $\mathbf{i}\mathbb{R}^n$ is very similar to the usual space $\mathbb{R}^n$. In a sense, the paramater already contains the imaginary unit.

[4] In Section 4.1 we give a short introduction to the space of positive definite matrices S_d^+ and symmetric matrices S_d.

for all $n \in \mathbb{N}$, $s_1, \ldots, s_n \in S$ and $c_1, \ldots, c_n \in \mathbb{C}$.
A function $\psi : S \to \mathbb{C}$ is called negative definite if it is hermitian, i.e. for any $s, t \in S$ holds $\psi(s + t^*) = \overline{\psi(s^* + t)}$, and

$$\sum_{j,k=1}^{n} c_j \bar{c}_k \psi(s_j^* + s_k) \leq 0 \tag{1.1}$$

for all $n \in \mathbb{N}$, $s_1, \ldots, s_n \in S$ and $c_1, \ldots, c_n \in \mathbb{C}$ with $\sum_{j=1}^{n} c_j = 0$,
or, equivalently, if ψ is hermitian, $\psi(0) \geq 0$ and

$$\sum_{j,k=1}^{n} c_j \bar{c}_k \left(\overline{\psi(s_j)} + \psi(s_k) - \psi(s_j^* + s_k) \right) \geq 0$$

for all $n \in \mathbb{N}$, $s_1, \ldots, s_n \in S$ and $c_1, \ldots, c_n \in \mathbb{C}$.

Positive definite functions naturally appear in probability theory. Bochner's theorem establishes a connection between positive definite functions and bounded Borel measures which include probability measures. We present a version based on Berg et al. [8, Section 4.2] suitable for our needs.

Corollary 1.2. *A function $\phi : \mathbb{C}_-^m \times \mathbf{i}\mathbb{R}^n \to \mathbb{C}$ is continuous bounded positive definite if and only if it is the Fourier-Laplace transform of a bounded Borel measure μ on $\mathbb{R}_+^m \times \mathbb{R}^n$. Furthermore, we have the representation*

$$\begin{aligned} \phi(\xi) &= \int_{\mathbb{R}_+^m \times \mathbb{R}^n} \mathrm{e}^{\sum_{k=1}^{m+n} x_k \xi_k} \mu(\mathrm{d}x) \\ &= \int_{\mathbb{R}_+^m \times \mathbb{R}^n} \mathrm{e}^{x^\top \xi} \mu(\mathrm{d}x), \qquad \xi \in \mathbb{C}_-^m \times \mathbf{i}\mathbb{R}^n. \end{aligned}$$

From a probabilistic point of view, this means that the characteristic function of a probability measure is a continuous positive definite function. The next lemma generalizes a result from Jacob [27, Corollary I.3.6.10]. It shows how to derive a negative definite function from a positive definite one. We present a general version for any semigroup S.

Lemma 1.3. *Let $\phi : S \to \mathbb{C}$ be a positive definite function. Then the function $\psi : S \to \mathbb{C}$ with $\psi(s) = a - \phi(s)$ is negative definite for all $a \in \mathbb{R}$ such that $a \geq \phi(0)$.*

Proof. The proof follows the lines of Jacob [27, Corollary I.3.6.10]. We just have to replace $\phi(0)$ by a. Let us mention that the condition $a \geq \phi(0)$ is essential. Otherwise we would have $\psi(0) = a - \phi(0) < 0$, which contradicts the definition. $\square$

We continue with the Lévy-Khintchine integral representation for a negative definite function, see Berg et al. [8, Theorem 4.3.19].

Theorem 1.4. *A continuous function* $\psi : (\mathbb{C}_-^m \times \mathbf{i}\mathbb{R}^n) \to \mathbb{C}$ *is negative definite with* $\operatorname{Re}\psi$ *bounded below if and only if there exists a triplet* (l, Q, ν) *such that*

$$\psi(\xi) = \psi(0) + l^\top \xi + \frac{1}{2}\xi^\top Q\xi + \int_{(\mathbb{R}_+^m \times \mathbb{R}^n)\setminus\{0\}} \left(1 - \mathrm{e}^{y^\top \xi} + \chi(y)^\top \xi\right) \nu(\,\mathrm{d}y), \tag{1.2}$$

where $l \in \mathbb{R}^{m+n}$, Q *is a symmetric positive semidefinite matrix in* $\mathbb{R}^{(m+n)\times(m+n)}$, $\chi : \mathbb{R}_+^m \times \mathbb{R}^n \to \mathbb{R}^{m+n}$ *is a truncation function componentwise given by*

$$\chi_k(y) = \begin{cases} 0 & k \in \{1, \dots, m\} \\ h(y_k) & k \in \{m+1, \dots, d\}, \end{cases}$$

with $h(x)$ *a bounded measurable function from* $\mathbb{R}$ *to* $\mathbb{R}$ *that behaves like* x *in a neighbourhood of zero, and* ν *is a Lévy measure on* $(\mathbb{R}_+^m \times \mathbb{R}^n)\setminus\{0\}$ *such that* $\int_{(\mathbb{R}_+^m \times \mathbb{R}^n)\setminus\{0\}} (1 - \operatorname{Re} \mathrm{e}^{y^\top \xi})\nu(\,\mathrm{d}y) < \infty$ *for all* $\xi \in \mathbb{C}_-^m \times \mathbf{i}\mathbb{R}^n$.

The integral condition is equivalent to[5] $\int_{(\mathbb{R}_+^m \times \mathbb{R}^n)\setminus\{0\}} \left((|y_I| + |y_{II}|^2) \wedge 1\right) \nu(\,\mathrm{d}y) < \infty$, where $y = (y_I, y_{II})$ with $y_I \in \mathbb{R}_+^m$ and $y_{II} \in \mathbb{R}^n$.

The above parameters are subject to further conditions. In Chapter 3, we specify them in detail by considering the subspaces. In the case of $\mathbf{i}\mathbb{R}^n$ we have the standard Lévy-Khintchine formula where the parameter ξ already contains the imaginary unit $\mathbf{i}$, cf. Jacob [27, Theorem. 3.7.7] or Schilling, Song and Vondraček [53, Theorem 4.15]. For $\mathbb{C}_-^m$, see Berg et al. [8, Theorem. 4.3.20], the drift term is positive, i.e. $l \geq 0$, the quadratic term vanishes, i.e. $Q = 0$, and no truncation function is required, i.e. $\chi \equiv 0$.

The next lemma gives estimates on the growth of continuous negative definite functions.

Lemma 1.5 (Growth of continuous negative definite functions)**.** *Let* $\psi : \mathbb{C}_-^m \times \mathbf{i}\mathbb{R}^n \to \mathbb{C}$ *be a continuous negative definite function with Lévy-Khintchine representation as in Theorem 1.4.*

1. *For all* $\xi \in \mathbb{C}_-^m \times \mathbf{i}\mathbb{R}^n$ *holds* $|\psi(\xi)| \leq c'|\xi|^2$, *where* $c' = \sup_{|\eta|\leq 2} |\psi(\eta)|$.
2. *For all* $\xi \in \mathbb{C}_-^m \times \mathbf{i}\mathbb{R}^n$ *holds* $|\psi(\xi)| \leq c_\psi(1 + |\xi|^2)$, *where* $c_\psi = 2\sup_{|\eta|\leq 1} |\psi(\eta)|$.

A proof of the these statements can be found in Jacob [27, Lemma I.3.6.22].

[5] The fact that the above condition is sufficient follows from applying Taylor's formula twice, cf. proof of Theorem 3.21.

1.2 Markov Processes

Our approch to define Markov processes and their transition functions is adapted from Ethier and Kurtz [19], Bauer [5] and Schilling and Partzsch [51]. In the following, we investigate two examples, a Lévy process and a generalized Ornstein-Uhlenbeck process, cf. Sato [46] and Behme and Lindner [6], respectively. To keep notation simple, we choose $\mathbb{R}^d$ as a state space. However, all subsequent definitions and statements are still valid for more general spaces, $D = \mathbb{R}_+^m \times \mathbb{R}^n$ and $D = S_d^+$, we will use in the succeeding chapters.

Definition 1.6 (Markov process)**.** Let $X = (X_t)_{t\geq 0}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^d$, and set $\mathcal{F}_t^X = \sigma(X_s;\ s \leq t)$. Then X is a Markov process if

$$\mathbb{P}(X_{t+s} \in B|\mathcal{F}_t^X) = \mathbb{P}(X_{t+s} \in B|X_t)$$

for all $s, t \geq 0$ and $B \in \mathcal{B}(\mathbb{R}^d)$.

Observe, that we do not impose any regularity conditions such as stochastic continuity so far. However, there exists a modification such that the process is separable, c.f. Wentzell [65, Section 5.2.9]. Without the assumption of separability, expressions as $\lim_{s\to t} X_s$ are not necessarily measurable or even meaningful. Obviously, stochastic continuity implies separability.

A function $p_t(x, B) : [0, \infty) \times \mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d) \to [0, 1]$ is called a **(time-homogeneous) transition** function if

1. $B \mapsto p_t(x, B)$ is for all $t \geq 0$ and all $x \in \mathbb{R}^d$ a probability measure;
2. $p_0(x, B) = \delta_x(B)$ holds for all x, where δ is the Dirac measure;
3. $(t, x) \mapsto p_t(x, B)$ is for all $B \in \mathcal{B}(\mathbb{R}^d)$ measurable;
4. $p_{t+s}(x, B) = \int_{\mathbb{R}^d} p_s(y, B)p_t(x,\, \mathrm{d}y)$ for all $s, t \geq 0$, $x \in \mathbb{R}^d$ and $B \in \mathcal{B}(\mathbb{R}^d)$.

The last equation is commonly known as Chapman-Kolmogorov equation.
We call a Markov process X time-homogeneous if a time-homogeneous transition function is associated with X, i.e.[6]

$$p_t(x, B) = \mathbb{P}^x(X_t \in B) = \mathbb{P}(X_t \in B|X_0 = x) \qquad \text{for all } t \geq 0,\ x \in \mathbb{R}^d, B \in \mathcal{B}(\mathbb{R}^d).$$

In the following, we always consider time-homogeneous Markov processes and, hence, briefly speak of Markov processes. One important subclass, which we frequently use, are Lévy processes.
A stochastic process $L = (L_t)_{t\geq 0}$ with values in $\mathbb{R}^d$ is called a **Lévy process** if it

0. starts in zero: $L_0 = 0$ a.s.;

[6] As usual, $\mathbb{P}^x$ and $\mathbb{E}^x$ are the probability measures $\mathbb{P}(\cdot|\ X_0 = x)$ and the corresponding expectation, respectively.

1. has independent increments: $L_t - L_s$ is independent of $\mathcal{F}_s^L$ for all $0 \leq s \leq t$;

2. has stationary increments: $L_t - L_s \sim L_{t-s}$ for all $0 \leq s \leq t$;

3. is stochastically continuous.

Well-known examples for Lévy processes are the Poisson process and Brownian motion. In the following, we always assume that a Lévy process L has càdlàg paths[7] since L has a modification $\tilde{L}$ for which $t \mapsto \tilde{L}_t$ is a.s. càdlàg, see Sato [46, Theorem 11.5]. A Lévy process is a Markov process with transition function $p_t(x, B) = p_t(B - x) := \mathbb{P}(L_t \in B - x)$, where $B - x = \{b - x;\ b \in B\}$. Observe that a Lévy process is homogeneous in space. In this work, we are interested in processes which are not homogenous in space. One example is the generalized Ornstein-Uhlenbeck process which can be defined by a Lévy process.

Example 1.7 (Generalized Ornstein-Uhlenbeck process)**.** Let $(\xi_t, \eta_t)_{t\geq 0}$ be a bivariate Lévy process and V_0 be a random variable on the same probability space. Then the process $V = (V_t)_{t\geq 0}$ defined by

$$V_t = \mathrm{e}^{-\xi_t} \left(\int_0^t \mathrm{e}^{\xi_{s-}} \,\mathrm{d}\eta_s + V_0 \right), \quad t \geq 0,$$

is called the generalized Ornstein-Uhlenbeck (GOU) process driven by $(\xi, \eta)^\top$ with starting random variable V_0. The generalized Ornstein-Uhlenbeck process driven by $(\xi, \eta)^\top$ is the unique solution of the stochastic differential equation

$$\mathrm{d}V_t = V_{t-} \,\mathrm{d}U_t + \mathrm{d}L_t, \quad t \geq 0, \tag{1.3}$$

where $(U, L)^\top$ is a bivariate Lévy process given by

$$\begin{pmatrix} U_t \\ L_t \end{pmatrix} = \begin{pmatrix} -\xi_t + \sum_{0<s\leq t} (\mathrm{e}^{-\Delta\xi_s} - 1 + \Delta\xi_s) + t\sigma_\xi^2/2 \\ \eta_t + \sum_{0<s\leq t} (\mathrm{e}^{-\Delta\xi_s} - 1)\Delta\eta_s - t\sigma_{\xi,\eta} \end{pmatrix}, \quad t \geq 0,$$

where σ_ξ^2 and $\sigma_{\xi,\eta}$ are the $(1,1)$ and $(1,2)$ elements of the Gaussian covariance matrix of the bivariate Lévy process $(\xi, \eta)^\top$.

If ξ is deterministic, i.e. $\xi_t = at$, we get the usual Ornstein-Uhlenbeck[-type] process driven by a Lévy process, see also Example 3.2.iv. If we additionally use the Brownian motion $B = (B_t)_{t\geq 0}$ as driving process, i.e. $(\xi_t, \eta_t) = (at, B_t)$, we obtain the classical Ornstein-Uhlenbeck process.

[7] A function is càdlàg if it is right-continuous with finite left-hand limits.

1.3 Semigroups

This section contains a short introduction to the theory of semigroups based on the monographs of Böttcher, Schilling and Wang [11], Jacob [27] and Rogers and Williams [45].

The analysis of semigroups highly depends on function spaces. Therefore, we briefly introduce the main definitions.
The support of a function $u : \mathbb{R}^d \to \mathbb{C}$ is defined by

$$\operatorname{supp}(u) := \overline{\{x \in \mathbb{R}^d;\ u(x) \neq 0\}}$$

and we say the function u vanishes at infinity if

$$\forall \varepsilon > 0\ \exists K \subseteq \mathbb{R}^d \text{ compact } \forall x \in \mathbb{R}^d \backslash K :\ |u(x)| < \varepsilon.$$

Let $\alpha \in \mathbb{N}_0^d$ be a multiindex. Then for $k \in \mathbb{N}_0 \cup \{\infty\}$, we define the spaces

$$\begin{aligned}
B_b(\mathbb{R}^d) &:= \{u : \mathbb{R}^d \to \mathbb{C};\ u \text{ is measurable and bounded}\},\\
C^k(\mathbb{R}^d) &:= \{u : \mathbb{R}^d \to \mathbb{C};\ u \text{ is } k\text{-times continuously differentiable}\},\\
C_c^k(\mathbb{R}^d) &:= \{u \in C^k(\mathbb{R}^d);\ \operatorname{supp}(u) \text{ is compact}\},\\
C_\infty(\mathbb{R}^d) &:= \{u \in C(\mathbb{R}^d);\ u \text{ vanishes at infinity}\},\\
C_\infty^k(\mathbb{R}^d) &:= \{u \in C^k(\mathbb{R}^d);\ \partial^\alpha u \in C_\infty(\mathbb{R}^d) \text{ for } |\alpha| \leq k\}.
\end{aligned}$$

Observe that these function spaces are also well defined if we replace $\mathbb{R}^d$ by an open subset $G \subseteq \mathbb{R}^d$. The space $C_\infty^k(\mathbb{R}^d)$ equipped with the norm

$$\|u\|_{(k)} := \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_\infty,$$

where

$$\|u\|_\infty := \sup_{x \in \mathbb{R}^d} |u(x)|,$$

is a Banach space. Elements of the test functions $C_c^\infty(\mathbb{R}^d)$ are smooth functions with compact support. It is well known that the test functions are dense in $C_\infty^k(\mathbb{R}^d)$ and $C_\infty(\mathbb{R}^d)$ with respect to the norm $\|\cdot\|_{(k)}$ and $\|\cdot\|_\infty$, respectively.

For a Markov process X, we get by

$$T_t u(x) := \mathbb{E}^x(u(X_t)) = \int_{\mathbb{R}^d} u(y) p_t(x, \mathrm{d}y) \quad \text{for } t \geq 0, u \in B_b(\mathbb{R}^d)$$

a family of linear operators $T = (T_t)_{t \geq 0}$ on the bounded Borel measurable functions $B_b(\mathbb{R}^d)$.[8] By the Markov property or, equivalently, by the Chapman-Kolmogorov equation we see that T is a **(one-parameter operator) semigroup**, i.e.

$$T_0 = \mathrm{id} \quad \text{and} \quad T_t T_s u = T_s T_t u = T_{t+s} u \quad \text{for all } u \in B_b(\mathbb{R}^d),\ s, t \geq 0.$$

Furthermore, it holds that a semigroup generated by a Markov process

[8] All subsequent definitions and statements are still valid for more general spaces, $D = \mathbb{R}_+^m \times \mathbb{R}^n$ and $D = S_d^+$.

1. is positivity preserving: $T_t u \geq 0$ for $u \in B_b(\mathbb{R}^d)$, $u \geq 0$;
2. has the sub-Markov property: $T_t u \leq 1$ for $u \in B_b(\mathbb{R}^d)$, $u \leq 1$.

Hence, we call such an operator semigroup a **sub-Markov semigroup**.
In the case of a Lévy process we have a nice representation of the semigroup. Using the homogeneity in space we get for $u \in C_\infty(\mathbb{R}^d)$

$$T_t u(x) = \int_{\mathbb{R}^d} u(y+x) p_t(\,\mathrm{d}y) = \tilde{p}_t * u(x),$$

where $\tilde{p}_t(B) = p_t(-B)$ for all $B \in \mathcal{B}(\mathbb{R}^d)$ and $t \geq 0$. In other words, we can write the Lévy semigroup as a convolution. Furthermore, we deduce from the semigroup property that $p_{t+s} = p_t * p_s$ and, hence, the semigroup of a Lévy process is a convolution semigroup.

Many semigroups generated by Markov processes satisfy further properties. We investigate several attributes in the next chapters and introduce them as required. We now focus on a very important subclass, Feller semigroups. In the literature, several definitions of Feller semigroups exist. We define a Feller semigroup and, hence, a Feller process according to Böttcher, Schilling and Wang [11, Definition 1.2].

Definition 1.8 (Feller semigroup and process)**.** A Feller semigroup is a sub-Markov semigroup $(T_t)_{t\geq 0}$, which satisfies the Feller property

$$T_t u \in C_\infty(\mathbb{R}^d) \quad \forall u \in C_\infty(\mathbb{R}^d),\ t > 0 \tag{1.4}$$

and which is strongly continuous

$$\lim_{t\to 0} \|T_t u - u\|_\infty = 0 \quad \forall u \in C_\infty(\mathbb{R}^d). \tag{1.5}$$

A Feller process is a time-homogeneous Markov process whose transition group $T_t u(x) = \mathbb{E}^x u(X_t)$ is a Feller semigroup.

Other definitions of a Feller semigroup distinguish themselves by the function space on which the semigroup is defined.

Definition 1.9 (C_b-Feller semigroup)**.** A sub-Markov semigroup $(T_t)_{t\geq 0}$ is called a C_b-Feller semigroup if it enjoys the C_b-Feller property, i.e.

$$T_t u \in C_b(\mathbb{R}^d) \quad \forall u \in C_b(\mathbb{R}^d),\ t > 0$$

and if $t \mapsto T_t u$ is continuous in the topology of locally uniform convergence in the space $C_b(\mathbb{R}^d)$.

Finally, we introduce the notion of a strong Feller semigroup.

Definition 1.10 (Strong Feller semigroup)**.** A sub-Markov semigroup $(T_t)_{t\geq 0}$ is called a strong Feller semigroup if $T_t : B_b(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$ for all $t > 0$.

For all these "Feller" semigroups exist examples which satisfy one but none of the other definitions, cf. Böttcher et al. [11, Section 1.1]. The semigroup $T = (T_t)_{t\geq 0}$ of a Lévy process is a Feller semigroup. Indeed, for a function $u \in C_\infty(\mathbb{R}^d)$ we have by homogeneity in space

$$T_t u(x) = \mathbb{E}^x\left(u(X_t)\right) = \mathbb{E}^0\left(u(X_t + x)\right).$$

An application of the dominated convergence theorem shows that $t \mapsto T_t u(x)$ is continuous and that $T_t u$ vanishes as $|x|$ tends to infinity. Since a Lévy process is stochastically continuous, the corresponding semigroup satisfies the strong continuity.
The semigroup of a generalized Ornstein-Uhlenbeck process is also a Feller semigroup. However, it is more involved to show this and we thus refer to Behme and Lindner [6, Theorem 3.1].

1.4 Generators and Pseudo-Differential Operators

This introduction to the theory of generators and pseudo-differential operators is adapted from Böttcher, Schilling and Wang [11], Jacob [27, 28] and Ethier and Kurtz [19]. The references [27, 28] are the standard work on pseudo-differential operators in the field of stochastic processes. However, this section is primarily adapted from [11].

As in the previous sections, we simplify the notation by considering the space $\mathbb{R}^d$ instead of more general spaces, e.g. $D = \mathbb{R}_+^m \times \mathbb{R}^n$ and $D = S_d^+$. In the following, we require several tools from Fourier analysis. Since there are several conventions for defining the Fourier transform, we introduce our notation. The **Fourier transform** of a (real-valued or even complex-valued) function $u \in L^1(\mathbb{R}^d, \mathrm{d}x)$ is defined as

$$\mathcal{F}(u)(\xi) = \hat{u}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathbf{i}x^\top \xi} u(x)\,\mathrm{d}x,$$

and the **inverse Fourier transform** is

$$\mathcal{F}^{-1}(u)(x) = \check{u}(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{\mathbf{i}\xi^\top x} u(\xi)\,\mathrm{d}\xi.$$

The Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{-1}$ are inverse operations on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of rapidly deacreasing C^∞-functions. By **Plancherel's identity**

$$\int_{\mathbb{R}^d} u(x)\hat{v}(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \hat{u}(x)v(x)\,\mathrm{d}x,$$

the Fourier transform has an extension to $L^2(\mathbb{R}^d)$.

The next definition establishes a connection between stochastic processes and operator theory.

Definition 1.11 (Generator)**.** Let $(T_t)_{t\geq 0}$ be a Feller semigroup on $C_\infty(\mathbb{R}^d)$ or $(X_t)_{t\geq 0}$ a Feller process on $\mathbb{R}^d$. Then the linear operator $(A, \mathcal{D}(A))$ defined by

$$Au := \lim_{t\to 0} \frac{T_t u - u}{t} \quad \Big(\text{the limit is taken in } (C_\infty(\mathbb{R}^d), \|\cdot\|_\infty)\Big), \tag{1.6}$$

$$\mathcal{D}(A) := \left\{ u \in C_\infty(\mathbb{R}^d);\ \exists g \in C_\infty(\mathbb{R}^d) :\ \lim_{t\to 0} \left\| \frac{T_t u - u}{t} - g \right\|_\infty = 0 \right\} \tag{1.7}$$

is the (infinitesimal) generator of the semigroup $(T_t)_{t\geq 0}$ or of the process $(X_t)_{t\geq 0}$.

In general it is not possible to determine the domain of the generator explicitly. However, for many applications it is sufficient to know the generator on a suitable dense subset.

Definition 1.12 (Operator core)**.** Let $(A, \mathcal{D}(A))$ be a densely defined, closed linear operator and $D \subseteq \mathcal{D}(A)$ be a dense subset. If D determines A in the sense that the closure of (A, D) is $(A, \mathcal{D}(A))$, then D is called an (operator) core. In other words, D is an operator core if

$$\forall u \in \mathcal{D}(A)\ \exists (u_n)_{n\geq 1} \subseteq D :\quad \lim_{n\to\infty} \big(\|u - u_n\|_\infty + \|Au - Au_n\|_\infty\big) = 0.$$

The test functions $C_c^\infty(\mathbb{R}^d)$ are a core of the generator of a Lévy process, see Sato [46, Theorem 31.5]. However, it is often difficult or even impossible to determine a core.

In the case of Lévy processes we have a representation of the generator for $u \in C_\infty^2(\mathbb{R}^d)$ given by

$$Au(x) = l^\top \nabla u(x) + \frac{1}{2}\mathrm{div} Q \nabla u(x) + \int_{\mathbb{R}^d\setminus\{0\}} \Big(u(x+y) - u(x) - y^\top \nabla u(x) \mathbb{1}_{\{|y|\leq 1\}}(y) \Big) \nu(\,\mathrm{d}x),$$

where $l \in \mathbb{R}^d$, $Q \in \mathbb{R}^{d\times d}$ is symmetric positive semidefinite and ν is a positive measure on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d\setminus\{0\}} (|y|^2 \wedge 1) \nu(\,\mathrm{d}x) < \infty$. Here we used $y\mathbb{1}_{\{|y|\leq 1\}}(y)$ as a truncation function for the integral term. However, it is possible to replace $y\mathbb{1}_{\{|y|\leq 1\}}(y)$ by a different truncation function $\chi : \mathbb{R}^d \to \mathbb{R}^d$, which is a bounded measurable function and behaves like y in a neighbourhood of zero.
For the generator of a generalized Ornstein-Uhlenbeck process we can give a similar representation. For the proof we refer to Behme et al. [6, Theorem 3.1].

Example 1.13 (Generator of a generalized Ornstein-Uhlenbeck process)**.** As in Example 1.7, let $(Z_t)_{t\geq 0} = \big((U_t, L_t)^\top\big)_{t\geq 0}$ be a bivariate Lévy process with characteristic triplet (l_Z, Q_Z, ν_Z), where $l_Z = (l_U, l_L)^\top$, $Q_Z = \begin{pmatrix} \sigma_U^2 & \sigma_{U,L} \\ \sigma_{U,L} & \sigma_L^2 \end{pmatrix}$ and $\nu_Z((\,\mathrm{d}z_1, \,\mathrm{d}z_2)^\top)$ such that $\nu_Z((-1, \,\mathrm{d}z_2)^\top) = 0$. Then the process $(V_t^x)_{t\geq 0}$ given by

$$V_t^x = x + \int_{(0,t]} V_{s-}^x \,\mathrm{d}U_s + L_t, \quad t \geq 0$$

is a Feller process. For any $u \in C_c^\infty(\mathbb{R})$, the generator can be written as

$$\begin{aligned} A^V u(x) &= (xl_U + l_L)u'(x) + \frac{1}{2}(x^2\sigma_U^2 + 2x\sigma_{U,L} + \sigma_L^2)u''(x) \\ &\quad + \int_{\mathbb{R}^2\setminus\{0\}} \big(u(x + xz_1 + z_2) - u(x) - (xz_1 + z_2)u'(x)\mathbb{1}_{\{|z|\le 1\}}\big)\, \nu_Z(\,\mathrm{d}z_1,\, \mathrm{d}z_2). \end{aligned} \tag{1.8}$$

The similarity of the structure of the last two examples is no coincidence as the next theorem shows.

Theorem 1.14 (Courrège; v. Waldenfels). *Let $(A, \mathcal{D}(A))$ be an infinitesimal generator of a Feller semigroup. If $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A)$, then A is of the following form*

$$\begin{aligned} Au(x) &= -c(x)u(x) + l(x)^\top \nabla u(x) + \frac{1}{2}\mathrm{div} Q(x)\nabla u(x) \\ &\quad + \int_{\mathbb{R}^d\setminus\{0\}} \Big(u(x+y) - u(x) - \chi(y)^\top \nabla u(x)\Big)\, N(x,\, \mathrm{d}y), \quad u \in C_c^\infty(\mathbb{R}^d), \end{aligned}$$

where $c(x) \ge 0$, $(l(x), Q(x), N(x,\cdot))$ is, for fixed $x \in \mathbb{R}^d$, a Lévy triplet, i.e. $l(x) \in \mathbb{R}^d$, $Q(x) \in \mathbb{R}^{d\times d}$ is symmetric positive semidefinite and $N(x,\cdot)$ is a positive Radon measure on $\mathbb{R}^d\setminus\{0\}$ satisfying $\int_{\mathbb{R}^d\setminus\{0\}}(|y|^2 \wedge 1)N(x,\, \mathrm{d}y) < \infty$, and $\chi : \mathbb{R}^d \to \mathbb{R}^d$ is a bounded measurable function such that $\chi(y)$ behaves like y in a neighbourhood of 0.

This result by Courrège [14] and von Waldenfels [60, 62, 61] gives a representation of the generator as an integro-differential operator if the test functions $C_c^\infty(\mathbb{R}^d)$ are contained in the domain. In the above representation of the operator A, we call the differential operator $-c(x)u(x) + l(x)^\top \nabla u(x) + \frac{1}{2}\mathrm{div}Q(x)\nabla u(x)$ the local part[9] of A and refer to the integral $\int_{\mathbb{R}^d\setminus\{0\}} \big(u(x+y) - u(x) - \chi(y)^\top \nabla u(x)\big)\, N(x,\, \mathrm{d}y)$ as the non-local part.

Using Fourier inversion, the previous theorem links Feller processes and their generators to pseudo-differential operators.

Definition 1.15 (Pseudo-differential operator, symbol)**.** Let $q : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be a function which is, for every $x \in \mathbb{R}^d$, continuous and negative definite. Then

$$-q(x, D)u(x) = -(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x^\top\xi} q(x,\xi)\hat{u}(\xi)\,\mathrm{d}\xi, \quad u \in C_c^\infty(\mathbb{R}^d) \tag{1.9}$$

is a pseudo-differential operator (with negative definite symbol) and $q(x,\xi)$ is called the symbol of the operator.

This allows to reformulate Theorem 1.14.

[9] An operator L is called local, if $\mathrm{supp}(Lu) \subseteq \mathrm{supp}(u)$.

Corollary 1.16. *Let $(A, \mathcal{D}(A))$ be an infinitesimal generator of a Feller semigroup. If $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A)$, then A is a pseudo-differential operator with symbol $q(x, \xi)$,*

$$Au(x) := -q(x, D)u(x) := -(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \mathrm{e}^{\mathbf{i}x^\top \xi} q(x, \xi) \hat{u}(\xi) \, \mathrm{d}\xi, \quad u \in C_c^\infty(\mathbb{R}^d),$$

where for every $x \in \mathbb{R}^d$ the function $q(x, \cdot)$ is a continuous negative definite function with a Lévy-Khintchine representation with an x-dependent triplet $(l(x), Q(x), N(x, \mathrm{d}y))$ relative to a truncation function χ, i.e.

$$\begin{aligned} q(x, \xi) = q(x, 0) - \mathbf{i}l(x)^\top \xi + \frac{1}{2} \xi^\top Q(x) \xi \\ + \int_{\mathbb{R}^d \setminus \{0\}} \left(1 - \mathrm{e}^{\mathbf{i}y^\top \xi} + \mathbf{i}\chi(y)^\top \xi\right) N(x, \mathrm{d}y). \end{aligned}$$

We now rewrite the generator of a Lévy process as

$$Au(x) = -(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \mathrm{e}^{\mathbf{i}x^\top \xi} q(x, \xi) \hat{u}(\xi) \, \mathrm{d}\xi, \quad u \in C_c^\infty(\mathbb{R}^d),$$

where

$$q(x, \xi) = \psi_L(\xi) = -\mathbf{i}l^\top \xi + \frac{1}{2} \xi^\top Q \xi + \int_{\mathbb{R}^d} \left(1 - \mathrm{e}^{\mathbf{i}y^\top \xi} + \mathbf{i}y^\top \xi \mathbb{1}_{\{|y| \le 1\}}(y)\right) \nu(\,\mathrm{d}y).$$

Obviously, the symbol $q(x, \xi)$ is independent of x. Furthermore, it coincides with the characteristic exponent $\psi_L : \mathbb{R}^d \to \mathbb{C}$ of the Lévy process X given by

$$\mathbb{E}\mathrm{e}^{\mathbf{i}\xi^\top X_t} = \mathrm{e}^{-t\psi_L(\xi)}.$$

In general, the coefficients of the symbol of a generator are x-dependent. Since the symbol has the same structure, i.e. an x-dependent Lévy-Khintchine representation, we use a similar notation to the Lévy case and call

$(l(x), Q(x), N(x, \mathrm{d}y)$	the x-dependent Lévy triplet
$l(x)$	the drift coefficient
$Q(x)$	the covariance matrix
$N(x, \mathrm{d}y)$	the jump or Lévy measure.

As an x-dependent continuous negative definite function, the symbol is subject to several properties. The next proposition shows a selection from Böttcher et al. [11, Proposition 2.27] together with a method to calculate the symbol from the generator.

Proposition 1.17. *Let $(A, \mathcal{D}(A))$ be an infinitesimal generator of a Feller semigroup satisfying $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A)$ and denote by $q(x, \xi)$ its symbol. Then*

1. *$q(x, 0)$ is locally bounded;*
2. *$-q(x, \xi) = e_{-\xi}(x) A e_\xi(x)$ where $e_\xi(x) := \mathrm{e}^{\mathbf{i}x^\top \xi}$;*

3. $|q(x,\xi)| \leq \gamma(x)(1+|\xi|^2)$ *for some locally bounded function* $\gamma : \mathbb{R}^d \to [0,\infty)$.

The last item states that the symbol is locally bounded in x. However, many applications based on the symbol require the stronger condition of global boundedness, i.e.

$$\sup_{x\in\mathbb{R}^d} |q(x,\xi)| \leq c(1+|\xi|^2) \qquad \text{for all } \xi \in \mathbb{R}^d,$$

where $c \geq 0$ is some suitable constant. Equivalent to this inequality we can demand that the coefficients $(l(x), Q(x), N(x, \mathrm{d}y))$ of the symbol satisfy

$$\sup_{x\in\mathbb{R}^d} |l(x)| + \sup_{x\in\mathbb{R}^d} |Q(x)| + \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d\setminus 0} (1\wedge |y|^2) N(x, \mathrm{d}y) < \infty.$$

If this holds we say that the symbol has **bounded coefficients**. In the same manner we speak of a symbol with unbounded coefficients if this conditions fails, i.e.

$$\sup_{x\in\mathbb{R}^d} |l(x)| + \sup_{x\in\mathbb{R}^d} |Q(x)| + \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d\setminus 0} (1\wedge |y|^2) N(x, \mathrm{d}y) = \infty.$$

Since the symbol of a generator of a Lévy process is independent of x, it has bounded coefficients. An example for a symbol with unbounded coefficients is the generator of a generalized Ornstein-Uhlenbeck process.

Example 1.18 (Symbol of a generalized Ornstein-Uhlenbeck process)**.** By Proposition 1.17, we can compute the symbol of the generator corresponding to the generalized Ornstein-Uhlenbeck process defined as in Example 1.7. Set $e_\xi(x) := \mathrm{e}^{\mathbf{i}x\xi}$. Then equation (1.8) yields

$$\begin{aligned}
q(x,\xi) &= -e_{-\xi}(x) A e_\xi(x) \\
&= -e_{-\xi}(x) \Bigg[(xl_U + l_L)\mathbf{i}\xi e_\xi(x) + \frac{1}{2}(x^2\sigma_U^2 + 2x\sigma_{U,L} + \sigma_L^2)(\mathbf{i}\xi)^2 e_\xi(x) \\
&\quad + \int_{\mathbb{R}^2\setminus\{0\}} \Big(e_\xi(x + xz_1 + z_2) - e_\xi(x) - (xz_1 + z_2)\mathbf{i}\xi e_\xi(x) \mathbb{1}_{\{|z|\leq 1\}} \Big) \nu_{U,L}(\mathrm{d}z_1, \mathrm{d}z_2) \Bigg] \\
&= -\mathbf{i}(xl_U + l_L)\xi + \frac{1}{2}(x^2\sigma_U^2 + 2x\sigma_{U,L} + \sigma_L^2)\xi^2 \\
&\quad + \int_{\mathbb{R}^2\setminus\{0\}} \Big(1 - \mathrm{e}^{\mathbf{i}\xi(xz_1+z_2)} + \mathbf{i}(xz_1 + z_2)\xi \mathbb{1}_{\{|z|\leq 1\}} \Big) \nu_{U,L}(\mathrm{d}z_1, \mathrm{d}z_2),
\end{aligned}$$

where $l_U, l_L \in \mathbb{R}$, $Q_Z = \begin{pmatrix} \sigma_U^2 & \sigma_{U,L} \\ \sigma_{U,L} & \sigma_L^2 \end{pmatrix}$ is a symmetric positive semidefinite matrix and $\nu_{U,L}((\mathrm{d}z_1, \mathrm{d}z_2)^\top)$ with $\nu_{U,L}((-1, \mathrm{d}z_2)^\top) = 0$ such that $\int (1 \wedge |z|^2)\nu_{U,L}(\mathrm{d}z) < \infty$ stem from the bivariate driving Lévy process $Z = (U, L)^\top$.

If the driving processes U and L are independent we have

$$q(x,\xi) = \psi_L(\xi) - \mathbf{i}xl_U\xi + \frac{1}{2}x^2\sigma_U^2\xi^2 + \int_{\mathbb{R}\setminus\{0\}} \Big(1 - \mathrm{e}^{\mathbf{i}\xi xy} + \mathbf{i}xy\xi \mathbb{1}_{\{|y|\leq 1\}} \Big) \nu_U(\mathrm{d}y).$$

Chapter 2

General Results for Generators with Unbounded Coefficients

There exist several applications based on the symbol of the generator. In most cases, however, it is required that the symbol has bounded coefficients, i.e. $\sup_{x\in\mathbb{R}^d}|q(x,\xi)| \leq c(1+|\xi|^2)$ for all $\xi \in \mathbb{R}^d$. This chapter covers relevant known and new results concerning path properties for unbounded coefficients. We close this chapter with an excursion to study the domain of the generator of a Feller process. A convergence result for a path approximation of Feller processes having a symbol with unbounded coefficients is presented later in Chapter 6.

2.1 Probability Estimates

In this section, we introduce two important probability estimates for Feller processes. First, we state the upper maximal inequality. In the next step, we present an extension for an upper bound of the tail probability. To this for the latter a global sector condition for the symbol is required in the literature, see inequality (2.2) below. We will show, however, that a local version is sufficient. In the end of this section, we apply the estimates and prove a law of iterated logarithm for a generalized Ornstein-Uhlenbeck process.
For $x \in \mathbb{R}^d$ and $r > 0$ we define the first exit time of a process $X = (X_t)_{t\geq 0}$ from the closed ball $\overline{B}(x,r)$ by

$$\tau_r^x := \tau_{\overline{B}(x,r)} := \inf\{t > 0;\ X_t \in \overline{B}^c(x,r)\}.$$

With this notation we state the first estimate.

Theorem 2.1 (Upper maximal inequality)**.** *Let X be a d-dimensional Feller process with generator $(A,\mathcal{D}(A))$, symbol $q(x,\xi)$ and $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A)$. Then we have for all x and every $r,t > 0$*

$$\mathbb{P}^x\Big(\tau_r^x \leq t\Big) \leq ct \sup_{|y-x|\leq r}\ \sup_{|\xi|\leq 1/r} |q(y,\xi)|$$

and

$$\mathbb{P}^x\Big(\sup_{s\le t}|X_s-x|>r\Big)\le ct\sup_{|y-x|\le r}\sup_{|\xi|\le 1/r}|q(y,\xi)|$$

holds with an absolute constant $c>0$.

For a proof we refer to Böttcher, Schilling and Wang [11, Theorem 5.1, Corollary 5.2]. The second key inequality presents an upper bound for the tail probability of the first exit time from a ball.

Theorem 2.2. *Let X be a d-dimensional Feller process with generator $(A,\mathcal{D}(A))$, symbol $q(x,\xi)$ and $C_c^\infty(\mathbb{R}^d)\subseteq\mathcal{D}(A)$. If the symbol satisfies a local sector condition for the ball $B(x,r)$, i.e.*

$$\big|\operatorname{Im} q(y,\xi)\big|\le c_{x,r}\operatorname{Re} q(y,\xi)\qquad \forall y\in B(x,r),\xi\in\mathbb{R}^d,$$

then we have for x and $r>0$ and every $t>0$

$$\mathbb{P}^x(\tau_r^x\ge t)\le c\left(t\sup_{|\xi|\le 1/(k_0r)}\inf_{|y-x|\le r}\operatorname{Re} q(y,\xi)\right)^{-1}$$

and

$$\mathbb{P}^x\Big(\sup_{s\le t}|X_s-x|\le r\Big)\le c\left(t\sup_{|\xi|\le 1/(k_0r)}\inf_{|y-x|\le r}\operatorname{Re} q(y,\xi)\right)^{-1}$$

with $k_0:=\Big(\arccos\sqrt{2/3}\Big)^{-1}\vee 2c_{x,r}$ and $c=4/\cos\sqrt{2/3}$.

Proof. This statement follows from a generalization of the proof of Böttcher et al. [11, Theorem 5.5]. Here, we focus on the modifications and give only the main steps of the remaining part.
Let $\varepsilon\in\mathbb{R}^d$ such that $|\varepsilon|\le\frac{1}{k_0}$, then

$$\begin{aligned}\mathbb{P}^x(\tau_r^x>t)&=\mathbb{P}^x(|X_t-x|\le r,\ \tau_r^x>t)\\&\le\mathbb{P}^x\left(\cos\frac{(X_{t\wedge\tau_r^x}-x)^\top\varepsilon}{r}\ge\cos\frac{1}{k_0}\right),\end{aligned}$$

because $\frac{(X_{t\wedge\tau_r^x}-x)^\top\varepsilon}{r}\le\frac{\pi}{4}$ and $x\mapsto\cos x$ is decreasing on $[0,\frac{\pi}{4}]$.
The function $\cos\frac{(\cdot-x)^\top\varepsilon}{r}$ is in the domain of the extended generator[1]. Especially, we have

$$\begin{aligned}A\left(\cos\frac{(\cdot-x)^\top\varepsilon}{r}\right)(z)&=A\left(\operatorname{Re}\left(\exp\frac{\mathbf{i}(\cdot-x)^\top\varepsilon}{r}\right)\right)(z)\\&=-\operatorname{Re}\left(\exp\frac{\mathbf{i}(z-x)^\top\varepsilon}{r}q(z,\varepsilon/r)\right).\end{aligned}$$

[1]The domain of the extended generator of a Feller process $(X_t)_{t\ge0}$ is given by $\hat{\mathcal{D}}(A):=\{f\in B(\mathbb{R}^d);\ \exists! g\in B(\mathbb{R}^d):\ (f,g)\in\hat{A}\}$, where $\hat{A}:=\{(f,g)\in B(\mathbb{R}^d)\times B(\mathbb{R}^d);\ (f(X_t)-f(X_0)-\int_0^t g(X_s)\,\mathrm{d}s,\mathcal{F}_t^X)_{t\ge0}$ is a local martingale$\}$, cf. Böttcher et al. [11, page 25].

If we combine these estimates, we get

$$\begin{aligned}
\mathbb{P}^x(\tau_r^x > t) &\le \frac{1}{\cos\frac{1}{k_0}} \mathbb{E}^x \left(\cos \frac{(X_{t\wedge\tau_r^x} - x)^\top \varepsilon}{r} \right) \\
&= \frac{1}{\cos\frac{1}{k_0}} \Bigg(1 - \mathbb{E}^x \int_0^{t\wedge\tau_r^x} \cos \frac{(X_s - x)^\top \varepsilon}{r} \operatorname{Re} q(X_s, \varepsilon/r) \\
&\qquad\qquad + \sin \frac{(X_s - x)^\top \varepsilon}{r} \operatorname{Im} q(X_s, \varepsilon/r) \,\mathrm{d}s \Bigg) \\
&\le \frac{1}{\cos\frac{1}{k_0}} \Bigg(1 - \mathbb{E}^x \int_0^{t\wedge\tau_r^x} \cos \frac{(X_s - x)^\top \varepsilon}{r} \\
&\qquad\qquad \cdot \left(\operatorname{Re} q(X_s, \varepsilon/r) - \frac{3|\varepsilon|}{2} |\operatorname{Im} q(X_s, \varepsilon/r)| \right) \mathrm{d}s \Bigg).
\end{aligned}$$

Before using the sector condition we restrict the relevant domain. Therefore, we change X_s to X_{s-} to avoid a jump at time τ_r^x. This is possible since we are integrating with respect to Lebesgue measure and a càdlàg process $(X_t)_{t\ge 0}$ has at most countably many jumps on $[0,t]$, i.e. $\{s \in [0,t];\ X_s \neq X_{s-}\}$ is almost surely a set of Lebesgue measure zero. In order to apply the estimate $\frac{(X_s - x)^\top \varepsilon}{r} \le |\varepsilon| \le \frac{1}{k_0}$ on $\{\tau_r^x < t\}$, we note that the integral below should be read $\int_0^\sigma = \int_{[0,\sigma)}$. Hence, we obtain

$$\begin{aligned}
\mathbb{P}^x(\tau_r^x > t) &\le \frac{1}{\cos\frac{1}{k_0}} \Bigg(1 - \mathbb{E}^x \int_0^{t\wedge\tau_r^x} \cos \frac{(X_s - x)^\top \varepsilon}{r} \\
&\qquad\qquad \cdot \left(\operatorname{Re} q(X_s, \varepsilon/r) - \frac{3|\varepsilon|}{2} |\operatorname{Im} q(X_s, \varepsilon/r)| \right) \mathrm{d}s \Bigg) \\
&\le \frac{1}{\cos\frac{1}{k_0}} \Bigg(1 - \cos\frac{1}{k_0} \mathbb{E}^x \int_0^{t\wedge\tau_r^x} \operatorname{Re} q(y, \varepsilon/r) - \frac{3|\varepsilon|}{2} |\operatorname{Im} q(y, \varepsilon/r)| \Big|_{y=X_s} \mathrm{d}s \Bigg) \\
&= \frac{1}{\cos\frac{1}{k_0}} - \mathbb{E}^x \Bigg(\int_0^{t\wedge\tau_r^x} \operatorname{Re} q(y, \varepsilon/r) - \frac{3|\varepsilon|}{2} |\operatorname{Im} q(y, \varepsilon/r)| \Big|_{y=X_{s-}} \mathrm{d}s \Bigg) \\
&= \frac{1}{\cos\frac{1}{k_0}} - \mathbb{E}^x \Bigg(\int_0^{t\wedge\tau_r^x} \Big(\operatorname{Re} q(y, \varepsilon/r) \\
&\qquad\qquad - \frac{3|\varepsilon|}{2} \mathbb{1}_{\{|y-x|<r\}} |\operatorname{Im} q(y, \varepsilon/r)| \Big) \Big|_{y=X_{s-}} \mathrm{d}s \Bigg).
\end{aligned}$$

Now the local sector condition for the ball $B(x,r)$ yields

$$\mathbb{1}_{\{|y-x|<r\}} \frac{\operatorname{Re} q(y,\xi)}{|\xi| |\operatorname{Im} q(y,\xi)|} \ge \frac{1}{|\xi| c_{x,r}}.$$

The right-hand side is greater than $2r$ if $|\xi| \leq \frac{1}{2rc_{x,r}}$. Since $|\varepsilon| \leq \frac{1}{k_0} \leq \frac{1}{2c_{x,r}}$, we get

$$\begin{aligned}
&\mathbb{P}^x(\tau_r^x > t) \\
&\leq \frac{1}{\cos\frac{1}{k_0}} - \mathbb{E}^x\left(\int_0^{t\wedge\tau_r^x} \left(\operatorname{Re} q(y,\varepsilon/r) - \frac{3|\varepsilon|}{2} c_{x,r} \operatorname{Re} q(y,\varepsilon/r)\right)\Big|_{y=X_{s-}} \mathrm{d}s\right) \\
&\leq \frac{1}{\cos\frac{1}{k_0}} - \frac{1}{4}\mathbb{E}^x\left(\int_0^{t\wedge\tau_r^x} \operatorname{Re} q(y,\varepsilon/r)\Big|_{y=X_{s-}} \mathrm{d}s\right) \\
&\leq \frac{1}{\cos\frac{1}{k_0}} - \frac{1}{4}\Big(\inf_{|y-x|\leq r} \operatorname{Re} q(y,\varepsilon/r)\Big)\mathbb{E}^x(t\wedge\tau_r^x) \\
&\leq \frac{1}{\cos\frac{1}{k_0}} - \frac{t}{4}\Big(\inf_{y-x|\leq r} \operatorname{Re} q(y,\varepsilon/r)\Big)\mathbb{P}^x(\tau_r^x > t).
\end{aligned}$$

Solving for $\mathbb{P}^x(\tau_r^x > t)$ we obtain

$$\begin{aligned}
\mathbb{P}^x(\tau_r^x > t) &\leq \frac{4}{\cos\frac{1}{k_0}}\Big(4 + t\inf_{|y-x|\leq r} \operatorname{Re} q(y,\varepsilon/r)\Big)^{-1} \\
&\leq \frac{4}{\cos\sqrt{2/3}}\Big(t\inf_{|y-x|\leq r} \operatorname{Re} q(y,\varepsilon/r)\Big)^{-1}.
\end{aligned}$$

Eliminating ε by taking the infimum with respect to $|\varepsilon| \leq \frac{1}{k_0}$ leads to

$$\mathbb{P}^x(\tau_r^x > t) \leq \frac{4}{\cos\sqrt{2/3}}\Big(t\sup_{|\xi|\leq 1/(k_0 r)}\inf_{|y-x|\leq r} \operatorname{Re} q(y,\xi)\Big)^{-1}.$$

The continuity of measures shows that the above inequality also holds for $\{\tau_r^x \geq t\} = \bigcup_{n\geq 1}\{\tau_r^x > t - \frac{1}{n}\}$. □

A first application of the above inequalities is a law of iterated logarithm type result for a generalized Ornstein-Uhlenbeck process. Similar to Knopova and Schilling [34, Proposition 9] we prove an upper bound using an adaption of Khintchine's criterion.

Example 2.3. Let V be a generalized Ornstein-Uhlenbeck process as defined in Example 1.7 which has the symbol

$$\begin{aligned}
q(x,\xi) &= -\mathbf{i}(xl_U + l_L)\xi + \frac{1}{2}(x^2\sigma_U^2 + 2x\sigma_{U,L} + \sigma_L^2)\xi^2 \\
&\quad + \int_{\mathbb{R}^2\setminus\{0\}} \Big(1 - \mathrm{e}^{\mathbf{i}(xz_1+z_2)\xi} + \mathbf{i}(xz_1+z_2)\xi\mathbb{1}_{\{|z|\leq 1\}}\Big)\nu_{U,L}(\mathrm{d}z_1, \mathrm{d}z_2)
\end{aligned}$$

as computed in Example 1.18. Then we have for every $x \neq 0$

$$\lim_{t\to 0}\frac{\sup_{0\leq s\leq t}|V_s - x|}{\sqrt{t|\log(t)|^{1+\epsilon}}} = 0 \quad \mathbb{P}^x - a.s.,$$

where $\epsilon > 0$.

Proof. To shorten notation we write $(V_\cdot - x)_t^* := \sup_{s\le t} |V_s - x|$ and $v(t) = \sqrt{t|\log(t)|^{1+\epsilon}}$. The process V satisfies the upper maximal inequality, cf. Theorem 2.1. Let $h \ll 1$ and set $t_k := \frac{h}{2^k}$. Then for $\theta_k \in [t_k, t_{k+1})$ we get

$$\begin{aligned}\mathbb{P}^x\Big((V_\cdot - x)^*_{\theta_k} > v(\theta_k)\Big) &\le \mathbb{P}^x\left((V_\cdot - x)^*_{\theta_k} > v(t_{k+1})\right)\\ &\le c\theta_k \sup_{|y-x|\le v(t_{k+1})} \sup_{|\xi|\le 1/v(t_{k+1})} |q(y,\xi)|, \end{aligned} \tag{2.1}$$

where we used the monotonicity in the first inequality. To simplify the calculations we separately discuss the diffusion part q_D and the jump part q_J of the symbol $q(x,\xi)$. Starting with the diffusion part we get

$$\begin{aligned} &\sup_{|y-x|\le v(t)} \sup_{|\xi|\le 1/v(t)} |q_D(y,\xi)|\\ &\quad\le \sup_{|y-x|\le v(t)} \sup_{|\xi|\le 1/v(t)} \left(|\mathbf{i}(yl_U + l_L)\xi| + |\frac{1}{2}(y^2\sigma_U^2 + 2y\sigma_{U,L} + \sigma_L^2)\xi^2|\right)\\ &\quad\le \frac{\big(|x| + v(t)\big)\,|l_U| + |l_L|}{|v(t)|} + \frac{1}{2}\frac{\big(|x|+v(t)\big)^2\sigma_U^2 + 2\left(|x|+v(t)\right)\sigma_{U,L} + \sigma_L^2}{|v(t)|^2}\\ &\quad\le c_l\left(\frac{1}{|v(t)|} + 1\right) + c_\sigma\left(\frac{1}{|v(t)|^2} + \frac{1}{|v(t)|} + 1\right)\\ &\quad\le C_1\frac{1}{|v(t)|^2} + C_2\frac{1}{|v(t)|} + C_3. \end{aligned}$$

Using Taylor's formula, we obtain

$$\begin{aligned} &\sup_{|y-x|\le v(t)} \sup_{|\xi|\le 1/v(t)} |q_J(y,\xi)|\\ &\le \sup_{|y-x|\le v(t)} \sup_{|\xi|\le 1/v(t)} \left|\int_{\mathbb{R}^2\setminus\{0\}} \Big(1 - \mathrm{e}^{\mathbf{i}(yz_1+z_2)\xi} + \mathbf{i}(yz_1+z_2)\xi\mathbb{1}_{\{|z|\le 1\}}\Big)\nu_{U,L}(\,\mathrm{d}z_1,\,\mathrm{d}z_2)\right|\\ &\le \sup_{|y-x|\le v(t)} \sup_{|\xi|\le 1/v(t)} \int_{|z|\ge 1} \left|1 - \mathrm{e}^{\mathbf{i}(yz_1+z_2)\xi}\right|\nu_{U,L}(\,\mathrm{d}z_1,\,\mathrm{d}z_2)\\ &\quad+ \sup_{|y-x|\le v(t)} \sup_{|\xi|\le 1/v(t)} \int_{B(0,1)\setminus\{0\}} \left|1 - \mathrm{e}^{\mathbf{i}(yz_1+z_2)\xi} + \mathbf{i}(yz_1+z_2)\xi\right|\nu_{U,L}(\,\mathrm{d}z_1,\,\mathrm{d}z_2)\\ &\le \int_{|z|\ge 1} 2\nu_{U,L}(\,\mathrm{d}z_1,\,\mathrm{d}z_2)\\ &\quad+ \sup_{|y-x|\le v(t)} \sup_{|\xi|\le 1/v(t)} \int_{B(0,1)\setminus\{0\}} \frac{1}{2}(yz_1+z_2)^2\xi^2\nu_{U,L}(\,\mathrm{d}z_1,\,\mathrm{d}z_2)\\ &\le c_\nu + \sup_{|y-x|\le v(t)} \frac{1}{v(t)^2}\int_{B(0,1)\setminus\{0\}} \frac{1}{2}(1+|y|)^2|z|^2\nu_{U,L}(\,\mathrm{d}z_1,\,\mathrm{d}z_2)\\ &\le c_\nu + \frac{\big(1+|x|+v(t)\big)^2}{v(t)^2}C_\nu\\ &\le C_4\frac{1}{v(t)^2} + C_5\frac{1}{v(t)} + C_6. \end{aligned}$$

Substituting these inequalities into (2.1), we obtain

$$\mathbb{P}^x\Big((V_\cdot - x)^*_{\theta_k} > v(\theta_k)\Big) \le c\theta_k\left(c_1\frac{1}{v(t_{k+1})^2} + c_2\frac{1}{v(t_{k+1})} + c_3\right).$$

Since the following integrals

$$\begin{aligned}
&\int_0^\delta \frac{1}{v(t)^2}\,\mathrm{d}t = \int_0^\delta \frac{1}{t|\log(t)|^{1+\epsilon}}\,\mathrm{d}t < \infty,\\
&\int_0^\delta \frac{1}{v(t)}\,\mathrm{d}t = \int_0^\delta \frac{1}{\sqrt{t|\log(t)|^{1+\epsilon}}}\,\mathrm{d}t < \infty,\\
&\int_0^\delta 1\,\mathrm{d}t < \infty
\end{aligned} \tag{*}$$

exist for suitable $\delta > 0$, we have

$$\sum_{k=1}^{\infty}\mathbb{P}^x\Big((V_\cdot - x)^*_{\theta_k} > v(t)\Big) < \infty.$$

The Borel-Cantelli lemma implies that

$$\limsup_{t\to 0}\frac{(V_\cdot - x)^*_t}{v(t)} \le 1 \quad \mathbb{P}^x - a.s.$$

For $\lambda \in (0,1)$ the above calculations also hold and lead

$$\begin{aligned}
\mathbb{P}^x\Big((V_\cdot - x)^*_{\theta_k} > \lambda v(\theta_k)\Big) &\le c\theta_k\left(c_1\frac{1}{\lambda^2 v(t_{k+1})^2} + c_2\frac{1}{\lambda v(t_{k+1})} + c_3\right)\\
&\le c\theta_k\frac{1}{\lambda^2}\left(c_1\frac{1}{v(t_{k+1})^2} + c_2\frac{1}{v(t_{k+1})} + c_3\right).
\end{aligned}$$

Thus the Borel-Cantelli lemma is valid and we get

$$\frac{1}{\lambda}\limsup_{t\to 0}\frac{(V_\cdot - x)^*_t}{v(t)} = \limsup_{t\to 0}\frac{(V_\cdot - x)^*_t}{\lambda v(t)} \le 1 \quad \mathbb{P}^x - a.s.$$

Now letting $\lambda \to 0$ finally proves the statement. □

Note that this result also holds for other norming functions $v(t)$. The only condition is that the integrals (*) converge. Although the symbol depends on x, the norming function is independent of x.

From Theorem 2.1 and 2.2 further estimates can be derived, see Böttcher, Schilling and Wang [11, Section 5.1] and the references given there.

2.2 Path Properties

In this section we compile relevant results on path properties of Feller processes. First, we consider the Blumenthal-Getoor-Pruitt indices introduced by Blumenthal and Getoor [9] and Pruitt [41]. These indices were generalized by Schilling [48] to Feller processes. We use a slight modification in accordance with Schnurr [57].

Before defining the various indices we introduce the following helpful quantities for $x \in \mathbb{R}^d$ and $R > 0$

$$\begin{aligned} H(x,R) &:= \sup_{|y-x|\le 2R} \sup_{|\epsilon|\le 1} \left| q\left(y, \frac{\epsilon}{R}\right) \right| \\ H(R) &:= \sup_{y\in\mathbb{R}^d} \sup_{|\epsilon|\le 1} \left| q\left(y, \frac{\epsilon}{R}\right) \right| \\ h(x,R) &:= \inf_{|y-x|\le 2R} \sup_{|\epsilon|\le 1} \operatorname{Re} q\left(y, \frac{\epsilon}{4\kappa R}\right) \\ h(R) &:= \inf_{y\in\mathbb{R}^d} \sup_{|\epsilon|\le 1} \operatorname{Re} q\left(y, \frac{\epsilon}{4\kappa R}\right), \end{aligned}$$

where $\kappa = (4\arctan(1/2c_0))^{-1}$ with $c_0 > 0$ in the last two equations is from the sector condition

$$|\operatorname{Im} q(x,\xi)| \le c_0 \operatorname{Re} q(x,\xi) \quad \text{for all } x,\xi \in \mathbb{R}^d. \tag{2.2}$$

In particular, the quantities $h(x,R)$ and $h(R)$ are non-trivial only if the sector condition is satisfied and only in this case they will be used. For example, an Ornstein-Uhlenbeck process driven by a Brownian motion with symbol $q(x,\xi) = \mathbf{i}\beta\xi x + \frac{1}{2}\xi^2$ does not satisfy the sector condition as for every $\xi \in \mathbb{R}$ we find an $x \in \mathbb{R}$ such that $|\operatorname{Im} q(x,\xi)| = |\beta x\xi| > c'(1+\xi^2) \ge c \operatorname{Re}\psi_L(\xi)$.

Definition 2.4. [2] Let $q(x,\xi)$ be a negative definite symbol. The generalized Blumenthal-Getoor-Pruitt indices (at infinity) are the numbers

$$\begin{aligned} \beta_\infty^x &:= \inf\left\{\lambda \ge 0;\ \limsup_{R\to 0} R^\lambda H(x,R) = 0\right\} \\ \underline{\beta}_\infty^x &:= \inf\left\{\lambda \ge 0;\ \liminf_{R\to 0} R^\lambda H(x,R) = 0\right\} \\ \overline{\delta}_\infty^x &:= \inf\left\{\lambda \ge 0;\ \limsup_{R\to 0} R^\lambda h(x,R) = 0\right\} \\ \delta_\infty^x &:= \inf\left\{\lambda \ge 0;\ \liminf_{R\to 0} R^\lambda h(x,R) = 0\right\}. \end{aligned}$$

[2] This definition is due to Schilling [48, Definition 4.2 and 4.5]. Note that Böttcher, Schilling and Wang [11, Definition 5.13 and 5.14] introduced a streamlined version of the indices without the quantities h and H. For our purpose however, the notation of this definition is more convenient.

The Blumenthal-Getoor-Pruitt indices (at zero) are the numbers

$$\begin{aligned}\beta_0 &:= \sup\left\{\lambda \geq 0;\ \limsup_{R\to\infty} R^\lambda H(R) = 0\right\}\\ \underline{\beta}_0 &:= \sup\left\{\lambda \geq 0;\ \liminf_{R\to\infty} R^\lambda H(R) = 0\right\}\\ \overline{\delta}_0 &:= \sup\left\{\lambda \geq 0;\ \limsup_{R\to\infty} R^\lambda h(R) = 0\right\}\\ \delta_0 &:= \sup\left\{\lambda \geq 0;\ \liminf_{R\to\infty} R^\lambda h(R) = 0\right\}.\end{aligned}$$

In the case of symbols with unbounded coefficients we have to be careful. For example take an Ornstein-Uhlenbeck process driven by a Brownian motion. Its symbol is given by $q(x,\xi) = \mathbf{i}\beta\xi x + \frac{1}{2}\xi^2$ with $\beta \neq 0$. Then we have $H(R) = \sup_{y\in\mathbb{R}} \sup_{|\epsilon|\leq 1} \left| q\left(y, \frac{\epsilon}{R}\right)\right| = \infty$. Hence, bounded coefficients are usually assumed for the indices at zero. However, in the next example we calculate the indices of a generalized Ornstein-Uhlenbeck process.

Example 2.5. Let V be a generalized Ornstein-Uhlenbeck process as in Example 1.18 driven by two independent rotationally stable Lévy processes. Then the symbol is given by

$$\begin{aligned}q(x,\xi) &= \psi_L(\xi) + \int_{\mathbb{R}\setminus\{0\}} \left(1 - \mathrm{e}^{\mathbf{i}xy\xi} + \mathbf{i}xy\xi \mathbb{1}_{\{|y|\leq 1\}}\right)\nu_U(\,\mathrm{d}y)\\ &= |\xi|^{\alpha_1} + \int_{\mathbb{R}\setminus\{0\}} \left(1 - \mathrm{e}^{\mathbf{i}(\xi x)y} + \mathbf{i}(\xi x)y\mathbb{1}_{\{|y|\leq 1\}}\right) c_{\alpha_2}|y|^{-1-\alpha}(\,\mathrm{d}y)\\ &= |\xi|^{\alpha_1} + |x\xi|^{\alpha_2}.\end{aligned}$$

Note that this symbol satisfies the sector condition for all $x \in \mathbb{R}$. Further we get

$$\begin{aligned}H(x,R) &= \sup_{|y-x|\leq 2R}\ \sup_{|\epsilon|\leq 1} \left(\left|\frac{\epsilon}{R}\right|^{\alpha_1} + \left|\frac{y\epsilon}{R}\right|^{\alpha_2}\right)\\ &\leq R^{-\alpha_1} + \left|\frac{|x| + 2R}{R}\right|^{\alpha_2}\\ &\leq R^{-\alpha_1} + c_{\alpha_2}|x|^{\alpha_2}R^{-\alpha_2} + c,\end{aligned}$$

where $c_{\alpha_2} = 2^{\alpha_2 - 1}$, if $\alpha_2 \geq 1$, and $c_{\alpha_2} = 2^{\frac{1-\alpha_2}{\alpha_2}}$, if $0 < \alpha_2 < 1$. Moreover, we obtain

$$\begin{aligned}h(x,R) &= \inf_{|y-x|\leq 2R}\ \sup_{|\epsilon|\leq 1} \operatorname{Re}\left(\left|\frac{\epsilon}{4\kappa R}\right|^{\alpha_1} + \left|\frac{y\epsilon}{4\kappa R}\right|^{\alpha_2}\right)\\ &= \inf_{|y-x|\leq 2R} \left(\left|\frac{1}{4\kappa R}\right|^{\alpha_1} + \left|\frac{y}{4\kappa R}\right|^{\alpha_2}\right)\\ &= \left(\frac{1}{4\kappa}\right)^{\alpha_1} R^{-\alpha_1} + \inf_{|y-x|\leq 2R} \left|\frac{y}{4\kappa R}\right|^{\alpha_2}.\end{aligned}$$

Hence, we have $\beta_\infty^x = \underline{\beta}_\infty^x = \min\{\alpha_1, \alpha_2\}$ and $\overline{\delta}_\infty^x = \min\{\alpha_1, \alpha_2\}$.
Since the symbol $q(x,\xi)$ has unbounded coefficients, the quantity $H(R)$ equals infinity and the indices at zero β_0 and $\underline{\beta}_0$ are set ∞. However, we can calculate the indices at zero δ_0 and $\overline{\delta}_0$ due to the positivity of $q(x,\xi)$. In particular, we have

$$\begin{aligned} h(R) &= \inf_{y\in\mathbb{R}^d} \sup_{|\epsilon|\le 1} \operatorname{Re}\left(\left|\frac{\epsilon}{4\kappa R}\right|^{\alpha_1} + \left|y\frac{\epsilon}{4\kappa R}\right|^{\alpha_2}\right) \\ &= \left|\frac{1}{4\kappa R}\right|^{\alpha_1} \end{aligned}$$

and hence $\delta_0 = \overline{\delta}_0 = \alpha_1$.

A variety of applications is based on these indices, see Böttcher et al. [11, Chapter 5] and the references given there. In this section we restrict ourselves to the path behaviour of Feller processes but we also refer to Section 3.6 which contains a discussion of path properties of affine processes.
A main application of the Blumenthal-Getoor-Pruitt indices is the asymptotic behaviour of the sample paths. For a full treatment of those growth and Hölder conditions for the paths of a process we refer to Schnurr [57, Theorem 3.11 and 3.12] and to Schilling [48, Theorem 4.3 and 4.6]. The proof of these results is based on Borel-Cantelli techniques, which can be seen in Example 2.6 where we investigate the path behaviour of an Ornstein-Uhlenbeck process. The sample path behaviour is based on two probability estimates, the upper maximal inequality and an upper bound of the tail probability. The latter estimate requires the global sector condition which is required in the quantities $h(R)$, $h(x,R)$ and, thus, in the indices δ_0, δ_∞^x. Based on our extension of the upper bound of the tail probability, see Theorem 2.2, the next example shows that it is sometimes enough to assume the local sector condition.

Example 2.6. Let V be an Ornstein-Uhlenbeck process with symbol $q(x,\xi) = \mathbf{i}\beta x\xi + \psi_L(\xi)$, where ψ_L is the characteristic exponent of the driving Lévy process, such that $|\operatorname{Im}\psi_L(\xi)| \le c_0 \operatorname{Re}\psi_L(\xi)$ and $|\beta\xi| \le c_1 \operatorname{Re}\psi_L(\xi)$ hold for some $c_0, c_1 > 0$. Then V satisfies the local sector condition but not the global sector condition. Indeed, we have

$$\begin{aligned} |\operatorname{Im} q(y,\xi)| &\le |y\beta\xi| + |\operatorname{Im}\psi_L(\xi)| \\ &\le c_1|y| \operatorname{Re}\psi_L(\xi) + c_0 \operatorname{Re}\psi_L(\xi) \\ &\le \underbrace{(c_1(|x|+r)+c_0)}_{=:c_{x,r}} \operatorname{Re}\psi_L(\xi) \qquad \forall y \in B(x,r). \end{aligned}$$

Furthermore, it holds that

$$\begin{aligned} \limsup_{t\to 0} t^{-\frac{1}{\lambda}} (V_\cdot - x)_t^* &= \infty \qquad && \text{for all } \overline{\delta}_\infty^x > \lambda \ge \delta_\infty^x \\ \lim_{t\to 0} t^{-\frac{1}{\lambda}} (V_\cdot - x)_t^* &= \infty && \text{for all } \lambda < \delta_\infty^x, \end{aligned}$$

where δ_∞^x and $\overline{\delta}_\infty^x$ are the indices derived from the negative definite function ψ_L, i.e. the symbol of the driving Lévy process.

Proof. We only prove the second statement and refer to Schilling [48, Theorem 4.3] for the first one since this method carries over to our case.
Let $\lambda < \alpha_2 < \alpha_1 < \delta_\infty^x$. Then by Theorem 2.2 we have

$$\begin{aligned}\mathbb{P}^x\Big((V_\cdot - x)_t^* < t^{1/\alpha_2}\Big) &\le c\left(t \sup_{|\xi|\le 1/(k_0 t^{1/\alpha_2})} \inf_{|y-x|\le r} \operatorname{Re} q(y,\xi)\right)^{-1} \\ &= c\left(t \sup_{|\xi|\le 1/(k_0 t^{1/\alpha_2})} \operatorname{Re}\psi_L(\xi)\right)^{-1} \\ &\le c\left(t \sup_{|\epsilon|\le 1} \operatorname{Re}\psi_L\Big(\frac{\epsilon}{k_0 t^{1/\alpha_2}}\Big)\right)^{-1}.\end{aligned}$$

For all $\lambda < \delta_\infty^x$ we find some $C \ge 0$ such that $R^\lambda h(x,R) \ge C$ as $R \to 0$. Since $\sup_{|\epsilon|\le 1}\operatorname{Re}\psi_L\Big(\frac{\epsilon}{k_0 t^{1/\alpha_2}}\Big) = h_L(x, 4\kappa k_0 t^{1/\alpha_2})$, where h_L is the quantity corresponding to the symbol ψ_L of the driving Lèvy process L, we get for α_1 with $\alpha_2 < \alpha_1 < \delta_\infty^x$

$$\begin{aligned}\mathbb{P}^x\Big((V_\cdot - x)_t^* < t^{1/\alpha_2}\Big) &\le c\left(t \sup_{|\epsilon|\le 1} \operatorname{Re}\psi_L\Big(\frac{\epsilon}{k_0 t^{1/\alpha_2}}\Big)\right)^{-1} \\ &\le c t^{-1} C (4\kappa k_0 t^{1/\alpha_2})^{\alpha_1} \\ &= c' t^{-1} k_0^{\alpha_1} t^{\alpha_1/\alpha_2}.\end{aligned}$$

Substituting $k_0 = \Big(\arccos\sqrt{2/3}\Big)^{-1} \vee 2c_{x,t^{1/\alpha_2}} = c_1(|x| + t^{1/\alpha_2}) + c_0$, we obtain

$$\begin{aligned}\mathbb{P}^x\Big((V_\cdot - x)_t^* < t^{1/\alpha_2}\Big) &\le c' t^{-1} k_0^{\alpha_1} t^{\alpha_1/\alpha_2} \\ &\le c'' t^{-1}(|x|^{\alpha_1} + t^{\alpha_1/\alpha_2}) t^{\alpha_1/\alpha_2} \\ &\le c''(|x|^{\alpha_1} t^{\alpha_1/\alpha_2 - 1} + t^{2\alpha_1/\alpha_2 - 1}).\end{aligned}$$

Now, we take $t_k := 2^{-k}$, $k \in \mathbb{N}$, and find

$$\sum_{k=1}^\infty \mathbb{P}^x\Big((V_\cdot - x)_{t_k}^* < t_k^{1/\alpha_2}\Big) \le \tilde{c} \sum_{k=1}^\infty \Big(2^{-k(\alpha_1/\alpha_2 - 1)} + 2^{-k(2\alpha_1/\alpha_2 - 1)}\Big) < \infty,$$

as $\alpha_1 > \alpha_2$. Hence by the Borel-Cantelli Lemma we know that $(V_\cdot - x)_{t_k}^* \ge t_k^{1/\alpha_2}$ for eventually all $k \in \mathbb{N}$. For fixed ω there exists an $N(\omega) \in \mathbb{N}$ such that for all $k \ge N(\omega)$ and $t \in (t_{k+1}, t_k]$ we have

$$(V_\cdot(\omega) - x)_t^* \ge (V_\cdot(\omega) - x)_{t_k}^* \ge t_k^{1/\alpha_2} \ge 2^{1/\alpha_2} t^{1/\alpha_2}.$$

Consequently, for $\lambda < \alpha_2$

$$t^{-1/\lambda}(V_\cdot - x)_t^* \ge 2^{1/\alpha_2} t^{1/\alpha_2 - 1/\lambda} \longrightarrow \infty$$

holds $\mathbb{P}^x - a.s.$ as $t \to 0$. □

Since t tends to 0, the radius of the sector condition decreases such that the inequality has to be valid for a small area. On the contrary, the radius of the sector condition increases as $t \to \infty$, for the path property related to the index δ_0. However, the sector condition does not hold globally which illustrates why a similar modification cannot be applied for the Blumenthal-Getoor-Pruitt index δ_0.

For many investigations it is important that the process does not explode in finite time. This behaviour is described in the following definition.

Definition 2.7. A Feller semigroup $(T_t)_{t\geq 0}$ is conservative if $T_t 1 = 1$ for all $t \geq 0$.

If the Feller semigroup is conservative, the corresponding stochastic process has *a.s.* infinite life-time, i.e.

$$\mathbb{P}^x\left(\inf\{t \geq 0;\ X_t \notin \mathbb{R}^d\} = \infty\right) = 1 \quad \text{for all } x \in \mathbb{R}^d.$$

There are two possibilities that the process has no infinite life-time. There can be a killing, i.e. $q(x,0) \neq 0$, or the coefficients grow too fast and cause an explosion. The next theorem presents a sufficient criterion based on the symbol to guarantee infinite life-time.

Theorem 2.8. *Suppose that the symbol $q(x,\xi)$ is locally bounded and that $q(x,0) = 0$ for every $x \in \mathbb{R}^d$. If in addition for all $x \in \mathbb{R}^d$,*

$$\liminf_{k\to\infty} \sup_{|y-x|\leq k} \sup_{|\xi|\leq \frac{1}{k}} |q(y,\xi)| < \infty,$$

then the process X is non-explosive.

For the proof we refer to Wang [63, Theorem 2.1].

Example 2.9. Let V be a generalized Ornstein-Uhlenbeck process as defined in Definition 1.7. Then the process is non-explosive or, in other words, the semigroup is conservative.
Indeed, by the triangle inequality we get

$$\begin{aligned}
&\liminf_{k\to\infty} \sup_{|y-x|\leq k} \sup_{|\xi|\leq \frac{1}{k}} |q(y,\xi)| \\
&\leq \liminf_{k\to\infty} \sup_{|y-x|\leq k} \sup_{|\xi|\leq \frac{1}{k}} \Big| -\mathbf{i}(yl_U + l_L)\xi + \frac{1}{2}(y^2\sigma_U^2 + 2y\sigma_{U,L} + \sigma_L^2)\xi^2 \\
&\quad + \int_{\mathbb{R}^2\setminus\{0\}} \left(1 - \mathrm{e}^{\mathbf{i}(yz_1+z_2)\xi} + \mathbf{i}(yz_1+z_2)\xi \mathbb{1}_{\{|z|\leq 1\}}\right)\nu_{U,L}(\,\mathrm{d}z)\Big| \\
&\leq |l_U| + \frac{1}{2}|\sigma_U^2| \\
&\quad + \liminf_{k\to\infty} \sup_{|y-x|\leq k} \sup_{|\xi|\leq \frac{1}{k}} \Big| \int_{|z|\leq 1} \left(1 - \mathrm{e}^{\mathbf{i}(yz_1+z_2)\xi} + \mathbf{i}(yz_1+z_2)\xi\right)\nu_{U,L}(\,\mathrm{d}z) \\
&\quad + \Big| \int_{|z|>1} \left(1 - \mathrm{e}^{\mathbf{i}(yz_1+z_2)\xi}\right)\nu_{U,L}(\,\mathrm{d}z)\Big|.
\end{aligned}$$

Using Taylor's formula yields

$$
\begin{aligned}
&\liminf_{k\to\infty} \sup_{|y-x|\le k} \sup_{|\xi|\le \frac{1}{k}} |q(y,\xi)| \\
&\le |l_U| + \frac{1}{2}|\sigma_U^2| \\
&\quad + \liminf_{k\to\infty} \sup_{|y-x|\le k} \sup_{|\xi|\le \frac{1}{k}} \Big(\int_{|z|\le 1} \frac{1}{2}\Big|(yz_1+z_2)\xi\Big|^2 \nu_{U,L}(\,\mathrm{d}z) \\
&\quad + \int_{|z|>1} \underbrace{\Big|1-\mathrm{e}^{\mathbf{i}\xi(yz_1+z_2)}\Big|}_{\le 2} \nu_{U,L}(\,\mathrm{d}z)\Big) \\
&\le |l_U| + \frac{1}{2}|\sigma_U^2| + \liminf_{k\to\infty} \sup_{|y-x|\le k} \sup_{|\xi|\le \frac{1}{k}} (|\xi|^2|y|^2+2) \int_{\mathbb{R}^2\setminus\{0\}} (|z|^2\wedge 1)\nu_{U,L}(\,\mathrm{d}z) \\
&\le |l_U| + \frac{1}{2}|\sigma_U^2| + 3\int_{\mathbb{R}^2\setminus\{0\}} (|z|^2\wedge 1)\nu_{U,L}(\,\mathrm{d}z) < \infty.
\end{aligned}
$$

Hence the condition of Theorem 2.8 is satisfied as $\nu_{U,L}$ is a Lévy measure. Consequently, a generalized Ornstein-Uhlenbeck process has infinite life-time.

2.3 Domain of a Generator

Describing the domain of a generator is a difficult task especially in the case of unbounded coefficients. It is even more complex to describe the domain as a Banach space. Since the coefficients of the symbol are unbounded, weighted norms are required to compensate the growth of the generator. In the following we introduce a weighted norm and show that it is equivalent to the unweighted norm of the weighted function. This result allows us to derive essential properties of the weighted function space.
The equivalence of weighted norms is motivated by Schmeisser and Triebel [55, Section 5.1] who treated weighted L^p function spaces.

Definition 2.10. A function $\rho \in C^p(\mathbb{R}^d, (0,\infty))$ is called an admissible weight if for all multi-indices α with $|\alpha| \le p$ there exists a constant c_α such that

$$
\Big|D^\alpha \rho(x)\Big| \le c_\alpha \rho(x) \qquad \forall x \in \mathbb{R}^d. \tag{**}
$$

The function $\rho(x) = (1+|x|)^r$ satisfies the above condition for all $r > 0$, i.e. it is an admissible weight.

Lemma 2.11. *Let $p \in \mathbb{N}_0$ and $\rho : \mathbb{R}^d \to (0,\infty)$ be an admissible weight. Then the norms*

$$
\|u\|_{(p),\rho} := \sum_{|\alpha|\le p} \Big\|\rho D^\alpha u\Big\|_\infty
$$

and

$$\|\rho u\|_{(p)} := \sum_{|\alpha|\le p} \Big\| D^\alpha(\rho u) \Big\|_\infty$$

are equivalent[3].

Proof. We begin by proving $\|\rho \cdot \|_{(p)} \le c\| \cdot \|_{(p),\rho}$. For a function u, we have

$$\begin{aligned}
\|\rho u\|_{(p)} &= \sum_{|\alpha|\le p} \Big\| D^\alpha(\rho u) \Big\|_\infty \\
&= \sum_{|\alpha|\le p} \Big\| \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \underbrace{D^\beta \rho}_{|\cdot|\le c_\beta \rho} D^{\alpha-\beta} u \Big\|_\infty \\
&\le \sum_{|\alpha|\le p} \bar{c}_\alpha \Big\| \rho D^\alpha u \Big\|_\infty \\
&\le c_p \Big\| \rho D^\alpha u \Big\|_{(p),\rho},
\end{aligned}$$

where $\bar{c}_\alpha := \max_{\beta\le\alpha} c_\beta$ and $c_p := \max_{|\alpha|\le p} \bar{c}_\alpha = \max_{|\beta|\le p} c_\beta$.
For the converse estimate

$$\|u\|_{(p),\rho} = \sum_{|\alpha|\le p} \Big\| \rho D^\alpha u \Big\|_\infty \le C \|\rho u\|_{(p)}$$

we separately consider the terms on the right-hand side. We will prove by induction that $\Big\| \rho D^\alpha u \Big\|_\infty \le C_\alpha \Big\| \rho u \Big\|_{(p)}$ for all $|\alpha| \le p$. The base case $|\alpha| = 0$ is clear. Now assume that the induction hypothesis holds for all $\beta < \alpha$. Then the reverse triangle inequality yields

$$\begin{aligned}
\Big\| \rho D^\alpha u \Big\|_\infty &\le \left| \Big\| \rho D^\alpha u \Big\|_\infty - \Big\| \sum_{\beta<\alpha} \binom{\alpha}{\beta} D^{\alpha-\beta} \rho D^\beta u \Big\|_\infty \right| + \Big\| \sum_{\beta<\alpha} \binom{\alpha}{\beta} D^{\alpha-\beta} \rho D^\beta u \Big\|_\infty \\
&\le \Big\| \rho D^\alpha u + \sum_{\beta<\alpha} \binom{\alpha}{\beta} D^{\alpha-\beta} \rho D^\beta u \Big\|_\infty + \sum_{\beta<\alpha} \binom{\alpha}{\beta} \Big\| \underbrace{D^{\alpha-\beta} \rho}_{|\cdot|\le c_{\alpha-\beta}\rho \quad (**)} D^\beta u \Big\|_\infty \\
&\le \Big\| D^\alpha(\rho u) \Big\|_\infty + \sum_{\beta<\alpha} \binom{\alpha}{\beta} c_{\alpha-\beta} \Big\| \rho D^\beta u \Big\|_\infty \\
&\le \Big\| \rho u \Big\|_{(p)} + \sum_{\beta<\alpha} \binom{\alpha}{\beta} c_{\alpha-\beta} C_\beta \Big\| \rho u \Big\|_{(p)} \\
&\le C_\alpha \Big\| \rho u \Big\|_{(p)},
\end{aligned}$$

where $C_\alpha = 1 + \sum_{\beta<\alpha} \binom{\alpha}{\beta} c_{\alpha-\beta} C_\beta$. Since by induction this holds for each term, we have shown

$$\|u\|_{(p),\rho} \le C \|\rho u\|_{(p)},$$

[3]Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if we have $c\|\cdot\|_1 \le \|\cdot\|_2 \le C\|\cdot\|_1$ for some $c, C > 0$.

where $C = \sum_{|\alpha|\le p} 1 + C_\alpha = \sum_{|\alpha|\le p}\Big(1 + \sum_{\beta<\alpha}\binom{\alpha}{\beta} c_{\alpha-\beta} C_\beta\Big)$. $\square$

From the equivalence of the weighted norms we can deduce the Banach property.

Lemma 2.12. *The function space*

$$C^p_{\rho,\infty} := \Big\{u \in C;\ \forall|\alpha| \le p:\ \rho u \in C^p \text{ and } \rho D^\alpha u \in C_\infty\Big\}$$

is a Banach space with respect to the norm $\|\cdot\|_{(p),\rho}$.

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset C^p_{\rho,\infty}$ be a Cauchy sequence with respect to $\|\cdot\|_{(p),\rho}$. Then due to the equivalence of the norms $(\rho u_n)_{n\in\mathbb{N}}$ is a $\|\cdot\|_{(p)}$ Cauchy sequence. In particular, we have $\rho u_n \in C^p_\infty$ since

$$\rho D^\alpha u \in C_\infty \quad \forall|\alpha| \le p \Longleftrightarrow D^\alpha(\rho u) \in C_\infty \quad \forall|\alpha| \le p.$$

We know that $(C^p_\infty, \|\cdot\|_{(p)})$ is a Banach space. Hence, there exist some $g \in C^p_\infty$ such that $\rho u_n \longrightarrow g$ for $n \longrightarrow \infty$ with respect to $\|\cdot\|_{(p)}$.
Now we set $u = \frac{1}{\rho} g \in C^p_{\rho,\infty}$ and get

$$\|\rho(u_n - u)\|_{(p)} = \|\rho u_n - \rho\frac{1}{\rho} g\|_{(p)} \longrightarrow 0 \quad (n \longrightarrow \infty).$$

The equivalence of the norms implies

$$\|u_n - u\|_{(2),\rho} \longrightarrow 0 \quad (n \longrightarrow \infty).$$

Thus $(C^p_{\rho.\infty}, \|\cdot\|_{(p),\rho})$ is a Banach space. $\square$

If a function space is a complete normed vector space, we can extend linear operators by the following important theorem, see for instance Reed and Simon [42, Theorem I.7, p.9].

Theorem 2.13 (B.L.T. theorem)**.** *Suppose T is a bounded linear transformation from a normed linear space $(V_1, \|\cdot\|_1)$ to a complete normed linear space $(V_2, \|\cdot\|_2)$. Then T can be uniquely extended to a bounded linear transformation, from the completion of V_1 to $(V_2, \|\cdot\|_2)$.*

Example 2.14. Let V be a generalized Ornstein-Uhlenbeck process as in Example 1.13 with generator A^V given by

$$\begin{aligned} A^V f(x) &= (x l_U + l_L) f'(x) + \frac{1}{2}(x^2\sigma_U^2 + 2x\sigma_{U,L} + \sigma_L^2) f''(x) \\ &\quad + \int_{\mathbb{R}^2\setminus\{0\}} \big(f(x + xz_1 + z_2) - f(x) - f'(x)(xz_1 + z_2)\mathbb{1}_{\{|z|\le 1\}}\big)\, \nu_Z(\,\mathrm{d}z_1, \,\mathrm{d}z_2) \end{aligned}$$

for $f \in C_c^\infty(\mathbb{R})$. By applying Taylor's theorem on the set $\{|z| \le \frac{1}{2}\}$, we find a $\zeta \in \mathbb{R}$ with $0 \le |\zeta| \le |xz_1 + z_2| \le (1+|x|)|z|$ such that for the integrand holds

$$\begin{aligned}
&|f(x+xz_1+z_2) - f(x) - f'(x)(xz_1+z_2)\mathbb{1}_{\{|z|\le 1\}}| \\
&\quad\le \frac{1}{2}|f''(x+\zeta)|(xz_1+z_2)^2\mathbb{1}_{\{|z|\le\frac{1}{2}\}} \\
&\qquad + 2\|f\|_\infty \mathbb{1}_{\{|z|>\frac{1}{2}\}} + \sup_{x\in\mathbb{R}}|f'(x)(xz_1+z_2)|\mathbb{1}_{\{\frac{1}{2}<|z|\le 1\}} \\
&\quad\le \frac{1}{2}\Big|f''(x+\zeta)(1+|x+\zeta|)^2\Big|\frac{(1+|x|^2)}{(1+|x+\zeta|)^2}|z|^2\mathbb{1}_{\{|z|\le\frac{1}{2}\}} \\
&\qquad + 2\|f\|_\infty \mathbb{1}_{\{|z|>\frac{1}{2}\}} + (\sup_{x\in\mathbb{R}}|xf'(x)| + \|f'\|_\infty)|z|\mathbb{1}_{\{\frac{1}{2}<|z|\le 1\}} \\
&\quad\le \frac{1}{4}K_2\Big(\sup_{x\in\mathbb{R}}|x^2 f''(x)| + \|f\|_{(2)}\Big)\mathbb{1}_{\{|z|\le\frac{1}{2}\}} + \|f'\|_\infty\mathbb{1}_{\{\frac{1}{2}<|z|\le 1\}} \\
&\qquad + 2\|f\|_\infty \mathbb{1}_{\{|z|>\frac{1}{2}\}} + \sup_{x\in\mathbb{R}}|xf'(x)|\mathbb{1}_{\{\frac{1}{2}<|z|\le 1\}},
\end{aligned}$$

where we set

$$K_2 := \frac{1}{2}\sup_{y\in\mathbb{R}}\sup_{|\zeta|\le(1+|y|)/2}\frac{(1+|y|)^2}{(1+|y+\zeta|)^2}.$$

As ν is a Lévy measure, we have

$$\|A^V f\|_\infty \le c\left(\|f\|_{(2)} + \sup_{x\in\mathbb{R}}|xf'(x)| + \sup_{x\in\mathbb{R}}|x^2 f''(x)|\right) =: c_q\|f\|_{GOU}.$$

It is clear that $\|\cdot\|_{GOU}$ is a norm. Furthermore, the space

$$C^2_{GOU}(\mathbb{R}) := \left\{ f \in C^2_\infty(\mathbb{R});\ \lim_{|x|\to\infty}\big(|xf'(x)| + |x^2 f''(x)|\big) = 0\right\}$$

equipped with $\|\cdot\|_{GOU}$ is a complete normed linear space[4]. Now let $\chi \in C_c^\infty(\mathbb{R})$ be a (smooth) cut-off function such that $\mathbb{1}_{B(0,1)} \le \chi \le \mathbb{1}_{B(0,2)}$ and set $\chi_n(\cdot) := \chi(\cdot/n)$ for $n \in \mathbb{N}$. Then for every $f \in C^2_{GOU}(\mathbb{R})$ the sequence $(f_n)_{n\ge 1}$ defined by $f_n := f\cdot\chi_n \in C_c^2(\mathbb{R})$ converges to f with respect to the norm $\|\cdot\|_{GOU}$.
Indeed, this is obvious for $\|\cdot\|_{(2)}$ and it also applies to the remaining terms. Note that an $N \in \mathbb{N}$ exists such that $|x^2 f''(x)| < \epsilon$, $|xf'(x)| < \epsilon$ and $|f(x)| < \epsilon$ for all $x \in K^c$ with K compact and $K \subseteq B(0,n)$ as $n \ge N$. Furthermore, we employ $\chi_n'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\chi_n(x) = \frac{\mathrm{d}}{\mathrm{d}x}\chi(\frac{x}{n}) = \chi'(\frac{x}{n})\frac{1}{n}$. As $\mathbb{1}_{B(0,n)} \le \chi_n \le \mathbb{1}_{B(0,2n)}$, we investigate each region separately and

[4] We refer to Lemma 2.16 below for a proof that this space is complete.

get

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\Big|xf'(x)-xf_n'(x)\Big| \\
&\quad= \sup_{|x|<n}\Big|xf'(x)-xf_n'(x)\Big| + \sup_{|x|\geq n}\Big|xf'(x)-xf_n'(x)\Big| \\
&\quad= \sup_{|x|<n}\Big|xf'(x)-xf'(x)\underbrace{\chi_n(x)}_{=1}-xf(x)\underbrace{\chi_n'(x)}_{=0}\Big| \\
&\qquad+ \sup_{|x|\geq n}\Big|xf'(x)-xf'(x)\chi_n(x)-xf(x)\chi_n'(x)\Big| \\
&\quad\leq \sup_{|x|<n}\Big|xf'(x)-xf'(x)\Big| \\
&\qquad+ \sup_{|x|\geq n}\Big|xf'(x)(1-\chi_n(x))\Big| + \sup_{|x|\geq n}\Big|xf(x)\chi_n'(x)\Big| \\
&\quad\leq 2\underbrace{\sup_{|x|\geq n}\Big|xf'(x)\Big|}_{\leq\epsilon} + \sup_{n\leq|x|<2n}\Big|xf(x)\chi_n'(x)\Big| + \sup_{|x|\geq 2n}\Big|xf(x)\underbrace{\chi_n'(x)}_{=0}\Big| \\
&\quad\leq 2\epsilon + \sup_{n\leq|x|<2n}\Big|2nf(x)\chi'(x/n)\frac{1}{n}\Big| \\
&\quad\leq 2\epsilon + 2\epsilon\sup_{n\leq|x|<2n}\Big|\chi'(x/n)\Big| \\
&\quad\leq C\epsilon
\end{aligned}$$

and similarly

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\Big|x^2f''(x)-x^2f_n''(x)\Big| \\
&\quad= \sup_{|x|<n}\Big|x^2\Big(f''(x)-f''(x)\underbrace{\chi_n(x)}_{=1}-f'(x)\underbrace{\chi_n'(x)}_{=0}-f(x)\underbrace{\chi_n''(x)}_{=0}\Big)\Big| \\
&\qquad+ \sup_{|x|\geq n}\Big|x^2\Big(f''(x)-f''(x)\chi_n(x)-f'(x)\chi_n'(x)-f(x)\chi_n''(x)\Big)\Big| \\
&\quad\leq \sup_{|x|\geq n}\Big|x^2f''(x)(1-\chi_n(x))| + \sup_{|x|\geq n}\Big|x^2\Big(f'(x)\chi_n'(x)-f(x)\chi_n''(x)\Big)\Big| \\
&\quad\leq 2\epsilon + \sup_{n\leq|x|<2n}\Big|x^2\Big(f'(x)\chi_n'(x)-f(x)\chi_n''(x)\Big)\Big| \\
&\qquad+ \sup_{|x|\geq 2n}\Big|x^2\Big(f'(x)\underbrace{\chi_n'(x)}_{=0}-f(x)\underbrace{\chi_n''(x)}_{=0}\Big)\Big| \\
&\quad\leq 2\epsilon + \sup_{n\leq|x|<2n}\Big|2nxf'(x)\chi'(x/n)\frac{1}{n}\Big| + \sup_{n\leq|x|<2n}\Big|4n^2f(x)\chi''(x/n)\frac{1}{n^2}\Big| \\
&\quad\leq C'\epsilon.
\end{aligned}$$

Since $\epsilon>0$ is arbitrary $C_c^2(\mathbb{R})$ is dense in $C_{GOU}^2(\mathbb{R})$ with respect to $\|\cdot\|_{GOU}$.
Using the Friedrichs mollifier, see Friedlander and Joshi [22, Theorem 1.2.1], we obtain

that $C_c^\infty(\mathbb{R})$ is dense in $C_c^2(\mathbb{R})$ with respect to $\|\cdot\|_{GOU}$, see Lemma 2.15 below. Hence, by the B.L.T. theorem 2.13 we conclude that the generator A^V extends to $\overline{C_c^\infty(\mathbb{R})}^{\|\cdot\|_{GOU}} \subseteq C^2_{GOU}(\mathbb{R})$.

It is important that the norm $\|\cdot\|_{(2)}$ is part of $\|\cdot\|_{GOU}$. Otherwise $\overline{C_c^\infty(\mathbb{R})}^{\|\cdot\|_{GOU}}$ would contain functions $f \notin C^2(\mathbb{R})$. To see this, consider a sequence of functions $(f_k)_{k\geq 1}$ that equals $\tanh(kx)$ for $|x|$ small enough. This sequence approximates around zero the signum function which equals -1 for negative x, 0 for $x=0$ and 1 for x positive. Indeed, for all $\epsilon>0$ we find k big enough such that $f_k(x)=\operatorname{sgn}(x)$ for $|x|\geq\alpha$, where $\alpha<\sqrt{\epsilon}$. Hence, the distance between f_k and sgn with respect to $\sup_x |x^2 f(x)|$ around zero is smaller than ϵ, i.e.

$$\sup_{|x|\leq\alpha}\Big|x^2(f_k(x)-\operatorname{sgn}(x))\Big| \leq \sup_{|x|\leq\alpha}|x^2|\cdot 1 < \alpha^2 < \epsilon.$$

This shows that the limit of the sequence with respect to the weighted seminorm need not to be continuous. As $\|\cdot\|_{(2)}$ is part of $\|\cdot\|_{GOU}$, it guarantees that the limit is in $C^2(\mathbb{R})$.
The next lemma is a modification of an approximation result in Friedlander and Joshi [22, Theorem 1.2.1] with respect to weighted norms.

Lemma 2.15. *Let $f\in C_c^k(\mathbb{R}^d)$ with $0\leq k\leq\infty$ and let $\rho\in C_c^\infty(\mathbb{R}^d)$ such that*

$$\rho\geq 0,\ \operatorname{supp}(\rho)\subseteq\overline{B}(0,1),\ \int\rho(x)\,\mathrm{d}x = 1.$$

Let $\epsilon>0$ and

$$f_\epsilon(x) := \epsilon^{-d}\int f(y)\rho\Big(\frac{x-y}{\epsilon}\Big)\,\mathrm{d}y$$

then $f_\epsilon\in C_c^\infty(\mathbb{R}^d)$, $\operatorname{supp}(f)\subseteq\operatorname{supp}(f_\epsilon)+\overline{B}(0,\epsilon)$ and for $|\alpha|\leq k$ we have

$$x^{|\alpha|}\partial^\alpha f_\epsilon \xrightarrow{\|\cdot\|_\infty} x^{|\alpha|}\partial^\alpha f \quad (\epsilon\to 0).$$

Proof. We see at once that $f_\epsilon\in C_c^\infty(\mathbb{R}^d)$ since repeated differentiation under the integral sign is permissible and $f_\epsilon=0$ when the distance of x from $\operatorname{supp} f$ exceeds ϵ. Following the lines of Friedlander and Joshi [22, Theorem 1.2.1] we write

$$f_\epsilon(x) := \int f(x-\epsilon z)\rho(z)\,\mathrm{d}z.$$

Then we conclude from the properties of ρ that

$$\begin{aligned}|xf_\epsilon(x)-xf(x)| &= \Big|x\int(f(x-\epsilon z)-f(x))\rho(z)\,\mathrm{d}z\Big|\\ &\leq |x|\int|f(x-\epsilon z)-f(x)|\rho(z)\,\mathrm{d}z\\ &\leq K\int|f(x-\epsilon z)-f(x)|\rho(z)\,\mathrm{d}z,\end{aligned}$$

with $K > 0$ such that for all $|x| \geq K$ we have $x \notin \operatorname{supp}(f)$ and $x \notin \operatorname{supp}(f_\epsilon)$. Note that

$$|f_\epsilon(x) - f(x)| \leq \sup \left\{ |f(x+y) - f(x)|;\ |y| \leq \epsilon \right\}.$$

This tends to zero uniformly as $\epsilon \to 0$, by uniform continuity of f.
Since we can differentiate under the integral sign, i.e. for $|\alpha| \leq k$

$$\partial^\alpha f_\epsilon(x) := \int \partial^\alpha f(x - \epsilon z)\rho(z)\,\mathrm{d}z,$$

the same arguments as above prove the statement. □

The next lemma shows that the function space $C^2_{GOU}(\mathbb{R})$ is complete with respect to the norm $\|\cdot\|_{GOU}$. The proof is similar to the well-known argument that uniform convergence implies differentiability.

Lemma 2.16. *Let $(f_n)_{n\in\mathbb{N}} \subseteq C^2(\mathbb{R})$ and $f, g \subseteq C(\mathbb{R})$ such that f_n converges to f with respect to $\|\cdot\|_{(2)}$ and xf_n' converges uniformly to g. Then obviously $f \in C^2(\mathbb{R})$ and $g(x) = xf'(x)$ for all $x \in \mathbb{R}$.*

Proof. Integration by parts yields

$$\int x f_n'(x)\,\mathrm{d}x = x f_n(x) - \int f_n(x)\,\mathrm{d}x.$$

Both integrands converge uniformly to g and f, respectively. Moreover, $xf_n(x)$ converges pointwise to $xf(x)$. Hence, we have $g(x) = xf'(x)$. □

Applying integration by parts twice, the same argument also holds for $x^2 f''(x)$.

Chapter 3

Affine Processes on Canonical State Space

In 2003, Duffie, Filipović and Schachermayer [18] published a seminal paper on affine processes with canonical state space $D = \mathbb{R}_+^m \times \mathbb{R}^n$. Special cases of this state space were already well-known in the literature , like Ornstein-Uhlenbeck[-type] process on $\mathbb{R}^d$ or continuous state branching processes with immigration (CBI), cf. Kawazu and Watanabe [29], Watanabe [64] or Pinsky [40]. Combining these two special cases to the canonical state space was necessary for applications in mathematical finance and lead to a broader class of processes.

In the next section, we introduce affine processes and examine various properties. A discussion of the Feller property and some consequences thereof follow in Section 3.2. In Section 3.3 and 3.4, we look more closely at the functional characteristics of an affine process. Using techniques from harmonic analysis we introduce a new approach to characterize the admissibility of the parameters. Section 3.5 establishes the relation between affine processes and pseudo-differential operators. In particular, we explicitly determine the symbol of an affine process. In the last section of this chapter we present some applications based on the symbol of an affine process.

3.1 Affine Processes

After introducing the definition of an affine process and showing some immediate properties, we present several examples which indicate the wide range of this class of processes. Throughout this chapter we have the following setting. We consider the so-called canonical state space for affine processes which is a product space $D = \mathbb{R}_+^m \times \mathbb{R}^n$, where $\mathbb{R}_+ = \{x \in \mathbb{R};\ x \geq 0\}$. Although this kind of state space seems artificial, Example 3.2.v shows that it appears naturally in the context of mathematical finance. In order to simplify some formulas, we introduce the following notations $d = m+n$, $I := \{1, \ldots, m\}$, $II := \{m+1, \ldots, d\}$ and x_I, x_{II} are the projections on the components with index in I and II, respectively. For vectors $x, y \in \mathbb{C}^d$ we define $x^\top y := \sum_{i=1}^d x_i y_i$ and write $\mathrm{e}_y(x) := \mathrm{e}^{x^\top y}$. Furthermore, the sets $\mathbb{C}_- := \{x \in \mathbb{C};\ \mathrm{Re}(x) \leq 0\}$ and $\mathcal{U} := \mathbb{C}_-^m \times \mathbf{i}\mathbb{R}^n$ are frequently used.

In accordance with Duffie et al. [18, Definition 2.1], we define an affine process as follows.

Definition 3.1 (Affine process)**.** A time-homogeneous Markov process $\big(X, (\mathbb{P}^x)_{x\in D}\big)$, and its semigroup $(T_t)_{t\geq 0}$ is called affine if for every $t \in \mathbb{R}_+$ the characteristic function of the transition function $p_t(x,\cdot)$ has exponential-affine dependence on x. That is, for every $(t,\xi) \in \mathbb{R}_+ \times \mathbf{i}\mathbb{R}^d$ there exist $\phi(t,\xi) \in \mathbb{C}$ and $\psi(t,\xi) \in \mathbb{C}^m \times \mathbb{C}^n$ such that

$$T_t \mathrm{e}_\xi(x) = \mathbb{E}^x\left(\mathrm{e}^{\xi^\top X_t}\right) = \mathrm{e}^{\phi(t,\xi)+x^\top\psi(t,\xi)} \quad \forall x \in D. \tag{3.1}$$

The next examples show some affine processes where the functions ϕ and ψ mostly are explicitly known. In general, ϕ and ψ do not have an explicit representation, cf. the Heston model (Example 3.2.v).

Example 3.2. i) Every Lévy process is an affine process.
A Lévy process $L = (L_t)_{t\geq 0}$ is uniquely determined by its characteristic exponent $\psi_L : \mathbb{R}^n \to \mathbb{C}$ and the relation

$$\mathbb{E}(\mathrm{e}^{\mathbf{i}\xi^\top L_t}) = \mathrm{e}^{-t\psi_L(\xi)}.$$

The function ψ_L is a continuous negative definite function and has the following Lévy-Khintchine representation

$$\psi_L(\xi) = -\mathbf{i}l^\top\xi + \frac{1}{2}\xi^\top Q\xi - \int_{\mathbb{R}^n\setminus\{0\}}\left(\mathrm{e}^{\mathbf{i}\xi^\top y} - 1 - \mathbf{i}\xi^\top y \mathbb{1}_{\{|y|\leq 1\}}\right)\nu(\,\mathrm{d}y),$$

where $l \in \mathbb{R}^n$, $Q \in \mathbb{R}^{n\times n}$ is a symmetric positive semi-definite matrix and ν is a measure on $\mathbb{R}^n\setminus\{0\}$ such that $\int_{\mathbb{R}^n\setminus\{0\}}(|y|^2 \wedge 1)\nu(\,\mathrm{d}y) < \infty$. The triplet (l, Q, ν) is called the generating triplet or Lévy triplet. If we rewrite the characteristic function,

$$\begin{aligned}\mathbb{E}^x\left(\mathrm{e}^{\mathbf{i}\xi^\top L_t}\right) &= \mathbb{E}^0\left(\mathrm{e}^{\mathbf{i}\xi^\top(L_t+x)}\right)\\ &= \mathrm{e}^{-t\psi_L(\xi)+x^\top\mathbf{i}\xi},\end{aligned}$$

we see that the Lévy process L fulfills the affine property with the functions

$$\phi(t,\mathbf{i}\xi) = -t\psi_L(\xi) \quad \text{and} \quad \psi(t,\mathbf{i}\xi) = \mathbf{i}\xi.$$

ii) The squared Bessel process on $\mathbb{R}_+$ given as strong solution of the stochastic differential equation

$$\mathrm{d}X_t = 2\sqrt{X_t}\,\mathrm{d}B_t + \delta t, \qquad X_0 = x,$$

where $\delta \geq 0$ is the dimension of the squared Bessel process and $(B_t)_{t\geq 0}$ is a Brownian motion, is an affine process with

$$\phi(t,\xi) = -\frac{\delta}{2}\ln(1-2\xi t), \qquad \psi(t,\xi) = \frac{\xi}{1-2\xi t},$$

see for instance Revuz and Yor [44, Corollary XI.1.3].

iii) The Cox-Ingersoll-Ross (CIR) process on $\mathbb{R}_+$ given as strong solution of the stochastic differential equation

$$\mathrm{d}X_t = (b - \beta X_t)\,\mathrm{d}t + \sigma\sqrt{X_t}\,\mathrm{d}B_t, \qquad X_0 = x,$$

where $b \geq 0$, $\beta \in \mathbb{R}$, $\sigma > 0$ and $(B_t)_{t\geq 0}$ is a Brownian motion, is an affine process with

$$\phi(t,\xi) = -\frac{2b}{\sigma^2}\ln(1 - \xi\frac{\sigma^2}{2\beta}(1 - \mathrm{e}^{-\beta t})), \qquad \psi(t,\xi) = \frac{\xi \mathrm{e}^{-\beta t}}{1 - \xi\frac{\sigma^2}{2\beta}(1 - \mathrm{e}^{-\beta t})}.$$

Note that the CIR process is a shifted and time-scaled squared Bessel process, i.e.

$$X_t = \mathrm{e}^{-\beta t} Y\Big(\frac{\sigma^2}{4\beta}(\mathrm{e}^{\beta t} - 1)\Big),$$

where $(Y(t))_{t\geq 0}$ is a squared Bessel process with dimension $\delta = \frac{4b}{\sigma^2}$. In Figure 3.1, we see a simulated path of a CIR process.[1]

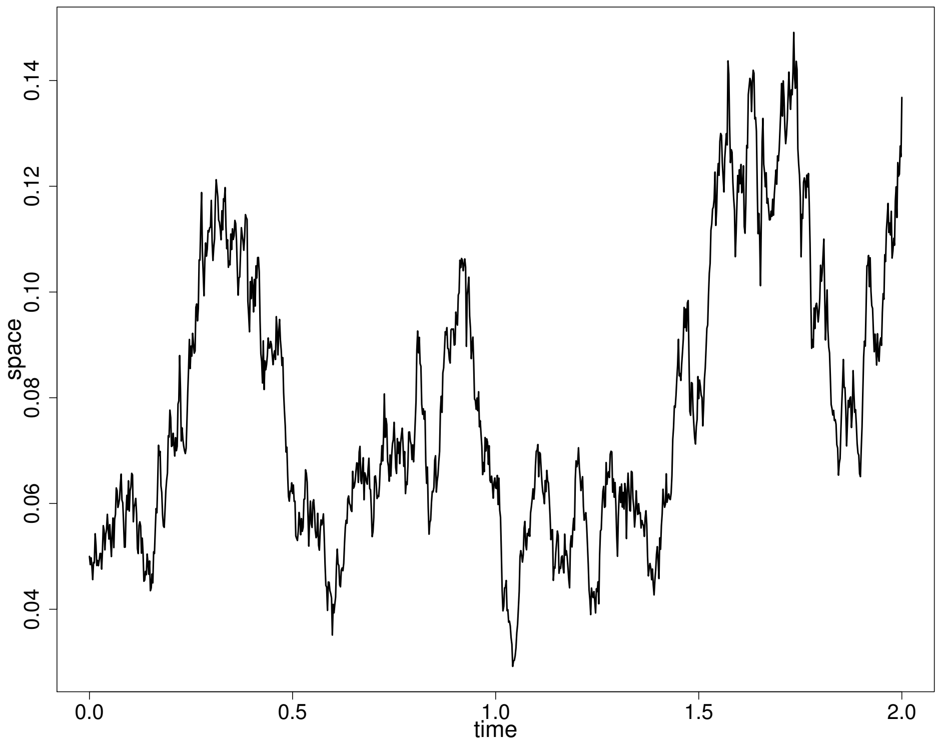

Figure 3.1: Path simulation of a CIR process starting at $x = 0.05$ with parameters $b = 0.2$, $\beta = -2$ and $\sigma^2 = 0.1$.

[1] All simulations presented in this chapter are generated by an approximation scheme whose convergence is shown in Chapter 6.

iv) The Ornstein-Uhlenbeck process on $\mathbb{R}$ given as strong solution of the stochastic differential equation

$$\mathrm{d}X_t = \mathrm{d}L_t + \beta X_t \,\mathrm{d}t, \qquad X_0 = x,$$

where $\beta \in \mathbb{R}$ and $(L_t)_{t\geq 0}$ is a Lévy process with characteristic exponent ψ_L, is an affine process with

$$\phi(t, \mathbf{i}\xi) = -\int_0^t \psi_L(\mathrm{e}^{\beta s}\xi)\,\mathrm{d}s, \qquad \psi(t, \mathbf{i}\xi) = \mathrm{e}^{\beta t}\mathbf{i}\xi.$$

This property can be found in Sato [46, Lemma 17.1]. A simulated sample path of an Ornstein-Uhlenbeck process driven by a Cauchy process is presented in Figure 3.2.

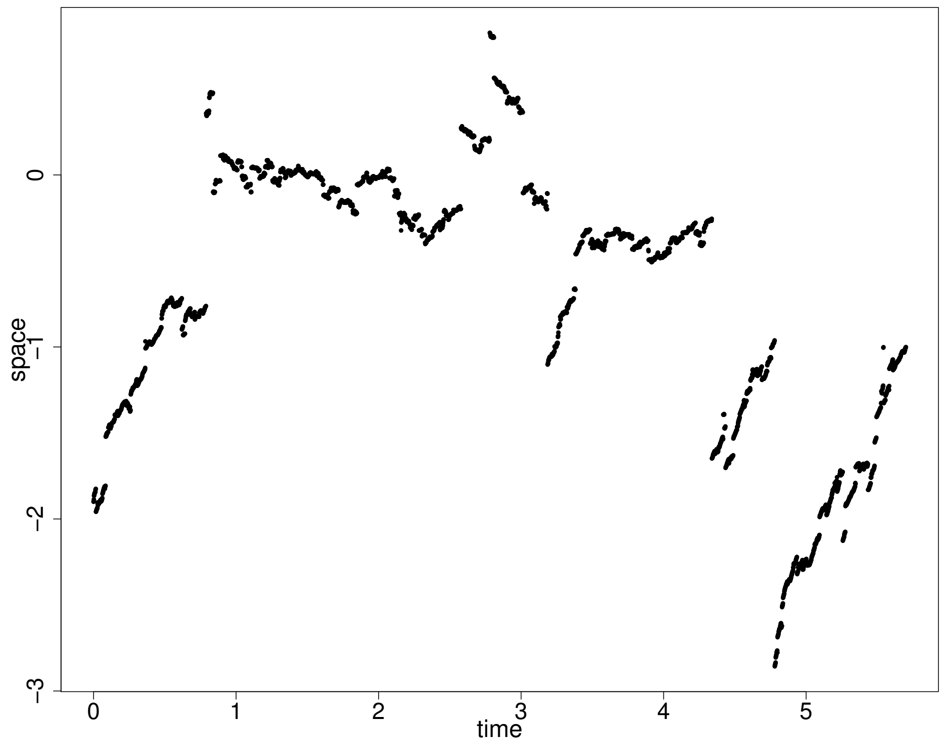

Figure 3.2: Path simulation of an Ornstein-Uhlenbeck process starting at $x = -1.9$ with parameter $\beta = -1.2$ and driven by a Cauchy process with characteristic exponent $\psi_L(\xi) = |\xi|$.

v) Heston [26] introduced the so-called Heston stochastic volatility model which is an extension of the well-known and widely used Black-Scholes model. Assuming a constant interest rate $r(t) \equiv r \geq 0$, the price of one risky asset $S = \mathrm{e}^{X_2}$, where $X = (X_1, X_2)$ is an affine process on the state space $\mathbb{R}_+ \times \mathbb{R}$, is determined by

$$\mathrm{d}X_1 = (k + \kappa X_1)\,\mathrm{d}t + \sigma\sqrt{2X_1}\,\mathrm{d}B_1$$
$$\mathrm{d}X_2 = (r - X_1)\,\mathrm{d}t + \sqrt{2X_1}(\rho\,\mathrm{d}B_1 + \sqrt{1-\rho^2}\,\mathrm{d}B_2)$$

for some constant parameters $k, \sigma \geq 0,\ \kappa \in \mathbb{R}$ and some $\rho \in [-1,1]$ and B_1, B_2 are one-dimensional independent Brownian motions.
If $\xi_1 \in \mathbb{C}_-$ and $0 \leq \operatorname{Re}\xi_2 \leq 1$, an explicit representation of the functions ϕ and ψ exists. Furthermore, for $\xi_1 = 0$ we have

$$\phi(t,\xi) = \frac{k}{\sigma^2}\log\left(\frac{2\lambda \mathrm{e}^{\frac{\lambda-(2\rho\sigma\xi_2+\kappa)}{2}t}}{\lambda(\mathrm{e}^{\lambda t}+1)-(2\rho\sigma\xi_2+\kappa)(\mathrm{e}^{\lambda t}-1)}\right)+r\xi_2 t$$
$$\psi(t,\xi) = -\frac{2(\xi_2-\xi_2^2)(\mathrm{e}^{\lambda t}-1)}{\lambda(\mathrm{e}^{\lambda t}-1)-(2\rho\sigma\xi_2+\kappa)(\mathrm{e}^{\lambda t}-1)},$$

where $\lambda = \sqrt{(2\rho\sigma\xi_2+\kappa)^2+4\sigma^2(\xi_2-\xi_2^2)}$, see Filipović and Mayerhofer [21, Section 6] for more details.

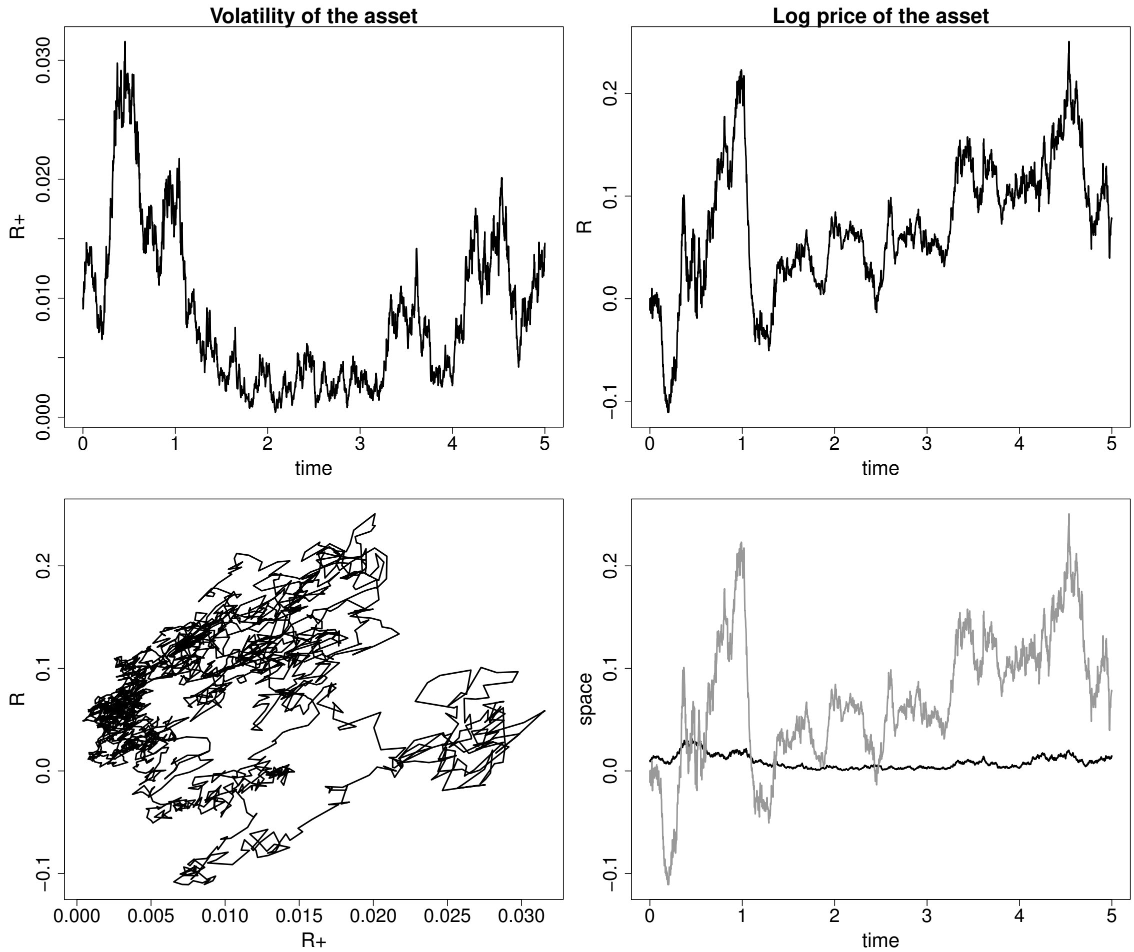

Figure 3.3: Path simulation of the Heston model starting at $x = (x_1, x_2) = (0.01, 0.0)$ with parameters $r = 0.02$, $k = 0.02$, $\kappa = -2$, $\sigma = 0.1$ and $\rho = 0.5$.

vi) Let $x_0 \in D$. Then the functions

$$\phi(t,\xi) = \begin{cases} 0, & \text{if } t = 0, \\ \xi^\top x_0, & \text{if } t > 0, \end{cases} \qquad \psi(t,\xi) = \begin{cases} \xi, & \text{if } t = 0, \\ 0, & \text{if } t > 0, \end{cases}$$

belong to the affine process with the transition function

$$p_t(x,\, \mathrm{d}\xi) = \begin{cases} \delta_x, & \text{if } t = 0, \\ \delta_{x_0}, & \text{if } t > 0, \end{cases}$$

where δ_x is the Dirac measure at x, cf. Duffie et al. [18, Remark 2.11].

The last example, originally from Kawazu and Watanabe [29], shows a special case of affine processes, for which the functions ϕ and ψ are not continuous. As we will see in the following, this is an essential property of affine processes. Therefore we introduce the next definition.

Definition 3.3 (Stochastic continuity)**.** An affine process (X) is called stochastically continuous if $p_s(x,\cdot) \to p_t(x,\cdot)$ weakly on D as $s \to t$ for $(t,x) \in \mathbb{R}_+ \times D$.

If X is affine, the continuity theorem of Lévy implies that $(X, (\mathbb{P}^x)_{x\in D})$ is stochastically continuous if and only if $\phi(t,\xi)$ and $\psi(t,\xi)$ are continuous in $t \in \mathbb{R}_+$ for every $\xi \in \mathbf{i}\mathbb{R}^{m+n}$. In the following, we always assume that an affine process is stochastically continuous. Under this assumption, we give an alternative characterization of affine processes in Section 3.5. Using Definition 3.1, we cannot identify the stochastic process of the Heston model from Example 3.2.v as an affine process. However, it follows from Example 3.20.iv and Corollary 3.23 that this process is indeed an affine process.

Lemma 3.4. *Let X be an affine process. Then*

$$\mathcal{O} = \big\{(t,\xi) \in \mathbb{R}_+ \times \mathcal{U};\ T_s \mathrm{e}_\xi(0) \neq 0 \ \forall s \in [0,t]\big\} \tag{3.2}$$

is open in $\mathbb{R}_+ \times \mathcal{U}$ and there exists a unique continuous extension of $\phi(t,\xi)$ and $\psi(t,\xi)$ to $\mathcal{O}$, such that (3.1) holds for all $(t,\xi) \in \mathcal{O}$.

A proof can be found in Duffie et al. [18, Lemma 3.1], which is an adaptation from Bauer [5, Lemma 23.7]. Although this proof is stated for regular affine processes, it is still valid in general since only stochastic continuity is required.

Example 3.2.v showed that the functions ϕ and ψ often have no explicit representation.*
Nevertheless, these functions are a key to examine affine processes as they possess several properties independent of their representation.

* ***Note Added:*** A complete explicit representation of the functions ϕ and ψ in the Heston model can be found in Aurélien Alfonsi. *Affine Diffusions and Related Processes: Simulation, Theory and Applications.* Springer, 2015, Proposition 4.2.1.

Proposition 3.5. *Let X be a stochastically continuous affine process. Then the functions ϕ and ψ have the following properties*

i) *ϕ maps $\mathcal{O}$ to $\mathbb{C}_-$.*

ii) *ψ maps $\mathcal{O}$ to $\mathcal{U}$.*

iii) *$\phi(0,\xi) = 0$ and $\psi(0,\xi) = \xi$ for all $\xi \in \mathcal{U}$.*

iv) *ϕ and ψ enjoy the "semi-flow property", i.e.*

$$\phi(t+s,\xi) = \phi(t,\xi) + \phi(s,\psi(t,\xi)), \tag{3.3}$$
$$\psi(t+s,\xi) = \psi(s,\psi(t,\xi)), \tag{3.4}$$

for all $t, s \geq 0$ with $(t+s,\xi) \in \mathcal{O}$.

v) *ϕ and ψ are jointly continuous on $\mathcal{O}$.*

vi) *With the remaining arguments fixed, $\xi_I \mapsto \phi(t,\xi)$ and $\xi_I \mapsto \psi(t,\xi)$ are analytic functions in $\{\xi_I;\ \mathrm{Re}(\xi_I) < 0, (t,\xi) \in \mathcal{O}\}$.*

vii) *Let $(t,\xi), (t,\zeta) \in \mathcal{O}$ with $\mathrm{Re}\,\xi_i \leq \mathrm{Re}\,\zeta_i$ for all $i = 1, \ldots, m$.[2] Then*

$$\begin{aligned}\mathrm{Re}\,\phi(t,\xi) &\leq \phi(t, \mathrm{Re}\,\zeta)\\ \mathrm{Re}\,\psi(t,\xi) &\leq \psi(t, \mathrm{Re}\,\zeta).\end{aligned}$$

Proof. (i) and (ii) stem from the fact that the semigroup maps bounded functions to bounded functions.
(iii) This is a consequence of $T_0 \mathrm{e}_\xi(x) = \mathrm{e}_\xi(x)$.
(iv) The semi-flow property follows immediately from the semigroup property. Let $t, s \geq 0$ with $(t+s,\xi) \in \mathcal{O}$. Then we have

$$\begin{aligned}\mathrm{e}^{\phi(t+s,\xi)+x^\top\psi(t+s,\xi)} &= T_{(t+s)}\mathrm{e}_\xi(x)\\ &= T_s T_t \mathrm{e}_\xi(x)\\ &= T_s(\mathrm{e}^{\phi(t,\xi)}\mathrm{e}_{\psi(t,\xi)}(\cdot))(x)\\ &= \mathrm{e}^{\phi(t,\xi)} T_s \mathrm{e}_{\psi(t,\xi)}(x)\\ &= \mathrm{e}^{\phi(t,\xi)}\mathrm{e}^{\phi(s,\psi(t,\xi))+x^\top\psi(s,\psi(t,\xi))}.\end{aligned}$$

(v) We show that for $(t,\xi) \in \mathcal{O}$

$$\begin{aligned}\left|\mathrm{e}^{\phi(t,\xi)+x^\top\psi(t,\xi)} - \mathrm{e}^{\phi(s,\eta)+x^\top\psi(s,\eta)}\right| &= \left|\mathbb{E}^x\left(\mathrm{e}^{\xi^\top X_t} - \mathrm{e}^{\eta^\top X_s}\right)\right|\\ &\leq \left|\mathbb{E}^x\left(\mathrm{e}^{\xi^\top X_t} - \mathrm{e}^{\xi^\top X_s}\right)\right| + \left|\mathbb{E}^x\left(\mathrm{e}^{\xi^\top X_s} - \mathrm{e}^{\eta^\top X_s}\right)\right|\\ &\longrightarrow 0 \quad \text{as } \mathcal{O} \ni (s,\eta) \to (t,\xi).\end{aligned}$$

[2] Observe that for $\xi \in \mathcal{U} = \mathbb{C}_-^m \times \mathbf{i}\mathbb{R}^n$ we have $\mathrm{Re}\,\xi_j = 0$ for all $j = m+1, \ldots, d$.

We start with the second term. As the integrand is bounded by 2, we apply the dominated convergence theorem

$$\lim_{\eta\to\xi}\left|\mathbb{E}^x\left(\mathrm{e}^{\xi^\top X_s}-\mathrm{e}^{\eta^\top X_s}\right)\right|\le \mathbb{E}^x\lim_{\eta\to\xi}\left|\mathrm{e}^{\xi^\top X_s}-\mathrm{e}^{\eta^\top X_s}\right|=0.$$

To deal with the first term, we rewrite it as an integral and split it up to obtain

$$\begin{aligned}
&\left|\mathbb{E}^x\left(\mathrm{e}^{\xi^\top X_t}-\mathrm{e}^{\xi^\top X_s}\right)\right|\\
&\quad\le\left|\int_{\{|X_t-X_s|>\delta\}}\underbrace{(\mathrm{e}^{\xi^\top X_t}-\mathrm{e}^{\xi^\top X_s})}_{|\cdot|\le 2}\,\mathrm{d}\mathbb{P}^x\right|+\left|\int_{\{|X_t-X_s|\le\delta\}}\left(\mathrm{e}^{\xi^\top X_t}-\mathrm{e}^{\xi^\top X_s}\right)\mathrm{d}\mathbb{P}^x\right|\\
&\quad\le 2\mathbb{P}^x(|X_t-X_s|>\delta)+\left|\int_{\{|X_t-X_s|\le\delta\}}\underbrace{(\mathrm{e}^{\xi^\top(X_t-X_s)}-1)}_{|\cdot|\le\epsilon}\underbrace{\mathrm{e}^{\xi^\top X_s}}_{|\cdot|\le 1}\,\mathrm{d}\mathbb{P}^x\right|\\
&\quad\le 2\mathbb{P}^x(|X_t-X_s|>\delta)+\epsilon\,\underbrace{\mathbb{P}^x(|X_t-X_s|\le\delta)}_{\le 1}\\
&\quad\longrightarrow\epsilon\quad\text{as } s\to t\\
&\quad\longrightarrow 0\quad\text{as } \epsilon\to 0.
\end{aligned}$$

Here we applied the stochastic continuity which tells us that $\mathbb{P}^x(|X_t-X_s|>\delta)\to 0$ as $s\to t$ for all $\delta>0$. In the second term, we use the continuity of the exp-function. Since Lemma 3.4 showed that ϕ and ψ are unique continuous extensions to $\mathcal{O}$, we deduce from the above calculation that ϕ and ψ are jointly continuous on $\mathcal{O}$.
(vi) Having fixed t and ξ_{II}, the mappings are essentially Laplace transforms. This yields the analyticity property by Widder [66, Theorem II.5a].
(vii) For all $x\in D$ we have by the monotonicity

$$\left|\mathbb{E}^x\left(\mathrm{e}^{\xi^\top X_t}\right)\right|\le\mathbb{E}^x\left(\left|\mathrm{e}^{\xi^\top X_t}\right|\right)=\mathbb{E}^x\left(\mathrm{e}^{\operatorname{Re}\xi^\top X_t}\right)\le\mathbb{E}^x\left(\mathrm{e}^{\operatorname{Re}\zeta^\top X_t}\right).$$

If (t,ξ) and (t,ζ) are in $\mathcal{O}$, we deduce from the affine property (3.1) that

$$\operatorname{Re}\phi(t,\xi)+x^\top\operatorname{Re}\psi(t,\xi)\le\phi(t,\operatorname{Re}\zeta)+x^\top\psi(t,\operatorname{Re}\zeta).$$

For a detailed proof we refer to Keller-Ressel [30, Proposition 1.3]. □

3.2 Feller Property

The Feller property of an affine process was first proven by Duffie et al. [18, Theorem 2.7], however, under the assumption of regularity. The more difficult part is to prove the Feller property for an affine process, in particular, to show that $T_t u$ vanishes at infinity for $u\in C_\infty(D)$. Having the Feller property, the strong continuity follows from the stochastic continuity of X. Additionally, we investigate the C_b- and the strong Feller property.

First, we state two auxiliary results from Keller-Ressel [30, Proposition 1.9 and 1.10].

Proposition 3.6. *Let $(X_t)_{t\geq 0}$ be an affine process on $D = \mathbb{R}_+^m \times \mathbb{R}^n$ and denote by $I\!I$ its $\mathbb{R}^n$-valued components. Then there exists a real $n \times n$-matrix β such that $\psi_{I\!I}(t,\xi) = \mathrm{e}^{t\beta}\xi_{I\!I}$ for all $(t,\xi) \in \mathcal{O}$.*

Proposition 3.7. *Suppose that $(t,u) \in \mathcal{O}$. If $u \in \mathcal{U}^\circ$, then $\psi(t,u) \in \mathcal{U}^\circ$.*

Based on these two propositions, it is possible to prove that the Feller property holds for an affine semigroup and, hence, that an affine process is a Feller process.

Theorem 3.8. *Every affine process is a Feller process.*

Sketch. We will present the main ideas and refer to Keller-Ressel [30, Theorem 1.11] for a detailed proof.
We first need to find a suitable dense subset of $C_\infty(D)$ such that we can apply the affine property (3.1). Therefore, we set

$$\Theta := \left\{ h_{(\xi_I, g)}(x) = \mathrm{e}^{\xi_I^\top x_I} \int_{\mathbb{R}^n} \mathrm{e}^{\mathbf{i} x_{I\!I}^\top z} g(z)\,\mathrm{d}z;\ \xi_I \in \mathbb{C}_-^m \text{ s.t. } \operatorname{Re}\xi_I < 0\ \ g \in C_c^\infty(\mathbb{R}^n) \right\},$$

where $\operatorname{Re}\xi^I < 0$ means that we have $\operatorname{Re}\xi_i < 0$ for all $i \in I = \{1,\ldots,m\}$. Note[3] that the linear span of the set $\left\{\mathrm{e}^{\xi_I^\top x_I}; \xi_I \in \mathbb{C}_-^m \text{ s.t. } \operatorname{Re}\xi_I < 0\right\}$ is a dense subset of $C_\infty(\mathbb{R}_+^m)$. The integral term is the inverse Fourier transform of a smooth function with compact support. We know that $C_c^\infty(\mathbb{R}^n)$ is dense in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. The inverse Fourier transform is a linear continuous operator from $\mathcal{S}(\mathbb{R}^n)$ into itself, i.e. $\mathcal{F}^{-1}(\mathcal{S}(\mathbb{R}^n)) = \mathcal{S}(\mathbb{R}^n)$, cf. Jacob [27, Theorem 3.1.6]. Therefore, the inverse Fourier transform of $C_c^\infty(\mathbb{R}^n)$ is dense in the Schwartz space. As the Schwartz space is a dense subset of $C_\infty(\mathbb{R}^n)$, we conclude that the inverse Fourier transform of $C_c^\infty(\mathbb{R}^n)$ is dense in $C_\infty(\mathbb{R}^n)$.
Hence the linear span of Θ is dense in $C_\infty(D)$.
Next, we verify the Feller property. Observe that Lemma 3.4 states that $T_t\mathrm{e}_\xi(x) = \mathrm{e}^{\phi(t,\xi)+x^\top\psi(t,\xi)}$ for $(t,\xi) \in \mathcal{O}$ and $T_t\mathrm{e}_\xi(x) = 0$ for $(t,\xi) \notin \mathcal{O}$. Let $h = h_{(\xi_I,g)} \in \Theta$, then

$$\begin{aligned}
T_t h(x) &= \mathbb{E}^x\left(\mathrm{e}^{\xi_I^\top X_I(t)} \int_{\mathbb{R}^n} \mathrm{e}_{\mathbf{i}z}(X_{I\!I}(t)) g(z)\,\mathrm{d}z\right) \\
&= \int_{\mathbb{R}^n} \mathbb{E}^x\left(\mathrm{e}_{(\xi_I,\mathbf{i}z)}(X(t))\right) g(z)\,\mathrm{d}z \\
&= \int_{\{z\in\mathbb{R}^n;(t,(\xi_I,z))\in\mathcal{O}\}} T_t\mathrm{e}_{(\xi_I,\mathbf{i}z)}(x) g(z)\,\mathrm{d}z \\
&= \int_{\{z\in\mathbb{R}^n;(t,(\xi_I,z))\in\mathcal{O}\}} \mathrm{e}^{\phi(t,(\xi_I,\mathbf{i}z))+x^\top\psi(t,(\xi_I,\mathbf{i}z))} g(z)\,\mathrm{d}z \\
&= \int_{\{z\in\mathbb{R}^n;(t,(\xi_I,z))\in\mathcal{O}\}} \mathrm{e}^{x_{I\!I}^\top \mathrm{e}^{t\beta}\mathbf{i}z}\, \underbrace{\mathrm{e}^{\phi(t,(\xi_I,\mathbf{i}z))}\mathrm{e}^{x_I^\top\psi_I(t,(\xi_I,\mathbf{i}z))} g(z)}_{=:\tilde{h}_{x_I}(z)}\,\mathrm{d}z,
\end{aligned}$$

[3]This follows by a Stone-Weierstrass argument. The functions $\mathrm{e}^{\xi_I^\top x_I}$ are point separating for each pair of disjoint points in $\mathbb{R}_+^m$, i.e. $\operatorname{span}(\Theta)$ separates $\mathbb{R}_+^m$. Since for each point $x \in \mathbb{R}_+^m$ we find a $x_I \in \mathbb{C}_-^m$ with $\operatorname{Re}\xi_I < 0$ such that $\mathrm{e}^{\xi_I^\top x_I}$ does not vanish, we can apply a Stone-Weierstrass theorem, cf. Semadeni [58, Corollary 7.3.9].

where we used Proposition 3.6 in the last line. Now, if x_I tends to infinity $\mathrm{e}^{x_I^\top \psi_I(t,(\xi_I,\mathbf{i}z))}$ vanishes because $\psi : \mathcal{U}^\circ \to \mathcal{U}^\circ$ and, especially, $\operatorname{Re}\psi_I(t,(\xi_I,\mathbf{i}z)) < 0$.
If $|x_{II}| \to \infty$, then $T_t h(x)$ also vanishes by the Riemann-Lebesgue lemma. Indeed, we have $T_t h(x_I, x_{II}) = \int \mathrm{e}^{x_{II}^\top \mathrm{e}^{t\beta}\mathbf{i}z} \tilde{h}_{x_I}(z)\,\mathrm{d}z = (2\pi)^{n/2}\mathcal{F}(\tilde{h}_{x_I})(\mathrm{e}^{t\beta^\top} x_{II})$. Hence, $x_{II} \mapsto T_t h(x_I, x_{II})$ is a Fourier transform and the integrand is a continuous function with compact support.
For $h \in \Theta$, the continuity of $T_t h(x)$ follows immediately by the dominated convergence theorem as the integrand is a continuous and bounded function. Since the linear span of Θ is dense in $C_\infty(D)$, we have $T_t C_\infty(D) \subseteq C_\infty(D)$.
It remains to show the strong continuity of the semigroup. However, this is implied by the stochastic continuity of the process X, cf. Böttcher, Schilling and Wang [11, Lemma 1.18]. □

Knowing that an affine process is a Feller process it is easy to show that it is also a C_b-Feller process, see Definition 1.9.

Corollary 3.9. *Every affine process is a C_b-Feller process.*

Proof. According to Böttcher, Schilling and Wang [11, Theorem 1.9], it is enough to show that the affine semigroup fulfills $T_t 1 \in C_b(D)$.
Indeed, as $1 = \mathrm{e}^{x^\top 0}$ for every $x \in D$ and $(t, 0) \in \mathcal{O}$ for all $t \geq 0$, we have due to the affine property (3.1)

$$T_t 1 = T_t \mathrm{e}_0(x) = \mathrm{e}^{\phi(t,0)+x^\top \psi(t,0)}.$$

We see that $T_t 1 \in C_b$ since $\phi(t, 0) \in \mathbb{C}_-$ and $\psi(t, 0) \in \mathcal{U}$ for all $t \geq 0$. □

We continue with strong Feller semigroups, see Definition 1.10.

Lemma 3.10. *The semigroup $(T_t)_{t\geq 0}$ of an affine process is strongly Feller if we have $\mathrm{e}^{Re\phi(t,\xi)} \in L^1(\mathrm{d}\xi)$ for all $t \geq 0$.*

Proof. We show that the semigroup is ultracontractive under the given assumption. Then the assertion follows from Schilling and Wang [54, Theorem 2.8]. Note that we consider the Young function $\Phi(x) = |x|$ and thus the ordinary L^p space with respect to Lebesgue measure.

Let $u \in C_c^\infty(D)$. Using Fourier inversion and $|\hat{u}(\xi)| \le (2\pi)^{-\frac{d}{2}}\|u\|_1$, we get

$$\begin{aligned}
\|T_t u\|_\infty &= \sup_{x\in D} |T_t u(x)| \\
&= \sup_{x\in D} |T_t \int \mathrm{e}^{\mathbf{i}x^\top \xi} \hat{u}(\xi)\,\mathrm{d}\xi| \\
&= \sup_{x\in D} |\int T_t \mathrm{e}_{\mathbf{i}\xi}(x)\hat{u}(\xi)\,\mathrm{d}\xi| \\
&\le \sup_{x\in D} \int |T_t \mathrm{e}_{\mathbf{i}\xi}(x)||\hat{u}(\xi)|\,\mathrm{d}\xi \\
&\le (2\pi)^{-\frac{d}{2}}\|u\|_1 \sup_{x\in D} \int |T_t \mathrm{e}_{\mathbf{i}\xi}(x)|\,\mathrm{d}\xi.
\end{aligned}$$

Now, we apply the affine property (3.1) to conclude from $\psi_{II} \in \mathbf{i}\mathbb{R}^n$ and $\psi_I \in \mathbb{C}_-^m$ that

$$\begin{aligned}
\|T_t u\|_\infty &\le (2\pi)^{-\frac{d}{2}}\|u\|_1 \sup_{x\in D} \int |\mathrm{e}^{\phi(t,\xi)+x^\top\psi(t,\xi)}|\,\mathrm{d}\xi \\
&\le (2\pi)^{-\frac{d}{2}}\|u\|_1 \sup_{x\in D} \int |\mathrm{e}^{\phi(t,\xi)}| \cdot \underbrace{|\mathrm{e}^{x_I^\top\psi_I(t,\xi)}|}_{\le 1} \cdot \underbrace{|\mathrm{e}^{x_{II}^\top\psi_{II}(t,\xi)}|}_{=1}\,\mathrm{d}\xi \\
&\le (2\pi)^{-\frac{d}{2}}\|u\|_1 \int |\mathrm{e}^{\phi(t,\xi)}|\,\mathrm{d}\xi.
\end{aligned}$$

Note that $\sup_{x_I\in\mathbb{R}_+^m} |\mathrm{e}^{x_I^\top\psi_I(t,\xi)}| = \mathrm{e}^{0^\top\psi_I(t,\xi)} = 1$ for all t, ξ. □

Remark 3.11. i) If a semigroup is strongly Feller then the transition function possesses a density with respect to some probability measure, cf. Schilling and Wang [54, Theorem 2.1].

ii) By Example 3.2.i we know that a Lévy process is an affine process such that $\phi(t,\mathbf{i}\xi) = -t\psi_L(\xi)$ and $\psi(t,\mathbf{i}\xi) = \mathbf{i}\xi$, where ψ_L is the characteristic exponent of the Lévy process. Hence, the condition of Lemma 3.10 reads $e^{-t\,\mathrm{Re}(\psi_L(\xi))} \in L^1(\mathrm{d}\xi)$.

Example 3.12. i) Consider the squared Bessel process X on $D = \mathbb{R}_+$ defined as in Example 3.2.ii. The function ϕ is given by

$$\phi(t,\xi) = -\frac{\delta}{2}\ln(1-2\xi t).$$

Using Lemma 3.10 we get

$$\begin{aligned}
\|T_t u\|_\infty &\le (2\pi)^{-\frac{1}{2}}\|u\|_1 \int_{\mathbb{R}} |\mathrm{e}^{\phi(t,\mathbf{i}\xi)}|\,\mathrm{d}\xi \\
&= (2\pi)^{-\frac{1}{2}}\|u\|_1 \int_{\mathbb{R}} |\mathrm{e}^{-\frac{\delta}{2}\ln(1-2\mathbf{i}\xi t)}|\,\mathrm{d}\xi \\
&= (2\pi)^{-\frac{1}{2}}\|u\|_1 \int_{\mathbb{R}} |(1-2\mathbf{i}\xi t)^{-\frac{\delta}{2}}|\,\mathrm{d}\xi \\
&= (2\pi)^{-\frac{1}{2}}\|u\|_1 \int_{\mathbb{R}} (1+4\xi^2t^2)^{-\frac{\delta}{4}}\,\mathrm{d}\xi.
\end{aligned}$$

We see that X is strong Feller for $\delta > 2$.

ii) For an Ornstein-Uhlenbeck semigroup, cf. Example 3.2.iv, the strong Feller property depends on the driving Lévy process. For sake of simplicity, we consider only the one-dimensional case. Since $x \mapsto \mathrm{e}^{-x}$ is a convex function, Jensen's inequality yields

$$\begin{aligned}\int_{\mathbb{R}} \left|\mathrm{e}^{\phi(t,\xi)}\right| \mathrm{d}\xi &= \int \left|\mathrm{e}^{-\int_0^t \psi_L(\mathrm{e}^{-\kappa s}\xi)\,\mathrm{d}s}\right| \mathrm{d}\xi \\ &\leq \int_{\mathbb{R}} \left|\frac{1}{t}\int_0^t \mathrm{e}^{-t\psi_L(\mathrm{e}^{-\kappa s}\xi)}\,\mathrm{d}s\right| \mathrm{d}\xi \\ &\leq \frac{1}{t}\int_{\mathbb{R}}\int_0^t \left|\mathrm{e}^{-t\psi_L(\mathrm{e}^{-\kappa s}\xi)}\right| \mathrm{d}s\,\mathrm{d}\xi.\end{aligned}$$

As the integrand is positive, we apply Tonelli's theorem, see Schilling [50, Satz 16.1], and then the substitution $\eta = \mathrm{e}^{-\kappa s}\xi$

$$\begin{aligned}\int_{\mathbb{R}} \left|\mathrm{e}^{\phi(t,\xi)}\right| \mathrm{d}\xi &\leq \frac{1}{t}\int_0^t\int_{\mathbb{R}} \left|\mathrm{e}^{-t\psi_L(\mathrm{e}^{-\kappa s}\xi)}\right| \mathrm{d}\xi\,\mathrm{d}s \\ &= \frac{1}{t}\int_0^t\int_{\mathbb{R}} \left|\mathrm{e}^{-t\psi_L(\eta)}\right| \mathrm{e}^{-\kappa s}\,\mathrm{d}\eta\,\mathrm{d}s \\ &= \frac{1}{t}\int_{\mathbb{R}} \left|\mathrm{e}^{-t\psi_L(\eta)}\right| \mathrm{d}\eta \int_0^t \mathrm{e}^{-\kappa s}\,\mathrm{d}s.\end{aligned}$$

The calculations shows that $\mathrm{e}^{\mathrm{Re}\,\phi(t,\xi)} \in L^1(\mathrm{d}\xi)$ if $\mathrm{e}^{-t\,\mathrm{Re}\,\psi_L(\xi)} \in L^1(\mathrm{d}\xi)$. Using the Lévy-Khintchine representation of the characteristic exponent of the driving Lévy process, Lemma 3.10 states that the semigroup T_t is strong Feller if

$$\int_{\mathbb{R}} \exp\Big\{ -\xi Q\xi - \int_{y\neq 0}\Big(1-\cos(\xi y)\Big)\nu(\mathrm{d}y)\Big\}\,\mathrm{d}\xi < \infty.$$

In particular, an Ornstein-Uhlenbeck semigroup is strongly Feller if for large ξ it holds that $Re\psi_L(\xi) \geq |\xi|^r$ with some constant $r > 0$. This is especially true if the driving Lévy process has a Brownian part. Another sufficient criteria for the above inequality is a logarithmic growth of the characteristic exponent. If we require

$$\lim_{|\xi|\to\infty} \frac{\mathrm{Re}\,\psi_L(\xi)}{\log(1+|\xi|)} > C$$

for some $C > 1$, then we get

$$\begin{aligned}\int \mathrm{e}^{-\,\mathrm{Re}\,\psi_L(\xi)}\,\mathrm{d}\xi &< \int \mathrm{e}^{-C\log(1+|\xi|)}\,\mathrm{d}\xi \\ &= \int (\log(1+|\xi|))^{-C}\,\mathrm{d}\xi < \infty.\end{aligned}$$

In the multidimensional case, an Ornstein-Uhlenbeck semigroup is strong Feller if the diffusion matrix of the driving Lévy process is (strictly) positive definite. This is necessary for the existence of the integral $\int \mathrm{e}^{-\xi Q\xi}\,\mathrm{d}\xi$.

We close this section with an excursion to regularity of affine processes.

Definition 3.13. An affine process which satisfies the affine property with the functions ϕ and ψ is called regular if the derivatives

$$F(\xi) := \frac{\partial}{\partial t}\phi(t,\xi)|_{t=0+} \quad \text{and} \quad R(\xi) := \frac{\partial}{\partial t}\psi(t,\xi)|_{t=0+} \tag{3.5}$$

exist for all $\xi \in \mathcal{U}$, and are continuous at $\xi = 0$.

A short calculation shows that this definition from Keller-Ressel [30, Definition 2.1] is equivalent to the one given by Duffie et al. [18, Definition 2.5], who based regularity on the differentiability of the affine semigroup T_t,

$$\frac{\partial}{\partial t}T_t \mathrm{e}_\xi(x)|_{t=0+} = \frac{\partial}{\partial t}\mathrm{e}^{\phi(t,\xi)+x^\top\psi(t,\xi)}|_{t=0+} = \frac{\partial}{\partial t}\Big(\phi(t,\xi) + x^\top\psi(t,\xi)\Big)\Big|_{t=0+}.$$

In their paper, regularity was a major assumption. Nevertheless, it was not clear under which conditions an affine process is regular. Note that in Example 3.2.vi a non-regular affine process is given which, however, is degenerate. Kawazu und Watanabe [29, Lemma 1.2 and 1.3] proved that a stochastically continuous affine process with state space $D = \mathbb{R}_+^m$ is regular. This statement was extended to the canonical state space $D = \mathbb{R}_+^m \times \mathbb{R}^n$ by Keller-Ressel, Schachermayer and Teichmann [32] by reducing the problem to the case $D = \mathbb{R}_+^m$ for which it was solved. Based on properties of the state space, regularity was shown for more general state spaces, in particular for positive semidefinite matrices by Cuchiero, Filipović, Mayerhofer and Teichmann [15] and for symmetric cones by Cuchiero, Keller-Ressel, Mayerhofer and Teichmann [16].
Further works generalized the above statements - Keller-Ressel, Schachermayer and Teichmann [33] by a probabilistic argument as well as Cuchiero and Teichmann [17] by theory of Markovian semimartingales.

Theorem 3.14. *For all possible state spaces, every stochastically continuous affine process is regular.*

It is worth mentioning that the regularity combined with the semi-flow property yields the so-called generalized Riccati equations

$$\begin{aligned} \frac{\partial}{\partial t}\phi(t,\xi) &= F(\psi(t,\xi)), \quad \phi(0,\xi) = 0 \\ \frac{\partial}{\partial t}\psi(t,\xi) &= R(\psi(t,\xi)), \quad \psi(0,\xi) = \xi. \end{aligned} \tag{3.6}$$

3.3 $-F$ and $-R$ as Negative Definite Functions

In this section, we will look more closely at the functions F and R, which were introduced in Definition 3.13. These functions are often refered to as functional characteristics of an

affine process in the literature. Although F and R are central elements in the theory of affine processes, they were never associated to negative definite functions. We will prove this statement in this section. This result gives an easier proof for the admissibility conditions for the parameters and, finally, for the representation of the affine generator as a pseudo-differential operator in the subsequent sections.
For a short overview containing the required arguments from harmonic analysis we refer to Section 1.1. The functions F and R are defined on $\mathcal{U} = \mathbb{C}^m_- \times \mathbf{i}\mathbb{R}^n$, which is an abelian semigroup equipped with the involution $\mathbb{C}^m_- \times \mathbf{i}\mathbb{R}^n \ni \xi \mapsto \bar{\xi}$.

Now let $\xi \in \mathcal{U}$ and set

$$\begin{aligned}\lambda_t(x,\xi) &:= \mathbb{E}^x(\mathrm{e}^{\xi^\top(X_t - x)}) = \mathrm{e}^{-\xi^\top x}\, \mathbb{E}^x(\mathrm{e}^{\xi^\top X_t}) \\ &= \mathrm{e}^{-\xi^\top x} \int_D \mathrm{e}^{\xi^\top y} p_t(x,\, \mathrm{d}y).\end{aligned}$$

As $\lambda_t(x,\xi)$ is the characteristic function of a probability measure $p_t(x,\cdot)$, $\xi \mapsto \lambda_t(x,\xi)$ is a positive definite function, cf. Corollary 1.2. According to the affine property (3.1), we have $\lambda_t(x,0) = \mathrm{e}^{\phi(t,0)+x^\top\psi(t,0)} \le 1$ as $\phi(t,0) \in \mathbb{C}_-$ and $\psi(t,0) \in \mathcal{U}$ for all $t \ge 0$. Lemma 1.3 now implies that the function $\xi \mapsto 1 - \lambda_t(x,\xi)$ is negative definite. The same is true for $\xi \mapsto \frac{1-\lambda_t(x,\xi)}{t}$. Whenever the following limit exists, the function

$$\xi \mapsto p(x,\xi) := \lim_{t\to 0} \frac{1-\lambda_t(x,\xi)}{t} \tag{3.7}$$

is a negative definite function for all $x \in D$ because the set of all negative definite functions is a convex cone which is closed under pointwise convergence, cf. Berg, Christensen and Ressel [8, §3.1.11]. We adapt the terminology from the literature and refer to $p(x,\xi)$ as the (probabilistic) symbol of the process X. The limit above exists for all $\xi \in \mathcal{U}$ because every stochastically continuous affine process is regular. Indeed, we have

$$\begin{aligned}p(x,\xi) &= \lim_{t\to 0} \frac{1-\lambda_t(x,\xi)}{t} \\ &= \lim_{t\to 0} \mathrm{e}^{-x^\top\xi} \frac{\mathrm{e}_\xi(x) - T_t \mathrm{e}_\xi(x)}{t} \\ &= -\partial_t^+ \mathrm{e}^{-x^\top\xi} T_t \mathrm{e}_\xi(x)|_{t=0} \\ &= -\partial_t^+ \mathrm{e}^{\phi(t,\xi)+x^\top(\psi(t,\xi)-\xi)}|_{t=0} \\ &= -\partial_t^+ \Big(\phi(t,\xi) + x^\top(\psi(t,\xi)-\xi)\Big)\Big|_{t=0}.\end{aligned}$$

Note that the derivative exists by regularity. This implies the existence of the limit in the first line.

Remark 3.15. The above calculation shows that regularity is equivalent to the existence of the probabilistic symbol of an affine process. Results on the existence of the probabilistic symbol have been obtained by Schnurr [56], using the semimartingale characteristics, as well as by Schilling and Schnurr [52], using the strong Markov property. Cuchiero and Teichmann [17] proved the regularity of an affine process by theory of semimartingales.

The main steps of this proof are to show right-continuity of the filtration, a cádlág version of an affine process, the strong Markov property, the semimartingale property and finally the regularity and representation. We conjecture that one can shorten the proof by extending the methods from Schilling and Schnurr [52].

For $x = 0$ we immediately see that

$$-F(\xi) = -\partial_t^+ \phi(t,\xi)|_{t=0} = \lim_{t\to 0} \frac{1-\lambda_t(0,\xi)}{t} = p(0,\xi)$$

is a negative definite function since $p(x,\xi)$ is negative definite for all $x \in D$ and, in particular, for $x = 0$. We also know that for all $x \in D$

$$-x^\top R(\xi) - F(\xi) = -\partial_t^+ \Big(\phi(t,\xi) + x^\top(\psi(t,\xi) - \xi)\Big)\Big|_{t=0} = p(x,\xi),$$

i.e. the right-hand side is negative definite for every x as well as the summand $-F(\xi)$ on the left-hand side. With the right choice of x, we can show for every $i = 1, \dots, d$ that $-R_i$ is a negative definite function.
For that purpose assume that $-R_i$ is not negative definite. According to Definition 1.1, there is a $k \in \mathbb{N}$ and $\xi^1, \dots, \xi^k \in \mathfrak{U}, \lambda_1, \dots, \lambda_k \in \mathbb{C}$ such that

$$\sum_{j,l=1}^k (-R_i(\xi^j) - \overline{R_i(\xi^l)} + R_i(\xi^j + \overline{\xi}^l))\lambda_j\overline{\lambda_l} < 0.$$

Substituting $x = re_i \in D$, where e_i is the ith unit vector with $i = 1, \dots, d$, we obtain

$$\begin{aligned}
0 &\le \sum_{j,l=1}^k \Big(p(re_i,\xi^j) + \overline{p(re_i,\xi^l)} - p(re_i,\xi^j + \overline{\xi}^l)\Big)\lambda_j\overline{\lambda_l} \\
&= \sum_{j,l=1}^k \Big(- F(\xi^j) - re_iR_i(\xi^j) + \overline{(-F(\xi^l) - re_iR_i(\xi^l))} \\
&\qquad\qquad + F(\xi^j - \xi^l) + re_iR_i(\xi^j + \overline{\xi}^l)\Big)\lambda_j\overline{\lambda_l} \\
&= \sum_{j,l=1}^k \Big(-F(\xi^j) - \overline{F(\xi^l)} + F(\xi^j + \overline{\xi}^l)\Big)\lambda_j\overline{\lambda_l} \\
&\qquad + re_i \sum_{j,l=1}^k \Big(-R_i(\xi^j) - \overline{R_i(\xi^l)} + R_i(\xi^j + \overline{\xi}^l)\Big)\lambda_j\overline{\lambda_l}.
\end{aligned}$$

The first sum is nonnegative as $-F$ is negative definite and the second sum is negative by assumption. This holds for all r, where $re_i \in D$. Thus, for r big enough, the right-hand side is negative. This contradicts that $p(re_i,\xi)$ is negative definite and shows that $-R_i$ is a negative definite function for all $i = 1, \dots, d$.
We know that $x = re_i \in D = \mathbb{R}_+^m \times \mathbb{R}^n$ as well as $x = -re_i \in D$ for $i = m+1, \dots, d$. Hence, the above argument for $x = -re_i$ yields that for $i = m+1, \dots, d$ the function

R_i is also negative definite[4]. By regularity, cf. Definition 3.13, F and R are continuous in $\xi = 0$. Bochner's theorem, cf. Ressel [43, Theorem 4], now implies that F and R are also continuous on the product space $\mathcal{U} = \mathbb{C}^m_- \times \mathbf{i}\mathbb{R}^n$. Thus we have proved the following theorem.

Theorem 3.16. *Let F and R be the functional characteristics of an affine process. Then $-F$ and $-R$ are continuous negative definite.*

3.4 Admissible Parameters of F and R

Having shown that the negative functional characteristics $-F$ and $-R$ of an affine process X are continuous negative definite, we can now give their Lévy-Khintchine representation. We will see that the affine property as well as the state space $D = \mathbb{R}^m_+ \times \mathbb{R}^n$ imply certain restrictions on the parameters. As these conditions follow from the negative definiteness of $-F$ and $-R$, which was derived from the probabilistic symbol of the process, every Feller process with an affine probabilistic symbol is subject to the admissible parameters; in other words the restrictions are necessary.

Theorem 3.17. *The functions F and R_i for $i = 1, \dots m+n$ may be written as*

$$F(\xi) = b^\top \xi + \frac{1}{2}\xi^\top a \xi - c + \int_{D\setminus\{0\}} \left(\mathrm{e}^{\xi^\top y} - 1 - \xi^\top \chi(y)\right) \mu(\,\mathrm{d}y)$$

$$R_i(\xi) = {\beta^i}^\top \xi + \frac{1}{2}\xi^\top \alpha^i \xi - \gamma^i + \int_{D\setminus\{0\}} \left(\mathrm{e}^{\xi^\top y} - 1 - \xi^\top \chi^i(y)\right) \mu^i(\,\mathrm{d}y),$$

where a, α^i are positive semidefinite matrices in $\mathbb{R}^{d\times d}$, $b, \beta^i \in \mathbb{R}^d$, $c, \gamma^i \in \mathbb{R}_+$, $\chi, \chi^i : D \to \mathbb{R}^d$ are truncation functions and μ, μ^i are Lévy measures on D. The parameters satisfy the following conditions:

$$a_{kl} = 0 \qquad \text{if } k \text{ or } l \in \{1, \dots, m\} \tag{3.8}$$

$$\alpha^j = 0 \qquad \text{for all } j \in \{m+1, \dots, d\} \tag{3.9}$$

$$\alpha^i_{kl} = 0 \qquad \text{if } k \text{ or } l \in \{1, \dots, m\}\setminus\{i\} \tag{3.10}$$

$$b \in D \tag{3.11}$$

$$\beta^i_k \geq 0 \qquad \text{for all } i \in \{1, \dots, m\} \text{ and } k \in \{1, \dots, m\}\setminus\{i\} \tag{3.12}$$

$$\beta^j_k = 0 \qquad \text{for all } j \in \{m+1, \dots, d\} \text{ and } k \in \{1, \dots, m\} \tag{3.13}$$

$$\gamma^j = 0 \qquad \text{for all } j \in \{m+1, \dots, d\} \tag{3.14}$$

$$\operatorname{supp}(\mu) \subseteq D \quad \text{and} \int_{D\setminus\{0\}} \left((|x_I| + |x_{II}|^2) \wedge 1\right) \mu(\,\mathrm{d}x) < \infty \tag{3.15}$$

$$\mu^j = 0 \qquad \text{for all } j \in \{m+1, \dots, d\} \tag{3.16}$$

[4] We will show later in the proof of Theorem 3.21 that this implies that the mapping R_i is linear for $i = m+1, \dots, d$.

$$\operatorname{supp}(\mu^i) \subseteq D \qquad \textit{for all } i \in \{1, \dots, m\} \tag{3.17}$$

$$\int_{D\setminus\{0\}} \big((|x_{I\setminus\{i\}}| + |x_{II\cup\{i\}}|^2) \wedge 1\big)\, \mu^i(\,\mathrm{d}x) < \infty \qquad \textit{for all } i \in \{1, \dots, m\}. \tag{3.18}$$

The parameters are called admissible if these conditions are satisfied. The truncation functions are componentwise given by

$$\chi_k(y) = \begin{cases} 0 & k \in \{1, \dots, m\} \\ h(y_k) & k \in \{m+1, \dots, d\} \end{cases}$$

$$\chi_k^i(y) = \begin{cases} 0 & k \in \{1, \dots, i-1, i+1, \dots, m\} \\ h(y_k) & k \in \{i, m+1, \dots, d\} \end{cases} \quad \textit{for all } i \in \{1, \dots, m\}$$

$$\chi^j \equiv 0 \quad \textit{for all } j \in \{m+1, \dots, d\},$$

where $h : \mathbb{R} \to \mathbb{R},\ z \mapsto h(z)$ *is a bounded measurable function from* $\mathbb{R}$ *to* $\mathbb{R}$*, that behaves like* z *in a neighbourhood of* 0*. Frequently used are* $h(z) = \frac{z}{1+|z|^2}$ *and* $h(z) = z\mathbb{1}_{\{|z|\le 1\}}$*. A truncation function that is continuous and has compact support is given by* $h(z) = z\mathbb{1}_{\{|z|\le 1\}} + z(2-|z|)\mathbb{1}_{\{1<|z|\le 2\}}$*. Note that a change of the truncation function* h *affects the linear terms* b, β^i*.*

Remark 3.18. Due to the admissibility conditions of the parameters for $j = m+1, \dots, d$, i.e. $\alpha^j = 0$, $\gamma^j = 0$ and $\mu^j = 0$, the representation of the characteristic function R simplifies for $j = m+1, \dots, d$ to

$$R_i(\xi) = {\beta^i}^\top \xi.$$

Remark 3.19. In Example 3.20 we present the admissible parameters of some affine processes. However, the examples given there are mostly one-dimensional. Hence, we now give a general visualization of the admissible parameters,

$$a = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & + \end{array}\right), \quad \alpha^i = \left(\begin{array}{ccccccc|ccc} & & & 0 & & & & & & \\ & & & \vdots & & & & & & \\ & & & 0 & & & & & & \\ 0 & \cdots & 0 & \alpha^i_{ii} & 0 & \cdots & 0 & * & \cdots & * \\ & & & 0 & & & & & & \\ & & & \vdots & & & & & & \\ & & & 0 & & & & & & \\ \hline & & & * & & & & & & \\ & & & \vdots & & & & & + & \\ & & & * & & & & & & \end{array}\right),$$

for $i = 1, \dots, m$, where $\alpha^i_{ii} \ge 0$. The symbol $+$ presents a positive semidefinite $n \times n$ matrix and $*$ can be an arbitrarily real entry.

Furthermore, we have $\alpha^j = 0$ for $j = m+1, \ldots, m+n$, and

$$b = \left(\begin{array}{c} + \\ \vdots \\ + \\ \hdashline * \\ \vdots \\ * \end{array}\right) \begin{matrix} \left.\vphantom{\begin{matrix} + \\ \vdots \\ + \end{matrix}}\right\} \in \mathbb{R}^m \\ \left.\vphantom{\begin{matrix} * \\ \vdots \\ * \end{matrix}}\right\} \in \mathbb{R}^n \end{matrix}, \quad \beta^i = \left(\begin{array}{c} + \\ \vdots \\ + \\ \beta^i_i \\ + \\ \vdots \\ + \\ \hdashline * \\ \vdots \\ * \end{array}\right) \begin{matrix} \left.\vphantom{\begin{matrix} + \\ \vdots \\ + \\ \beta^i_i \\ + \\ \vdots \\ + \end{matrix}}\right\} \in \mathbb{R}^m \\ \left.\vphantom{\begin{matrix} * \\ \vdots \\ * \end{matrix}}\right\} \in \mathbb{R}^n \end{matrix}, \quad \beta^j = \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hdashline * \\ \vdots \\ * \end{array}\right) \begin{matrix} \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}}\right\} \in \mathbb{R}^m \\ \left.\vphantom{\begin{matrix} * \\ \vdots \\ * \end{matrix}}\right\} \in \mathbb{R}^n \end{matrix}$$

for $i = 1, \ldots, m$, where $\beta^i_i \in \mathbb{R}$, and $j = m+1, \ldots, m+n$. Now, $+$ is a non-negative real and $*$ an arbitrary real number.

Proof. We start with the function F. Since $-F$ is negative definite, it has a Lévy-Khintchine representation, cf. Theorem 1.4 with $b \in \mathbb{R}^d$, a is a positive semidefinite $d \times d$ matrix and μ is a Lévy measure on D.
Now we take a look at the subspace $\mathbb{C}^m_-$. Therefore we write $\xi = (\xi_I, \xi_{II})$ with $\xi_I \in \mathbb{C}^m_-$ and $\xi_{II} \in \mathbf{i}\mathbb{R}^n$. Setting $\xi_{II} = 0$, we see that $\xi_I \mapsto -F(\xi)$ is a continuous negative definite function. We thus get a Lévy-Khintchine representation, see Berg et al. [8, Theorem 4.3.20], for which the parameters are given by $b_I \in \mathbb{R}^m_+$, $a_{II} = 0$, $\chi_I \equiv 0$ and m_I is a measure on $\mathbb{R}^m_+$ such that $\int_{\mathbb{R}^m_+ \setminus \{0\}} (1 \wedge |y|) \mu_I(\,\mathrm{d}y) < \infty$. Especially, as a is positive semidefinite, it follows from the Cauchy-Schwarz inequality, $|a_{kl}| \leq \sqrt{a_{kk} a_{ll}}$, that $a_{kl} = 0$ for $k, l \in \{1, \ldots, m\}$ implies $a_{kl} = 0$ for k or $l \in \{1, \ldots, m\}$.
Next we set $\xi_I = 0$ and consider the continuous negative function $\xi_{II} \mapsto -F(\xi)$. According to the Lévy-Khintchine formula on $\mathbf{i}\mathbb{R}^d$, cf. Schilling, Song and Vondracek [53, Theorem 4.15], we get $b_J \in \mathbb{R}^n$, a_{II} is a positive semidefinite $n \times n$ matrix, χ_{II} is a truncation function on $\mathbb{R}^n$ and the Lévy measure μ_{II} on $\mathbb{R}^n$ satisfies $\int_{\mathbb{R}^n \setminus \{0\}} (1 \wedge |y|^2) \mu_{II}(\,\mathrm{d}y) < \infty$. Combining the two cases, we see that the parameters of F fulfill the above conditions.

For R_i with $i \in \{1, \ldots, m\}$ we proceed similarly to above. However, we have to take special care of the ith component. Therefore, we set $I^- = \{1, \ldots, i-1, i+1, \ldots, m\}$ and $II = \{m+1, \ldots d\}$; i.e. $\xi_{I^-} \in \mathbb{C}^{m-1}_-$ and $\xi_{II} \in \mathbf{i}\mathbb{R}^n$. Then $\xi_{I^-} \mapsto -R_i(\xi)$ is a multidimensional continuous negative definite function on the semigroup $\mathbb{R}^{m-1}_+$ equipped with identical involution. Consequently, the corresponding parameters of the Lévy-Khintchine representation are $\beta^i_{I^-} \in \mathbb{R}^{m-1}_+$, $\alpha^i_{I^- I^-} = 0$, $\chi_{I^-} \equiv 0$ and $\mu^i_{I^-}$ is a Lévy measure on $\mathbb{R}^{m-1}_+$ such that $\int_{\mathbb{R}^{m-1}_+ \setminus \{0\}} (1 \wedge |y|) \mu_{I^-}(\,\mathrm{d}y) < \infty$. Again, by the Cauchy-Schwarz inequality $\alpha^i_{kl} = 0$ for $k, l \in I^- = \{1, \ldots, i-1, i+1, \ldots, m\}$ implies $\alpha^i_{kl} = 0$ for k or $l \in I^- = \{1, \ldots, i-1, i+1, \ldots, m\}$.

Considering the mapping $\xi_{II} \mapsto -R_i(\xi)$ implies that $\beta^i_{II} \in \mathbb{R}^n$, α^i_{II} is a positive semidefinite $n \times n$ matrix, χ^i_{II} is a truncation function on $\mathbb{R}^n$ and the Lévy measure μ^i_{II} on $\mathbb{R}^n$ satisfies $\int_{\mathbb{R}^n\setminus\{0\}}(1+|y|^2)\mu^i_{II}(\,\mathrm{d}y) < \infty$.
Now, we set $\xi_{I^-} = 0$ and $\xi_{II} = 0$ and consider $\xi_i \mapsto -R_i(\xi)$. Recall that $-R_i(\xi) = -e_i^\top R(\xi) = \frac{1}{r}F(\xi) + p(re_i,\xi) = \frac{1}{r}F(\xi) + \lim_{t\to 0} \frac{1-\mathbb{E}^{re_i}(\mathrm{e}^{(X_i(t)-re_i)\xi_i})}{t}$. Since $r > 0$, the term re_i shifts the support of the integral.[5] Hence, in the ith component, $-R_i$ has the parameters of a continuous negative function on $\mathbb{R}$, i.e. $\alpha^i_{ii} \geq 0$, $\beta^i_i \in \mathbb{R}$, χ_i is a truncation function and the Lévy measure μ^i_i satisfies $\int_{\mathbb{R}_+}(1+|y|^2)\mu^i_i(\,\mathrm{d}y) < \infty$. Again, combining the factors yields that the parameters of R_i for $i = 1,\dots,m$ are admissible.

Finally, we examine R_j for $j \in II$. At the end of the last paragraph we showed that $-R_j$ as well as R_j is negative definite. Especially, we know that

$$\sum_{i,l=1}^{k}(-R_j(\xi^i) - \overline{R_j(\xi^l)} + R_j(\xi^i + \overline{\xi}^l))\lambda_i\overline{\lambda_l} = 0.$$

Setting $k = 1$ and $\lambda_1 = 1$, we immediately see that

$$-R_j(\xi) - \overline{R_j(\xi)} + R_j(0) = 0 \qquad \text{for all } \xi \in \mathbf{i}\mathbb{R}^d,$$

and consequently that $\operatorname{Re} R_j \equiv 0$; i.e. R_j is purely imaginary. Now we set $k = 2$, $\lambda_1 = 1/\sqrt{2}, \lambda_2 = \mathbf{i}/\sqrt{2}$. Since R_j is purely imaginary, we have $R_j(\overline{\xi}) = \overline{R_j(\xi)} = -R_j(\xi)$ for $\xi \in \mathbf{i}\mathbb{R}^d$. Thus we get for $\xi^1 = \xi, \xi^2 = -\eta \in \mathbf{i}\mathbb{R}^d$ the functional equation

$$\begin{aligned}
0 &= \sum_{i,l=1}^{2}(-R_j(\xi^i) - \overline{R_j(\xi^l)} + R_j(\xi^i + \overline{\xi}^l))\lambda_i\overline{\lambda_l} \\
&= (-R_j(\xi) + R_j(-\eta) + R_j(\xi+\eta))(-\mathbf{i})/2 + (-R_j(\eta) + R_j(+\xi) + R_j(-\eta-\xi))\mathbf{i}/2 \\
&= \mathbf{i}(R_j(\xi) + -R_j(\eta) - R(\eta+\xi)).
\end{aligned}$$

The continuous solution of this equation is $R_j(\xi) = {\beta^j}^\top\xi$, cf. Aczél [1, §5.1]; i.e. $\alpha^j = 0$, $\gamma^j = 0$, $\mu^j = 0$ and thus $\chi^j \equiv 0$ for $j \in \{m+1,\dots,m+n\}$. Moreover, choosing again $k = 1$, $\lambda_1 = 1$, we obtain for $\xi \in \mathbb{R}^m_- \times \{0\}$

$$\begin{aligned}
-R_j(\xi) - \overline{R_j(\xi)} + R_j(\xi + \overline{\xi}) &= -R_j(\xi) + R_j(\xi) + R_j(2\xi) \\
&= R_j(2\xi) = 0 \qquad \text{for all } \xi \in \mathbb{R}^m_- \times \{0\}.
\end{aligned}$$

Hence, we have $\beta^j_I = 0 \in \mathbb{R}^m$. □

Example 3.20. i) Let X be a one-dimensional Lévy process as in Example 3.2.i and (l, Q, ν) be its Lévy triplet. Then the admissible parameters are given by

$$a = Q,\ \alpha^1 = 0,\ b = l,\ \beta^1 = 0,\ \mu = \nu,\ \mu^1 = 0.$$

[5] Observe that by substitution we have $\mathbb{E}(u(X(t)-x)) = \int_D u(y-x)\mathbb{P}^{X_t}(\,\mathrm{d}y) = \int_{D-x} u(\tilde{y})\mathbb{P}^{X_t}(\,\mathrm{d}\tilde{y})$. Since $D - x \subset \mathbb{R}^d$ and $D - x \not\subseteq D$ for $x \in D$, the shift creates a integral over $\mathbb{R}^d$.

ii) Let X be a Cox-Ingersoll-Ross (CIR) process on $\mathbb{R}_+$ defined by

$$\mathrm{d}X_t = (b - \beta X_t)\,\mathrm{d}t + \sigma\sqrt{X_t}\,\mathrm{d}B_t, \qquad X_0 = x,$$

then the admissible parameters are

$$a = 0,\ \alpha^1 = \sigma^2,\ b = b,\ \beta^1 = -\beta,\ \mu = 0,\ \mu^1 = 0.$$

iii) For an Ornstein-Uhlenbeck process on $\mathbb{R}$ driven by a Lévy process L with Lévy triplet (l, Q, ν), i.e.

$$\mathrm{d}X_t = \mathrm{d}L_t + \beta X_t\,\mathrm{d}t, \qquad X_0 = x,$$

we have

$$a = Q,\ \alpha^1 = 0,\ b = l,\ \beta^1 = \beta,\ \mu = \nu,\ \mu^1 = 0.$$

iv) As in Example 3.2.v, let the Heston model be given by

$$\begin{aligned}
\mathrm{d}X_1 &= (k + \kappa X_1)\,\mathrm{d}t + \sigma\sqrt{2X_1}\,\mathrm{d}B_1\\
\mathrm{d}X_2 &= (r - X_1)\,\mathrm{d}t + \sqrt{2X_1}(\rho\,\mathrm{d}B_1 + \sqrt{1-\rho^2}\,\mathrm{d}B_2).
\end{aligned}$$

Then the admissible parameters are

$$a = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \alpha^1 = \begin{pmatrix} 2\sigma^2 & 2\sigma\rho \\ 2\sigma\rho & 2 \end{pmatrix}, \qquad \alpha^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$b = \begin{pmatrix} k \\ r \end{pmatrix}, \qquad \beta^1 = \begin{pmatrix} \kappa \\ -1 \end{pmatrix}, \qquad \beta^2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and $\mu,\ \mu^1,\ \mu^2 = 0$.

3.5 Representation as Pseudo-Differential Operator

In the following section, we give a representation of the generator of an affine process as a pseudo-differential operator. The operator of an affine process has been studied by Duffie et al. [18, Theorem 2.7]. Our viewpoint sheds some new light on the class of symbols with unbounded coefficients since the affine symbol has linear growing coefficients. Furthermore, our proof provides a shorter and more accessible approach to the representation and characterization of the generator. Using weighted norms, see Section 2.3, we determine the domain of the generator by the B.L.T. theorem. We also present a simplified proof for the fact that the test functions[6] $C_c^\infty(D)$ are a core of the generator, which is originally shown by Duffie et al. [18, Proposition 8.2].

As we have pointed out in Section 3.3, the functions $-F$ and $-R$ are continuous negative definite. We will use this fact and some properties in the following proof.

Theorem 3.21. *The infinitesimal generator $(A, \mathcal{D}(A))$ of a stochastically continuous affine process restricted on the test functions $C_c^\infty(D)$ has a representation as a pseudo-differential operator with symbol*

$$\begin{aligned} q(x,\xi) &= -F(\mathbf{i}\xi) - x^\top R(\mathbf{i}\xi) \\ &= -\mathbf{i}b^\top \xi + \frac{1}{2}\xi^\top a\xi + c + \int_{D\setminus\{0\}} \left(1 - \mathrm{e}^{\mathbf{i}\xi^\top y} + \mathbf{i}\xi^\top \chi(y)\right) \mu(\,\mathrm{d}y) \\ &\quad - \mathbf{i}\sum_{i=1}^{m+n} x_i {\beta^i}^\top \xi + \sum_{i=1}^{m} x_i \frac{1}{2}\xi^\top \alpha^i \xi + \sum_{i=1}^{m} x_i \gamma^i \\ &\quad + \sum_{i=1}^{m} x_i \int_{D\setminus\{0\}} \left(1 - \mathrm{e}^{\mathbf{i}\xi^\top y} + \mathbf{i}\xi^\top \chi^i(y)\right) \mu^i(\,\mathrm{d}y) \end{aligned}$$

for $(x,\xi) \in (\mathbb{R}_+^m \times \mathbb{R}^n) \times \mathbb{R}^d$.

Proof. We consider the pointwise generator A_p, i.e. the pointwise limit in x. For a function $f \in C_c^\infty(D)$, the pointwise calculation leads to

$$\begin{aligned} A_p f(x) &= \lim_{t\to 0} \frac{T_t f(x) - f(x)}{t} \\ &= \lim_{t\to 0} (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \frac{T_t \mathrm{e}_{\mathbf{i}y}(x) - \mathrm{e}^{\mathbf{i}x^\top y}}{t} \hat{f}(y)\,\mathrm{d}y \\ &= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \lim_{t\to 0} \frac{T_t \mathrm{e}_{\mathbf{i}y}(x) - \mathrm{e}^{\mathbf{i}x^\top y}}{t} \hat{f}(y)\,\mathrm{d}y \\ &= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \mathrm{e}^{\mathbf{i}x^\top y} (F(\mathbf{i}y) + x^\top R(\mathbf{i}y)) \hat{f}(y)\,\mathrm{d}y, \end{aligned}$$

where we have to verify that the interchange of the limit and the integral is allowed by the dominated convergence theorem. Substituting the affine property (3.1) into the

[6] Observe that $D = \mathbb{R}_+^m \times \mathbb{R}^n$ differs from the usual open domain on which smooth functions with compact support are investigated. We refer to the next subsection for a discussion of this definition.

equation and applying Taylor's theorem we obtain

$$\begin{aligned}
&\left|\frac{T_t \mathrm{e}_{\mathbf{i}y}(x) - \mathrm{e}^{\mathbf{i}x^\top y}}{t}\right| \\
&\quad = \left|\frac{\mathrm{e}^{\phi(t,\mathbf{i}y)+x^\top \psi(t,\mathbf{i}y)} - \mathrm{e}^{\mathbf{i}x^\top y}}{t}\right| \\
&\quad = \left|\frac{1}{t}\left(\underbrace{\mathrm{e}^{\phi(0,\mathbf{i}y)+x^\top \psi(0,\mathbf{i}y)}}_{\mathrm{e}^{0+\mathbf{i}y^\top x}=\mathrm{e}^{\mathbf{i}y^\top x}} + t\partial_t\Big|_{t=0} \mathrm{e}^{\phi(t,\mathbf{i}y)+x^\top \psi(t,\mathbf{i}y)} + O(t) - \mathrm{e}^{\mathbf{i}x^\top y}\right)\right| \\
&\quad = \left|\partial_t\Big|_{t=0} \mathrm{e}^{\phi(t,\mathbf{i}y)+x^\top \psi(t,\mathbf{i}y)} + \frac{1}{t}O(t)\right|.
\end{aligned}$$

By regularity, cf. (3.5), we continue

$$\begin{aligned}
&\left|\frac{T_t \mathrm{e}_{\mathbf{i}y}(x) - \mathrm{e}^{\mathbf{i}x^\top y}}{t}\right| \\
&\quad \leq \left|F(\mathbf{i}y) + x^\top R(\mathbf{i}y)\right| + c \\
&\quad \leq \left(c_F(1+|y|^2) + |x| c_R(1+|y|^2)\right) + c \\
&\quad \leq c_{F,R}(1+|x|)(1+|y|^2),
\end{aligned}$$

where we used in the second to last line that negative definite functions are bounded by quadratic functions, see Lemma 1.5.

Obviously, $f \in C_c^\infty(D)$ is sufficiently smooth, so that $\frac{T_t \mathrm{e}_{\mathbf{i}y}(x) - \mathrm{e}^{\mathbf{i}x^\top y}}{t}\hat{f}(y)$ admits an integrable dominating function. Based on this result, we next show that $A_p f(x) = -q(x,D)f(x)$ vanishes at infinity for $f \in C_c^\infty(D)$. Therefore we rewrite the pseudo-differential operator as an integro-differential operator

$$\begin{aligned}
-q(x,D)f(x) = &\sum_{j,k=1}^{d}\left(a_{jk} + \sum_{l=1}^{m} x_l \alpha_{jk}^l\right)\partial_j\partial_k f(x) \\
&+ \left(b + \sum_{l=1}^{d}(x_l\beta^l)\right)^\top \nabla f(x) + \left(c + \gamma^\top x\right) f(x) \\
&+ \int_{D\setminus\{0\}} \left(f(x+y) - f(x) - \chi(y)^\top \nabla f(x)\right)\mu(\,\mathrm{d}y) \\
&+ \sum_{l=1}^{m} x_l \int_{D\setminus\{0\}} \left(f(x+y) - f(x) - \chi^l(y)^\top \nabla f(x)\right)\mu^l(\,\mathrm{d}y).
\end{aligned} \tag{3.19}$$

If we choose $|x|$ large enough, we have $x \notin \operatorname{supp}(f)$ and the above equation simplifies to

$$-q(x,D)f(x) = \int_{D\setminus\{0\}} f(x+y)\mu(\,\mathrm{d}y) + \sum_{l=1}^{m} x_l \int_{D\setminus\{0\}} f(x+y)\mu^l(\,\mathrm{d}y).$$

As before, we want to apply the dominated convergence theorem. We get an estimate of the integrand by applying Taylor's theorem twice. We first consider the constant part. Denote by

$$\tilde{y}_l = \begin{cases} 0 & \text{for } l = 1, \dots, m \\ y_l & \text{for } l = m+1, \dots, d, \end{cases}$$

then $\chi(y) = \tilde{y}\mathbb{1}_{\{|\tilde{y}|\leq 1\}}$ and we get for $y \in D\backslash\{0\} \cap \overline{B}(0,1)$ that

$$\begin{aligned} &f(x+y) - f(x) - \tilde{y}^\top \nabla f(x) \\ &\quad= f(x+y) - f(x+\tilde{y}) + f(x+\tilde{y}) - f(x) - \tilde{y}^\top \nabla f(x) \\ &\quad= \nabla f(x+\tilde{y}+\theta(y-\tilde{y}))(y-\tilde{y}) + \frac{1}{2}\sum_{j,k=m+1}^{d} \partial_j\partial_k f(x+\tilde{\theta}\tilde{y})y_j y_k, \end{aligned}$$

where $\theta, \tilde{\theta} \in (0,1)$. For $x \notin \operatorname{supp}(f)$ this leads to

$$\begin{aligned} &\Big|\int_{D\backslash\{0\}} f(x+y)\mu(\,\mathrm{d}y)\Big| \\ &\quad\leq \Big|\int_{\overline{B}(0,1)\backslash\{0\}} f(x+y) - f(x) - y^\top\nabla f(x)\mu(\,\mathrm{d}y)\Big| + \Big|\int_{D\backslash\overline{B}(0,1)} f(x+y)m(\,\mathrm{d}y)\Big| \\ &\quad\leq \underbrace{\left(\int_{\overline{B(0,1)}\backslash\{0\}} (|y-\tilde{y}| + |\tilde{y}|^2)\mu(\,\mathrm{d}y) + \int_{D\backslash B[0,1]} 1\mu(\,\mathrm{d}y)\right)}_{=\int_{D\backslash\{0\}}[(|y-\tilde{y}|+|\tilde{y}|^2)\wedge 1]\mu(\,\mathrm{d}y)=M<\infty} \|f\|_{(2)} < \infty. \end{aligned}$$

Note that the estimate can be chosen stronger. By a similar argument we get an integrable dominating function which allows us to apply the dominating convergence theorem for the linear integral parts.

Now let $|x|$ tend to infinity. First consider that $x_j \to \infty$ for some $j \in \{1,\dots,m\}$. We can choose x_j large enough such that $x \notin \operatorname{supp}(f)$. Since $D = \mathbb{R}_+^m \times \mathbb{R}^n$, this implies that $x + y \notin \operatorname{supp}(f)$ for all $y \in D$. Hence, if x_j is sufficiently large, we have

$$x_j \int_{D\backslash\{0\}} \underbrace{f(x+y)}_{=0} \mu^j(\,\mathrm{d}y) = 0.$$

For the same reason, the other integrals also vanish for $x_j \to \infty$. Next, consider that $x_j \leq K$ for some constant K and for all $j = 1,\dots,m$ and that $|x_l| \to \infty$ for some $l \in \{m+1,\dots,d\}$. Since the first m coordinates of x are bounded and $f \in C_c^\infty(D)$, the integrands are always bounded. Due to the above estimates, we can apply the dominated convergence theorem for $j = 1,\dots,m$

$$\lim_{|x|\to\infty} x_j \int_{D\backslash\{0\}} f(x+y)\mu^j(\,\mathrm{d}y) = \int_{D\backslash\{0\}} \underbrace{\lim_{|x|\to\infty} x_j f(x+y)}_{=0} \mu^j(\,\mathrm{d}y) = 0.$$

This shows that $x \mapsto -q(x,D)f(x) \in C_\infty(D)$ vanishes for all functions $f \in C_c^\infty(D)$ as $|x|$ tends to infinity. According to Sato [46, Lemma 31.7], the pointwise generator equals the generator. As a result, $C_c^\infty(D) \subseteq \mathcal{D}(A)$ and for $f \in C_c^\infty(D)$ we have

$$Af(x) = A_p f(x) = -(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \mathrm{e}^{\mathbf{i}x^\top y}(-F(\mathbf{i}y) - x^\top R(\mathbf{i}y))\hat{f}(y)\,\mathrm{d}y \quad \text{for all } x \in D.$$

Hence, $-A\Big|_{C_c^\infty}$ is a pseudo-differential operator with symbol $q(x,\xi) = -F(\mathbf{i}\xi) - x^\top R(\mathbf{i}\xi)$.

□

Example 3.22. i) The symbol of a one-dimensional Ornstein-Uhlenbeck process, see Example 3.2.iv, is given by

$$q(x,\xi) = \mathbf{i}\beta\xi x + \psi_L(\xi),$$

where ψ_L is the characteristic exponent of the driving Lévy process.

ii) The symbol approach is advantageous in the case of jump processes. For instance, the symbol of an CIR process which is not driven by a Brownian motion but by a symmetric α-stable process with $\alpha \in (0;2)$ is given by

$$q(x,\xi) = \mathbf{i}b\xi + \mathbf{i}\beta x\xi + x|\xi|^\alpha.$$

Note that by solving the generalized Riccati equations[7], we can give an explicit representation of the functions ψ and ϕ

$$\begin{aligned}\psi(t,\xi) &= \left(\xi^{1-\alpha}\mathrm{e}^{(1-\alpha)\beta t} + \frac{1}{\beta}\mathrm{e}^{(1-\alpha)\beta t} - \frac{1}{\beta}\right)^{\frac{1}{1-\alpha}},\\ \phi(t,\xi) &= \int_0^t b\cdot\psi(s,\xi)\,\mathrm{d}s.\end{aligned}$$

The symbol of an affine process is subject to several restrictions, cf. Theorem 3.17. One might ask whether more general symbols with affine x-dependent coefficients give rise to pseudo-differential operators or even stochastic processes.

Corollary 3.23. *Let $q(x,\xi)$ be the negative definite symbol of a pseudo-differential operator with state space $D = \mathbb{R}_+^m \times \mathbb{R}^n$ such that its coefficients are affine dependent of x. Then the symbol has a representation as illustrated in Theorem 3.21 such that it satisfies the admissible parameter condition and gives rise to an affine process. In particular, there are no other symbols with an affine structure than those from affine processes.*

[7] Sketch: Using the method of characteristics with $t = t(x) = x + c, \xi = \xi(x) = c$ such that $\psi(t,\xi) = \psi(x)$, we obtain the differential equation $y' = b\cdot y + y^\alpha$. With $u = y^{1-\alpha}$, this is transformed into a Bernoulli differential equation for which the solution is known.

Proof. The symbol of a pseudo-differential operator has to be continuous negative definite. In Section 3.3 we have shown that the functions $-F$ and $-R_i$, $i = 1, \dots, m$ satisfy this property. In the proof we only used the assumption that $F + x^\top R$ is an affine function along with the geometry of the state space. Therefore, affine processes are the only processes with symbols whose coefficients are affine x-dependent.
It is well known that the admissible parameters and, hence, a symbol with an affine structure give rise to an affine process, cf. Duffie et al. [18, Theorem 2.7]. □

3.5.1 Domain of the Generator

The above theorem shows that the generator A of an affine process is a pseudo-differential operator on the test functions $C_c^\infty(D)$. This operator $(-q(x, D), C_c^\infty(D))$ can easily be extended to an operator on $C_c^2(D)$, see Schnurr [56, Theorem 3.8], by approximating $C_c^2(D)$ functions with a sequence of test functions obtained by a Friedrichs mollifier. This also shows that $C_c^2(D) \subset \mathcal{D}(A)$. Using the B.L.T. theorem we even extend the representation to the weighted space $C^{2,1}_{(1+x_I),|x_{II}|,\infty}(D)$, see Lemma 3.24. In this section, we will characterize the maximal domain of the generator. Using results of Section 2.3, we give a representation of the maximal domain as a Banach space.

Since we investigate function spaces related to the affine generator, we now give a proper introduction to spaces with domain $D = \mathbb{R}_+^m \times \mathbb{R}^n$. The literature, cf. Jacob [27, Section 2.1], defines for an open set $G \subseteq \mathbb{R}^d$ and $k \in \mathbb{N}_0 \cup \{\infty\}$

$$C^k(G) := \{f : G \to \mathbb{C};\ f \text{ is } k\text{-times continuously differentiable}\}.$$

This function space can be extended to the closed domain $\overline{G}$ if we require $\partial^\delta f(x) = 0$ for $x \in \partial G = \overline{G} \backslash G$ and $|\delta| \leq k$. In contrast to that, we assume that a function in $C^k(D)$ takes a finite value on the boundary. Furthermore, we only require ones-sided continuity and one-sided differentiability from the inside at the boundary. For simplicity of notation, we write Df and $\partial_x f$ instead of $D_+ f$ and $\partial_x^+ f$, respectively. Hence, we define

$$C^k(D) := \{f : D \to \mathbb{C};\ D^\delta f \in C(D) \wedge |D^\delta f(0, x_{II})| < \infty \text{ for all } |\delta| \leq k\}.$$

The spaces $C_b(D), C_c(D), C_\infty(D)$ and variations thereof are defined in the same way.

Lemma 3.24. *Let X be an affine process with generator A and the usual admissible parameters as in Theorem 3.17. Then for a function $f \in C^{2,1}_{(1+x_I),|x_{II}|,\infty}(D)$, where*

$$C^{2,1}_{(1+x_I),|x_{II}|,\infty}(D) = \Big\{ f \in C^2(D);\ (1 + x_I) D^\delta f(x) \in C_\infty(D)\ \forall \delta \in \mathbb{N}^d,\ |\delta| \leq 2,$$
$$x_j \frac{\partial}{\partial x_j} f(x) \in C_\infty(D)\ \forall j \in II \Big\},$$

the generator has a representation as an integro-differential operator given by

$$\begin{aligned}
Af(x) &= \sum_{j,k=1}^{d} \Big(a_{jk} + \sum_{l=1}^{m} x_l \alpha^l_{jk}\Big) \partial_j \partial_k f(x) \\
&+ \Big(b + \sum_{l=1}^{d} x_l \beta^l\Big)^\top \nabla f(x) + \Big(c + \gamma^\top x\Big) f(x) \\
&+ \int_{D\setminus\{0\}} \Big(f(x+y) - f(x) - \chi(y)^\top \nabla f(x)\Big) \mu(\,\mathrm{d}y) \\
&+ \sum_{l=1}^{m} x_l \int_{D\setminus\{0\}} \Big(f(x+y) - f(x) - \chi^l(y)^\top \nabla f(x)\Big) \mu^l(\,\mathrm{d}y).
\end{aligned}$$

Proof. The idea is to take the bounded linear transform (B.L.T.) theorem, see for instance Reed and Simon [42, Theorem I.7] or Theorem 2.13. Therefore, we have to prove that $C^{2,1}_{(1+x_I),|x_{II}|,\infty}(D)$ equipped with the norm

$$\begin{aligned}
\|f\|_{(2),(1),(1+|x_I|),|x_{II}|} &:= \|f\|_{(2),(1+|x_I|)} + \sum_{j=m+1}^{d} \Big\| x_j \frac{\partial}{\partial x_j} f \Big\|_\infty \\
&= \sum_{|\delta|\le 2} \Big\| (1+|x_I|) D^\delta f \Big\|_\infty + \sum_{j=m+1}^{d} \Big\| x_j \frac{\partial}{\partial x_j} f \Big\|_\infty
\end{aligned}$$

is a complete normed linear space.[8] Applying the same reasoning as in Example 2.14, i.e. using Lemma 2.15 and 2.16, we see that the above function space is a complete normed linear space. It remains to prove that $C_c^\infty(D)$ is a dense subset.

This argument will be divided into two steps. We first show that $C_c^{2,1}(D)$ is dense in the set $C^{2,1}_{(1+x_I),|x_{II}|,\infty}(D)$. Applying the Friedrichs mollifier gives that $C_c^\infty(D)$ is dense in $C_c^{2,1}(D)$.

Let $\chi \in C_c^\infty(\mathbb{R}^d)$ be a smooth cut-off function such that $\mathbb{1}_{B(0,1)} \le \chi \le \mathbb{1}_{B(0,2)}$ and set $\chi_n(\cdot) := \chi(\cdot/n)$. This function is defined for all $x \in \mathbb{R}^d$ but we restrict the domain to D. Then a similar calculation as in Example 2.14 shows that for every $f \in C^{2,1}_{(1+x_I),|x_{II}|,\infty}(D)$ the sequence $(f_n)_{n\ge1}$ defined by $f_n := f \cdot \chi_n \in C_c^{2,1}(D)$ converges to f with respect to the norm $\|\cdot\|_{(2),(1),(1+|x_I|),|x_{II}|}$.

We now proceed analogously to Example 2.14 and apply a Friedrichs mollifier, see Lemma 2.15, to show that $C_c^\infty(D)$ is dense in $C_c^{2,1}(D)$ with respect to $\|\cdot\|_{(2),(1),(1+|x_I|),|x_{II}|}$. Observe that the mollified functions have to be restricted to the domain D such that they are an element of $C_c^\infty(D)$.

The requirements for the B.L.T. theorem are satisfied and the lemma follows. □

From the representation as an integro-differential operator we see that if a function f lacks one of the conditions required in $C^{2,1}_{(1+x_I),|x_{II}|,\infty}(D)$ the generator does not map this

[8] The weighted norm is defined as in Section 2.3.

function into $C_\infty(D)$. Therefore, we call $C^{2,1}_{(1+x_I),|x_{II}|,\infty}(D)$ the maximal domain of the generator.[9]

For many applications based on the symbol it is important to know that the test functions $C_c^\infty(D)$ are a core of the generator, cf. Definition 1.12. Based on the norms and spaces previously introduced, we show that $C_c^\infty(D)$ are a core. Therefore, we verify that the affine semigroup satisfies the following criterion.

Lemma 3.25. *Let $(T_t)_{t\geq 0}$ be a Feller semigroup, $(A, \mathcal{D}(A))$ the generator and $\mathfrak{D}_0 \subseteq \mathfrak{D} \subseteq \mathcal{D}(A)$ be dense subsets of $C_\infty(D)$. Then $\mathfrak{D}$ is an operator core for $(A, \mathcal{D}(A))$ if*

$$T_t(\mathfrak{D}_0) \subseteq \mathfrak{D} \quad \textit{for all } t \geq 0.$$

This result can be found in Ethier and Kurtz [19, Proposition 3.3, p. 16]. Now we will apply it to the generator of an affine process.

Proposition 3.26. *Let X be an affine process with semigroup $(T_t)_{t\geq 0}$ and generator $(A, \mathcal{D}(A))$. Then $C^{2,1}_{(1+|x_I|),|x_{II}|,\infty}(D)$ is a core of $(A, \mathcal{D}(A))$.*

Our approach to prove this result is based on the proof of Duffie et al. [18, Proposition 8.2] with several simplifications. Especially, the approximation of the functions in our case is easily accessible due to an application of a reflection argument.

Proof. The main idea of the proof is to apply Lemma 3.25. Therefore, we set $\mathfrak{D}_0$ as the linear span of the set

$$\Theta = \left\{ h_{(\xi_I, g)}(x) = \mathrm{e}^{x_I^\top \xi_I} \int_{\mathbb{R}^n} \mathrm{e}^{\mathbf{i} x_{II}^\top z} g(z)\,\mathrm{d}z;\ \xi_I \in \mathbb{C}^m_- \text{ s.t.} \operatorname{Re}\xi_I < 0,\ g \in C_c^\infty(\mathbb{R}^n) \right\}$$

and $\mathfrak{D}_1 = C^{2,1}_{(1+|x_I|),|x_{II}|,\infty}(D)$. Now we have to show that $\operatorname{span}(\Theta) \subseteq C^{2,1}_{(1+|x_I|),|x_{II}|,\infty}(D) \subseteq \mathcal{D}(A)$ is dense in $C_\infty(D)$ and that the semigroup T_t maps Θ into $C^{2,1}_{(1+|x_I|),|x_{II}|,\infty}(D)$.
From Theorem 3.8 we know that the linear span of Θ is dense in $C_\infty(D)$. In order to show that $\mathfrak{D}_0$ is a subset of $\mathfrak{D}_1$, we choose a function $f \in \Theta$. As $\Theta \subseteq C_\infty(D)$, we only need to consider the cases $|x_I| D^{\delta_I} f$ with $\delta_I \in \mathbb{N}_0^m \times \{0\}^n$ and $|\delta_I| \leq 2$ as well as $x_j \frac{\partial}{\partial x_j} f(x) \in C_\infty(D)$ for all $j \in II$.
As $\operatorname{Re}\xi_I < 0$, we see immediately that $x_i f(x) = x_i \mathrm{e}^{x_I^\top \xi_I} \int_{\mathbb{R}^n} \mathrm{e}^{\mathbf{i} x_{II}^\top z} g(z)\,\mathrm{d}z$ vanishes for $x_i \to \infty$, for arbitrary $i \in \{1, \dots, m\}$. The terms $x_I D^{\delta_I} f$ can be treated in a similar way. For the second case, let $j, k \in \{m+1, \dots, d\}$ and, as usual, denote by $\partial_j = \frac{\partial}{\partial x_j}$ the partial derivative with respect to the variable x_j. Then we have by the Riemann-Lebesgue Lemma for Fourier transforms

$$\begin{aligned} \left| x_j \partial_k \int_{\mathbb{R}^n} \mathrm{e}^{\mathbf{i} x_{II}^\top z} g(z)\,\mathrm{d}z \right| &= \left| x_j \int_{\mathbb{R}^n} \mathrm{e}^{\mathbf{i} x_{II}^\top z} z_k g(z)\,\mathrm{d}z \right| \\ &= \left| \int_{\mathbb{R}^n} \mathrm{e}^{\mathbf{i} x_{II}^\top z} \underbrace{\partial_j (z_k g(z))}_{\in C_c^\infty(\mathbb{R}^n)}\,\mathrm{d}z \right| \longrightarrow 0 \quad \text{as } |x_j| \to \infty. \end{aligned}$$

[9] Note that the integro-differential operator is defined on a larger space, in particular, on $C_b^2(D)$.

Hence, we conclude that $\operatorname{span}(\Theta) \subseteq C^{2,1}_{(1+|x_I|),|x_{II}|,\infty(D)} \subseteq C_\infty(D)$.
It remains to show that $T_t f \in C^{2,1}_{(1+|x_I|),|x_{II}|,\infty}(D)$ for $f \in \Theta$. We recall from the proof of Theorem 3.8 that for $f \in \Theta$ we have

$$T_t f(x) = \int \mathrm{e}^{x_{II}^\top (\mathrm{e}^{t\beta}\mathbf{i}z)} \mathrm{e}^{\phi(t,(v,\mathbf{i}z))} \mathrm{e}^{x_I^\top \psi_I(t,(\xi_I,\mathbf{i}z))} g(z)\,\mathrm{d}z,$$

and, of course, that $T_t f \in C_\infty(D)$ as T_t is Feller. Since $\operatorname{Re} v < 0$, Proposition 3.7 shows that $\operatorname{Re}\psi_I(t,(\xi_I,\mathbf{i}z)) < 0$, hence that $x_i \mathrm{e}^{x_I^\top \psi_I(t,(v,\mathbf{i}z))}$ is bounded for $i \in \{1,\dots,m\}$, and finally that we can apply the dominated convergence theorem. This yields

$$\lim_{x_i\to\infty} x_i T_t f(x) = \int \mathrm{e}^{\mathbf{i}x_{II}^\top (\mathrm{e}^{t\beta} z)} \mathrm{e}^{\phi(t,(\xi_I,\mathbf{i}z))} \underbrace{\lim_{x_i\to\infty} x_i \mathrm{e}^{x_I^\top \psi_I(t,(\xi_I,\mathbf{i}z))}}_{=0} g(z)\,\mathrm{d}z = 0.$$

In a similar way, we obtain for $i \in \{1,\dots,m\}$, $l \in \{1,\dots,m\}$

$$\begin{aligned}\lim_{x_i\to\infty} x_i \partial_l T_t f(x) &= \lim_{x_i\to\infty} \int \mathrm{e}^{\mathbf{i}x_{II}^\top (\mathrm{e}^{t\beta} z)} \mathrm{e}^{\phi(t,(\xi_I,\mathbf{i}z))} x_i \partial_l \mathrm{e}^{x_I^\top \psi_I(t,(\xi_I,\mathbf{i}z))} g(z)\,\mathrm{d}z \\ &= \lim_{x_i\to\infty} \int \mathrm{e}^{\mathbf{i}x_{II}^\top (\mathrm{e}^{t\beta} z)} \mathrm{e}^{\phi(t,(\xi_I,\mathbf{i}z))} x_i \mathrm{e}^{x_I^\top \psi_I(t,(\xi_I,\mathbf{i}z))} \psi_l(t,(\xi_I,\mathbf{i}z)) g(z)\,\mathrm{d}z.\end{aligned}$$

Due to the boundedness of $x_i \mathrm{e}^{x_I^\top \psi_I(t,(\xi_I,\mathbf{i}z))}$, we can use the dominated convergence theorem and, as $x_i \mathrm{e}^{x_I^\top \psi_I(t,(v,\mathbf{i}z))}$ vanishes for $x_i \to \infty$, so does $x_i \partial_l T_t f(x)$.
Next, we show that $x_j T_t f(x) \to 0$ for $|x_j| \to \infty$ with $j \in II$. Therefore, note that $T_t f \in \mathcal{D}(A)$ and $AT_t f = \frac{\mathrm{d}}{\mathrm{d}t} T_t f$, cf. Schilling and Partzsch [51, Lemma 7.10]. Using the above representation of $T_t f$ and the differentiation of parameter dependent integrals – note that its requirements are met as ϕ and ψ are differentiable in t and jointly continuous – we get

$$\begin{aligned} AT_t f(x) &= \frac{\mathrm{d}}{\mathrm{d}t} T_t f(x) \\ &= \int \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{\phi(t,(\xi_I,\mathbf{i}z))} \mathrm{e}^{x^\top \psi(t,(\xi_I,\mathbf{i}z))} g(z)\,\mathrm{d}z \\ &= \int \Big(F\left(\psi(t,(\xi_I,\mathbf{i}z))\right) + x^\top R\left(\psi(t,(\xi_I,\mathbf{i}z))\right) \Big) \underbrace{\mathrm{e}^{\phi(t,(\xi_I,\mathbf{i}z))} \mathrm{e}^{x^\top \psi(t,(\xi_I,\mathbf{i}z))}}_{=T_t \mathrm{e}_{(\xi_I,\mathbf{i}z)}(x)} g(z)\,\mathrm{d}z \\ &= \int \Big(\underbrace{F\left(\psi(t,(\xi_I,\mathbf{i}z))\right) + x_I^\top R_I\left(\psi(t,(\xi_I,\mathbf{i}z))\right)}_{=:h_{x_I,\xi_I}(z) \in C(\mathbb{R}^n)} \\ &\qquad + x_{II}^\top (\beta \mathrm{e}^{\beta t} \mathbf{i}z) \Big) T_t \mathrm{e}_{(\xi_I,\mathbf{i}z)}(x) g(z)\,\mathrm{d}z \\ &= \int h_{x_I,\xi_I}(z) T_t \mathrm{e}_{(\xi_I,\mathbf{i}z)}(x) g(z)\,\mathrm{d}z + \int x_{II}^\top (\beta \mathrm{e}^{\beta t}\mathbf{i}z) T_t \mathrm{e}_{(\xi_I,\mathbf{i}z)}(x) g(z)\,\mathrm{d}z \\ &= \int \mathrm{e}^{x_{II}^\top (\mathrm{e}^{t\beta}\mathbf{i}z)} \underbrace{h_{x_I,\xi_I}(z) \mathrm{e}^{\phi(t,(\xi_I,\mathbf{i}z))} \mathrm{e}^{x_I^\top \psi_I(t,(\xi_I,\mathbf{i}z))} g(z)}_{=:\tilde{h}_{x_I,\xi_I}(z) \in C_c(\mathbb{R}^n)}\,\mathrm{d}z + x_{II}^\top \beta \nabla_{II} T_t f(x). \end{aligned}$$

As $T_t f \in \mathcal{D}(A)$, we have $AT_t f \in C_\infty(D)$. The integral in the last line is a shifted Fourier transform and, thus, by the Riemann-Lebesgue lemma it vanishes as $|x_j| \to \infty$. Since the left-hand side as well as the integral term on the right-hand side is in $C_\infty(D)$, we deduce that $x_{II}^\top \beta \nabla_{II} T_t f(x) \in C_\infty(D)$ and, in particular, that $x_j \partial_j T_t f(x)$ vanishes as $|x_j| \to \infty$. Hence, we have shown that $T_t f \in C^{2,1}_{(1+|x_I|),|x_{II}|,\infty}$. Now Lemma 3.25 applies and the assertion that $C^{2,1}_{(1+|x_I|),|x_{II}|,\infty}(D)$ is a core holds.

□

Lemma 3.27. *Let X be an affine process with generator $(A, \mathcal{D}(A))$. Then $C_c^\infty(D)$ is a core of $(A, \mathcal{D}(A))$.*

Proof. By the definition of the operator core, cf. Definition 1.12, it is sufficient to show that the smooth functions with compact support are dense in the core $C^{2,1}_{(1+|x_I|),|x_{II}|,\infty}(D)$ with respect to the graph norm $\|\cdot\|_\infty + \|A\cdot\|_\infty$.
Let us first remark that the generator has a representation as an integro-differential operator for all functions $f \in C^{2,1}_{(1+|x_I|),|x_{II}|,\infty}(D)$, see Lemma 3.24. For $f \in C^{2,1}_{(1+|x_I|),|x_{II}|,\infty}(D)$, we obtain

$$\begin{aligned}
\|Af\|_\infty \le &\sum_{|\delta|=2} \Big\|(1+|x_I|)D^\delta f\Big\|_\infty (\|a\| + \sum_{i=1}^m \|\alpha^i\|) \\
&+ \sum_{|\delta|=1} \Big\|(1+|x_I|)D^\delta f\Big\|_\infty (\|b\| + \sum_{i=1}^m \|\beta^i\|) + \sum_{j=m+1}^d \Big\|x_j \frac{\partial}{\partial x_j} f\Big\|_\infty \|\beta^j\| \\
&+ \|(1+|x_I|)f\|_\infty \\
&+ \|(1+|x_I|)f\|_{(2)} \Big(\int_{D\setminus\{0\}} \big((|x_I| + |x_{II}|^2) \wedge 1\big)\, \mu(\,\mathrm{d}x) \\
&\qquad + \sum_{i=1}^m \int_{D\setminus\{0\}} \big((|x_{I\setminus\{i\}}| + |x_{II\cup\{i\}}|^2) \wedge 1\big)\, \mu^i(\,\mathrm{d}x) \Big) \\
&\le c_A \Big(\sum_{|\delta|\le 2} \Big\|(1+|x_I|)D^\delta f\Big\|_\infty + \sum_{j=m+1}^d \Big\|x_j \frac{\partial}{\partial x_j} f\Big\|_\infty \Big) \\
&= \|f\|_{(2),(1),(1+|x_I|),|x_{II}|}.
\end{aligned}$$

The estimate of the integral terms is based on an application of Taylor's theorem as used in the proof of Theorem 3.21. Applying the same argument as in Example 2.14, we deduce that $C_c^\infty(D)$ is dense in $C^{2,1}_{(1+|x_I|),|x_{II}|,\infty}(D)$ with respect to the norm $\|\cdot\|_{(2),(1),(1+|x_I|),|x_{II}|}$. Therefore, the smooth functions with compact support, $C_c^\infty(D)$, are a core. □

3.6 Path Properties

In this section, we take a short look at path properties of affine processes. We will use some of the results from Section 2.2. However, we obtain further results due to the explicit structure of the symbol and knowledge of the semigroup of the affine process. In particular, we attain an easily usable criterion based on the symbol for the conservativeness of affine processes.

Mayerhofer, Muhle-Karbe and Smirnov [39, Theorem 3.4] have given a full characterization of conservativeness for affine processes.

Theorem 3.28. *An affine process X is conservative if and only if its functional characteristics holds that $F(0) = 0$ and that there exists no non-trivial $\mathbb{R}^m_-$-valued local solution $g(t)$ of*

$$\partial_t g(t) = R_I(g(t), 0)$$

with $g(0) = 0$.

It is quite interesting to see that the real-valued coordinates have no influence on the conservativeness condition. Note that on the real space an affine process is an Ornstein-Uhlenbeck process which does not explode. Hence, it is not surprising that only the $\mathbb{R}^m_+$ components affect the conservativeness. However, it is not clear whether the half-space coordinates can induce explosion to the real-space coordinates. This problem also appears in the criterion for conservativeness based on the symbol, cf. Theorem 2.8.

Corollary 3.29. *An affine process X with admissible parameters as in Theorem 3.17 is conservative if $c = 0$, $\gamma^i = 0$ and*

$$\int_{D\setminus\{0\}} |y| \mu^i(\,\mathrm{d}y) < \infty$$

for all $i = 1, \dots, m$.

Proof. We use the criterion of Theorem 2.8 which states that a stochastic process is non-explosive if the corresponding symbol $q(x, \xi)$ is locally bounded, satisfies $q(x, 0) = 0$ and

$$\liminf_{k\to\infty} \sup_{|y-x|\le k} \sup_{|\xi|\le \frac{1}{k}} |q(y, \xi)| < \infty$$

for all $x \in D$.
We start by considering the constant part of the affine symbol $q(x, \xi) = F(\mathbf{i}\xi) + x^\top R(\mathbf{i}\xi)$,

$$\begin{aligned}
&\liminf_{k\to\infty} \sup_{|y-x|\le k} \sup_{|\xi|\le \frac{1}{k}} |-F(\mathbf{i}\xi)| \\
&\quad\le \lim_{k\to\infty} \frac{1}{2}|a|\frac{1}{k^2} + |b|\frac{1}{k} + \sup_{|\xi|\le\frac{1}{k}} \left| \int_{D\setminus\{0\}} \left(1 - \mathrm{e}^{\mathbf{i}\xi^\top z} + \mathbf{i}\xi^\top \chi(z)\right) \mu(\,\mathrm{d}z)\right|.
\end{aligned}$$

Since the integral exists and is finite for all ξ, we get by dominated convergence that $\lim_{k\to\infty} \sup_{|y-x|\le k} \sup_{|\xi|\le\frac{1}{k}} |-F(\mathbf{i}\xi)| = 0$.

We continue with the linear part of the affine symbol, namely $x^\top R(\mathbf{i}\xi)$, which we split up in a local and a non-local part,

$$\begin{aligned}
&\liminf_{k\to\infty} \sup_{|y-x|\le k} \sup_{|\xi|\le\frac{1}{k}} |\sum_{i=1}^{m} y_i \frac{1}{2}\xi^\top\alpha^i\xi - \mathbf{i}\sum_{i=1}^{m+n} y_i {\beta^i}^\top\xi| \\
&\le \liminf_{k\to\infty} \sum_{i=1}^{m}(|x|+k)\frac{1}{2}|\alpha^i|\frac{1}{k^2} + \sum_{i=1}^{m+n}(|x|+k)|\beta^i|\frac{1}{k} \\
&\le \|\beta\| < \infty.
\end{aligned}$$

Hence, the local part always meets the requirement.
For the non-local part, fix $i\in\{1,\dots,m\}$. If we use $\chi^i(z) = \mathbb{1}_{\{|z_{II+}|\le 1\}} z_{II+}$ as truncation function, we get[10]

$$\begin{aligned}
&1 - \mathrm{e}^{\mathbf{i}\xi^\top z} + \mathbf{i}\xi^\top\chi^i(z) \\
&\quad = \mathrm{e}^{\mathbf{i}\xi^\top (z - z_{I-})} - \mathrm{e}^{\mathbf{i}\xi^\top z} + 1 - \mathrm{e}^{\mathbf{i}\xi^\top (z - z_{I-})} + \mathbf{i}\xi^\top\chi^i(z) \\
&\quad = \mathrm{e}^{\mathbf{i}\xi^\top z_{II+}}\Big(1 - \mathrm{e}^{\mathbf{i}\xi^\top z_{I-}}\Big) \\
&\qquad + \mathbb{1}_{\{|z_{II+}|\le 1\}}\Big(1 - \mathrm{e}^{\mathbf{i}\xi^\top z_{II+}} + \mathbf{i}\xi^\top z_{II+}\Big) + \mathbb{1}_{\{|z_{II+}|> 1\}}\Big(1 - \mathrm{e}^{\mathbf{i}\xi^\top z_{II+}}\Big).
\end{aligned}$$

Taking the absolute value and applying Taylor's theorem for each term yields

$$\begin{aligned}
&\Big|1 - \mathrm{e}^{\mathbf{i}\xi^\top z} + \mathbf{i}\xi^\top\chi^i(z)\Big| \\
&\quad \le \Big|\mathrm{e}^{\mathbf{i}\xi^\top z_{II+}}\Big|\Big|1 - \mathrm{e}^{\mathbf{i}\xi^\top z_{I-}}\Big| \\
&\qquad + \mathbb{1}_{\{|z_{II+}|\le 1\}}\Big|1 - \mathrm{e}^{\mathbf{i}\xi^\top z_{II+}} + \mathbf{i}\xi^\top z_{II+}\Big| + \mathbb{1}_{\{|z_{II+}|> 1\}}\Big|1 - \mathrm{e}^{\mathbf{i}\xi^\top z_{II+}}\Big| \\
&\quad \le c_1|z_{I-}|\cdot|\xi| + \mathbb{1}_{\{|z_{II+}|\le 1\}}|z_{II+}|^2\cdot|\xi|^2 + \mathbb{1}_{\{|z_{II+}|> 1\}}|z_{II+}|\cdot|\xi|.
\end{aligned}$$

Substituting this into the criterion, we obtain for $i\in\{1,\dots,m\}$

$$\begin{aligned}
&\liminf_{k\to\infty} \sup_{|y-x|\le k} \sup_{|\xi|\le\frac{1}{k}} \Big|y_i\int_{D\setminus\{0\}}\Big(1 - \mathrm{e}^{\mathbf{i}z^\top\xi} + \mathbf{i}\xi^\top\chi^i(z)\Big)\mu^i(\,\mathrm{d}z)\Big| \\
&\quad \le \liminf_{k\to\infty} \sup_{|\xi|\le\frac{1}{k}} (|x|+k)\int_{D\setminus\{0\}}\Big(c_1|z_{I-}|\cdot|\xi| \\
&\qquad + \mathbb{1}_{\{|z_{II+}|\le 1\}}|z_{II+}|^2\cdot|\xi|^2 + \mathbb{1}_{\{|z_{II+}|> 1\}}|z_{II+}|\cdot|\xi|\Big)\mu^i(\,\mathrm{d}z) \\
&\quad \le \liminf_{k\to\infty} c(|x|+k)\frac{1}{k}\int_{D\setminus\{0\}}\Big(|z_{I-}| + |z_{II+}|\Big)\mu^i(\,\mathrm{d}z) \\
&\quad \le C\int_{D\setminus\{0\}}|z|\mu^i(\,\mathrm{d}z),
\end{aligned}$$

where the second term in the second line can be neglected because of the integrability condition for μ^i. $\square$

[10] Again, for $i\in\{1,\dots,m\}$ we denote $II+ = II\cup\{i\} = \{i, m+1,\dots,m+n\}$ and $I- = I\setminus\{i\} = \{1,\dots,i-1,i+1,\dots,m\}$.

This criterion is interesting because it is easier to check than the condition of Theorem 3.28. However, we should mention that the criterion based on the symbol is sufficient but not necessary as the following example, see Mayerhofer et al. [39, Example 3.6] for more details, shows.

Example 3.30. Let δ_k be the Dirac measure supported by the one-point set $\{k\}$. We define the parameters of an affine process on the state space $D = \mathbb{R}_+$, i.e. $m = 1$ and $n = 0$, by

$$a = 0, \quad \alpha^1 = 0, \quad b = 0, \quad \beta^1 = \sum_{k=1}^{\infty} \frac{1}{k^2}, \quad c = 0, \quad \gamma^1 = 0, \quad \mu = 0, \quad \mu^1 = \sum_{k=1}^{\infty} \frac{\delta_k}{k^2}.$$

Then the corresponding affine process X does not satisfy the condition of Corollary 3.29 as

$$\int_{\mathbb{R}_+\setminus\{0\}} |y|\mu^1(\,\mathrm{d}y) = \sum_{k=1}^{\infty} \frac{|k|}{k^2} = \infty.$$

However, Mayerhofer et al. [39, Example 3.6] have shown that this process is conservative.

In Section 2.2 we have introduced the so-called Blumenthal-Getoor-Pruitt indices, see Definition 2.4. Now we will use these indices to characterize sample path properties of affine processes.

Example 3.31. Consider an Ornstein-Uhlenbeck process driven by a Lévy process L, i.e. $\mathrm{d}X = \sum_{j=1}^n X_j \beta^j \,\mathrm{d}t + \,\mathrm{d}L$.
Then X is an affine process on $\mathbb{R}^n$ with symbol $q(x,\xi) = -F(\mathbf{i}\xi) - \mathbf{i}\sum_{j=1}^n \mathbf{i}x_j\xi^\top\beta^j$, where $-F(\mathbf{i}\xi) = \psi_L(\xi)$ with ψ_L as the characteristic exponent of the Levy process L. The quantity $H(x,R)$ is given by

$$\begin{aligned} H(x,R) &:= \sup_{\|y-x\|\le 2R} \sup_{\|\varepsilon\|\le 1} \left| q\Big(y, \frac{\varepsilon}{R}\Big)\right| \\ &\le \sup_{\|y-x\|\le 2R} \sup_{\|\varepsilon\|\le 1} \left| \sum_{j=1}^n x_j \frac{\varepsilon^\top \beta^j}{R} \right| + \sup_{\|\varepsilon\|\le 1} \left| \psi_L\Big(\frac{\varepsilon}{R}\Big)\right| \\ &\le (|x| + 2R)\|\beta\| \frac{1}{R} + \sup_{\|\varepsilon\|\le 1} \left| \psi_L\Big(\frac{\varepsilon}{R}\Big)\right|. \end{aligned}$$

Hence we get for $x \neq 0$ the indices β_∞^x and $\underline{\beta_\infty^x}$

$$\begin{aligned} &\beta_\infty^x = 1 \vee \beta_\infty^L \\ &\underline{\beta_\infty^x} = 1 \vee \underline{\beta_\infty^L}, \end{aligned}$$

where β_∞^L and $\underline{\beta_\infty^L}$ are the corresponding indices of the Lévy process L.

Example 3.32. The symbol of an affine process is given by

$$q(x,\xi) = -F(\mathbf{i}\xi) - \sum_{i=1}^{m} x_i R_i(\mathbf{i}\xi) - \sum_{j=m+1}^{m+n} \mathbf{i} x_j {\beta^j}^\top \xi.$$

Considering the terms separately, we get for the quantity $H(x,R)$ that

$$H(x,R) \leq H^F(R) + \sum_{i=1}^{m}(|x_i| + 2R)H^{R_i}(R) + (|x_{II}| + 2R)\frac{1}{R},$$

where H^F and H^{R_i} are the quantities corresponding to the negative definite functions F and R_i for $i = 1, \dots, m$. Thus we get for the index of $q(x,\xi)$ at infinity for $x \neq 0$

$$\beta_\infty^x = \beta_\infty^F \vee (\sup_{i\in\{1,\dots,m\}} \beta_\infty^{R_i}) \vee 1.$$

We can describe the asymptotic sample path behaviour of an affine process similar to Example 2.6 since the same reasoning applies here.
Having the Blumenthal-Getoor-Pruitt indices, we can determine the strong variation of a process. Therefore, let $p \in (0,\infty)$ and $f : [0,\infty) \to \mathbb{R}^d$ be a (non-random) càdlàg function. Then the (strong) p-variation is defined as

$$V_p(f,[0,t]) = \sup \sum_{l=0}^{k-1} |f(t_{l+1}) - f(t_l)|^p,$$

where the supremum is taken over all finite partitions $0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = t$, $k \geq 1$, of the interval $[0,t]$.
The following criterion for the p-variation of a Feller process is given by Böttcher, Schilling and Wang [11, Proposition 5.21].

Proposition 3.33 (p-variation)**.** *Let $(X_t)_{t\geq 0}$ be a Feller process with symbol $q(x,\xi)$ and denote by $\beta^* := \sup_K \beta_\infty^K$, where the supremum ranges over all compact sets $K \subset \mathbb{R}^d$. Then*

$$\mathbb{P}^x(V_p(X,[0,t]) < \infty) = 1 \quad \forall p > \beta^*,\ x \in \mathbb{R}^d,\ t > 0.$$

As we have seen in Example 3.32, the index is independent of x for $x \neq 0$. Hence, we get for an affine process $\beta^* = \beta_\infty^F \vee (\sup_{i\in\{1,\dots,m\}} \beta_\infty^{R_i}) \vee 1$, It also shows that for any $p > \beta_\infty^F \vee (\sup_{i\in\{1,\dots,m\}} \beta_\infty^{R_i}) \vee 1$ the p-variation of the sample function $(X_t)_{t\geq 0}$ is finite almost surely.
The Besov regularity is concluded for affine processes in the same way as the p-variation, cf. Böttcher et al. [11, Proposition 5.31].

We close this section with an improvement of the upper bound for an affine process. The following result is an extension of Knopova and Schilling [34, Proposition 9] for affine processes similar to Example 2.3.

Proposition 3.34. *Let X be an affine process with corresponding symbol $q(x,\xi)$ and admissible parameters as in Theorem 3.17 such that, in addition,*

$$\int_{D\setminus\{0\}} |y|\mu(\,\mathrm{d}y) < \infty$$
$$\int_{D\setminus\{0\}} |y|\mu^i(\,\mathrm{d}y) < \infty$$

for all $i = 1, \ldots, m$ and $q(x,0) = 0$ holds. Then

$$\lim_{t\to 0} \frac{\sup_{0\leq s\leq t} |X_s - x|}{\sqrt{t|\log(t)|^{1+\varepsilon}}} = 0 \quad (\mathbb{P}^x - a.s.),$$

where $\varepsilon > 0$.

We see that the small time sample path behaviour does not depend on x, i.e. it is independent of the current position of the process.
We skip the proof as its outline is analogous to the argument used in Example 2.3, only a minor change in the estimation of the jump part of the symbol is necessary. These calculations are analogous to the proof of Corollary 3.29.

Chapter 4

Affine Processes on Positive Semidefinite Matrices

In recent years, research in stochastic processes has laid more and more focus on positive semidefinite matrices as a non-trivial state space. This development was triggered by the need for and existence of new models, especially in mathematical finance, which applies affine processes defined on positive semidefinite matrices. There are a lot of differences to the canonical state space $\mathbb{R}_+^m \times \mathbb{R}^n$, which we will point out in the following. If possible, however, i.e. if the proofs are analogous to the canonical state space, we will refer to the corresponding results in Chapter 3.
In Section 4.1, we compile relevant basic facts concerning the space of positive semidefinite matrices. Section 4.2 provides an introduction to affine processes on this state space including some properties of affine processes, in particular, the Feller property. Using techniques from harmonic analysis, we examine in Section 4.3 the characteristic functions of an affine process and characterize the admissibility of the parameters. Section 4.4 provides the representation of the generator of an affine process as a pseudo-differential operator. Furthermore, we determine the symbol of an affine process and derive a further condition for the parameters. The last section outlines some applications based on the symbol of an affine process.

4.1 Some Facts on S_d^+

The space of positive semidefinite matrices differs in many aspects from the vector space $\mathbb{R}^d$ and from the half-space $\mathbb{R}_+^d$. Therefore, we give a brief introduction to this space and show some properties. Further, we set up notation and terminology. Observe that it is possible to regard the space of positive semidefinite matrices as an example of a symmetric cone. However, we will not develop this point here. At the end of the section, we indicate this connection and give references.

Definition 4.1. The space of $d \times d$ dimensional positive semidefinite matrices is given by

$$\begin{aligned} S_d^+ &= \Big\{x \in \mathbb{R}^{d\times d};\ x \text{ symmetric positive semidefinite}\Big\} \\ &= \Big\{x \in \mathbb{R}^{d\times d};\ x = x^\top,\ \forall y \in \mathbb{R}^d :\ y^\top x y \geq 0\Big\}. \end{aligned}$$

Obviously, S_d^+ is a subspace of the space of symmetric $d \times d$ dimensional matrices, $S_d = \big\{x \in \mathbb{R}^{d\times d};\ x \text{ symmetric}\big\}$.

The standard basis in S_d is given by $\{c^{ij},\ i \leq j\}$ with $c_{kl}^{ij} = \delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}(1-\delta_{ij})$, where δ_{ij} denotes the Kronecker symbol. Furthermore, there exists a basis consisting of positive semidefinite matrices $\{e^{ij},\ i \leq j\}$, where

$$e^{ij} = \begin{cases} c^{ii} & \text{if } i = i \\ c^{ii} + c^{ij} + c^{jj} & \text{if } i \neq j. \end{cases}$$

Example 4.2. Let $d = 3$. Then basis matrices are given by

$$c^{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad c^{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad c^{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and by

$$e^{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e^{12} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e^{13} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

On S_d, we define a scalar product,

$$\langle x, y\rangle := \mathrm{Tr}(xy) = \sum_{i=1}^d \sum_{k=1}^d x_{ik} y_{ki}.$$

Note that S_d is isomorphic, but not isometric, to the Euclidean space $\mathbb{R}^{d(d+1)/2}$. This can be realized by the mapping

$$\begin{aligned} \mathrm{vec} : S_d &\to \mathbb{R}^{d(d+1)/2}; \\ x = (x_{ik})_{i,k=1,\dots,d} &\mapsto (x_{11}, \sqrt{2}x_{21}, \dots, \sqrt{2}x_{d1}, x_{22}, \sqrt{2}x_{32}, \dots, \sqrt{2}x_{d(d-1)}, x_{dd})^\top. \end{aligned}$$

With this mapping, we have

$$\langle x, y\rangle_{S_d} = Tr(xy) = \mathrm{vec}(x)^\top \mathrm{vec}(y).$$

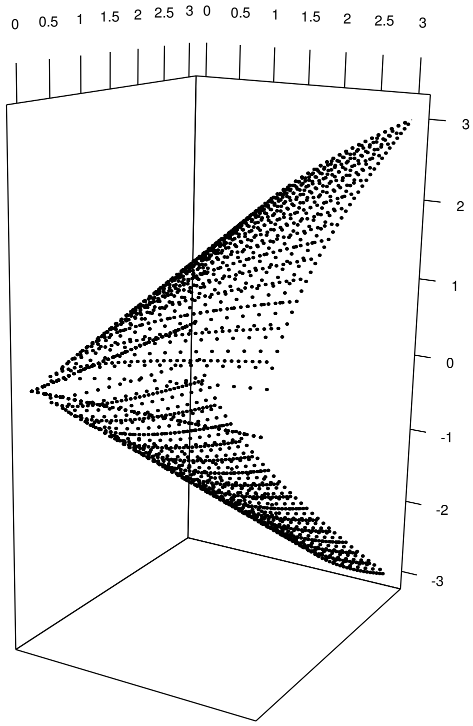

Figure 4.1: Isomorphic representation of S_2^+ in $\mathbb{R}^3$

For example, the isomorphic representation of S_2^+ in $\mathbb{R}^3$ is an open cone. Figure 4.1 shows that even in this simple example the boundary of the state space is curved and non-trivial, getting more complex in higher dimensions. The boundary of S_d^+ is given by $\partial S_d^+ = S_d^+ \backslash S_d^{++}$, where S_d^{++} are the symmetric strictly positive definite matrices. A partial and strict order relation is induced on S_d by the two cones, S_d^+ and S_d^{++},

$$x \preceq y \text{ if } y - x \in S_d^+ \text{ and } x \prec y \text{ if } y - x \in S_d^{++}.$$

Without further proof, there are more representations of S_d^+,

$$\begin{aligned} S_d^+ &= \left\{ x \in S_d;\ \forall y \in S_d^+ :\ \langle x, y \rangle \geq 0 \right\} \\ &= \left\{ x^2;\ x \in S_d \right\}. \end{aligned}$$

The first equality shows that S_d^+ is self-dual. The second representation is valuable for further generalizations of the state space of an affine process.[1] The cone S_d^+ can be defined in the same way as general symmetric cones. In a Euclidean vector space V with an inner product $\langle \cdot, \cdot \rangle$ and a multiplication $\circ$, i.e. V is an algebra over a field, these symmetric cones K are characterized by $K = \{x \circ x;\ x \in V\}$. Hence, S_d^+ is an accessible example of a more general class of state spaces. Especially, most results carry over to the general case with only few modifications.

[1] Cuchiero, Keller-Ressel, Mayerhofer and Teichmann [16] studied affine processes on symmetric cones.

We use similar function spaces as for the canonical state space and, therefore, adopt the usual notation. Observe that the boundary $\partial S_d^+ = S_d^+ \backslash S_d^{++}$ belongs to S_d^+. Hence, we assume that a function in $C^k(S_d^+)$ as well as its derivatives take finite values on the boundary. Moreover, we only require one-sided continuity and one-sided differentiability from the inside at the boundary.[2] If not mentioned otherwise, the function spaces are equipped with the supremum norm, $\|u\|_\infty := \sup_{x \in S_d^+} |u(x)|$. Furthermore, the Fourier transform for an integrable function $u \in L^1(S_d, \mathrm{d}x)$ is defined as

$$\mathcal{F}(u)(\xi) = \hat{u}(\xi) = (2\pi)^{-(d+1)d/2} \int_{S_d} \mathrm{e}^{-\langle x, \mathbf{i}\xi \rangle} u(x)\, \mathrm{d}x.$$

Observe that the isomorphic representation of S_d as a vector space $\mathbb{R}^{(d+1)d/2}$ allows us to reduce the above definition to a Fourier transform on a vector space

$$\mathcal{F}(u)(\xi) = \hat{u}(\xi) = (2\pi)^{-(d+1)d/2} \int_{\mathbb{R}^{(d+1)d/2}} \mathrm{e}^{-\mathbf{i}x^\top \mathrm{vec}(\xi)} u(\mathrm{vec}^{-1}(x))\, \mathrm{d}x.$$

Therefore, we can take two different viewpoints. We can reduce the space S_d^+ to a vector space and derive many results from the known case or we can directly work within the space of symmetric positive semidefinite matrices and use its characteristics.
The next example, a quadratic form $A : S_d \mapsto S_d$, reveals these different approaches. By the isomorphism we can write the quadratic form as $A(x) = \mathrm{vec}(x)^\top A \mathrm{vec}(x)$, where $A \in \mathbb{R}^{d(d+1)/2 \times d(d+1)/2}$ is a symmetric matrix. This yields $A(x) = \sum_{i,j,k,l=1}^{d} x_{ij} A_{ijkl} x_{kl}$, where $A \in \mathbb{R}^{d \times d \times d \times d}$, and thus, by abuse of notation, $A(x) = \sum_{i,j=1}^{d} x_{ij} (\sum_{k,l=1}^{d} A_{ijkl} x_{kl}) = \langle x, Ax \rangle$. The last expression again is similar to the well-known $\mathbb{R}^d$ case where we have $A(x) = x^\top Ax$.

4.2 Introduction to Affine Processes on S_d^+

The definition of an affine process on S_d^+ is very similar to the canonical state space. The Fourier-Laplace transform of the transition probability has to have exponential-affine dependence on the initial state. We look at some examples which will show the motivation of an affine process on positive semidefinite matrices. After that, we present several basic properties and observe that they are obtained by minor modifications of the methods from Chapter 3. In particular, the Feller property is derived by the same argument.

Consider a time-homogeneous Markov process $X = (X_t)_{t \geq 0}$ with state space S_d^+, then the semigroup $(T_t)_{t \geq 0}$ is given by

$$T_t f(x) = \int_{S_d^+} f(y) p_t(x, \mathrm{d}y), \quad x \in S_d^+$$

for bounded Borel measurable functions on S_d^+, $f \in B_b(S_d^+)$.

[2]This definition is similar to that of Chapter 3, see page 55.

Definition 4.3 (Affine process)**.** A Markov process X on S_d^+ and its semigroup $(T_t)_{t\geq 0}$ is called affine if it is stochastically continuous, i.e. $\lim_{s\to t} p_s(x,\cdot) = p_t(x,\cdot)$ weakly on S_d^+ for every $t \geq 0$ and $x \in S_d^+$, and if its Fourier-Laplace transform has exponential-affine dependence on the initial spate, i.e.

$$T_t \mathrm{e}^{\langle \cdot,\xi\rangle}(x) = \int_{S_d^+} \mathrm{e}^{\langle y,\xi\rangle} p_t(x,\,\mathrm{d}y) = \mathrm{e}^{\phi(t,\xi)+\langle\psi(t,\xi),x\rangle} \tag{4.1}$$

for all $t \geq 0$ and $x \in S_d^+$ and $\xi \in \mathcal{U} := \{-S_d^+ + \mathbf{i}S_d\}$, for some functions $\phi : \mathbb{R}_+ \times \mathcal{U} \to \mathbb{C}_-$ and $\psi : \mathbb{R}_+ \times \mathcal{U} \to \mathcal{U}$.

In the literature, the definition of an affine process on a symmetric cone, especially, on positive semidefinite matrices, is sometimes based on the Laplace transform, i.e. $\xi \in \mathcal{U} = S_d^+$ and $T_t \mathrm{e}^{-\langle\cdot,\xi\rangle}(x) = \mathrm{e}^{-\phi(t,\xi)-\langle\psi(t,\xi),x\rangle}$. Although an affine process is fully characterized the Laplace transform we choose the more general approach as we want to consider pseudo-differential operators.
Moreover, we include the stochastic continuity in the definition as it is an essential assumption for our further study of the generator of an affine process.

Example 4.4. In the following examples, $\sqrt{x}$ denotes the symmetric positive semidefinite square root of $x \in S_d^+$.

i) The Wishart process was first defined by Bru [13]. A stochastic process $S = (S_t)_{t\geq 0}$ is called a Wishart process if it is governed by the stochastic differential equation[3]

$$\mathrm{d}S_t = \sqrt{S_t}\,\mathrm{d}B_t + \mathrm{d}B_t^\top \sqrt{S_t} + \delta I\,\mathrm{d}t,$$

where $S_0 = s \in S_d^+$ is the initial value, $\delta > d-1$ and $(B_t)_{t\geq 0}$ is a $d\times d$ matrix-valued Brownian motion. The Fourier-Laplace transform is given by

$$\mathbb{E}^s\Big(\mathrm{e}^{\langle S_t,\xi\rangle}\Big) = \big(\det(I+2t\xi)\big)^{-\delta/2} \exp\Big(-\langle s,(I+2t\xi)^{-1}\xi\rangle\Big).$$

Hence, we have

$$\phi(t,\xi) = \frac{\delta}{2}\log\big(\det(I+2t\xi)\big), \quad \psi(t,\xi) = (I+2t\xi)^{-1}\xi.$$

Comparing this with Example 3.2.ii, we see that the Wishart process is a matrix-valued generalization of the squared Bessel process.

ii) Another example for an affine diffusion process is the non-central Wishart process on S_d^+. This process is determined by the stochastic differential equation

$$\mathrm{d}X_t = \sqrt{X_t}\,\mathrm{d}B_t Q + Q^\top\,\mathrm{d}B_t^\top\sqrt{X_t} + (2pQ^\top Q + \beta X_t + X_t\beta^\top)\,\mathrm{d}t,$$

[3]For $\delta > d-1$, it is defined as a weak solution and for $\delta \geq d+1$ as a strong solution.

where $X_0 = x \in S_d^+$ is the initial value, $Q \in \mathbb{R}^{d\times d}$ satisfies $Q^\top Q = \alpha$, $\beta \in \mathbb{R}^{d\times d}$ and B is a $d \times d$ matrix-valued Brownian motion, see for instance Mayerhofer [38]. The affine property is satisfied by the functions

$$\phi(t,\xi) = p \log\left(\det(I + \xi\sigma_t^\beta(\alpha))\right), \quad \psi(t,\xi) = \mathrm{e}^{\beta^\top t}(\xi^{-1} + \sigma_t^\beta(\alpha))^{-1}\mathrm{e}^{\beta t},$$

with $\sigma_t^\beta(x) := 2\int_0^t \mathrm{e}^{\beta s} x \mathrm{e}^{\beta^\top s}\,\mathrm{d}s$ for all $t \geq 0$ and $x \in S_d^+$.
Setting $\beta = 0$, $\delta = p/2$ and $Q = I$, we see that this is an extension of the first example.

iii) A multivariate stochastic volatility model with an application to credit risk, due to Gouriéroux and Sufana [24], models the logarithm of a price process S with a stochastic volatility matrix Σ. The processes are given by the stochastic differential system

$$\begin{aligned}
\mathrm{d}\log S_t &= \left(\mu + \begin{pmatrix} \mathrm{Tr}(D_1\Sigma_t) \\ \vdots \\ \mathrm{Tr}(D_d\Sigma_t) \end{pmatrix}\right)\mathrm{d}t + \sqrt{\Sigma_t}\,\mathrm{d}B_t^S, \\
\mathrm{d}\Sigma_t &= (\Omega\Omega^\top + M\Sigma_t + \Sigma_t M^\top)\,\mathrm{d}t + \sqrt{\Sigma_t}\,\mathrm{d}B_t^\Sigma Q + Q^\top(\,\mathrm{d}B_t^\Sigma)^\top\sqrt{\Sigma_t},
\end{aligned}$$

where B_t^S and B_t^Σ are a d-dimensional vector and a $d\times d$ matrix, respectively, whose elements are independent one-dimensional Brownian motions, $\mu \in \mathbb{R}^d$, $D_1, \ldots, D_d$, $\Omega, M, Q \in \mathbb{R}^{d\times d}$ with Ω invertible.
Then the joint process $(\log S_t, \Sigma_t)$ is an affine process[4], see Gouriéroux and Sufana [24, Section 2.2]. Similar to the Heston model 3.2.v the functions ψ and ϕ cannot be explicitly given.[5]

We continue with several properties of the functions ϕ and ψ.

Proposition 4.5. *Let X be an affine process on S_d^+. Then the following statements hold:*

i) The functions ϕ and ψ satisfy the semi-flow property

$$\phi(t+s,\xi) = \phi(t,\xi) + \phi(s,\psi(t,\xi)), \tag{4.2}$$
$$\psi(t+s,\xi) = \psi(s,\psi(t,\xi)), \tag{4.3}$$

for all $t \in \mathbb{R}_+$ and $\xi \in \mathcal{U}$.

ii) For all $\xi, \zeta \in \mathcal{U}$ with $\mathrm{Re}\,\xi \preceq \mathrm{Re}\,\zeta$ and for all $t \geq 0$, the order relations

$$\mathrm{Re}\,\phi(t,\xi) \leq \phi(t,\mathrm{Re}\,\zeta) \quad \textit{and} \quad \mathrm{Re}\,\psi(t,\xi) \preceq \psi(t,\mathrm{Re}\,\zeta)$$

hold true.

[4] It is an affine process on the product space $\mathbb{R}^d \times S_d^+$.
[5] However, we can numerically calculate ψ and ϕ by recursively solving the corresponding generalized Riccati equations.

iii) The functions ϕ and ψ are jointly continuous in $\mathbb{R}_+ \times \mathcal{U}$. Furthermore, $\xi \mapsto \phi(t,\xi)$ and $\xi \mapsto \psi(t,\xi)$ are analytic on S_d^{++}.

iv) Let $\psi : \mathbb{R}_+ \times \mathcal{U} \to \mathcal{U}$ be arbitrary mapping satisfying $\psi(0,\xi) = \xi$ with the above properties (regarding ψ). Then $\psi(t,\xi) \in \mathcal{U}^\circ$ for all $(t,\xi) \in \mathbb{R}_+ \times \mathcal{U}^\circ$.

The statements are analogous to the canonical state space, cf. Proposition 3.5. Hence, they can be proved in a similar way. We refer to Proposition 3.5 for a sketch of the main ideas and to Cuchiero et al. [15, Proposition 3.2 and 3.3] for a detailed proof. Observe that the last assertion is the key result showing the Feller property of an affine process on positive semidefinite matrices.

Theorem 4.6 (Feller property)**.** *Let X be an affine process with state space S_d^+. Then X is a Feller process or, equivalently, the affine semigroup is a Feller semigroup.*

The proof is very similar to that of the canonical state space. For this reason, we only give a sketch of the proof.
Based on a Stone-Weierstrass theorem, it suffices to consider the set of exponential functions, $\mathrm{e}^{\langle \xi, x\rangle}$ with $\xi \in -S_d^{++} \times \mathbf{i}S_d$, since their linear span is dense in $C_\infty(S_d^+)$. By Proposition 4.5.iv, we have $\langle \psi(t,\xi), x\rangle < 0$ for $\xi \in -S_d^{++} \times \mathbf{i}S_d$ and all $x \neq 0$. Hence, $T_t \mathrm{e}^{\langle \cdot, \xi\rangle}(x) = \mathrm{e}^{\phi(t,\xi) + \langle \psi(t,\xi), x\rangle}$ vanishes for $x \to \infty$. Proposition 4.5.iii now implies the continuity of $T_t \mathrm{e}^{\langle \cdot, \xi\rangle}(x)$. By the density of the linear span, we conclude that $T_t C_\infty(S_d^+) \subseteq C_\infty(S_d^+)$. For a detailed proof, we refer to Cuchiero et al. [15, Proposition 3.4] or to Cuchiero, Keller-Ressel, Mayerhofer and Teichmann [16, Proposition 3.3], which also covers the case of symmetric cones.

Similar to Chapter 3, cf. Definition 3.13, we call an affine process regular, if the derivatives

$$F(\xi) := \frac{\partial}{\partial t}\phi(t,\xi)|_{t=0+} \quad \text{and} \quad R(\xi) := \frac{\partial}{\partial t}\psi(t,\xi)|_{t=0+} \tag{4.4}$$

exist for all $\xi \in \mathcal{U}$ and are continuous at $\xi = 0$. The functions F and R are referred to as functional characteristics of an affine process.

In contrast to the canonical state space, we have already included in Definition 4.3 the assumption that an affine process X is stochastically continuous since we focus on regular affine processes. As shown in Cuchiero and Teichmann [17] as well as in Keller-Ressel, Schachermayer and Teichmann [33], an affine process is regular if and only if it is stochastically continuous.

4.3 Admissible Parameters

The isomorphic representation of S_d^+ in $\mathbb{R}^{d(d+1)/2}$ suggests that the parameters describing an affine process on positive semidefinite matrices are subject to the same constraints as in the real vector-valued case. In this section, we will show that many conditions

are indeed similar to those of the canonical state space. However, there is one major difference. On S_d^+ the linear diffusion coefficient has to be smaller than the constant drift term. This constraint follows from the nonlinear curved boundary. Furthermore, the dimension d also affects this inequality.
The next theorem states all conditions for the admissibility of the parameters. The proof is divided in several parts. The main idea is to use methods from harmonic analysis to examine the functional characteristics. Note that the linear diffusion condition is proven in the next section, see Proposition 4.17, since it is based on the infinitesimal generator of the affine process.

Theorem 4.7. *For $\xi \in -S_d^+ + \mathbf{i}S_d = \mathcal{U}$, the functions $\xi \mapsto -F(\xi)$ and $\xi \mapsto -\langle R(\xi), x\rangle$ are continuous negative definite for all $x \in S_d^+$ and may be written as*

$$F(\xi) = -\langle b, \xi\rangle - c + \int_{S_d^+\setminus\{0\}} \left(\mathrm{e}^{\langle \xi, y\rangle} - 1\right) m(\,\mathrm{d}y)$$
$$R(\xi) = -2\xi\alpha\xi - B^\top(\xi) - \gamma + \int_{S_d^+\setminus\{0\}} \left(\mathrm{e}^{\langle \xi, y\rangle} - 1 - \langle \chi(y), \xi\rangle\right) \mu(\,\mathrm{d}y),$$

where $\alpha \in S_d^+$, $b \in S_d^+$, $B_{ij}^\top(\xi) = \langle \beta^{ij}, \xi\rangle$ with $\beta^{ij} \in S_d$, $c \in \mathbb{R}_+$, $\gamma \in S_d^+$, $\chi : S_d^+ \to S_d^+$ is a truncation function and m, μ are Lévy measures on S_d^+. Additionally, the parameters satisfy the following conditions:

$$(d-1)\alpha \preceq b,$$
$$B : S_d \to S_d \quad \text{with } B(x) = \sum_{i,j=1}^{d} \beta^{ij} x_{ij}, \text{ where}$$
$$\beta^{ij} = \beta^{ji} \in S_d,$$
$$\operatorname{supp}(m) \subseteq S_d^+ \quad \text{and} \int_{S_d^+\setminus\{0\}} \left(\|x\| \wedge 1\right) m(\,\mathrm{d}x) < \infty,$$
$$\mu = (\mu_{ij})_{i,j=1,\ldots,d} \quad \text{such that } \operatorname{supp}(\mu_{ij}) \subseteq S_d^+,$$
$$\mu(E) \in S_d^+ \quad \text{for all } E \in \mathcal{B}(S_d^+\setminus\{0\}),$$
$$M(x, dy) := \frac{\langle x, \mu(\,\mathrm{d}y)\rangle}{\|y\|^2 \wedge 1} \quad \text{satisfies}$$
$$\int_{S_d^+\setminus\{0\}} \langle \chi(y), \xi\rangle M(x, \,\mathrm{d}y) < \infty \qquad \text{for all } x \in S_d^+, \xi \in \mathcal{U} \text{ s.t. } \langle x, \xi\rangle = 0,$$
$$\langle B(x), \xi\rangle + \int_{S_d^+\setminus\{0\}} \langle \chi(y), \xi\rangle M(x, \,\mathrm{d}y) \geq 0 \qquad \text{for all } x \in S_d^+, \xi \in \mathcal{U} \text{ s.t. } \langle x, \xi\rangle = 0.$$

The parameters are called admissible if these conditions are satisfied.

Outline of the proof. We first show that $-F$ and $-\langle R, x\rangle$ are continuous negative definite functions, see Proposition 4.8. Applying Berg, Christensen and Ressel [8, Theorem 3.19] gives us the Lévy-Khintchine representation of F and the corresponding conditions, cf. Corollary 4.10. To get the admissibility conditions for the parameters of R, we start with a Lévy-Khintchine representation of $\langle R, x\rangle$ but have to consider the linear shift due to x. In Proposition 4.13, a representation of R independent of the choice of x is shown. For the condition on the linear diffusion coefficient we refer to Proposition 4.17. $\square$

Proposition 4.8. *The mappings $\xi \mapsto -F(\xi)$ and $\xi \mapsto -\langle R(\xi), x\rangle$ are continuous negative definite for all $x \in S_d^+$.*

Proof. It is known that the Fourier-Laplace transform of a measure is a positive definite function, hence

$$\xi \mapsto T_t \mathrm{e}^{\langle \cdot, \xi\rangle}(x) = \int_{S_d^+} \mathrm{e}^{\langle y,\xi\rangle} p_t(x,\, \mathrm{d}y) = \mathrm{e}^{\phi(t,\xi)+\langle \psi(t,\xi), x\rangle}$$

is positive definite for all $x \in S_d^+$. Setting

$$\lambda_t(x,\xi) := \mathrm{e}^{-\langle x,\xi\rangle} \int_{S_d^+} \mathrm{e}^{\langle y,\xi\rangle} p_t(x,\, \mathrm{d}y) = \mathrm{e}^{-\langle x,\xi\rangle} T_t \mathrm{e}^{\langle \cdot,\xi\rangle}(x)$$

for $x \in S_d^+$ and $\xi \in \mathcal{U}$ yields that $\xi \mapsto \lambda_t(x,\xi)$ is positive definite for all $x \in S_d^+$. From Lemma 1.3 we deduce that $\xi \mapsto 1 - \lambda_t(x,\xi)$ is negative definite and thus

$$\begin{aligned}
p(x,\xi) &= \lim_{t\to 0} \frac{1-\lambda_t(x,\xi)}{t} = \lim_{t\to 0} \mathrm{e}^{-\langle x,\xi\rangle} \frac{\mathrm{e}^{\langle x,\xi\rangle} - T_t \mathrm{e}^{\langle \cdot,\xi\rangle}(x)}{t} \\
&= -\partial_t^+ \mathrm{e}^{-\langle x,\xi\rangle} T_t \mathrm{e}^{\langle x,\xi\rangle}\Big|_{t=0} = -\partial_t^+ \left(\mathrm{e}^{\phi(t,\xi)+\langle \psi(t,\xi)-\xi, x\rangle}\right)\Big|_{t=0} \\
&= -\partial_t^+ \left(\phi(t,\xi) + \langle \psi(t,\xi) - \xi, x\rangle\right)\Big|_{t=0} \\
&= -F(\xi) - \langle R(\xi), x\rangle
\end{aligned}$$

is negative definite for all $x \in S_d^+$. Since an affine process is regular, F and R are continuous in 0.
Setting $x = 0$ immediately implies that $\xi \mapsto -F(\xi)$ is a continuous negative definite function. Applying the similar argument as in Section 3.3, i.e. inserting $\tilde{x}_r = r \cdot x$ in the above equation for some sufficiently large $r \geq 0$, shows that $\xi \mapsto -\langle R(\xi), x\rangle$ is also a continuous negative definite function for every $x \in S_d^+$. $\square$

Remark 4.9. Using the order relation induced on S_d and the self-duality of S_d^+, it follows directly that the continuous negative definiteness of $\xi \mapsto -\langle R(\xi), x\rangle$ for all $x \in S_d^+$ implies the negative definiteness of $\xi \mapsto -R(\xi)$ with respect to the order relation $\preceq$, i.e. R is hermitian and

$$\sum_{j,k=1}^{n} c_j \bar{c}_k R(\bar{\xi}_j + \xi_k) \preceq 0$$

for all $n \in \mathbb{N}$, $\xi_1, \ldots, \xi_n \in \mathcal{U}$ and $c_1, \ldots, c_n \in \mathbb{C}$ with $\sum_{j=1}^n c_j = 0$.

As the function $-F$ is continuous negative definite, the next result quite naturally states the Lévy-Khintchine representation for the semigroup S_d^+.

Corollary 4.10. *The function $-F : \mathcal{U} \to \mathbb{C}$ is given by a Lévy-Khintchine formula*

$$-F(\xi) = c + \langle b, \xi \rangle + \int_{S_d^+ \setminus \{0\}} (1 - \mathrm{e}^{\langle y, \xi \rangle}) m(\,\mathrm{d}y) \tag{4.5}$$

where $c \geq 0$, $b \in S_d^+$, $\mathrm{supp}(m) \subseteq S_d^+$ and $\int_{S_d^+ \setminus \{0\}} \big(\|x\| \wedge 1\big)\, m(\,\mathrm{d}x) < \infty$.

Proof. This follows from Berg et al. [8, Theorem 3.20]. □

We can deduce almost the same result for $-\langle R, x \rangle$ for all $x \in S_d^+$. Due to a shift in the Fourier-Laplace transform, the state space of the Lévy-Khintchine representation, if directly applied, is S_d. Hence, in contrast to $-F$, a quadratic term and a truncation function appear in the Lévy-Khintchine formula. However, the parameters have to satisfy several admissibility conditions as the next proposition shows.

Proposition 4.11. *For all $x \in S_d^+$ the mapping $\xi \mapsto -\langle R(\xi), x \rangle$ is given by a Lévy-Khintchine representation*

$$-\langle R(\xi), x \rangle = \gamma(x) + \frac{1}{2} A(x)(\xi) + \langle B(x), \xi \rangle + \int_{S_d^+ \setminus \{0\}} (1 - \mathrm{e}^{\langle y, \xi \rangle} + \langle \chi(y), \xi \rangle) M(x, \,\mathrm{d}y) \tag{4.6}$$

where $\gamma(x)$ is non-negative, $A(x)$ is a symmetric positive form, $B(x) \in S_d$ and $M(x, \cdot)$ is a measure on S_d^+ such that

$$\begin{aligned} \langle \xi, A(x)(\xi) \rangle &= 0 \qquad \text{for all } \xi \in S_d \text{ with } \xi x = x \xi = 0, \\ \int_{S_d^+ \setminus \{0\}} \langle \chi(y), \xi \rangle M(x, \,\mathrm{d}y) &< \infty, \\ \langle B(x), \xi \rangle + \int_{S_d^+ \setminus \{0\}} \langle \chi(y), \xi \rangle M(x, \,\mathrm{d}y) &\geq 0 \qquad \text{for all } x, \xi \in S_d^+ \text{ with } \langle \xi, x \rangle = 0. \end{aligned}$$

Proof. We restrict the function R to the domain $\mathrm{Re}\, \mathcal{U} = -S_d^+$ for the proof since the representation (4.6) for a real variable $\xi \in -S_d^+$ extends to $-S_d^+ \times \mathbf{i} S_d = \mathcal{U}$.
From the proof of Proposition 4.8,

$$-\langle R(\xi), x \rangle = F(\xi) + p(x, \xi) = F(\xi) + \lim_{t \to 0} \frac{1 - \mathbb{E}^x(\mathrm{e}^{\langle X_t - x, \xi \rangle})}{t},$$

follows that the support of the integral is shifted[6]. Hence, $-\langle R(\cdot), x \rangle$ behaves like a negative definite function with domain S_d. Applying Berg et al. [8, Theorem 3.19] to the mapping $\xi \mapsto -\langle R(\xi), x \rangle$, we get representation (4.6) such that the parameters satisfy

- $A(x)(\cdot)$ is a symmetric positive semidefinite form,
- $B(x) \in S_d$,

[6]Observe that by substitution we have $\mathbb{E}(u(X_t - x)) = \int_{S_d^+} u(y - x) \mathbb{P}^{X_t}(\,\mathrm{d}y) = \int_{S_d^+ - x} u(\tilde{y}) \mathbb{P}^{X_t}(\,\mathrm{d}\tilde{y})$. Since $S_d^+ - x \subseteq S_d$ and $S_d^+ - x \nsubseteq S_d^+$ for $x \in S_d^+$, the shift creates an integral over S_d.

- $\gamma(x) \geq 0$,
- $M(x)$ is a measure on S_d such that $\int_{S_d \setminus \{0\}} \langle \chi(y), \xi \rangle M(x, \mathrm{d}y) < \infty$.

Next, we show the constraints of the parameters. Recall that $-\langle R(\cdot), x\rangle$ is a negative definite function on $S_d^+ - \mathbb{R}x \subseteq S_d$ with a Lévy-Khintchine representation. The main idea now is to apply a suitable shift and examine the composed function which is negative definite on S_d^+. Thus we obtain new restrictions which we retransfer to the original function $-\langle R(\cdot), x\rangle$. This approach is based on the concept of Schoenberg triples, cf. Berg et al. [8, Section 5.1]. Let $T : S_d \to S_d$ be a linear mapping such that $T(S_d^+ - \mathbb{R}x) \subseteq S_d^+$. Observe that $(S_d^+ - \mathbb{R}x, S_d^+, T)$ is a Schoenberg triplet, cf. Berg et al. [8, Definition 5.1.4][7]. Therefore, a result from Berg et al. [8, Corollary 5.1.8] ensures that $-\langle R \circ T^\top(\cdot), x\rangle$ is a negative definite function on S_d^+ with representation

$$
\begin{aligned}
-\langle x, R \circ T^\top(\eta)\rangle = \gamma(x) + \frac{1}{2} A(x)(T^\top(\eta)) + \langle B(x), T^\top(\eta)\rangle \\
+ \int_{S_d^+ \setminus \{0\}} \langle \chi(y), T^\top(\eta)\rangle M(x, \mathrm{d}y) + \int_{S_d^+ \setminus \{0\}} (1 - \mathrm{e}^{\langle T^\top(\eta), y\rangle}) M(x, \mathrm{d}y).
\end{aligned}
$$

Since this is a negative definite function on S_d^+, the parameters have to satisfy the conditions from the Lévy-Khintchine representation for semigroups with an identical involution, see Berg et al. [8, Theorem 4.3.20][8]. In other words, the quadratic term vanishes, the drift is positive and $M(x)$ is a measure on S_d^+, i.e. for all $x \in S_d^+$ we have

$$
\begin{aligned}
\langle T^\top \eta, A(x) T^\top \eta\rangle &= 0, & \forall \eta \in S_d, \\
\langle B(x), T^\top \eta\rangle + \int_{S_d^+ \setminus \{0\}} \langle \chi(y), T^\top \eta\rangle M(x, \mathrm{d}y) &\geq 0, & \forall \eta \in S_d^+, \\
\int_{S_d^+ \setminus \{0\}} (\|Ty\| \wedge 1) M(x, \mathrm{d}y) &< \infty.
\end{aligned}
$$

For $x \in \partial S_d^+$, there exists an $\xi \in S_d$ with $\xi x = x \xi = 0$. The function $T_\xi : S_d \to S_d, \eta \mapsto \xi \eta \xi$ defines a mapping from $S_d^+ - \mathbb{R}x$ onto S_d^+. Especially, for all $\eta \in S_d$ or $\eta \in S_d^+$ exists a $\xi \in S_d$ or $\xi \in S_d^+$, respectively, such that $T_\xi \eta = \xi$, see Cuchiero et al. [15, Lemma 4.3]. Hence, the above equations with such a function T_ξ are equivalent to

$$
\begin{aligned}
\langle \xi, A(x)\xi\rangle &= 0, & \forall \xi \in S_d \text{ with } \xi x = x\xi = 0, \\
\langle B(x), \xi\rangle + \int_{S_d^+ \setminus \{0\}} \langle \chi(y), \xi\rangle M(x, \mathrm{d}y) &\geq 0, & \forall \xi \in S_d^+ \text{ with } \xi x = x\xi = 0, \\
\int_{S_d^+ \setminus \{0\}} \langle \chi(y), \xi\rangle M(x, \mathrm{d}y) &< \infty, & \forall \xi \in S_d^+ \text{ with } \xi x = x\xi = 0.
\end{aligned}
$$

□

[7] Our triplet meets the requirements of the definition since $T(0) = 0$, $T(\bar{\xi}) = \overline{T(\xi)}$, $T(S_d^+ - \mathbb{R}x) = S_d^+$ and $T(\mathrm{e}^{\langle y, \cdot\rangle})$ is positive definite for all $y \in S_d^+ - \mathbb{R}x$.

[8] These conditions are similar to that of a Bernstein function, i.e. a negative definite function on $\mathbb{R}_+$.

The linearity of $\langle R, x\rangle$ in x implies the linearity in x of the corresponding parameters. This allows us to rewrite the parameters independently of x

$$\gamma(x) = \sum_{i,j=1}^{d} x_{ij}\gamma^{ij} \quad \text{with } \gamma^{ij} = \gamma^{ji} = (1+\delta_{ij})\frac{\gamma(c^{ij})}{2} \in \mathbb{R},$$

$$A(x) = \sum_{i,j=1}^{d} x_{ij}a^{ij} \quad \text{with } a^{ij} = a^{ji} = (1+\delta_{ij})\frac{A(c^{ij})}{2} : S_d \to S_d \text{ linear},$$

$$B(x) = \sum_{i,j=1}^{d} x_{ij}\beta^{ij} \quad \text{with } \beta^{ij} = \beta^{ji} = (1+\delta_{ij})\frac{B(c^{ij})}{2} \in S_d,$$

$$\int_E (\|y\|^2 \wedge 1) M(x,\, \mathrm{d}y) = \langle x, \mu(E)\rangle = \sum_{i,j=1}^{d} x_{ij}\mu_{ij}(E) \quad \text{for every } E \in \mathcal{B}(S_d^+\backslash\{0\}),$$

$$E \mapsto \mu_{ij}(E) = \mu_{ji}(E) = (1+\delta_{ij})\frac{\int_E (\|y\|^2 \wedge 1) M(c^{ij},\, \mathrm{d}y)}{2},$$

where μ_{ij} are finite measures on S_d^+.
Since the killing term has to be non-negative for all $x \in S_d^+$, the self-duality of S_d^+ implies $(\gamma^{ij})_{i,j=1\ldots,d} \in S_d^+$. Therefore, we write $\gamma(x) = \sum_{i,j=1}^d x_{ij}\gamma^{ij} = \langle \gamma, x\rangle$ in the sequel.

Remark 4.12. However, we cannot conclude that the linear drift B has to map into S_d^+. Due to the shift, the linear jump term also effects the drift. This effect is similar to the canonical state space $\mathbb{R}_+^m \times \mathbb{R}^n$, where for $i \in \{1, \ldots, m\}$ the ith coordinate of the linear drift vector β^i is an element of $\mathbb{R}$ and does not have to be non-negative.

Writing B as the sum of matrices allows us to reformulate the linear drift term. In fact, for $x \in S_d^+$ and $\xi \in \mathfrak{U}$ we get

$$\begin{aligned}\langle B(x), \xi\rangle &= \langle \sum_{i,j=1}^{d} x_{ij}\beta^{ij}, \xi\rangle = \sum_{k,l=1}^{d}\sum_{i,j=1}^{d} x_{ij}\beta^{ij}_{kl}\xi_{kl} \\ &= \langle x, \sum_{k,l=1}^{d} \beta_{kl}\xi_{kl}\rangle = \langle x, B^\top(\xi)\rangle.\end{aligned}$$

The next proposition allows us to write the linear diffusion coefficient in a compressed form.

Proposition 4.13. *Let $A(x) : S_d \to S_d$ be the linear diffusion coefficient of an affine process as above. Then there exists $\alpha \in S_d^+$ such that for all $\xi \in \mathfrak{U}$ and $x \in S_d^+$*

$$\langle \xi, A(x)\xi\rangle = 4\langle \xi\alpha\xi, x\rangle.$$

This result is analogous to the state space $D = \mathbb{R}_+^m$, where only one entry appears in each linear diffusion matrix, cf. Remark 3.19. The proof is based on Hilbert space methods and can be found in Cuchiero et al. [15, p. 24].

Using the above result, we are able to give a representation of F and R which is independent of x,

$$-F(\xi) = c + \langle b, \xi \rangle + \int_{S_d^+ \setminus \{0\}} (1 - \mathrm{e}^{\langle y, \xi \rangle}) m(\,\mathrm{d}y), \tag{4.7}$$

$$-R(\xi) = 2\xi\alpha\xi + B^\top(\xi) + \gamma + \int_{S_d^+ \setminus \{0\}} \left(1 - \mathrm{e}^{\langle \xi, y \rangle} + \langle \chi(y), \xi \rangle\right) \mu(\,\mathrm{d}y). \tag{4.8}$$

Furthermore, all admissibility conditions for the parameters except the linear diffusion condition have been shown.

4.4 Representation as Pseudo-Differential Operator

Most authors deal with infinitesimal generators on $\mathbb{R}^d$ since stochastic processes are mainly considered on this state space. Obviously, many results are valid on more general spaces. For a treatment of our case, i.e. a locally compact Hausdorff space with a countable base, which includes the cone S_d^+, we refer to Rogers and Williams [45, Section III.2].

In analysis, pseudo-differential operators have been extended from $\mathbb{R}^d$ to more general state spaces like Lie groups. In stochastic analysis, however, the focus for pseudo-differential operators has remained on $\mathbb{R}^d$ for a long time. Only in recent years some approaches to pseudo-differential operators on Lie groups have been done in probability, see for instance Applebaum [3]. The next definitions are a step in a slightly different direction to cone states spaces. Since we choose S_d^+ as a state space, many results can be derived from $\mathbb{R}^n$ by using the isomorphic embedding of S_d^+ in $\mathbb{R}^{(d+1)d/2}$. Furthermore, we can easily access the characteristics of cone state spaces and use this approach as a basis for further extensions of the state space. Therefore, we recall some definitions from Section 1.4 in the light of the state space S_d^+. Then we continue with the main theorem of this section, the representation of the generator of an affine process as a pseudo-differential operator and the determination of the symbol. Our proof also provides the representation as an integro-differential operator. Based on this, we finish the proof of the admissibility of the parameters, see Theorem 4.7, by verifying the linear diffusion condition.

Definition 4.14 (Generator)**.** Let $(T_t)_{t\geq 0}$ be a Feller semigroup on $C_\infty(S_d^+)$. Then

$$Au := \lim_{t \to u} \frac{T_t u - u}{t} \quad \left(\text{the limit is taken in } (C_\infty(S_d^+), \|\cdot\|_\infty)\right), \tag{4.9}$$

$$\mathcal{D}(A) := \left\{ u \in C_\infty(S_d^+);\ \exists g \in C_\infty(S_d^+) :\ \lim_{t \to 0} \left\| \frac{T_t u - u}{t} - g \right\|_\infty = 0 \right\} \tag{4.10}$$

is the (infinitesimal) generator of the semigroup $(T_t)_{t\geq 0}$.

Definition 4.15 (Pseudo-differential operator, symbol)**.** Let $q : S_d^+ \times \mathbf{i}S_d \to \mathbb{C}$ be a function which is, for every $x \in S_d^+$, continuous and negative definite. Then

$$-q(x, D)u(x) = -(2\pi)^{-\frac{(d+1)d/2}{2}} \int_{S_d} \mathrm{e}^{\langle x, \mathbf{i}\xi\rangle} q(x, \mathbf{i}\xi)\hat{u}(\xi)\,\mathrm{d}\xi, \quad u \in C_c^\infty(S_d^+) \tag{4.11}$$

is a pseudo-differential operator (with negative definite symbol) and $q(x, \xi)$ is called the symbol of the operator.

The next theorem ensures the existence of the pseudo-differential operator and the symbol of an affine process. Furthermore, we get an explicit representation of the generator.

Theorem 4.16. *The infinitesimal generator of an affine process restricted to the test functions $C_c^\infty(S_d^+)$ has a representation as a pseudo-differential operator which has the symbol $q(x, \xi) = -F(\xi) - \langle x, R(\xi)\rangle$, for $(x, \xi) \in S_d^+ \times \mathbf{i}S_d$.*

Proof. Using a similar approach as in Section 3.5, we explicitly calculate the pointwise generator A_p. If it vanishes as x tends to infinity, we deduce that the pointwise generator coincides with the infinitesimal generator, cf. Sato [11, Theorem 1.33]. The calculations immediately give us a representation as a pseudo-differential operator and, hence, the symbol.
We first determine the pointwise generator for a function $f \in C_c^\infty(S_d^+)$,

$$\begin{aligned}
A_p f(x) &= \lim_{t\to 0} \frac{T_t f(x) - f(x)}{t} \\
&= \lim_{t\to 0} (2\pi)^{-\frac{(d+1)d/2}{2}} \int_{S_d} \frac{T_t \mathrm{e}^{\langle \mathbf{i}y, \cdot\rangle}(x) - \mathrm{e}^{\langle \mathbf{i}y, x\rangle}}{t} \hat{f}(y)\,\mathrm{d}y \\
&= (2\pi)^{-\frac{(d+1)d/2}{2}} \int_{S_d} \lim_{t\to 0} \frac{T_t \mathrm{e}^{\langle \mathbf{i}y, \cdot\rangle}(x) - \mathrm{e}^{\langle \mathbf{i}y, x\rangle}}{t} \hat{f}(y)\,\mathrm{d}y \\
&= (2\pi)^{-\frac{(d+1)d/2}{2}} \int_{S_d} \mathrm{e}^{\langle \mathbf{i}y, x\rangle} (F(\mathbf{i}y) + \langle R(\mathbf{i}y), x\rangle)\hat{f}(y)\,\mathrm{d}y,
\end{aligned}$$

where we have to verify that the interchange of the limit and the integral is allowed using dominated convergence. Therefore, we start with the affine property (4.1) and apply Taylor's theorem to obtain

$$
\begin{aligned}
&\left|\frac{T_t e^{\langle \mathbf{i}y,\cdot\rangle}(x) - e^{\langle \mathbf{i}y,x\rangle}}{t}\right| \\
&\quad = \left|\frac{e^{\phi(t,\mathbf{i}y)+\langle x,\psi(t,\mathbf{i}y)\rangle} - e^{\langle \mathbf{i}y,x\rangle}}{t}\right| \\
&\quad = \left|\frac{1}{t}\left(e^{\phi(0,\mathbf{i}y)+\langle x,\psi(0,\mathbf{i}y)\rangle} + t\partial_t\Big|_{t=0} e^{\phi(t,\mathbf{i}y)+\langle x,\psi(t,\mathbf{i}y)\rangle} + O(t) - e^{\langle \mathbf{i}y,x\rangle}\right)\right| \\
&\quad = \left|e^{\langle \mathbf{i}y,x\rangle}\partial_t\Big|_{t=0} e^{\phi(t,\mathbf{i}y)+\langle x,\psi(t,\mathbf{i}y)\rangle} + tO(t)\right| \\
&\quad \le \left|e^{\langle \mathbf{i}y,x\rangle}\big(F(\mathbf{i}y) + \langle x, R(\mathbf{i}y)\rangle\big)\right| + c \\
&\quad \le c_F(1+\|y^2\|) + \|x\| c_R(1+\|y^2\|) + c \\
&\quad \le c_{F,R}(1+\|x\|)(1+\|y^2\|),
\end{aligned}
$$

where we used the growth property of continuous negative definite functions, cf. Lemma 1.5.
Obviously, $f \in C_c^\infty(S_d^+)$ is sufficiently smooth such that $\frac{T_t e^{\langle \mathbf{i}y,\cdot\rangle}(x) - e^{\langle \mathbf{i}y,x\rangle}}{t}\hat{f}(y)$ admits an integrable dominating function.
Based on this result, we next show that $A_p f(x) = -q(x,D)f(x)$ vanishes at infinity for $f \in C_c^\infty(S_d^+)$. Therefore, we rewrite the pseudo-differential operator as an integro-differential operator

$$
\begin{aligned}
-q(x,D)f(x) = {} & \frac{1}{2}\sum_{i,j,k,l=1}^{d} A_{ijkl}(x)\partial_{ij}\partial_{kl} f(x) \\
& + \langle b + B(x), \nabla f(x)\rangle + \big(c + \langle \gamma, x\rangle\big) f(x) \\
& + \int_{S_d^+\setminus\{0\}} \big(f(x+y) - f(x)\big)\, m(\mathrm{d}y) \\
& + \int_{S_d^+\setminus\{0\}} \big(f(x+y) - f(x) + \langle \chi(y), \nabla f(x)\rangle\big)\, M(x,\mathrm{d}y).
\end{aligned}
\tag{4.12}
$$

A similar argument as in Theorem 3.21 shows that this converges to zero for $\|x\| \to \infty$. The result is that $x \mapsto -q(x,D)f(x) \in C_\infty(S_d^+)$ and $Af = A_p f = -q(\cdot,D)f$ for $f \in C_c^\infty(S_d^+)$. This finally shows that $C_c^\infty(S_d^+) \subseteq \mathcal{D}(A)$ and, in particular, that $-A|_{C_c^\infty}$ is a pseudo-differential operator with symbol $q(x,\mathbf{i}\xi) = -F(\mathbf{i}\xi) - \langle x, R(\mathbf{i}\xi)\rangle$. □

Observe that we have given an integro-differential representation of the generator, see (4.12). This representation is based on the isomorphism of S_d^+ and $\mathbb{R}^{d(d+1)/2}$. In this sense, one can interpret $(A_{ij,kl})_{i,j,k,l=1,\ldots d}$ as a matrix. In order to write the integro-differential operator as an S_d^+ based version, we recall some notation. Using the tensor product $\otimes$, defined by $(u \otimes v)x = \langle x, v\rangle u$, and the trace for linear operators Tr, defined by $\mathrm{Tr}(AB) = \sum_{i\le j}\langle A\frac{1}{\sqrt{2}}c^{ij}, B\frac{1}{\sqrt{2}}c^{ij}\rangle = \frac{1}{2}\sum_{i\le j}\langle Ac^{ij}, Bc^{ij}\rangle$, where $(c^{ij})_{i\le j}$ is the basis of S_d as given in Section 4.1, we obtain the following integro-differential representation

$$\begin{aligned}
\mathcal{A}f(x) &= -q(x,D)f(x) \\
&= \frac{1}{2}\mathrm{Tr}\left(A(x)\Big(\frac{\partial}{\partial x}\otimes\frac{\partial}{\partial x}\Big)\right)f(x) + \langle b + B(x), \nabla f(x)\rangle + (c + \langle\gamma, x\rangle)f(x) \\
&\quad + \int_{S_d^+\setminus\{0\}} \big(f(x+y) - f(x)\big)\, m(\,\mathrm{d}y) \qquad (4.13)\\
&\quad + \int_{S_d^+\setminus\{0\}} \big(f(x+y) - f(x) + \langle\chi(y), \nabla f(x)\rangle\big)\, M(x,\,\mathrm{d}y).
\end{aligned}$$

Now, we use this to show the linear diffusion condition.

Proposition 4.17. *Let X be an affine process with generator $\mathcal{A}$ which has a representation* (4.12). *Then* $(d-1)\alpha \preceq b$.

Proof. We first define a set of functions in $C_c^\infty(S_d^+)$ which depend on a parameter $v \in \mathbb{R}_+$. Applying the generator $\mathcal{A}$ on such a function, we get a Laplace transform depending on v and hence a one-dimensional continuous negative definite function on $\mathbb{R}_+$, i.e. a Bernstein function. The parameter restrictions for Bernstein functions then imply the linear diffusion condition. This proof is motivated by Cuchiero et al. [16, Proposition 4.5].
Let $y \in \partial S_d^+$ and $f \in C_c^\infty(S_d^+)$ such that $f \ge 0$ and $f(x) = \det(x)$ for all x in a neighbourhood of y. For instance, this can be done by mollifying the det function. For any $v \in \mathbb{R}_+$, $x \mapsto \mathrm{e}^{-vf(x)} - 1$ is a smooth function with compact support, i.e. an element of $C_c^\infty(S_d^+)$.
Since y is positive semidefinite but not strictly positive definite we have $f(y) = \det(y) = 0$, $\mathrm{e}^{-vf(y)} - 1 = 0$ and $\nabla(\mathrm{e}^{-vf(y)} - 1) = \mathrm{e}^{-vf(y)}(-v\nabla f(y)) = -v\nabla f(y)$. Furthermore, the second derivative becomes

$$\begin{aligned}
\partial_{ij}\partial_{kl}(\mathrm{e}^{-vf(y)} - 1) &= \partial_{ij}\left(\mathrm{e}^{-vf(y)}(-v\partial_{kl}f(y))\right) \\
&= \mathrm{e}^{-vf(y)}(-v\partial_{ij}f(y))(-v\partial_{kl}f(y)) - \mathrm{e}^{-vf(y)}\left(-v\partial_{ij}\partial_{kl}f(y)\right) \\
&= v^2\partial_{ij}f(y)\partial_{kl}f(y) + v\partial_{ij}\partial_{kl}f(y).
\end{aligned}$$

Applying the generator $\mathcal{A}$ with representation (4.12), the above calculations yield

$$\begin{aligned}
\mathcal{A}(\mathrm{e}^{-vf(\cdot)} - 1)(y) &= \frac{1}{2}\sum_{i,j,k,l}^{d} A_{ijkl}(y)(v^2\partial_{ij}f(y)\partial_{kl}f(y) - v\partial_{ij}\partial_{kl}f(y)) \\
&\quad - v\langle b + B(y), \nabla f(y)\rangle \\
&\quad + \int_{S_d^+\setminus\{0\}} (\mathrm{e}^{-vf(y+z)} - 1)m(\,\mathrm{d}z) \\
&\quad + \int_{S_d^+\setminus\{0\}} (\mathrm{e}^{-vf(y+z)} - 1 - v\langle\chi(z), \nabla f(y)\rangle)M(y,\,\mathrm{d}z).
\end{aligned}$$

However, the function $v \mapsto -A(\mathrm{e}^{-vf(\cdot)} - 1)(y)$ is a negative definite function on $\mathbb{R}_+$. To show this, let $n \in \mathbb{N}$, $v_1, \dots, v_n \in \mathbb{R}_+$ and $c_1, \dots, c_n \in \mathbb{C}$ satisfying $\sum_{j=1}^n c_j = 0$. Then

$$\begin{aligned}
&\sum_{j,k=1}^n c_j \bar{c}_k \left(-A(\mathrm{e}^{-(v_j+v_k)f(\cdot)} - 1)(y)\right) \\
&\quad = -\sum_{j,k=1}^n c_j \bar{c}_k \lim_{t\to 0} \frac{1}{t} \int_{S_d^+} \left(\mathrm{e}^{-(v_j+v_k)f(z)} - 1\right) p_t(x,\, \mathrm{d}z) \\
&\quad = -\lim_{t\to 0} \frac{1}{t} \int_{S_d^+} \sum_{j,k=1}^n c_j \bar{c}_k \left(\mathrm{e}^{-(v_j+v_k)f(z)} - 1\right) p_t(y,\, \mathrm{d}z) \\
&\quad = -\lim_{t\to 0} \frac{1}{t} \int_{S_d^+} \sum_{j,k=1}^n c_j \bar{c}_k \mathrm{e}^{-v_j f(z)} \mathrm{e}^{-v_k f(z)} p_t(y,\, \mathrm{d}z) \\
&\quad = -\lim_{t\to 0} \frac{1}{t} \int_{S_d^+} \left(\sum_{j=1}^n c_j \mathrm{e}^{-v_j f(z)}\right) \overline{\left(\sum_{k=1}^n c_k \mathrm{e}^{-v_k f(z)}\right)} p_t(y,\, \mathrm{d}z) \\
&\quad = -\lim_{t\to 0} \frac{1}{t} \int_{S_d^+} \left|\sum_{j=1}^n c_j \mathrm{e}^{-v_j f(z)}\right|^2 p_t(y,\, \mathrm{d}z).
\end{aligned}$$

This expression is negative for every $t > 0$, and, consequently, the function is negative definite. Since the set of negative definite functions is a convex cone and the above limit exists, the mapping $v \mapsto -A(\mathrm{e}^{-vf(\cdot)} - 1)(y)$ is negative definite. In particular, this function has a representation as a Bernstein function, see Berg et al. [8, Theorem 4.3.20]. Therefore, the parameters of the above Lévy-Khintchine representation satisfy the usual conditions of a Bernstein function. In particular, there is no quadratic term, i.e.

$$\sum_{i,j,k,l}^d A_{ijkl}(y) \partial_{ij} \det(y) \partial_{kl} \det(y) = 0.$$

Furthermore, the linear term has to be positive, i.e.

$$\begin{aligned}
&\frac{1}{2} \sum_{i,j,k,l} A_{ijkl}(y) \partial_{ij} \partial_{kl} \det(y) + \langle b, \nabla \det(y) \rangle \\
&\qquad + \langle B(y), \nabla \det(y) \rangle + \int_{S_d^+ \setminus \{0\}} \langle \chi(z), \nabla \det(y) \rangle M(y,\, \mathrm{d}z) \geq 0.
\end{aligned}$$

Proposition 4.11 states that the sum of the last two terms is positive for all $y \in \partial S_d^+$, since Jacobi's formula shows that

$$\langle \nabla \det(y), y \rangle = \mathrm{Tr}(\mathrm{adj}(y) y) = \det(y) = 0,$$

and hence that $y \nabla \det(y) = 0$ for $y \in \partial S_d^+$.
The Leibniz formula for the determinant yields that the first two terms of the above

inequality are polynomials of degree $d-1$, whereas the last two terms are polynomials of degree d for all $y \in \partial S_d^+$. Consequently, for small $|y| \to 0$ the first two terms govern the inequality and we get

$$\frac{1}{2} \sum_{i,j,k,l} A_{i,j,k,l}(y) \partial_{ij} \partial_{kl} \det(y) + \langle b, \nabla \det(y) \rangle \geq 0,$$

or, equivalently,

$$\frac{1}{2} \mathrm{Tr} \left(A(y) \left(\frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} \right) \right) \det(y) + \langle b, \nabla \det(y) \rangle \geq 0. \tag{4.14}$$

In the next step, we deduce from this inequality that $(d-1)\alpha \preceq b$. For the following calculation, we need the derivative of the determinant, in particular the representation for invertible matrices $x \in S_d^{++}$ given by

$$\nabla \det(x) = \det(x) x^{-1}.$$

In order to use this representation of the derivative, we consider in the following calculation $x \in S_d^{++}$ instead of $y \in S_d^+$. Remark that c^{ij} is the standard basis in S_d as introduced in Section 4.1. Hence, we obtain by the chain rule

$$\begin{aligned}
&\mathrm{Tr} \left(A(x) \left(\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} \right) \right) \det(x) \\
&= \frac{1}{2} \sum_{i \leq j} \left\langle A(x) c^{ij}, \frac{\partial}{\partial x} \left\langle \frac{\partial}{\partial x}, c^{ij} \right\rangle \det(x) \right\rangle \\
&= \frac{1}{2} \sum_{i \leq j} \left\langle A(x) c^{ij}, \frac{\partial}{\partial x} \det(x) \left\langle x^{-1}, c^{ij} \right\rangle \right\rangle \\
&= \frac{1}{2} \sum_{i \leq j} \left\langle A(x) c^{ij}, \det(x) x^{-1} \left\langle x^{-1}, c^{ij} \right\rangle + \det(x) \frac{\partial}{\partial x} \left\langle x^{-1}, c^{ij} \right\rangle \right\rangle \\
&= \frac{1}{2} \sum_{i \leq j} \left\langle A(x) c^{ij}, \det(x) (x^{-1} \otimes x^{-1}) c^{ij} + \det(x) (-x^{-1} c^{ij} x^{-1}) \right\rangle.
\end{aligned}$$

Substituting the representation formula of the linear diffusion term $\mathrm{Tr}\big(A(x)(z \otimes z)\big) = 4\langle x, z^{-1} \alpha z^{-1} \rangle$ and applying cyclic permutation[9] $\langle x, yz \rangle = \langle y, zx \rangle = \langle z, xy \rangle$ yields

$$\begin{aligned}
&= \frac{1}{2} \det(x) \left(\mathrm{Tr}\left(A(x)(x^{-1} \otimes x^{-1}) \right) - \sum_{i \leq j} \left\langle A(x) c^{ij}, x^{-1} c^{ij} x^{-1} \right\rangle \right) \\
&= 2 \det(x) \left\langle x, x^{-1} \alpha x^{-1} \right\rangle - \frac{1}{2} \det(x) \sum_{i \leq j} \left(\left\langle c^{ij} x^{-1}, A(x) c^{ij} x^{-1} \right\rangle \right)
\end{aligned}$$

[9] The cyclic permutation leaves the trace and, hence, the scalar product unchanged.

$$
\begin{aligned}
&= 2\det(x)\left\langle x^{-1}, \alpha\right\rangle - \frac{1}{2}\det(x)\sum_{i\leq j}\left(4\left\langle x, c^{ij}x^{-1}\alpha c^{ij}x^{-1}\right\rangle\right) \\
&= 2\det(x)\left\langle x^{-1}, \alpha\right\rangle - 2\det(x)\sum_{i\leq j}\left(\left\langle \alpha c^{ij}, c^{ij}x^{-1}\right\rangle\right) \\
&= 2\det(x)\left\langle x^{-1}, \alpha\right\rangle - 2\det(x)\sum_{i\leq j}\left(\left\langle x^{-1}, \alpha c^{ij}c^{ij}\right\rangle\right).
\end{aligned}
$$

For $i = j$, c^{ii} has only entries on the main diagonal and hence $c^{ii}c^{ii} = c^{ii}$. In the case $i \neq j$, a short calculation shows that $c^{ij}c^{ij} = c^{ii} + c^{jj}$. This gives

$$
\begin{aligned}
&= 2\det(x)\left(\left\langle x^{-1}, \alpha\right\rangle - \sum_{i=1}^{d}\left\langle x^{-1}, \alpha c^{ii}\right\rangle - \sum_{i<j}\left\langle x^{-1}, \alpha(c^{ii}+c^{jj})\right\rangle\right) \\
&= 2\det(x)\left(\left\langle x^{-1}, \alpha\right\rangle - \left\langle x^{-1}, \alpha I_d\right\rangle - (d-1)\sum_{i=1}^{d}\left\langle x^{-1}, \alpha c^{ii}\right\rangle\right) \\
&= 2\det(x)\left(\left\langle x^{-1}, \alpha\right\rangle - \left\langle x^{-1}, \alpha\right\rangle - (d-1)\left\langle x^{-1}, \alpha I_d\right\rangle\right) \\
&= 2\det(x)\left(-(d-1)\left\langle x^{-1}, \alpha\right\rangle\right) \\
&= -2(d-1)\left\langle \nabla\det(x), \alpha\right\rangle.
\end{aligned}
$$

For $y \in \partial S_d^+$, however, $\nabla\det(y) \in S_d^+$ is also well-defined as a derivative of a polynomial function $\det : S_d^+ \to \mathbb{R}$. Hence, by an approximation argument $(x_n)_{n\in\mathbb{N}} \subseteq S_d^{++}$, $x_n \to y$, it also holds for $y \in S_d^+$ that

$$
\operatorname{Tr}\left(A(y)\left(\frac{\partial}{\partial y}\otimes\frac{\partial}{\partial y}\right)\right)\det(y) = -2(d-1)\left\langle \nabla\det(y), \alpha\right\rangle.
$$

Substituting this result into equation (4.14), we conclude

$$
\begin{aligned}
&-(d-1)\left\langle \nabla\det(y), \alpha\right\rangle + \langle b, \nabla\det(y)\rangle \\
&\quad= \left\langle \nabla\det(y), b-(d-1)\alpha\right\rangle \geq 0.
\end{aligned}
$$

Now, the self-duality of S_d^+ implies that the last inequality holds if and only if $b-(d-1)\alpha \in S_d^+$. Finally, we have $(d-1)\alpha \preceq b$. □

This linear diffusion condition does not appear in the canonical state space, see Chapter 3. However, in the case of general symmetric cones, a similar condition applies. For details we refer to Cuchiero et al. [16, Proposition 4.5].

4.5 Further Properties

It is possible to derive several properties of the previous chapter for an affine process on positive semidefinite matrices. Using the B.L.T. theorem 2.13, we can immediately extend the infinitesimal generator to the weighted space $(C^2_{(1+\|x\|),\infty}(S_d^+), \|\cdot\|_{(2),(1+\|x\|)})$, cf. Section 2.3, as the arguments are still valid due to the isomorphism of S_d^+ and $\mathbb{R}^{d(d+1)/2}$. Furthermore, we show that the test functions $C_c^\infty(S_d^+)$ are a core of the generator.
Again, using the isomorphism, we can immediately get several path properties from Section 3.6. For example we obtain an accessible criterion for the conservativeness of an affine semigroup. However, we are not going to study these properties in detail. At last, we state a result on the boundary attainment of an affine process on S_d^+.

Proposition 4.18. *Let X be an affine process with semigroup $(T_t)_{t\geq 0}$ and generator $(A, \mathcal{D}(A))$. Then $C_c^\infty(S_d^+)$ is a core of $(A, \mathcal{D}(A))$.*

Proof. The main idea of the proof is to apply Lemma 3.25 with $D_0 = C_c^\infty(S_d^+)$ and $D = C^2_{(1+\|x\|),\infty}(S_d^+)$. It is obvious that these sets are dense subsets of $C_\infty(S_d^+)$ and contained in the domain of the generator. It remains to show that the semigroup $(T_t)_{t\geq 0}$ maps a function $f \in C_c^\infty(S_d^+)$ into $C^2_{(1+\|x\|),\infty}(S_d^+)$. This can be shown in the same manner as in Proposition 3.27. □

Similar to Mayerhofer et al. [39], there exists an equivalent condition for conservativeness of an affine semigroup on S_d^+. It requires that there is no killing, i.e. $q(x, 0) = 0$, and that a unique local solution of the matrix-variate generalized Riccati equation exists. However, the last condition is often difficult to verify. The next proposition presents an accessible and sufficient criterion.

Proposition 4.19. *An affine semigroup is conservative. In other words, an affine process has infinite life-time if $c = 0$, $\gamma = 0$ and*

$$\Big\| \int_{S_d^+\setminus\{0\}} \|y\|\mu(\,\mathrm{d}y)\Big\| < \infty.$$

Theorem 3.29 establishes this criterion when applied to representation (4.7) and (4.8).
One may ask whether an affine process hits the boundary of the cone S_d^+. The following result is an analogue of the Feller condition for $\mathbb{R}_+$-valued square root processes.

Proposition 4.20 (Boundary non-attainment)**.** *Let X be a conservative affine process on S_d^+ such that*

$$\int_{S_d^+} (\|y\| \wedge 1)\langle \mu(dy), x\rangle < \infty$$

for all $x \in S_d^+$. If $((d-1)+2)\alpha \preceq b$, then for each $x \in S_d^{++}$, we have $\mathbb{P}^x$-a.s.

$$T_x := \inf\{t > 0;\ X_{t-} \notin S_d^{++}\} = \infty.$$

This statement was proved for a more general setting, in particular for irreducible symmetric cones, by Cuchiero et al. [16, Proposition 6.1].

Chapter 5

Ornstein-Uhlenbeck Processes in L^p Space

Having examined affine processes on the space of positive semidefinite matrices $D = S_d^+$, we now consider the real state space $D = \mathbb{R}^n$. However, we change the function space of the operators. This means, that we will discuss in this chapter the semigroup of an Ornstein-Uhlenbeck process as an operator on the L^p space. Our specific interest is to show that the semigroup is an L^2 sub-Markovian semigroup. This, and several related results, are contained in Section 5.1. In Section 5.2 we will be concerned with the invariance and symmetry of operators. For this purpose, we focus on the state space $D = \mathbb{R}$, i.e. we consider an operator corresponding to a one-dimensional Ornstein-Uhlenbeck process. However, we will see that the results extend to perturbed operators.

5.1 Ornstein-Uhlenbeck Processes and Dirichlet Operators

In this section we examine an Ornstein-Uhlenbeck semigroup and its generator defined as in the previous chapter, cf. Definition 3.1, as operators on an L^p space[1] , $2 \leq p < \infty$. The main result, Theorem 5.2, states that an Ornstein-Uhlenbeck process generates an L^2 sub-Markov semigroup. This enables further extensions, e.g. that its generator is a Dirichlet operator. The proof gives more, namely the exponential convergence of the semigroup in L^2.
We start with an auxiliary lemma, a monotonicity result for $\operatorname{Re}\psi(\cdot,\xi)$ and $\operatorname{Re}\phi(\cdot,\xi)$.

Lemma 5.1. *Let $X = (X_t)_{t\geq 0}$ be an affine process, as defined in Definition 3.1, with corresponding functions ϕ and ψ. Then the mappings $t \mapsto \operatorname{Re}\phi(t,\xi)$ and $t \mapsto \operatorname{Re}\psi(t,\xi)$ are monotonically decreasing for all $\xi \in \mathbb{C}_-^m \times \mathbf{i}\mathbb{R}^n$.*

[1]In the following section the L^p space is equipped with the Lebesgue measure if not mentioned otherwise. Hence, we write $L^p(\mathbb{R}^n)$ instead of $L^p(\mathbb{R}^n, \mathrm{d}x)$.

Proof. By the Riccati equations (3.6), we have

$$\frac{\partial}{\partial t}\operatorname{Re}\phi(t,\xi) = \operatorname{Re}F(\psi(t,\xi)) \le F(0) \le 0,$$
$$\frac{\partial}{\partial t}\operatorname{Re}\psi(t,\xi) = \operatorname{Re}R(\psi(t,\xi)) \le R(0) \le 0,$$

where the second to last inequalities stem from the negative definiteness of $-F$ and $-R$, respectively. The t-derivatives are non-positive which shows the assertion. □

This monotonicity is a key element to the next theorem that an Ornstein-Uhlenbeck semigroup maps L^2 functions into L^2.

Theorem 5.2. *The semigroup $(T_t)_{t\ge0}$ of an Ornstein-Uhlenbeck process on $\mathbb{R}^n$ with symbol*[2] *$q(x,\xi) = \psi_L(\xi) + \mathbf{i}x^\top B\xi$ is a strongly continuous contraction semigroup on $L^2(\mathbb{R}^n)$ if* $\operatorname{trace}B \ge 0$.

Proof. It is well known that $C_c^\infty(\mathbb{R}^n)$ is dense in the Schwartz space, see Jacob [27, Corollary I.2.6.1] and that the Fourier transform is a linear bijective and continuous operator from the Schwartz space into itself, cf. Jacob [27, Theorem I.3.1.6]. Since the Schwartz space is dense in $L^2(\mathbb{R}^n)$, so are the functions $x \mapsto \int_{\mathbb{R}^n} \mathrm{e}^{\mathbf{i}x^\top z} g(z)\,\mathrm{d}z$ for $g \in C_c^\infty(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$. Hence, it is sufficient to show that $\|T_t f\|_{L^2} \le \|f\|_{L^2}$ for $f(x) = \int_{\mathbb{R}^n} \mathrm{e}^{\mathbf{i}x^\top z} g(z)\,\mathrm{d}z$ with some $g \in C_c^\infty(\mathbb{R}^n)$. Using the affine property (3.1) and Proposition 3.6 we get

$$\begin{aligned}
&\|T_t f\|^2_{L^2(\mathbb{R}^n)} &(5.1)\\
&= \int_{\mathbb{R}^n} \left|T_t f(x)\right|^2 \mathrm{d}x\\
&= \int_{\mathbb{R}^n} \left|T_t \int_{\mathbb{R}^n} \mathrm{e}^{\mathbf{i}x^\top z} g(z)\,\mathrm{d}z\right|^2 \mathrm{d}x\\
&= \int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} T_t \mathrm{e}_{\mathbf{i}z}(x) g(z)\,\mathrm{d}z\right|^2 \mathrm{d}x\\
&= \int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} \mathrm{e}^{\mathbf{i}x^\top \mathrm{e}^{tB} z} \mathrm{e}^{\phi(t,\mathbf{i}z)} g(z)\,\mathrm{d}z\right|^2 \mathrm{d}x. &(5.2)
\end{aligned}$$

Since $\mathrm{e}^{\phi(t,\mathbf{i}z)} g \in C_c(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$, by Plancherel's identity and a substitution for multiple

[2]The function ψ_L is the characteristic exponent of the driving Lèvy process.

variables, $\bar{x} = -(\mathrm{e}^{tB})^\top x$, see Schilling [50, Satz 20.1], the integral term becomes

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \mathrm{e}^{\mathbf{i}\left((\mathrm{e}^{tB})^\top x\right)^\top z} \mathrm{e}^{\phi(t,\mathbf{i}z)} g(z)\,\mathrm{d}z \right|^2 \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \mathrm{e}^{-\mathbf{i}\bar{x}^\top z} \mathrm{e}^{\phi(t,\mathbf{i}z)} g(z)\,\mathrm{d}z \right|^2 |\det(\mathrm{e}^{tB})^\top|\,\mathrm{d}\bar{x} \\
&\quad = (2\pi)^{n/2} \underbrace{\mathrm{e}^{-t\,\mathrm{trace}(B)}}_{=:c} \int_{\mathbb{R}^n} \left| \mathcal{F}(\mathrm{e}^{\phi(t,\mathbf{i}\cdot)} g)(\bar{x}) \right|^2 \mathrm{d}\bar{x} \\
&\quad = (2\pi)^{n/2} c \int_{\mathbb{R}^n} \left| \mathrm{e}^{\phi(t,\mathbf{i}x)} g(x) \right|^2 \mathrm{d}x.
\end{aligned}$$

Using this result and Plancherel's identity again finally gives

$$\begin{aligned}
\|T_t f\|^2_{L^2(\mathbb{R}^n)} &\le (2\pi)^{n/2} c \int_{\mathbb{R}^n} \Big| \underbrace{\mathrm{e}^{\phi(t,\mathbf{i}x)}}_{|\cdot|^2 \le 1} g(x) \Big|^2 \mathrm{d}x \\
&\le (2\pi)^{n/2} c \int_{\mathbb{R}^n} \left| g(x) \right|^2 \mathrm{d}x \\
&= c \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \mathrm{e}^{\mathbf{i}x^\top z} g(z)\,\mathrm{d}z \right|^2 \mathrm{d}x \\
&= \mathrm{e}^{-t\,\mathrm{trace}(B)} \int_{\mathbb{R}^n} \left| f(x) \right|^2 \mathrm{d}x \\
&= \mathrm{e}^{-t\,\mathrm{trace}(B)} \|f\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

The above calculations prove that every Ornstein-Uhlenbeck semigroup maps $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. Furthermore, it is a contraction semigroup if $\mathrm{trace}(B) \ge 0$.
It remains to prove the strong continuity. Let f be as before. Then $f \in C_\infty(\mathbb{R}^n)$ and by dominated convergence theorem we get

$$\begin{aligned}
\lim_{t\to 0} \|T_t f - f\|_{L^2(\mathbb{R}^n)} &= \lim_{t\to 0} \int_{\mathbb{R}^n} |T_t f(x) - f(x)|^2\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \lim_{t\to 0} |T_t f(x) - f(x)|^2\,\mathrm{d}x \\
&= 0.
\end{aligned}$$

Note that for continuous functions we have already shown the continuity, see Theorem 3.8. A standard density argument proves the assertion. □

The above proof gives more, namely we have explicitly determined the contraction factor

$$\|T_t f\|_{L^2(\mathbb{R}^n)} \le \mathrm{e}^{-\frac{t}{2}\mathrm{trace}(B)} \|f\|_{L^2(\mathbb{R}^n)}.$$

This exponential decay of the semigroup is sharp as the following example of a one-dimensional Ornstein-Uhlenbeck process driven by a Brownian motion shows.

Example 5.3. Let $(X_t)_{t\geq 0}$ be an Ornstein-Uhlenbeck process defined by the stochastic differential equation

$$X_t = x_0 + b\int_0^t X_s \, \mathrm{d}s + \int_0^t \mathrm{d}B_s, \qquad x_0 \in \mathbb{R},$$

where $b \in \mathbb{R}$ is a constant and $(B_t)_{t\geq 0}$ is a one-dimensional Brownian motion. The solution is given by

$$X_t = \mathrm{e}^{bt}x_0 + \mathrm{e}^{bt}\int_0^t \mathrm{e}^{-bs} \, \mathrm{d}B_s, \qquad t \geq 0.$$

This can be found in Schilling [51, Example 19.5]. Hence the calculation of the L^2-norm of the semigroup is straightforward. Using the substitution $y = \mathrm{e}^{bt}x + \int_0^t \mathrm{e}^{b(t-s)} \, \mathrm{d}B_s$ and $\mathrm{d}y = \mathrm{e}^{bt} \, \mathrm{d}x$ we get

$$\begin{aligned}
\|\mathbb{E}(f(X_t))\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \left|\mathbb{E}\Big(f(\mathrm{e}^{bt}x + \mathrm{e}^{bt}\int_0^t \mathrm{e}^{-bs} \, \mathrm{d}B_s)\Big)\right|^2 \mathrm{d}x \\
&= \mathrm{e}^{-bt}\int_{\mathbb{R}} \left|\mathbb{E}\left(f(y)\right)\right|^2 \mathrm{d}y \\
&= \mathrm{e}^{-bt}\|f\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

We see that the Ornstein-Uhlenbeck semigroup is a contraction with factor $\mathrm{e}^{-\frac{t}{2}b}$ if $b > 0$.

Remark 5.4. Ornstein-Uhlenbeck processes driven by a Brownian motion are usually defined on the space $L^p(\mu)$, where μ is an invariant measure of the semigroup, see below Definition 5.11. For a treatment of this case, we refer to Bakry, Gentil and Ledoux [4].

The natural question arises whether general affine processes also correspond to L^2 semigroups. Therefore, we take a look at a squared Bessel process. The transition probability of the squared Bessel semigroup is explicitly known, see Revuz and Yor [44, Section XI.1]. Hence, the semigroup is given by

$$T_t f(x) = \int_0^\infty f(y)\frac{1}{2}\left(\frac{y}{x}\right)^{\nu/2} \mathrm{e}^{-\frac{x+y}{2t}} I_\nu\Big(\frac{\sqrt{xy}}{t}\Big) \, \mathrm{d}y, \tag{5.3}$$

where $I_\nu(x) = \sum_{m=0}^\infty \frac{1}{m!\Gamma(m+\nu+1)}\left(\frac{x}{2}\right)^{2m+\nu}$ is the modified Bessel function of first kind of index ν. For simplicity, we set $t=1$ and $\nu=4$ to obtain

$$
\begin{aligned}
&\|T_t f\|^2_{L^2(\mathbb{R}_+,\,\mathrm{d}x)}\\
&\quad= \int_{\mathbb{R}^+} \left| \int_0^\infty f(y)\frac{1}{2}\left(\frac{y}{x}\right)^2 \mathrm{e}^{-\frac{x+y}{2}} I_4(\sqrt{xy})\,\mathrm{d}y \right|^2 \mathrm{d}x\\
&\quad= \int_{\mathbb{R}_+} \left| \int_0^\infty \sum_{m=0}^\infty \frac{1}{m!(m+4)!} f(y)\frac{1}{2}\left(\frac{y}{x}\right)^2 \mathrm{e}^{-\frac{x+y}{2}} \left(\frac{\sqrt{xy}}{2}\right)^{2m+4} \mathrm{d}y \right|^2 \mathrm{d}x\\
&\quad= \int_{\mathbb{R}_+} \left| \int_0^\infty \sum_{m=0}^\infty \frac{1}{m!(m+4)!} f(y)\frac{1}{2}\left(\frac{y}{x}\right)^2 \mathrm{e}^{-\frac{x+y}{2}} \left(\frac{xy}{4}\right)^{m+2} \mathrm{d}y \right|^2 \mathrm{d}x\\
&\quad= \int_{\mathbb{R}_+} \int_0^\infty \int_0^\infty \sum_{m=0}^\infty \sum_{k=0}^\infty \frac{1}{m!(m+4)!}\frac{1}{k!(k+4)!} \left[f(y)\frac{1}{2}\left(\frac{y}{x}\right)^2 \mathrm{e}^{-\frac{x+y}{2}} \left(\frac{xy}{4}\right)^{m+2} \right]\\
&\qquad\qquad \cdot \left[f(z)\frac{1}{2}\left(\frac{z}{x}\right)^2 \mathrm{e}^{-\frac{x+z}{2}} \left(\frac{xz}{4}\right)^{k+2} \right] \mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x\\
&\quad= \int_{\mathbb{R}_+} \int_0^\infty \int_0^\infty \sum_{m=0}^\infty \sum_{k=0}^\infty \frac{1}{m!(m+4)!}\frac{1}{k!(k+4)!} \left[f(y)\frac{16}{2}\mathrm{e}^{-\frac{y}{2}} \left(\frac{y}{4}\right)^{m+4} \right]\\
&\qquad\qquad \cdot \left[f(z)\frac{16}{2}\mathrm{e}^{-\frac{z}{2}} \left(\frac{z}{4}\right)^{k+4} \right] \mathrm{e}^{-x} x^{m+k}\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x.
\end{aligned}
$$

Observe that $\int_{\mathbb{R}_+} x^{(m+k+1)-1}\mathrm{e}^{-x}\,\mathrm{d}x = \Gamma(m+k+1) = (m+k)!$ since the integrand is the density of a Gamma distribution. Hence, the above calculation becomes

$$
\begin{aligned}
&\|T_t f\|^2_{L^2(\mathbb{R}_+,\,\mathrm{d}x)}\\
&\quad= 16 \int_0^\infty \int_0^\infty \sum_{m=0}^\infty \sum_{k=0}^\infty \frac{(m+k)!}{m!k!} \frac{1}{(m+4)!}\frac{1}{(k+4)!} \left[f(y)\mathrm{e}^{-\frac{y}{2}} \left(\frac{y}{4}\right)^{m+4} \right]\\
&\qquad\qquad \cdot \left[f(z)\mathrm{e}^{-\frac{z}{2}} \left(\frac{z}{4}\right)^{k+4} \right] \mathrm{d}y\,\mathrm{d}z.
\end{aligned}
$$

By the Stirling formula, see Krengel [35, Satz 5.1], we have the following upper and lower bound for the factorial,

$$
\sqrt{2\pi n}n^n\mathrm{e}^{-n} \le n! \le \sqrt{2\pi n}n^n\mathrm{e}^{-n}\mathrm{e}^{1/(12n)}.
$$

Therefore, we obtain

$$
\begin{aligned}
\frac{(m+k)!}{m!k!} &\le \frac{\sqrt{2\pi}\sqrt{m+k}(m+k)^m(m+k)^k\mathrm{e}^{-(m+k)}\mathrm{e}^{1/(12(m+k))}}{\sqrt{2\pi}\sqrt{m}m^m\mathrm{e}^{-m}\sqrt{2\pi}\sqrt{k}k^k\mathrm{e}^{-k}}\\
&\le \frac{\mathrm{e}^{1/24}}{\sqrt{2\pi}} \underbrace{\sqrt{\frac{m+k}{mk}}}_{\le\sqrt{2}} \underbrace{\left(1+\frac{m}{k}\right)^k}_{\le \mathrm{e}^m} \underbrace{\left(1+\frac{k}{m}\right)^m}_{\le \mathrm{e}^k}\\
&\le \underbrace{\frac{\sqrt{2}\mathrm{e}^{1/24}}{\sqrt{2\pi}}}_{=:c} \mathrm{e}^k\mathrm{e}^m.
\end{aligned}
$$

Substituting this into the above equation yields

$$
\begin{aligned}
&\|T_t f\|^2_{L^2(\mathbb{R}_+,\,\mathrm{d}x)} \\
&\quad \le 16 \int_0^\infty \int_0^\infty \sum_{m=0}^\infty \sum_{k=0}^\infty c\mathrm{e}^k \mathrm{e}^m \frac{1}{(m+4)!} \frac{1}{(k+4)!} f(y) \mathrm{e}^{-\frac{y}{2}} \left(\frac{y}{4}\right)^{m+4} \\
&\qquad\quad \cdot f(z) \mathrm{e}^{-\frac{z}{2}} \left(\frac{z}{4}\right)^{k+4} \mathrm{d}y \,\mathrm{d}z \\
&\quad = c' \int_0^\infty \sum_{m=0}^\infty \frac{1}{(m+4)!} f(y) \mathrm{e}^{-\frac{y}{2}} \left(\frac{\mathrm{e}y}{4}\right)^{m+4} \\
&\qquad\quad \cdot \int_0^\infty f(z) \mathrm{e}^{-\frac{z}{2}} \sum_{k=0}^\infty \frac{1}{(k+4)!} \left(\frac{\mathrm{e}z}{4}\right)^{k+4} \mathrm{d}y \,\mathrm{d}z \\
&\quad \le c' \int_0^\infty \sum_{m=0}^\infty \frac{1}{m!} f(y) \mathrm{e}^{-\frac{y}{2}} \left(\frac{\mathrm{e}y}{4}\right)^{m} \cdot \int_0^\infty f(z) \mathrm{e}^{-\frac{z}{2}} \sum_{k=0}^\infty \frac{1}{k!} \left(\frac{\mathrm{e}z}{4}\right)^{k} \mathrm{d}y \,\mathrm{d}z \\
&\quad \le c' \left[\int_0^\infty f(y) \mathrm{e}^{\frac{ey}{4} - \frac{y}{2}} \,\mathrm{d}y \right]^2 .
\end{aligned}
$$

This calculation shows that for functions of a weighted space, $L^2(\mathbb{R}_+, \rho(x)\,\mathrm{d}x)$, with an exponential weight $\rho(x) = \mathrm{e}^{Cx}$, the squared Bessel semigroup maps into L^2. However, this does not prove that the semigroup does not map L^2 into L^2. We did not manage to construct a suitable lower bound and numerical estimates failed due to the involved Bessel function.

We now return to the Ornstein-Uhlenbeck semigroup. By a variant of the Riesz-Thorin theorem, we can extend the above theorem.

Corollary 5.5. *The Ornstein-Uhlenbeck semigroup $(T_t)_{t\ge 0}$ defined on $L^2(\mathbb{R}^n)$ extends to a strongly continuous contraction semigroup $(T_t^{(p)})_{t\ge 0}$ on $L^p(\mathbb{R}^n)$ for all $2 < p < \infty$, if* $\mathrm{trace}B \ge 0$, *where B is the linear drift coefficient.*

Proof. From Theorem 5.2 and Theorem 3.8 we know that

$$
\begin{aligned}
\|T_t f\|_{L^2(\mathbb{R}^n)} &\le \mathrm{e}^{-\frac{t}{2}\mathrm{trace}(B)} \|f\|_{L^2(\mathbb{R}^n)} \\
\|T_t f\|_\infty &\le \|f\|_\infty .
\end{aligned}
$$

In particular both extensions coincide on $L^1(\mathbb{R}^n) \cap B_b(\mathbb{R}^n)$. Hence from Farkas, Jacob and Schilling [20, Theorem 1.10] it follows that $(T_t)_{t\ge 0}$ extends to a strongly continuous contraction semigroup on $L^p(\mathbb{R}^n)$ for $2 < p < \infty$. Furthermore, setting $p = \frac{1-s}{2}$ for some[3] $s \in (0,1)$ we obtain the following estimate, see Stein and Weiß [59, Theorem V.1.3],

$$
\|T_t f\|_{L^p(\mathbb{R}^n)} \le \mathrm{e}^{-\frac{t}{2}\mathrm{trace}(B)(1-s)} \|f\|_{L^p(\mathbb{R}^n)} = \mathrm{e}^{-\frac{t}{p}\mathrm{trace}(B)} \|f\|_{L^p(\mathbb{R}^n)},
$$

which shows the assertion. □

[3] We adopt the usual convention and let $\frac{1}{\infty} = 0$.

To state a consequence of Theorem 5.2 we introduce the following notion.

Definition 5.6. Let $(T_t)_{t\geq 0}$ be a strongly continuous contraction semigroup on $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. We call $(T_t)_{t\geq 0}$ a sub-Markov semigroup on $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$ if for $f \in L^p(\mathbb{R}^n)$ such that $0 \leq f \leq 1$ almost everywhere it follows that $0 \leq T_t f \leq 1$ almost everywhere.

Corollary 5.7. *The semigroup $(T_t)_{t\geq 0}$ of an Ornstein-Uhlenbeck process on $\mathbb{R}^n$ is sub-Markovian on $L^p(\mathbb{R}^n)$ with $p \geq 2$ if $\mathrm{trace} B \geq 0$, where B is the linear drift coefficient.*

Now we want to turn towards the generator of an Ornstein-Uhlenbeck process. The first result is a consequence of the above theorem and its corollary.

Corollary 5.8. *The infinitesimal generator $(A, \mathcal{D}(A))$ of an Ornstein-Uhlenbeck process on $\mathbb{R}^n$ is a Dirichlet operator on $L^2(\mathbb{R}^n)$, i.e.*

$$\int_{\mathbb{R}^n} Af(x) \cdot (f-1)^+(x)\,\mathrm{d}x \leq 0 \quad \textit{for all } f \in \mathcal{D}(A),$$

if its parameter B satisfies $\mathrm{trace} B \geq 0$.

Proof. In Theorem 5.2 we have proven that an Ornstein-Uhlenbeck semigroup is a strongly continuous contraction semigroup on $L^2(\mathbb{R}^n)$. This statement is equivalent to $(A, \mathcal{D}(A))$ being an L^2 Dirichlet operator. A proof of this can be found in Ma and Röckner [37, Proposition I.4.3]. $\square$

Lemma 5.9. *Let $(A, \mathcal{D}(A))$ be the generator of a stochastically continuous affine process on $\mathbb{R}_+^m \times \mathbb{R}^n$. Denote by $q(x, \xi)$ its corresponding symbol such that A is given by*

$$\begin{aligned} Af(x) = -q(x,D)f(x) &= \sum_{j,k=1}^{d} \Big(a_{jk} + \sum_{l=1}^{m} x_l \alpha_{jk}^l\Big) \partial_j \partial_k f(x) \\ &+ \Big(b + \sum_{l=1}^{d} \beta^l x_l\Big)^\top \nabla f(x) + \Big(c + \gamma^\top x\Big)\, f(x) \\ &+ \int_{D\setminus\{0\}} \Big(f(x+y) - f(x) - \chi(y)^\top \nabla f(x)\Big)\, \mu(\,\mathrm{d}y) \\ &+ \sum_{l=1}^{m} x_l \int_{D\setminus\{0\}} \Big(f(x+y) - f(x) - \chi^l(y)^\top \nabla f(x)\Big)\, \mu^l(\,\mathrm{d}y), \end{aligned}$$

for $f \in C_c^\infty(\mathbb{R}_+^m \times \mathbb{R}^n)$. Then A maps $C_c^\infty(\mathbb{R}_+^m \times \mathbb{R}^n)$ into $L^p(\mathbb{R}_+^m \times \mathbb{R}^n)$ for all $p \geq 1$. In particular, we have

$$\|Af\|_{L^p} \leq c\|f\|_{W^2_{p,(1+|x|)}},$$

where

$$W^2_{p,(1+|x|)}(\mathbb{R}_+^m \times \mathbb{R}^n) = \Big\{ f \in L^p(\mathbb{R}_+^m \times \mathbb{R}^n); \|f\|_{W^2_{p,(1+|x|)}} = \sum_{|\alpha|\leq 2} \|(1+|x|)D^\alpha f\|_{L^p} < \infty \Big\}$$

is a weighted Sobolev space.

Proof. For $f \in C_c^\infty(\mathbb{R}_+^m \times \mathbb{R}^n)$ the estimate follows by the same method as in Schilling [49, Lemma 3.4]. Since $C_c^\infty(\mathbb{R}_+^m \times \mathbb{R}^n) \subseteq W^2_{p,(1+|x|)}(\mathbb{R}_+^m \times \mathbb{R}^n)$ is a dense subset, a standard argument finishes the proof. □

Remark 5.10. Schmeißer and Triebel [55, Theorem 5.1.4] show that the norm of the weighted Sobolev space $\|f\|_{W^2_{p,(1+|x|)}} = \sum_{|\delta|\le 2} \|(1+|x|)D^\delta f\|_{L^p}$ is equivalent to the norm $\|(1+|x|)f\|_{W^2_p} = \sum_{|\delta|\le 2} \|D^\delta\big((1+|x|)f\big)\|_{L^p}$.

One may ask whether affine processes are related to Dirichlet forms. For Ornstein-Uhlenbeck processes driven by a Brownian motion this is well known, cf. Barky, Gentil and Ledoux [4, Section 4.1]. For general Ornstein-Uhlenbeck processes there are no results in the literature. Since we have shown that its generator is a Dirichlet operator according to Jacob [27, Theorem I.4.7.5 and Corollary I.4.7.6], it is left to show that the generator $(A, \mathcal{D}(A))$ of an Ornstein-Uhlenbeck process satisfies the sector condition

$$|\langle -Af, g\rangle_{L^2}| \le c(\langle -Af, f\rangle_{L^2})^{\frac{1}{2}}(\langle -Ag, g\rangle_{L^2})^{\frac{1}{2}}, \qquad f, g \in \mathcal{D}(A).$$

It would be desirable to verify this inequality but we have not been able to do this.

5.2 Invariant Measures and the Symmetry Property

Invariant measures and symmetry appear quite naturally in the examination of semigroups and operators on L^p spaces. In this section we focus on affine processes with state space $D = \mathbb{R}^n$, in other words general Ornstein-Uhlenbeck processes but we will extend the results to Lévy-type processes. For affine processes on $D = \mathbb{R}_+^m$ we refer the reader to Keller-Ressel and Mijatovic [31] and Handa [25]. For Ornstein-Uhlenbeck processes driven by a Brownian motion it is well-known that $\mathrm{e}^{-x^2}\,\mathrm{d}x$ is an invariant measure and that the semigroup is symmetric with respect to that measure. We will extend this result and consider general Ornstein-Uhlenbeck processes. First we examine the existence of invariant measure. To this end, we use an approach based on the symbol of the generator.[4] In a next step, we prove that Ornstein-Uhlenbeck processes are μ-symmetric if and only if they have no jumps. This result extends to perturbed Ornstein-Uhlenbeck processes.

Definition 5.11 (Invariant measure)**.** A probability measure μ on $\mathbb{R}^d$ is invariant (or stationary) for the Feller process X with semigroup $(T_t)_{t\ge 0}$ if

$$\int_{\mathbb{R}^d} T_t f(x)\mu(\,\mathrm{d}x) = \int_{\mathbb{R}^d} f(x)\mu(\,\mathrm{d}x), \quad \forall f \in C_\infty(\mathbb{R}^d),\ t \ge 0.$$

In the following, we will use a criterion for invariant measures that is based on the symbol.

[4]There exist other methods to study invariant measures of Ornstein-Uhlenbeck semigroups, e.g. Sato and Yamazato [47] used an operator based approach and Fuhrmann and Röckner [23] recognized them as a special case of Mehler semigroups.

Theorem 5.12. *Let $(A, \mathcal{D}(A))$ be the infinitesimal generator of a d-dimensional Feller process X such that $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A)$ and symbol $q(x,\xi)$.*
If μ is an invariant measure for X such that $\int_{\mathbb{R}^d} |q(x,\xi)|\mu(\mathrm{d}x) < \infty$, then

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathbf{i}x^\top \xi} q(x,\xi)\mu(\mathrm{d}x) = 0 \quad \forall \xi \in \mathbb{R}^d.$$

Conversely, if there exists a probability measure μ such that $\int_{\mathbb{R}^d} |q(x,\xi)|\mu(\mathrm{d}x) < \infty$ and $\int_{\mathbb{R}^d} \mathrm{e}^{\mathbf{i}x^\top \xi} q(x,\xi)\mu(\mathrm{d}x) = 0$, then

$$\int_{\mathbb{R}^d} Af(x)\mu(\mathrm{d}x) = 0 \quad \forall f \in C_c^\infty(\mathbb{R}^d).$$

In particular, μ is an invariant measure for X if $C_c^\infty(\mathbb{R}^d)$ is a core of the generator A.

Both statements, the necessity and the sufficiency, can be found in Behme and Schnurr [7, Theorem 3.1 and Theorem 4.1]. The addition is due to an equivalent condition for invariant measures, cf. Liggett [36, Theorem 3.37], which states that for Feller semigroups a measure μ is invariant if and only if

$$\int Af \,\mathrm{d}\mu = 0 \quad \forall f \in D,$$

where D is a core of the generator A.
The key to our proof of the existence of an invariant measure for Ornstein-Uhlenbeck processes is the following lemma.

Lemma 5.13. *Let $\psi : \mathbb{R} \longrightarrow \mathbb{C}$ be a continuous negative definite function with Lévy-Khintchine representation*

$$\psi(\xi) = -\mathbf{i}l\xi + \frac{1}{2}Q\xi^2 + \int_{y\neq 0} \left(1 - \mathrm{e}^{\mathbf{i}y\xi} + \mathbf{i}\xi y \mathbb{1}_{\{|y|\leq 1\}}\right)\nu(\mathrm{d}y),$$

such that $\int_{|y|\geq 1} \ln|y|\nu(\mathrm{d}y) < \infty$. Then the function

$$\xi \mapsto \int_0^\xi \frac{\psi(\eta)}{\eta}\,\mathrm{d}\eta$$

is continuous negative definite.

Proof. Starting with the local part of ψ we have

$$\begin{aligned}\int_0^\xi \frac{-\mathbf{i}l\eta + \frac{1}{2}Q\eta^2}{\eta}\,\mathrm{d}\eta &= \int_0^\xi -\mathbf{i}l + \frac{1}{2}Q\eta\,\mathrm{d}\eta \\ &= -\mathbf{i}l\xi + \frac{1}{4}Q\xi^2,\end{aligned}$$

which is obviously continuous negative definite.
Before calculating the non-local part, we verify the existence of the integral. To do this, we use Taylor's theorem and Tonelli's theorem,

$$\begin{aligned}
&\int_0^\xi \int_{y\neq 0} \left|\frac{1}{\eta}\left(1-\mathrm{e}^{\mathbf{i}y\eta}+\mathbf{i}y\eta\mathbb{1}_{\{|y|\leq 1\}}\right)\right| \nu(\,\mathrm{d}y)\,\mathrm{d}\eta \\
&= \int_0^\xi \int_{0<|y|\leq 1} \left|\frac{1}{\eta}\left(1-\mathrm{e}^{\mathbf{i}y\eta}+\mathbf{i}y\eta\right)\right| \nu(\,\mathrm{d}y) + \int_{|y|>1} \left|\frac{1}{\eta}\left(1-\mathrm{e}^{\mathbf{i}y\eta}\right)\right| \nu(\,\mathrm{d}y)\,\mathrm{d}\eta \\
&\leq \frac{1}{2}\int_0^\xi \int_{0<|y|\leq 1} \frac{1}{|\eta|}|y\eta|^2 \nu(\,\mathrm{d}y)\,\mathrm{d}\eta + \int_{|y|>1}\int_0^\xi \left|\frac{1}{\eta}\left(1-\mathrm{e}^{\mathbf{i}y\eta}\right)\right| \mathrm{d}\eta\nu(\,\mathrm{d}y) \\
&\leq \frac{1}{4}|\xi|^2 \underbrace{\int_{0<|y|\leq 1} |y|^2\nu(\,\mathrm{d}y)}_{<\infty} + \int_{|y|>1}\int_0^\xi \left|\frac{1}{\eta}\left(1-\mathrm{e}^{\mathbf{i}y\eta}\right)\right| \mathrm{d}\eta\nu(\,\mathrm{d}y).
\end{aligned}$$

The first integral poses no problem because ν is a Lévy measure. Note that in the first inequality we used Fubini's theorem which is allowed as shown now. For this we substitute $z = y\eta,\ \mathrm{d}z = \frac{\mathrm{d}\eta}{y}$ and obtain

$$\begin{aligned}
&\int_{|y|>1}\int_0^\xi \left|\frac{1}{\eta}\left(1-\mathrm{e}^{\mathbf{i}y\eta}\right)\right| \mathrm{d}\eta\nu(\,\mathrm{d}y) \\
&= \int_{|y|>1}\int_0^{y\xi} \left|\frac{1}{z}\left(1-\mathrm{e}^{\mathbf{i}z}\right)\right| \mathrm{d}z\nu(\,\mathrm{d}y) \\
&= \int_{|y|>1}\left[\int_0^\varepsilon \frac{1}{|z|}\underbrace{\left|1-\mathrm{e}^{\mathbf{i}z}\right|}_{\leq|z|}\,\mathrm{d}z + \int_\varepsilon^{y\xi} \frac{1}{|z|}\underbrace{\left|1-\mathrm{e}^{\mathbf{i}z}\right|}_{\leq 2}\,\mathrm{d}z\right]\nu(\,\mathrm{d}y) \\
&\leq \int_{|y|>1}\left[\int_0^\varepsilon \mathrm{d}z + 2\int_\varepsilon^{y\xi}\frac{1}{|z|}\,\mathrm{d}z\right]\nu(\,\mathrm{d}y) \\
&\leq \int_{|y|>1}\Big(\varepsilon + 2\ln|y\xi| - 2\ln|\varepsilon|\Big)\nu(\,\mathrm{d}y) \\
&\leq 2\int_{|y|>1}\ln|y|\nu(\,\mathrm{d}y) + \Big(\varepsilon + 2\ln|\xi| - 2\ln|\varepsilon|\Big)\int_{|y|>1}\nu(\,\mathrm{d}y) < \infty \quad \forall\xi\in\mathbb{R},
\end{aligned}$$

where $\varepsilon \in (0, y\xi)$ is a constant. This allows us to apply Fubini's theorem and the substitution $\eta = \frac{\xi}{y}t,\ \frac{\mathrm{d}\eta}{\eta} = \frac{\mathrm{d}t}{t}$,

$$\begin{aligned}
&\int_0^\xi \int_{y\neq 0} \frac{1}{\eta}\left(1-\mathrm{e}^{\mathbf{i}y\eta}+\mathbf{i}y\eta\mathbb{1}_{\{|y|\leq 1\}}\right) \nu(\,\mathrm{d}y)\,\mathrm{d}\eta \\
&= \int_{y\neq 0}\int_0^\xi \left(1-\mathrm{e}^{\mathbf{i}y\eta}+\mathbf{i}y\eta\mathbb{1}_{\{|y|\leq 1\}}\right)\frac{\mathrm{d}\eta}{\eta}\nu(\,\mathrm{d}y) \\
&= \int_{y\neq 0}\int_0^y \left(1-\mathrm{e}^{\mathbf{i}\xi t}+\mathbf{i}\xi t\mathbb{1}_{\{|y|\leq 1\}}\right)\frac{\mathrm{d}t}{t}\nu(\,\mathrm{d}y).
\end{aligned}$$

It is clear that the integrand $\xi \mapsto \left(1 - \mathrm{e}^{\mathbf{i}\xi t} + \mathbf{i}\xi t \mathbb{1}_{\{|y|\leq 1\}}\right) \frac{1}{t}$ is a negative definite function. Since both measures are positive, the non-local part and thus $\xi \mapsto \int_0^\xi \frac{\psi(\eta)}{\eta}\,\mathrm{d}\eta$ is continuous negative definite. □

The above lemma is already known if the state space is $\mathbb{R}_+$. In this case, negative definite functions are Bernstein functions and for all Bernstein functions $\lambda \mapsto f(\lambda)/\lambda$ is a completely monotone function, cf. Schilling, Song and Vondraček [53, Corollary 3.7]. Hence, $\xi \mapsto \int_0^\xi f(\lambda)/\lambda\,\mathrm{d}\lambda$ again is a Bernstein function if the integral exists. This holds for a Bernstein function $f(\lambda) = a + b\lambda + \int_{(0,\infty)}(1 - \mathrm{e}^{-\lambda t})\nu(\,\mathrm{d}t)$ if $\int \ln(|y|)\nu(\,\mathrm{d}y) < \infty$. In the case of α-stable processes, i.e. $\psi(\xi) = |\xi|^\alpha$, we can directly calculate

$$\int_0^\xi \frac{|\eta|^\alpha}{\eta}\,\mathrm{d}\eta = \frac{|\xi|^\alpha}{\alpha}.$$

We continue with a result on invariant measures of Ornstein-Uhlenbeck processes.

Theorem 5.14. *Let X be an Ornstein-Uhlenbeck process on $\mathbb{R}$ with symbol $q(x,\xi) = \psi_L(\xi) + \mathbf{i}\beta x\xi$, where $\beta > 0$ and $\psi_L : \mathbb{R} \longrightarrow \mathbb{C}$ is the characteristic exponent of the driving Lévy process.*[5] *Assume that ψ_L satisfies the log-moment condition, $\int_{|y|\geq 1} \ln|y|\nu(\,\mathrm{d}y) < \infty$. If the characteristic function $\phi_\mu(\xi) = \exp\{-\int_0^\xi \frac{\psi_L(\eta)}{\beta\eta}\,\mathrm{d}\eta\}$ defines a probability measure μ such that $\int |x|\mu(\,\mathrm{d}x) < \infty$, then μ is an invariant measure for X.*

Proof. The main idea of the proof is to take a continuous negative definite function, defined as in the previous lemma using ψ_L. We show that this function generates a probability measure μ which is invariant.
By the lemma above, we know that the function $\xi \mapsto \int_0^\xi \frac{\psi_L(\eta)}{\beta\eta}\,\mathrm{d}\eta$ is negative definite. Then it follows by Schoenberg's theorem that $\xi \mapsto \exp\{-t\int_0^\xi \frac{\psi_L(\eta)}{\beta\eta}\,\mathrm{d}\eta\}$ is positive definite for every $t > 0$, in particular, $\phi_\mu(\xi) = \exp\{-\int_0^\xi \frac{\psi_L(\eta)}{\beta\eta}\,\mathrm{d}\eta\}$. Hence, by Bochner's theorem it is the characteristic function of a probability measure μ, as $\phi_\mu(0) = 1$. Note that the derivative ϕ'_μ exists which gives

$$-\int_0^\xi \frac{\psi_L(\eta)}{\beta\eta}\,\mathrm{d}\eta = \ln\phi_\mu(\xi) = \int_0^\xi \frac{\phi'_\mu(\eta)}{\phi_\mu(\eta)}\,\mathrm{d}\eta$$

and, in consequence, for almost all $\xi \in \mathbb{R}$

$$\frac{\phi'_\mu(\xi)}{\phi_\mu(\xi)} = -\frac{\psi_L(\xi)}{\beta\xi}.$$

[5] Such a process is given by the stochastic differential equation $\mathrm{d}X_t = -\beta X_t\,\mathrm{d}t + \mathrm{d}L_t$, where $(L_t)_{t\geq 0}$ is the driving Lévy process.

In order to apply the criterion of Theorem 5.12, we have to check two conditions. The first one follows by a short calculation

$$\begin{aligned}
0 &= \psi_L(\xi)\phi_\mu(\xi) + \beta\xi\phi'_\mu(\xi) \\
&= \psi_L(\xi)\phi_\mu(\xi) + \beta\xi\int_{\mathbb{R}} \partial_\xi \mathrm{e}^{\mathbf{i}x\xi}\mu(\,\mathrm{d}x) \\
&= \int_{\mathbb{R}} \mathrm{e}^{\mathbf{i}x\xi}(\psi_L(\xi) + \mathbf{i}x\beta\xi)\mu(\,\mathrm{d}x) \\
&= \int_{\mathbb{R}} \mathrm{e}^{\mathbf{i}x\xi} q(x,\xi)\mu(\,\mathrm{d}x).
\end{aligned}$$

Next we have to show that $\int_{\mathbb{R}^d} |q(x,\xi)|\mu(\,\mathrm{d}x) < \infty$ which reduces by

$$\begin{aligned}
\int_{\mathbb{R}} |q(x,\xi)|\mu(\,\mathrm{d}x) &\le \int_{\mathbb{R}} (|\psi_L(\xi)| + |x\beta\xi|)\mu(\,\mathrm{d}x) \\
&\le |\psi_L(\xi)| + |\beta\xi| \int_{\mathbb{R}} |x|\mu(\,\mathrm{d}x)
\end{aligned}$$

to the question whether $\int_{\mathbb{R}} |x|\mu(\,\mathrm{d}x) < \infty$. The assumptions of the theorem ensure this last condition, and the proof is complete. □

The existence of the first moment of the invariant measure cannot be dropped as it is essential to Theorem 5.12. Although we know that the first derivative of the characteristic function exists, we cannot conclude that the measure μ possesses a first moment.
The above technique extends to some special cases of affine processes on $\mathbb{R}_+$. This class has already been studied by Keller-Ressel and Mijatović [31].
Furthermore, our approach allows us to extend the result of Albeverio, Rüdiger and Wu [2] who studied invariant measures for perturbed Ornstein-Uhlenbeck processes with α-stable jumps. Therefore we are interested in regular invariant measures, i.e. we additionally require that the measure μ is absolutely continuous with respect to the Lebesgue measure.

Corollary 5.15. *Let X be an Ornstein-Uhlenbeck process on $\mathbb{R}$ satisfying the conditions of Theorem 5.14. In addition, we assume for $\psi_L : \mathbb{R} \longrightarrow \mathbb{C}$ that*

$$\operatorname{Re}\psi_L(\xi) \ge c|\xi|^r$$

holds with some constants $c > 0$, $r > 0$ for large $|\xi|$, $\xi \in \mathbb{R}$. Then an invariant probability measure μ and a function $\rho \ge 0$ almost everywhere with $\mu(\,\mathrm{d}x) = \rho(x)\,\mathrm{d}x$ exists.

Proof. Due to Theorem 5.14, we get an invariant measure μ and in particular its characteristic function $\phi_\mu(\xi) = \exp\{-\int_0^\xi \frac{\psi_L(\eta)}{\beta\eta}\,\mathrm{d}\eta\}$. If ϕ_μ is integrable there exists an almost everywhere positive function ρ such that $\phi_\mu(\xi) = \hat{\rho}(\xi)$ and $\mu(\,\mathrm{d}x) = \rho(x)\,\mathrm{d}x$. The integrability indeed holds as

$$\begin{aligned}\int |\phi_\mu(\xi)|\,\mathrm{d}\xi &= \int \exp\{-\operatorname{Re}\int_0^\xi \frac{\psi_L(\eta)}{\beta\eta}\,\mathrm{d}\eta\}\,\mathrm{d}\xi \\ &\le c_1 + \int_{|\xi|>k} \exp\{-\int_0^\xi \frac{c|\eta|^r}{\beta\eta}\,\mathrm{d}\eta\}\,\mathrm{d}\xi \\ &\le c_1 + \int_{|\xi|>k} \exp\{-c\operatorname{sgn}(\xi)\int_0^\xi \frac{|\eta|^{r-1}}{\beta}\,\mathrm{d}\eta\}\,\mathrm{d}\xi < \infty.\end{aligned}$$

□

This corollary holds, in particular for $\psi_L(\xi) = |\xi|^\alpha$. Furthermore, we can state the result for perturbed Ornstein-Uhlenbeck processes[6].

Theorem 5.16. *Let L be an operator given by*

$$Lf(x) = a_1 f''(x) + \beta(x)f'(x) + \int_{y\neq 0} \big(f(x+y) - f(x) - f'(x)y\mathbb{1}_{\{|y|\le 1\}}\big)\,\nu(\,\mathrm{d}y),$$

where $f \in C_c^\infty(\mathbb{R})$ and ν is a measure such that an Ornstein-Uhlenbeck process driven by a Lévy process with characteristic exponent $\psi_L(\xi) = \int_{y\neq 0}\Big(1 - \mathrm{e}^{\mathbf{i}y\xi} + \mathbf{i}\xi y\mathbb{1}_{\{|y|\le 1\}}\Big)\nu(\,\mathrm{d}y)$ satisfies the assumptions of Corollary 5.15. Further, set $\beta_2(x) = -x$ and ρ_2 is an almost everywhere nonnegative function such that $\rho_2\,\mathrm{d}x$ is an invariant measure of an Ornstein-Uhlenbeck process with symbol $q(x,\xi) = \psi_L(\xi) + \mathbf{i}x\xi$. Then for all

$$\beta \in \left\{\frac{(\beta_1\rho_1) * \rho_2(x) + \rho_1 * (\beta_2\rho_2)(x)}{\rho_1 * \rho_2(x)};\ \beta_1 \in \mathcal{A}_1,\ \rho \in \mathcal{Q}(\beta_1)\right\},$$

where

$$\begin{aligned}\mathcal{A}_1 &:= \left\{\beta;\ \hat{\beta} \text{ exists },\exists\gamma > 0\ \exists\theta > \frac{1}{2\gamma^2} \text{ s.t. } \int \mathrm{e}^{\theta|\beta(x)+\gamma x|^2-(1/2)|x|^2}\,\mathrm{d}x < \infty\right\},\\ \mathcal{Q}(\beta) &= \left\{\rho;\ \rho \ge 0,\ \rho \not\equiv 0,\ -a_1|\xi|^2\hat{\rho}(\xi) + \mathbf{i}\xi\widehat{(\beta\rho)}(\xi) = 0\right\},\end{aligned}$$

there exists a function ρ such that $\mu(\,\mathrm{d}x) = \rho(x)\,\mathrm{d}x$ is an invariant probability measure for L.

For the proof we refer to Albeverio et al. [2, Theorem 3.1] as it follows from theirs by adopting the extended result for invariant measures of general Ornstein-Uhlenbeck processes, Corollary 5.15. The idea of the proof is to construct an invariant measure of the perturbed process as a convolution of the measure of an Ornstein-Uhlenbeck process driven by a pure jump Lévy process and the one of a Brownian motion with

[6] In our context the perturbation affects the drift coefficient, i.e. we look at drift coefficients $\beta(x)$ instead of $\beta x + l$.

drift parameter $\beta(x)$. To achieve this, the invariant measure of the perturbed Brownian motion has to exist and the nonlinear drift $\beta(x)$ has to satisfy a variation condition. These requirements are represented by the sets $\mathcal{Q}$ and $\mathcal{A}_1$, respectively.
Assume that the function β is bounded and locally Lipschitz. Then this is a special case of a result by Behme and Schnurr [7, Proposition 3.8]. Consider a process X given as the unique solution of the stochastic differential equation

$$\mathrm{d}X_t = b\,\mathrm{d}Z_t + \beta(X_{t-})\,\mathrm{d}L_t, \qquad t \geq 0,$$

where L and Z are independent Lévy processes with characteristic functions ϕ_Z and ϕ_L, respectively. Then μ is an invariant measure for X if

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathbf{i}x^\top \xi}\Big(\phi_L(\beta(x)^\top \xi) + \phi_Z(b\xi)\Big)\mu(\,\mathrm{d}x) = 0. \tag{5.4}$$

In the next part of this section we give a condition when an Ornstein-Uhlenbeck process is μ-symmetric and extend this result to perturbed Ornstein-Uhlenbeck processes.

Definition 5.17. An operator $(A, \mathcal{D}(A))$ in $L^2(\mathbb{R}^d, \,\mathrm{d}\mu)$ is said to be μ-symmetric on $D \subseteq D(A)$ if

$$\int f \cdot Ag \,\mathrm{d}\mu = \int Af \cdot g \,\mathrm{d}\mu \quad \forall f, g \in D.$$

If $1 \in D$ and[7] $A1 = 0$, then the μ-symmetry of an operator implies that μ is an invariant measure for A. Taking $g = 1$, we conclude from the conservativeness that $Ag = A1 = 0$. Hence, the definition above gives $\int Af \,\mathrm{d}\mu = 0$.
The next theorem expresses the equivalence of μ-symmetry of Ornstein-Uhlenbeck processes and the absence of jumps. A similar result for affine processes on $D = \mathbb{R}_+^m$ is due to Handa who proved that under μ-symmetry for a nondegenerate probability measure the process coincides with a Cox-Ingersoll-Ross process, cf. Example 3.2.3. The first steps of our proof are adapted from Handa [25, Theorem 2.3]. However, their requirement that the functions F and R are analytical fails in our case. Our main idea is to derive and solve a functional equation.

Theorem 5.18. *The generator of an Ornstein-Uhlenbeck process is μ-symmetric for a nondegenerate probability measure if and only if the process is driven by a Brownian motion.*
In this case, the symbol of the Ornstein-Uhlenbeck process is given by $q(x,\xi) = x\mathbf{i}\beta\xi + \frac{1}{2}q\xi^2 - \mathbf{i}b\xi$ and the characteristic function of μ by $\phi_\mu(\xi) = \exp\{-\frac{q}{4\beta}\xi^2 + \mathbf{i}\frac{b}{\beta}\xi\}$.

Proof. It can be found in Bakry, Gentil and Ledoux [4, Section 2.7.1, p. 103] that the Ornstein-Uhlenbeck process driven by a Brownian motion is μ-symmetric.
It remains to prove that μ-symmetry implies that the driving process is a Brownian motion. The symbol of an Ornstein-Uhlenbeck process is given by $q(x,\xi) = \mathbf{i}\beta\xi x + \psi_L(\xi)$,

[7] For a Feller generator this is true if the semigroup is conservative.

where ψ_L is the characteristic exponent of the driving Lévy process. Due to the μ-symmetry, we know that μ is an invariant measure. Therefore we have

$$\int \mathrm{e}^{\mathbf{i}x\xi}\mu(\mathrm{d}x) = \phi_\mu(\xi) = \mathrm{e}^{-\psi_\mu(\xi)},$$

where $\psi_\mu(\xi) = \int_0^\xi \frac{\psi_L(\eta)}{\beta\eta}\,\mathrm{d}\eta$, cf. Theorem 5.14. By differentiation of the last equality we get

$$\psi_\mu'(\xi)\beta\xi = \psi_L(\xi) \tag{5.5}$$

and by the differentiation rules of the Fourier transform

$$\int x\mathrm{e}^{\mathbf{i}(\xi+\eta)x}\mu(\mathrm{d}x) = -\mathbf{i}\phi_\mu'(\xi+\eta) = \mathbf{i}\psi_\mu'(\xi+\eta)\phi_\mu(\xi+\eta).$$

Now Proposition 1.17.2 combined with the last two equalities yields

$$\begin{aligned}\int \Big(A\mathrm{e}_{\mathbf{i}\xi}(x)\Big)\mathrm{e}^{\mathbf{i}\eta x}\mu(\mathrm{d}x) &= \int \left(\mathbf{i}\beta\xi x + \psi_L(\xi)\right)\mathrm{e}^{\mathbf{i}\xi x}\mathrm{e}^{\mathbf{i}\eta x}\mu(\mathrm{d}x)\\ &= \mathbf{i}\beta\xi\left(\mathbf{i}\psi_\mu'(\xi+\eta)\phi_\mu(\xi+\eta)\right) + \psi_L(\xi)\phi_\mu(\xi+\eta)\\ &= \beta\xi\left(-\psi_\mu'(\xi+\eta) + \psi_\mu'(\xi)\right)\phi_\mu(\xi+\eta).\end{aligned}$$

The same calculation holds true if we interchange the roles of ξ and η,

$$\int \Big(A\mathrm{e}_{\mathbf{i}\eta}(x)\Big)\mathrm{e}^{\mathbf{i}\xi x}\mu(\mathrm{d}x) = \beta\eta\left(-\psi_\mu'(\eta+\xi) + \psi_\mu'(\eta)\right)\phi_\mu(\eta+\xi).$$

By μ-symmetry we have $\langle f, Ag\rangle_{L^2(\mathrm{d}\mu)} = \langle Af, g\rangle_{L^2(\mathrm{d}\mu)}$. Hence the left-hand sides of the last two equations coincide, and we obtain

$$\beta\xi\left(-\psi_\mu'(\xi+\eta) + \psi_\mu'(\xi)\right)\phi_\mu(\xi+\eta) = \beta\eta\left(-\psi_\mu'(\xi+\eta) + \psi_\mu'(\eta)\right)\phi_\mu(\xi+\eta).$$

Rewriting this as

$$\beta(\xi+\eta)\psi_\mu'(\xi+\eta)(\xi-\eta) = \left(\psi_\mu'(\xi)\xi - \psi_\mu'(\eta)\eta\right)\beta(\xi+\eta)$$

allows us to substitute (5.5) into this equality

$$\psi_L(\xi+\eta)(\xi-\eta) = \left(\psi_L(\xi) - \psi_L(\eta)\right)(\xi+\eta).$$

This is a functional equation for ψ_L, which we will solve now.
Let $\eta = 0$ and ξ be arbitrary. Then $\psi_L(\xi)\xi = \left(\psi_L(\xi) - \psi_L(0)\right)\xi$ implies $\psi_L(0) = 0$.
Choosing $\xi = 1-\zeta$ and $\eta = \zeta$ we get

$$\psi_L(1)(1-\zeta-\zeta) = \psi_L(1-\zeta) - \psi_L(\zeta).$$

If we set $f(\zeta) := \psi_L(\zeta) - \psi_L(1)\zeta$, the functional equation simplifies to

$$f(1-\zeta) = f(\zeta), \tag{5.6}$$

and from $\psi_L(0) = 0$ we deduce $f(0) = 0$ and $f(1) = 0$. In particular, the function f also satisfies the functional equation from above

$$\begin{aligned} f(\xi+\eta)(\xi-\eta) &= (\psi_L(\xi+\eta) - \psi_L(1)(\xi+\eta))(\xi-\eta) \\ &= \big(\psi_L(\xi) - \psi_L(\eta)\big)(\xi+\eta) - \psi_L(1)(\xi-\eta)(\xi+\eta) \\ &= \big(f(\xi) - f(\eta)\big)(\xi+\eta). \end{aligned} \tag{5.7}$$

By setting $\xi = \lambda$ and $\eta = 2-\lambda$ in (5.7) we get

$$\begin{aligned} & f(2)\big(\lambda - (2-\lambda)\big) = 2\big(f(\lambda) - f(2-\lambda)\big) \\ \Longleftrightarrow \quad & \frac{f(2)}{2}(2\lambda - 2) = f(\lambda) - f(2-\lambda). \end{aligned} \tag{5.8}$$

To simplify the right-hand side we start again with equation (5.7) and now choose $\xi = 1$ and $\eta = 1-\lambda$, which, combined with (5.6), leads to

$$\begin{aligned} & f(2-\lambda)\big(1-(1-\lambda)\big) = \Big(\underbrace{f(1)}_{=0} - \underbrace{f(1-\lambda)}_{=f(\lambda)}\Big)(2-\lambda) \\ \Longleftrightarrow \quad & f(2-\lambda)\lambda = -f(\lambda)(2-\lambda). \end{aligned} \tag{5.9}$$

Substituting (5.9) into (5.8) we conclude

$$\begin{aligned} & \frac{f(2)}{2}(2\lambda-2) = f(\lambda) - f(2-\lambda) \\ \Longleftrightarrow \quad & \frac{f(2)}{2}(2\lambda-2)\lambda = f(\lambda)\lambda + f(\lambda)(2-\lambda) \\ \Longleftrightarrow \quad & \frac{f(2)}{2}\lambda(\lambda-1) = f(\lambda) \\ \Longleftrightarrow \quad & f(\lambda) = \frac{f(2)}{2}(\lambda^2 - \lambda). \end{aligned}$$

Consequently, we get $\psi_L(\xi) = \frac{f(2)}{2}(\xi^2 - \xi) + \psi_L(1)\xi$. As ψ_L is a negative definite function, we have a Lévy Khintchine representation, i.e.

$$\psi_L(\xi) = \frac{1}{2}q\xi^2 - \mathbf{i}b\xi + \int_{y\neq 0}\Big(1 - e^{\mathbf{i}\xi y} + \mathbf{i}\xi y \mathbb{1}_{\{|y|\leq 1\}}\Big)\nu(\,dy),$$

with $q \geq 0$, $b \in \mathbb{R}$ and ν is a Lévy measure such that $\int_{y\neq 0}(1\wedge|y|^2)\nu(\,dy) < \infty$. Comparing this representation with the solution of the functional equation, we see that the integral part of ψ_L drops out. Furthermore, it follows that $\operatorname{Im}\frac{f(2)}{2} = 0$ and $\operatorname{Re}\frac{f(2)}{2} = \operatorname{Re}\psi_L(1)$, since the imaginary part of ψ_L is linear and the real part of ψ_L is quadratic. Hence, we have $\psi_L(\xi) = \frac{1}{2}q\xi^2 - \mathbf{i}b\xi$, where q is the diffusion and b is the drift parameter of the

driving Lévy process.
The additional assertion has already been shown in Theorem 5.14. The characteristic function of μ is given by

$$\begin{aligned}\phi_\mu(\xi) &= \exp\Big\{-\int_0^\xi \frac{\psi_L(\eta)}{\beta\eta}\,\mathrm{d}\eta\Big\}\\ &= \exp\Big\{-\int_0^\xi \frac{\frac{1}{2}q\eta^2 - \mathbf{i}b\eta}{\beta\eta}\,\mathrm{d}\eta\Big\}\\ &= \exp\Big\{-\frac{q}{4\beta}\xi^2 + \mathbf{i}\frac{b}{\beta}\xi\Big\},\end{aligned}$$

in other words, μ is a Gaussian measure. □

Since μ is an invariant measure, the linear drift parameter has to be positive, $\beta > 0$.
This result also extends to perturbed Ornstein-Uhlenbeck processes. For the proof we need the following result which is a counterpart to Schnurr [56, Proposition 6.2] who showed that the only continuous negative definite function vanishing at infinity is constantly zero. In particular, our statement follows by the same method.

Proposition 5.19. *The only continuous negative definite function vanishing in a neighbourhood of zero is constantly zero.*

Proof. Let ψ_L be a continuous negative definite function which vanishes in a neighbourhood of zero, i.e. for $\varepsilon > 0$ we have $\psi_L(\xi) = 0$ if $\xi \in B(0,\varepsilon)$. For every γ there exists an integer $k \in \mathbb{N}$ such that $\gamma/k \in B(0,\varepsilon)$. By sub-additivity of $\sqrt{|\psi_L|}$, cf. Jacob [27, Lemma I.3.6.21], we obtain

$$\sqrt{|\psi_L(\gamma)|} = \sqrt{\Big|\psi_L\Big(\sum_{l=1}^{k}\gamma/k\Big)\Big|} \leq \sum_{l=1}^{k}\underbrace{\sqrt{|\psi_L(\gamma/k)|}}_{=0} = 0,$$

which completes the proof. □

By another method Jacob [27, Lemma I.3.6.28] showed that a negative definite function that is zero on an arbitrary interval $[a,b]$ with $a < b$ is identical to zero.
The next theorem is an extension of Alberevio et al. [2, Proposition 4.4]. In contrast to their approach, we do not restrict ourselves to α-stable jumps. Furthermore, we use a different technique for the proof which allows us to drop the assumption that the support of the invariant measure and its density function equals the real line.

Theorem 5.20. *Let L be an operator given by*

$$Lf(x) = a_1 f''(x) + \beta(x)f'(x) + a_2\int_{y\neq 0}\big(f(x+y) - f(x) - f'(x)y\mathbb{1}_{\{|y|\leq 1\}}\big)\,\nu(\mathrm{d}y),$$

where $f \in C_c^\infty(\mathbb{R})$, with symbol $q(x,\xi)$ whose domain $\mathcal{D}(L)$ contains the test functions $C_c^\infty(\mathbb{R})$. Assume μ is an infinitesimal invariant probability measure on $\mathbb{R}$ such that $\mu(\mathrm{d}x) = \rho(x)\,\mathrm{d}x$, $\int|\beta(x)|\,\mathrm{d}\mu(x) < \infty$ and $\widehat{\beta\rho}$ exists. If L is μ-symmetric, then $a_2 = 0$. If in addition $\mathrm{supp}(\rho) = \mathbb{R}$, we have $\beta = a_1\nabla\rho/\rho$.

Proof. We denote the symbol of the operator L by $q(x,\xi) = \mathbf{i}\beta(x)\xi + \psi_L(\xi)$, where $\psi_L(\xi) = a_1\xi^2 + a_2\int_{y\neq 0}\left(\mathrm{e}^{\mathbf{i}\xi y} - 1 - \mathbf{i}y\xi\mathbb{1}_{\{|y|\leq 1\}}\right)\nu(\,\mathrm{d}y)$. Then by Theorem 5.12 we have

$$
\begin{aligned}
& & \int \mathrm{e}^{\mathbf{i}x\xi} q(x,\xi)\mu(\,\mathrm{d}x) &= 0 \\
&\Longleftrightarrow & \int \mathrm{e}^{\mathbf{i}x\xi}\left(\mathbf{i}\beta(x)\xi + \psi_L(\xi)\right)\mu(\,\mathrm{d}x) &= 0 \\
&\Longleftrightarrow & \mathbf{i}\xi\int \mathrm{e}^{\mathbf{i}x\xi}\beta(x)\mu(\,\mathrm{d}x) &= -\psi_L(\xi)\phi_\mu(\xi).
\end{aligned} \tag{5.10}
$$

By Proposition 1.17 it follows that

$$
\begin{aligned}
\int \left(L\mathrm{e}^{\mathbf{i}\xi\cdot}(x)\right)\mathrm{e}^{\mathbf{i}\eta x}\mu(\,\mathrm{d}x) &= \int q(x,\xi)\mathrm{e}^{\mathbf{i}x\xi}\mathrm{e}^{\mathbf{i}x\eta}\mu(\,\mathrm{d}x) \\
&= \int \mathbf{i}\xi\beta(x)\mathrm{e}^{\mathbf{i}x(\xi+\eta)}\mu(\,\mathrm{d}x) + \psi_L(\xi)\phi_\mu(\xi+\eta) \\
&= \frac{\xi}{\xi+\eta}\int \mathbf{i}(\xi+\eta)\beta(x)\mathrm{e}^{\mathbf{i}x(\xi+\eta)}\mu(\,\mathrm{d}x) + \psi_L(\xi)\phi_\mu(\xi+\eta) \\
&= \frac{\xi}{\xi+\eta}\left(-\psi_L(\xi+\eta)\right)\phi_\mu(\xi+\eta) + \psi_L(\xi)\phi_\mu(\xi+\eta) \\
&= \left(-\frac{\xi}{\xi+\eta}\psi_L(\xi+\eta) + \psi_L(\xi)\right)\phi_\mu(\xi+\eta),
\end{aligned}
$$

where we used equation (5.10) in the second to last line. By μ-symmetry we can interchange ξ and η which yields

$$
\left(-\frac{\xi}{\xi+\eta}\psi_L(\xi+\eta) + \psi_L(\xi)\right)\phi_\mu(\xi+\eta) = \left(-\frac{\eta}{\xi+\eta}\psi_L(\eta+\xi) + \psi_L(\eta)\right)\phi_\mu(\xi+\eta).
$$

The characteristic function ϕ_μ is continuous and $\phi_\mu(0) = 1$. Hence, there exists an $\varepsilon > 0$ such that $\phi_\mu(\zeta) > 0$ for all $\zeta \in B(0,\varepsilon)$. Therefore, for all ξ, η such that $|\xi+\eta| < \varepsilon$, we can divide the above equality by $\phi_\mu(\xi+\eta)$ to obtain

$$
-\frac{\xi}{\xi+\eta}\psi_L(\xi+\eta) + \psi_L(\xi) = -\frac{\eta}{\xi+\eta}\psi_L(\eta+\xi) + \psi_L(\eta) \qquad |\xi+\eta| \leq \varepsilon.
$$

We proceed in the same manner as in the proof of Theorem 5.18. Therefore, we set $\varepsilon_k = \varepsilon/k$ for large $k \in \mathbb{N}$ and $\eta = \varepsilon_k - \varepsilon_k\zeta$, $\xi = \varepsilon_k\zeta$ with arbitrary $\zeta \in \mathbb{R}$. Since $|\xi+\eta| = \varepsilon_k < \varepsilon$, we get for all $\zeta \in \mathbb{R}$

$$
-\frac{\varepsilon_k\zeta}{\varepsilon_k}\psi_L(\varepsilon_k) + \psi_L(\varepsilon_k\zeta) = -\frac{\varepsilon_k - \varepsilon_k\zeta}{\varepsilon_k}\psi_L(\varepsilon_k) + \psi_L(\varepsilon_k - \varepsilon_k\zeta).
$$

Introducing $f(\zeta) := \varepsilon_k\psi_L(\varepsilon_k\zeta) - \varepsilon_k\zeta\psi_L(\varepsilon_k)$, we have for all $\zeta \in \mathbb{R}$

$$
f(1-\zeta) = f(\zeta),
$$

and $f(0) = f(1) = 0$ as $\psi_L(0) = 0$. A calculation similar to (5.7) yields that the function f satisfies the functional equation for all $|\xi + \eta| \leq k$. Furthermore, (5.8) and (5.9) hold for all $\lambda \in \mathbb{R}$ and all $|2 - \lambda| \leq k$, respectively. Hence, on $|\lambda| \leq k - 2$ we have the solution

$$f(\lambda) = \frac{f(2)}{2}(\lambda^2 - \lambda).$$

Substituting this into ψ_L, we conclude for $|\lambda| \leq k - 2$

$$\psi_L(\varepsilon\lambda) = \frac{f(2)}{2\varepsilon}\lambda^2 + \Big(\psi_L(\varepsilon) - \frac{f(2)}{2\varepsilon}\Big)\lambda.$$

Since by assumption ψ_L has no linear part, we have $\frac{f(2)}{2\varepsilon} = \psi_L(\varepsilon) = \varepsilon^2 a_1$ and thus for $|\xi| \leq \varepsilon\frac{k-2}{k}$ we have $\psi_L(\xi) = a_1\xi^2$. The integral part $\xi \mapsto a_2 \int_{y \neq 0} \big(\mathrm{e}^{\mathbf{i}\xi y} - 1 - \mathbf{i}y\xi\mathbb{1}_{\{|y| \leq 1\}}\big)\, \nu(\,\mathrm{d}y)$ is a continuous negative definite function and vanishes in a neighbourhood of zero. Hence, Proposition 5.19 gives $a_2 = 0$.

Furthermore, we now suppose that $\operatorname{supp}(\rho) = \mathbb{R}$. Then substituting $\psi_L(\xi) = a_1\xi^2$ and $\mu(\,\mathrm{d}x) = \rho(x)\,\mathrm{d}x$ into (5.10) gives

$$\mathbf{i}\xi \int \mathrm{e}^{\mathbf{i}x\xi}\beta(x)\rho(x)\,\mathrm{d}x = -\int \mathrm{e}^{\mathbf{i}x\xi}a_1\xi^2\rho(x)\,\mathrm{d}x.$$

This is equivalent to

$$\begin{aligned}\int \mathrm{e}^{\mathbf{i}x\xi}\beta(x)\rho(x)\,\mathrm{d}x &= a_1\mathbf{i}\xi \int \mathrm{e}^{\mathbf{i}x\xi}\rho(x)\,\mathrm{d}x \\ &= a_1 \int \nabla_x \mathrm{e}^{\mathbf{i}x\xi}\rho(x)\,\mathrm{d}x \\ &= a_1 \int \mathrm{e}^{\mathbf{i}x\xi}\nabla\rho(x)\,\mathrm{d}x.\end{aligned}$$

We conclude from the uniqueness of the Fourier transform that

$$\beta(x) = a_1\frac{\nabla\rho(x)}{\rho(x)}$$

almost everywhere. □

As we observed before, if β is a bounded locally Lipschitz function, such a process is a special case of the stochastic differential equation

$$\mathrm{d}X_t = b\,\mathrm{d}Z_t + \beta(X_{t-})\,\mathrm{d}L_t, \qquad t \geq 0. \tag{5.11}$$

The next corollary shows that we can treat another special case of this type if we assume that L is a Brownian motion without drift. Then we can proceed similar as in the previous proof.

Corollary 5.21. *Let X be the unique solution of the stochastic differential equation (5.11), where $\beta : \mathbb{R} \to \mathbb{R}$ is a bounded locally Lipschitz function and L is a one-dimensional Brownian motion without drift, independent of the one-dimensional Lévy process Z with the characteristic exponent ψ_Z. If the invariant probability measure μ is symmetric then the process Z has no jumps.*

Proof. Starting in the same manner as in Theorem 5.20, i.e. using the invariance of the measure μ, cf. (5.4), and the μ-symmetry, cf. Definition 5.17, gives for $\xi, \eta \in \mathbb{R}$

$$\Big(-\frac{\xi^2}{(\xi+\eta)^2}\psi_Z(b(\xi+\eta))+\psi_z(b\xi)\Big)\phi_\mu(\xi+\eta) = \Big(-\frac{\eta^2}{(\xi+\eta)^2}\psi_Z(\xi+\eta)+\psi_Z(\eta)\Big)\phi_\mu(\xi+\eta).$$

By rearranging the terms we get

$$\begin{aligned}\Big(\psi_z(b\xi)-\psi_Z(\eta)\Big)\phi_\mu(\xi+\eta) &= \frac{\xi^2-\eta^2}{(\xi+\eta)^2}\psi_Z(b(\xi+\eta))\phi_\mu(\xi+\eta)\\ &= \frac{\xi-\eta}{\xi+\eta}\psi_Z(b(\xi+\eta))\phi_\mu(\xi+\eta).\end{aligned}$$

This is the same functional equation as in Theorem 5.20 and we conclude that $\psi_Z(\xi) = \mathbf{i}l\xi + \frac{1}{2}\sigma^2\xi^2$, i.e. the process Z has no jumps. $\square$

This method cannot be further exploited in the case of the stochastic differential (5.11) since we need that the x-dependent part of the symbol $q(x,\xi)$ splits into a product of a function depending on x and a function depending on ξ.

Chapter 6

Simulation of Feller Processes

There are several approaches to simulate stochastic processes, in particular Feller processes. The most popular method is the Euler scheme which is based on the representation of the stochastic process as a solution of a stochastic differential equation. In this chapter, however, we present a Markov chain approximation. This approach is based on the symbol. In many cases, in particular for jump processes, the symbol is easily accessible. Böttcher and Schilling [10] introduced the Markov chain approximation for Feller processes with bounded symbols which is extended to Feller processes with sublinear growth conditions, cf. Böttcher, Schilling and Wang [11]. As the Euler scheme suggests the Markov chain approximation should converge for **symbols subject to linear growth conditions**. In Section 6.1 we show that this is valid for the drift and diffusion parameters of the symbol and that in this case the approximation coincides with the Euler scheme. Although affine processes have linearly x-dependent parameters, we treat this case separately in section 6.2. Using the special geometry of the affine processes, the Markov chain approximation even applies to a linearly growing jump part. Furthermore our scheme allows general state spaces, like positive semidefinite matrices, cf. Chapter 4.

6.1 Simulation of Feller Processes

We start with the main result of this section, a proof of the convergence of the Markov chain approximation. As an application of the theory, we simulate paths of a generalized Ornstein-Uhlenbeck process. Finally, we establish a relation between the Markov chain approximation and the Euler scheme.

Theorem 6.1. *Let $(X_t)_{t\geq 0}$ be a d-dimensional Feller process with infinitesimal generator $(A, \mathcal{D}(A))$ such that*

$$C_c^\infty(\mathbb{R}^d) \quad \textit{is an operator core of } A,$$

i.e. the closure of $A|_{C_c^\infty(\mathbb{R}^d)}$ is $(A, \mathcal{D}(A))$. The corresponding symbol $q(x, \xi)$ is given by

$$q(x,\xi) = q(x,0) - \mathbf{i}l(x)^\top \xi + \frac{1}{2}\xi^\top Q(x)\xi + \int_{\mathbb{R}^d\setminus\{0\}} \left(1 - \mathrm{e}^{\mathbf{i}\xi^\top y} + \mathbf{i}\xi^\top \chi(y)\right) N(x,\,\mathrm{d}y).$$

Assume that for all $x \in \mathbb{R}^d$ and some constants c_l, c_Q

$$\begin{aligned} q(x,0) &= 0, \\ \|l(x)\|_{\max} &\le c_l(1+|x|), \\ \|Q(x)\|_{\max} &\le c_Q(1+|x|^2) \\ \lim_{|x|\to\infty} \sup_{|\xi|\le\frac{1}{|x|}} &\left| \int_{\mathbb{R}^d\setminus\{0\}} \left(1 - \mathrm{e}^{\mathbf{i}\xi^\top y} + \mathbf{i}\xi^\top \chi(y)\right) N(x, \mathrm{d}y)\right| = 0. \end{aligned} \tag{6.1}$$

For each $n \ge 1$ define a Markov chain $(Y^n(k))_{k\ge1}$ with $Y^n(0) := x_0$ and transition kernel $\mu_{x,\frac{1}{n}}(\mathrm{d}y)$ given by

$$\int \mathrm{e}^{\mathbf{i}y^\top\xi} \mu_{x,\frac{1}{n}}(\mathrm{d}y) = \mathrm{e}^{\mathbf{i}x^\top\xi - \frac{1}{n}q(x,\xi)}, \quad x \in D, \xi \in \mathbb{R}^d, n \ge 1.$$

Then

$$Y^n(\lfloor \cdot n \rfloor) \xrightarrow{d} X \quad (n\to\infty).$$

Here $\lfloor x \rfloor = \max\{k \in \mathbb{Z};\ k \le x\}$ and $\xrightarrow{d}$ denotes convergence in distribution in the space of right continuous functions with left limits equipped with the Skorohod J_1 topology.

Proof. Our proof starts with the observation that the convergence of the Markov chain is equivalent to the uniform convergence of the difference quotient of the Markov chain transition operator to the generator of X.
Let $(T_t)_{t\ge0}$ be the semigroup associated to the Feller process $(X_t)_{t\ge0}$ having the generator A. For $u \in C_\infty(\mathbb{R}^d)$ we have $\mathbb{E}(u(Y^n(k))) = W_{\frac{1}{n}}^k u(x_0)$, where

$$W_{\frac{1}{n}} u(x) = \int_{\mathbb{R}^d\setminus\{0\}} u(y)\mu_{x,\frac{1}{n}}(\mathrm{d}y)$$

is the operator generated by the transition kernel $\mu_{x,\frac{1}{n}}$. For $u \in C_c^\infty(\mathbb{R}^d)$ we get by Fourier inversion

$$W_{\frac{1}{n}} u(x) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathbf{i}x^\top\xi}\mathrm{e}^{-\frac{1}{n}q(x,\xi)}\hat{u}(\xi)\,\mathrm{d}\xi. \tag{6.2}$$

We will now show that

$$\lim_{n\to\infty} \left\| \frac{W_{\frac{1}{n}}u - u}{\frac{1}{n}} - Au \right\|_\infty = 0 \qquad \forall u \in C_c^\infty(\mathbb{R}^d). \tag{6.3}$$

For this, we introduce some further notation. Let $\epsilon > 0$, $u \in C_c^\infty(\mathbb{R}^d)$ and $r = r(u) > 0$ such that $\mathrm{supp}(u) \subseteq B(0,r)$. Equation (6.2) shows that the transition operator $W_{\frac{1}{n}}$ corresponds to a Lévy process $L = (L_t^{(x)})_{t\ge0}$ with characteristic exponent $\xi \mapsto q(x,\xi)$ and starting point $L_0^{(x)} = x$. Note that the symbol $q(x,\xi)$ with fixed x is a continuous negative definite function. Hence the characteristic function of the random variable $L_t^{(x)}$ is given by $\mathrm{e}^{\mathbf{i}x^\top\xi - tq(x,\xi)}$. For $x \in \mathbb{R}^d$ and $r > 0$ we denote the exit time of a process $Y = (Y_t)_{t\geq 0}$ starting in x by

$$\tau_{B(x,r)}^{Y} := \inf\{t > 0;\ Y_t \notin B(x,r)\}.$$

First, we verify equation (6.3) on compact sets. This part is identical to the proof in Böttcher, Schilling and Wang [10]. Let $K \geq 0$ and $u \in C_c^\infty(\mathbb{R}^d)$ then

$$\begin{aligned}
&\sup_{|x|\leq K}\left|\frac{W_{\frac{1}{n}}u(x) - u(x)}{\frac{1}{n}} - Au(x)\right| \\
&\quad= \sup_{|x|\leq K}\left|\int \mathrm{e}^{\mathbf{i}x^\top\xi}\left(\frac{\mathrm{e}^{-\frac{1}{n}q(x,\xi)} - 1}{\frac{1}{n}} + q(x,\xi)\right)\hat{u}(\xi)\,\mathrm{d}\xi\right| \\
&\quad\leq \sup_{|x|\leq K}\int s|q(x,\xi)|^2\mathrm{e}^{-h\operatorname{Re}q(x,\xi)}|\hat{u}(\xi)|\,\mathrm{d}\xi && (0 < h, s < \tfrac{1}{n}) \\
&\quad\leq \frac{c_K^2}{n}\int |(1+|\xi|^2)|^2|\hat{u}(\xi)|\,\mathrm{d}\xi && (q \text{ is locally bounded}) \\
&\quad\leq \frac{1}{n}\tilde{C},
\end{aligned}$$

where $\tilde{C} = \tilde{C}(c_K, u)$ is some constant. Note that the mean value theorem is used twice in the first inequality and that the symbol of a Feller process is locally bounded. If we choose an N large enough, possibly dependent on $\tilde{C}$ and ϵ, we get

$$\frac{1}{n}\tilde{C} \leq \frac{\epsilon}{3} \quad \forall n \geq N.$$

Next, we look at (6.3) outside of a compact set. Therefore we consider each term separately.
Since $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}(A)$ (it is even a core of $(A, \mathcal{D}(A))$), we know that $Au \in C_\infty(\mathbb{R}^d)$ for $u \in C_c^\infty(\mathbb{R}^d)$. Hence there exists a constant $K_A = K(A,u)$ such that

$$|Au(x)| \leq \frac{\epsilon}{3} \quad \forall |x| \geq K_A.$$

It remains to show that $\frac{W_{\frac{1}{n}}u - u}{\frac{1}{n}}$ converges strongly to zero outside of a sufficiently large compact set. Let $|x| \geq K$ such that $x \notin \operatorname{supp}(u)$. Since $L^{(x)}$ is the Lévy process corresponding to the transition operator $W_{\frac{1}{n}}$, we get

$$\begin{aligned}
\left|\frac{W_{\frac{1}{n}}u(x) - u(x)}{\frac{1}{n}}\right| &= n\left|W_{\frac{1}{n}}u(x)\right| \\
&= n\left|\mathbb{E}u\left(L_{\frac{1}{n}}^{(x)}\right)\right| \\
&\leq n\|u\|_\infty\mathbb{P}\left(L_{\frac{1}{n}}^{(x)} \in B(0,r)\right),
\end{aligned}$$

where in the last step $r = r(u)$ was chosen such that $\operatorname{supp}(u) \subseteq B(0,r)$. Instead of showing that the probability of starting in x and hitting the ball $B(0,r)$ is sufficiently small, we consider the probability of exiting from the ball $B(x,|x|-r)$. As Figure 6.1 depicts, the ball $B(0,r)$ is contained in $B^c(x,|x|-r)$. Hence the probability of exiting from $B(x,|x|-r)$ is bigger than that of hitting $B(0,r)$, i.e. we have

$$n\|u\|_\infty \mathbb{P}\left(L^{(x)}_{\frac{1}{n}} \in B(0,r)\right) \leq n\|u\|_\infty \mathbb{P}\left(L^{(x)}_{\frac{1}{n}} \in B^c(x,|x|-r)\right).$$

As $L^{(x)}$ is a Lévy process, we can use the Lévy-Itô decomposition and get

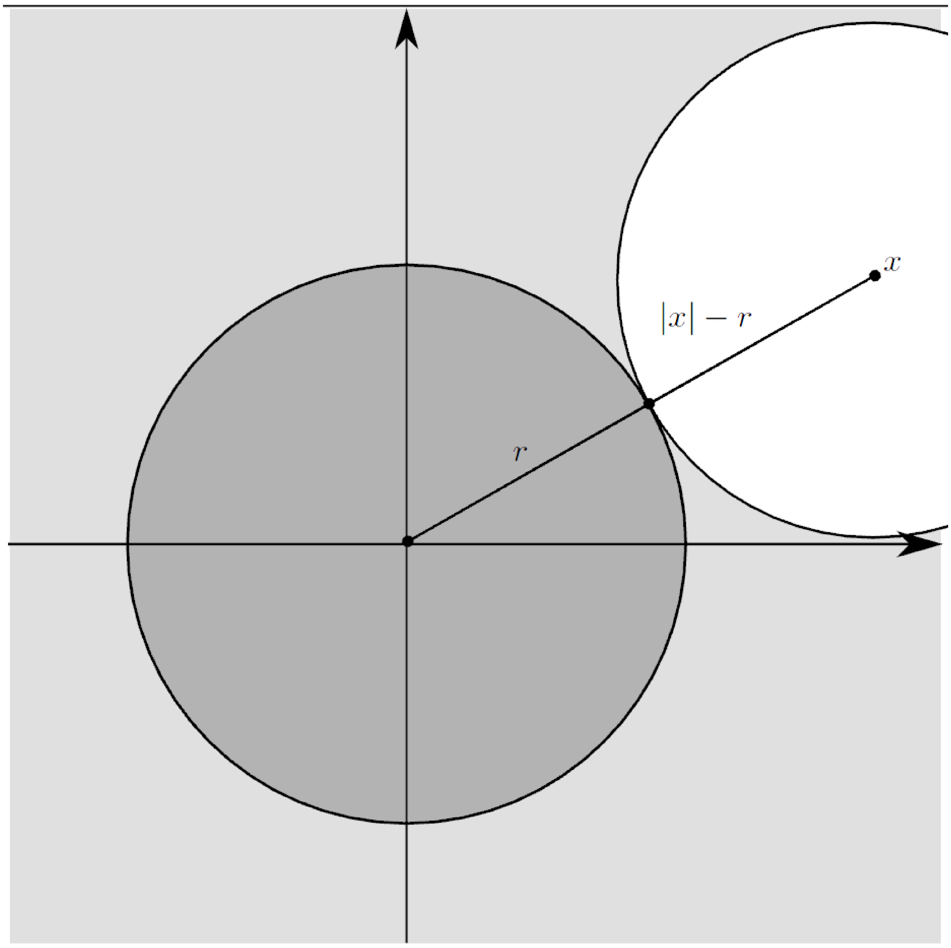

Figure 6.1: Ball $B(0,r)$ is contained in the set $B^c(x,|x|-r)$

$$L^{(x)}_t = x + l(x)t + D(x)_t + J(x)_t,$$

where x is the deterministic starting point, $l(x)t$ is a deterministic drift process with fixed drift $l(x)$, $D(x)$ is a $Q(x)$-Brownian motion starting in 0 with covariance matrix $Q(x)$ and $J(x)$ is a pure jump Lévy process starting in 0 with corresponding jump measure

$N(x, \mathrm{d}y)$. Note that in order to exit the ball at least one of three independent parts of the process has to leave the smaller ball with one third of the radius.

$$
\begin{aligned}
&n\mathbb{P}\left(L^{(x)}_{\frac{1}{n}} \in B^c(x, |x| - r)\right) \\
&\leq n\mathbb{P}\left(x + l(x)\frac{1}{n} + D(x)_{\frac{1}{n}} + J(x)_{\frac{1}{n}} \in B^c(x, |x| - r)\right) \\
&= n\mathbb{P}\left(l(x)\frac{1}{n} + D(x)_{\frac{1}{n}} + J(x)_{\frac{1}{n}} \in B^c(0, |x| - r)\right) \\
&\leq n\mathbb{P}\Bigg(\left(l(x)\frac{1}{n} \in B^c(0, \tfrac{1}{3}(|x| - r))\right) \cup \left(D(x)_{\frac{1}{n}} \in B^c(0, \tfrac{1}{3}(|x| - r))\right) \\
&\qquad\qquad \cup \left(J(x)_{\frac{1}{n}} \in B^c(0, \tfrac{1}{3}(|x| - r))\right)\Bigg) \\
&\leq n\mathbb{P}\left(l(x)\frac{1}{n} \in B^c(0, \tfrac{1}{3}(|x| - r))\right) + n\mathbb{P}\left(D(x)_{\frac{1}{n}} \in B^c(0, \tfrac{1}{3}(|x| - r))\right) \\
&\quad + n\mathbb{P}\left(J(x)_{\frac{1}{n}} \in B^c(0, \tfrac{1}{3}(|x| - r))\right).
\end{aligned}
$$

Now we are able to examine each term on its own. Note that the d-dimensional hypercube $\frac{1}{\sqrt{d}}Q(0, 1)$ is contained in the ball $B(0, 1)$. Using the linear growth condition for the drift coefficient, we get the following estimate for the drift process

$$
\begin{aligned}
n\mathbb{P}\left(l(x)\frac{1}{n} \in B^c(0, \tfrac{1}{3}(|x| - r))\right) &\leq n\mathbb{P}\left(l(x)\frac{1}{n} \in \frac{1}{\sqrt{d}}Q^c(0, \tfrac{1}{3}(|x| - r))\right) \\
&= n\mathbb{P}\left(\|l(x)\|_{\max}\frac{1}{n} > \frac{1}{\sqrt{d}}\frac{1}{3}(|x| - r)\right) \\
&\leq n\mathbb{P}\left(c_l(1 + |x|)\frac{1}{n} > \frac{1}{\sqrt{d}}\frac{1}{3}(|x| - r)\right).
\end{aligned}
$$

It is obvious that we can find some $N = N(l, K_l, d, r)$ independent of x such that for every $|x| \geq K_l$ the above probability equals 0 for all $n \geq N$.
The increment of the $Q(x)$-Brownian motion is normally distributed with mean zero and covariance matrix $\frac{1}{n}Q(x)$, i.e. $D(x)_{\frac{1}{n}} \sim N\left(0, \frac{1}{n}Q(x)\right)$. Hence, there exists a d-dimensional vector $Z = (Z_1, \ldots, Z_d)^\top$, whose components are standard normal distributions, and a matrix $A(x)$ such that $D(x)_{\frac{1}{n}} = \sqrt{\frac{1}{n}}A(x)^\top Z \sim N\left(0, \frac{1}{n}Q(x)\right)$. Since the covariance matrix $Q(x)$ is positive semidefinite, there exists a matrix $A(x)$ such that $Q(x) = A(x)A(x)^\top$. Inserting this in the subsequent calculation, we obtain

$$
\begin{aligned}
&n\mathbb{P}\left(D(x)_{\frac{1}{n}} \in B^c(0, \tfrac{1}{3}(|x| - r))\right) \\
&\leq n\mathbb{P}\left(D(x)_{\frac{1}{n}} \in \frac{1}{\sqrt{d}}Q^c(0, \tfrac{1}{3}(|x| - r))\right) \\
&= n\mathbb{P}\left(\sqrt{\frac{1}{n}}A(x)^\top Z \in \frac{1}{\sqrt{d}}Q^c(0, \tfrac{1}{3}(|x| - r))\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq n\mathbb{P}\left(\sqrt{\frac{1}{n}}\|A(x)\|_{\max}(|Z_1|+\ldots+|Z_d|) > \frac{1}{\sqrt{d}}\frac{1}{3}(|x|-r)\right)\\
&\leq n\sum_{i=1}^{d}\mathbb{P}\left(\sqrt{\frac{1}{n}}\|A(x)\|_{\max}|Z_i| > \frac{1}{d}\frac{1}{\sqrt{d}}\frac{1}{3}(|x|-r)\right)\\
&\leq nd\mathbb{P}\Bigg(|Z_1| > \sqrt{n}\,\underbrace{\frac{1}{d\sqrt{d}}\frac{1}{3}}_{:=c_d}\underbrace{\frac{|x|-r}{\|A(x)\|_{\max}}}_{:=r(x)}\Bigg).
\end{aligned}
$$

In the last step we used the fact that all $Z_i,\ i=1,\ldots,n$, are identically distributed. Since the standard normal distribution Z_1 is symmetric, we can rewrite the above term as a tail probability and use the standard upper bound for normal random variables, cf. Schilling and Partzsch [51, Lemma 10.5]. The growth condition for the covariance matrix implies $\|A(x)\|_{\max} \leq c_A(1+|x|)$. In particular, since r is fixed, we have for all $|x| \geq K_Q$, $|x| \gg r$

$$
r(x) = \frac{|x|-r}{\|A(x)\|_{\max}} \geq \frac{|x|-r}{c_A(1+|x|)} > c_r.
$$

Combining these inequalities we obtain for all $|x| > K_Q$

$$
\begin{aligned}
n\mathbb{P}\left(D(x)_{\frac{1}{n}} \in B^c(0,\tfrac{1}{3}(|x|-r))\right) &\leq nd\mathbb{P}\left(|Z_1| > \sqrt{n}c_d r(x)\right)\\
&\leq 2nd\mathbb{P}\left(Z_1 > \sqrt{n}c_d c_r\right)\\
&\leq 2nd\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{n}}\frac{1}{c_d}\frac{1}{c_r}\exp\left(-\frac{nc_d^2c_r^2}{2}\right)\\
&\xrightarrow{n\to\infty} 0.
\end{aligned}
$$

For the estimation of the jump process $J(x)_{\frac{1}{n}}$ we use the upper maximal inequality, cf. Theorem 2.1,

$$
\begin{aligned}
&n\mathbb{P}\left(J(x)_{\frac{1}{n}} \in B^c(0,t\frac{1}{3}(|x|-r))\right)\\
&\leq n\mathbb{P}\left(\tau^{J(x)}_{B(0,t\frac{1}{3}(|x|-r))} \leq \frac{1}{n}\right)\\
&\leq n\frac{c}{n}\sup_{|\xi|\leq\frac{3}{|x|-r}}\left|\int_{\mathbb{R}^d\setminus\{0\}}\left(1-\mathrm{e}^{\mathbf{i}\xi^\top y}+\mathbf{i}\xi^\top\chi(y)\right)N(x,\,\mathrm{d}y)\right|\\
&\leq \varepsilon \qquad \forall |x| \geq K_q,
\end{aligned}
$$

where $K_q = K_q(r,\varepsilon,q)$ is chosen suitable and independent of x. Note that $J(x)$ is a Lévy proccess, i.e. x is fixed when we use the upper maximal inequality. Hence the supremum of x drops out.

Combining these three estimates, we find a constant $K = \max\{K_l, K_Q, K_q\}$ independent of x such that

$$\sup_{|x|\geq K}\left|\frac{W_{\frac{1}{n}}u(x)-u(x)}{\frac{1}{n}}\right| \leq \sup_{|x|\geq K} n\|u\|_\infty \mathbb{P}\left(L^{(x)}_{\frac{1}{n}} \in B(0,r)\right) \xrightarrow{n\to\infty} 0.$$

Now we can choose some sufficiently large N such that

$$\begin{aligned}
&\left\|\frac{W_{\frac{1}{n}}u-u}{\frac{1}{n}}-Au\right\|_\infty \\
&\leq \sup_{|x|\leq K}\left|\frac{W_{\frac{1}{n}}u(x)-u(x)}{\frac{1}{n}}-Au(x)\right| + \sup_{|x|\geq K}\left|\frac{W_{\frac{1}{n}}u(x)-u(x)}{\frac{1}{n}}\right| + \sup_{|x|\geq K}\left|Au(x)\right| \\
&\leq \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon,
\end{aligned}$$

for all $n \geq N$.
By Ethier and Kurtz [19, Theorem 6.5, p.31] we know that

$$\lim_{n\to\infty}\left\|\frac{W_{\frac{1}{n}}u-u}{\frac{1}{n}}-Au\right\|_\infty = 0$$

is equivalent to

$$\lim_{n\to\infty}\left\|W^{\lfloor tn\rfloor}_{\frac{1}{n}}u-T_tu\right\|_\infty = 0 \qquad \forall t>0, \forall u \in C_\infty(\mathbb{R}^d).$$

It follows by Böttcher, Schilling and Wang [11, Theorem 7.1] that this is equivalent to the convergence in distribution of the Markov chain, i.e. the assertion $Y^n(\lfloor\cdot n\rfloor) \xrightarrow{d} X \quad (n\to\infty)$. □

This extension allows us to simulate a generalized Ornstein-Uhlenbeck process whose driving process U has no jumps[1].

Example 6.2. Let V be a generalized Ornstein-Uhlenbeck process as in Example 1.18 with the additional condition that the driving process U has no jumps, i.e. $\nu_U = 0$. In other words, the symbol has the representation

$$\begin{aligned}
q(x,\xi) = &-\mathbf{i}(xl_U+l_L)\xi + \frac{1}{2}(x^2\sigma_U^2+2x\sigma_{U,L}+\sigma_L^2)\xi^2 \\
&+\int_{\mathbb{R}\setminus\{0\}}\left(1-\mathrm{e}^{\mathbf{i}\xi z}+\mathbf{i}\xi z\mathbb{1}_{\{|z|\leq 1\}}\right)\nu_L(\,\mathrm{d}z).
\end{aligned}$$

[1] This restriction is necessary since we can only show that $\lim_{|x|\to\infty}\sup_{|\xi|\leq\frac{1}{|x|}}|\int_{\mathbb{R}\setminus\{0\}}(1-\mathrm{e}^{\mathbf{i}\xi xz} - \mathbf{i}\xi xz\mathbb{1}_{\{|z|\leq 1\}})\nu_U(\,\mathrm{d}z)| \leq c,\ c>0$, cf. Example 2.9.

Obviously we have

$$|xl_U + l_L| \le \max(|l_U|, |l_L|)(1+|x|),$$
$$|x^2\sigma_U^2 + 2x\sigma_{U,L} + \sigma_L^2| \le 2\max(|\sigma_U^2|, |\sigma_L^2|, \sigma_{U,L})(1+|x|^2).$$

In order to show the last condition of (6.1) we use Taylor's formula twice

$$\begin{aligned}
&\left| \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{\mathbf{i}\xi z} + \mathbf{i}\xi z \mathbb{1}_{\{|z|\le 1\}}\right) \nu_L(\,dz) \right| \\
&\quad\le \int_{0<|z|\le 1} \left|1 - e^{\mathbf{i}\xi z} + \mathbf{i}\xi z\right| \nu_L(\,dz) \\
&\qquad + \int_{1<|z|\le\sqrt{|x|}} \left|1 - e^{\mathbf{i}\xi z}\right| \nu_L(\,dz) + \int_{|z|\ge\sqrt{|x|}} \left|1 - e^{\mathbf{i}\xi z}\right| \nu_L(\,dz) \\
&\quad\le \frac{1}{2}\int_{0<|z|\le 1} |\xi|^2|z|^2 \nu_L(\,dz) + \int_{1<|z|\le\sqrt{|x|}} |\xi||z| \nu_L(\,dz) + \int_{|z|\ge\sqrt{|x|}} 2\nu_L(\,dz) \\
&\quad\le (|\xi|^2 + |\xi|\sqrt{|x|}) \int_{0<|z|\le\sqrt{|x|}} (|z|^2 \wedge 1)\nu_L(\,dz) + 2\nu_L\left(B^c(0,\sqrt{|x|})\right).
\end{aligned}$$

Since $\nu_L(\,dz)$ is a Lévy measure, it holds that $\nu_L\left(B^c(0,R)\right) \longrightarrow 0$ for $R \to \infty$ and $\int_{\mathbb{R}\setminus\{0\}}(|z|^2 \wedge 1)\nu_L(\,dz) < \infty$. As a result, we have shown that

$$\begin{aligned}
&\lim_{|x|\to\infty} \sup_{|\xi|\le\frac{1}{|x|}} \left| q(x,\xi) - \mathbf{i}l(x)\xi - \frac{1}{2}\xi^2 Q(x) \right| \\
&\quad= \lim_{|x|\to\infty} \sup_{|\xi|\le\frac{1}{|x|}} \left| \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{\mathbf{i}\xi z} + \mathbf{i}\xi z \mathbb{1}_{\{|z|\le 1\}}\right) \nu_L(\,dz) \right| \\
&\quad\le \lim_{|x|\to\infty} \left(\frac{1}{|x|^2} + \frac{1}{|x|}\sqrt{|x|}\right) \int_{\mathbb{R}\setminus\{0\}} (|z|^2 \wedge 1)\nu_L(\,dz) + 2\nu_L\left(B^c(0,\sqrt{|x|})\right) = 0,
\end{aligned}$$

and that the assumptions of Theorem 6.1 are fulfilled. Figure 6.2 shows the result of the simulation of such a generalized Ornstein-Uhlenbeck process.

Böttcher and Schnurr [12] have shown that the Markov chain approximation coincides with the Euler scheme if the symbol has bounded coefficients, i.e. $|q(x,\xi)| \le c(1+|\xi|^2)$ for all x and ξ, where $c > 0$ is a constant. For the Euler scheme the coefficients are usually required to satisfy a linear growth condition. The above theorem also ensures convergence of the Markov chain approximation for symbols with drift and diffusion having linear growth. Furthermore, both approaches still coincide under these extended conditions as our next theorem shows.

Theorem 6.3. *Let $(X)_{t\ge 0}$ be a d-dimensional Feller process with generator $(A, \mathcal{D}(A))$. Assume that*

$$C_c^\infty(\mathbb{R}^d) \quad \text{is an operator core of } A,$$

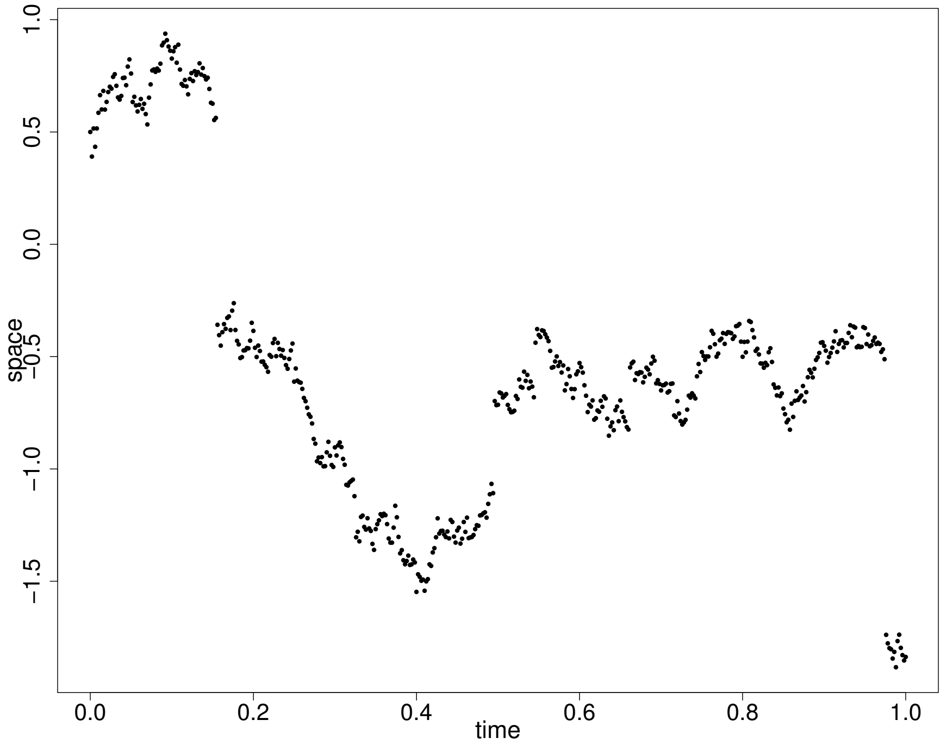

Figure 6.2: Simulation of a Generalized Ornstein-Uhlenbeck process starting at $x = 0.5$ with parameters $l_U = 0.3$, $l_L = 0.2$, $\sigma_U^2 = 0.4$, $\sigma_{U,L} = 0,2$, $\sigma_L^2 = 1$ and ν_L is the Lévy measure of a Cauchy process

and that the corresponding symbol $q(x, \xi)$ satisfies the growth conditions of Theorem 6.1. Then the Euler scheme for the corresponding SDE converges to $(X_t)_{t\geq 0}$ weakly in $D([0,\infty), \mathbb{R}^d)$. Moreover, given that $\bar{X}_{m\cdot h} = x$, the next step of the Euler scheme $\bar{X}_{(m+1)\cdot h}$ has the characteristic function

$$\mathrm{e}^{\mathbf{i}x^\top \xi}\mathrm{e}^{-hq(x,\xi)}.$$

Proof. This theorem is an extension of the main theorem of Böttcher and Schnurr [12]. Their proof is divided in three steps:

1. Show the convergence of the Markov chain approximation;
2. Calculate the SDE of the Feller process explicitly;
3. The characteristic functions of the increments of the Euler scheme coincide with the characteristic functions of the Markov chain increments.

The first step of the proof is covered by Theorem 6.1. The second part only requires the process to be conservative. Indeed, a conservative Feller process is an Itô process, cf. Schnurr [56, Theorem 3.14], and an Itô process has a representation as a stochastic differential equation, cf. Schnurr [56, Proposition 5.6]. In our case, conservativeness follows from Wang [63, Theorem 2.1], see also Theorem 2.8, due to the growth conditions of the symbol. The proof of the last part is identical to that in Böttcher and Schnurr [12]. □

Since the Markov chain approximation coincides with the Euler scheme, we conjecture that the growth conditions for the jump part of the symbol can be replaced by a linear growth condition.

6.2 Simulation of Affine Processes

Although the drift and diffusion coefficients of affine processes meet the assumptions of the previous convergence result, Theorem 6.1, the jump part fails to do so as it is linearly dependent of x. Using the structure of the state space we are however able to extend the approximation. Furthermore, we show that the approach is also valid on the state space of positive semidefinite matrices, cf. Corollary 6.5.

Theorem 6.4. *Let X be an affine process on $D = \mathbb{R}_+^m \times \mathbb{R}^n$ without killing, i.e. $q(x, 0) = 0$ for all $x \in D$, such that the corresponding symbol is given by*

$$\begin{aligned} q(x,\xi) = &\frac{1}{2}\xi^\top a\xi - \mathbf{i}b^\top\xi + \int_{D\setminus\{0\}} \left(1 - \mathrm{e}^{\mathbf{i}\xi^\top y} + \mathbf{i}\xi^\top\chi(y)\right)\mu(\,\mathrm{d}y) \\ &+ \sum_{i=1}^m x_i \frac{1}{2}\xi^\top\alpha^i\xi - \mathbf{i}\sum_{i=1}^{m+n}(x_i\beta^i)^\top\xi \\ &+ \sum_{i=1}^m x_i \int_{D\setminus\{0\}} \left(1 - \mathrm{e}^{\mathbf{i}\xi^\top y} + \mathbf{i}\xi^\top\chi^i(y)\right)\mu^i(\,\mathrm{d}y), \end{aligned}$$

where $(a, \alpha, b, \beta, \mu, \mu^i)$ satisfy the usual admissibility conditions, see Theorem 3.17. For each $n \geq 1$ define a Markov chain $(Y^n(k))_{k\geq 1}$ with $Y^n(0) := x_0$ and transition kernel $\mu_{x,\frac{1}{n}}(\,\mathrm{d}y)$ as in Theorem 6.1. Then

$$Y^n(\lfloor \cdot n \rfloor) \xrightarrow{d} X \quad (n \to \infty).$$

Here $\lfloor x \rfloor = \max\{k \in \mathbb{Z};\ k \leq x\}$ and $\xrightarrow{d}$ denotes convergence in distribution in the space of right continuous functions with left limits equipped with the Skorohod J_1 topology.

Proof. Before proving the result for general spaces $D = \mathbb{R}_+^m \times \mathbb{R}^n$, we consider the subspace $\mathbb{R}_+^m$ individually. Therefore we set $D = \mathbb{R}_+^m$ and $n = 0$. The symbol q of the affine process corresponds to a continuous time branching process with immigration

(CBI) and simplifies to

$$\begin{aligned}q(x,\xi) = &\sum_{i=1}^{m} x_i \frac{1}{2}\xi^\top \alpha^i \xi - \mathbf{i} b^\top \xi - \mathbf{i}\sum_{i=1}^{m}(x_i\beta^i)^\top \xi \\ &+ \int_{\mathbb{R}_+^m\setminus\{0\}} \left(1 - \mathrm{e}^{\mathbf{i}\xi^\top y}\right)\mu(\,\mathrm{d}y) \\ &+ \sum_{i=1}^{m} x_i \int_{\mathbb{R}_+^m\setminus\{0\}} \left(1 - \mathrm{e}^{\mathbf{i}\xi^\top y} + \mathbf{i}\xi^\top \chi^i(y)\right)\mu^i(\,\mathrm{d}y).\end{aligned}$$

Following the lines of the proof of Theorem 6.1, for $u \in C_c^\infty(D)$ fix some $r = r(u) > 0$ with $\operatorname{supp}(u) \subseteq B(0,r)$. Then the main step is to show that for every $\epsilon > 0$ there is some constant $K \gg r$ such that for all $|x| \geq K$ the increment of the Markov chain approximation satisfies

$$n\mathbb{P}\left(L_{\frac{1}{n}}^{(x)} \in B(0,r)\right) \leq \epsilon,$$

where $L = (L_t^{(x)})_{t\geq 0}$ is a Lévy process with characteristic exponent $\xi \mapsto q(x,\xi)$ and starting point $L_0^{(x)} = x$, i.e. characteristic function $\mathrm{e}^{\mathbf{i}\xi^\top x - tq(x,\xi)}$. Applying the Lévy-Itô decomposition we split the Lévy process $L^{(x)}$ into a drift process $l(x)$, a diffusion process $D(x)_t$, and a jump process $J(x)_t$, i.e. $L_t^{(x)} = x + l(x)t + D(x)_t + J(x)_t$. First, we use that $l(x) = b + \sum_{i=1}^m x_i\beta^i$ is affine and deterministic. Now we choose n sufficiently large such that $\left|\frac{b}{n}\right|, \left|\sum_{i=1}^m \frac{\beta^i}{n}\right| \leq \delta \ll \frac{1}{\sqrt{m}}$ to get

$$\begin{aligned}n\mathbb{P}\left(L_{\frac{1}{n}}^{(x)} \in B(0,r)\right) &= n\mathbb{P}\left(x + l(x)_{\frac{1}{n}} + D(x)_{\frac{1}{n}} + J(x)_{\frac{1}{n}} \in B(0,r)\right) \\ &= n\mathbb{P}\Big(\sum_{i=1}^{m} x_i \underbrace{\beta^i\frac{1}{n}}_{|\cdot|\leq\delta} + \underbrace{b\frac{1}{n}}_{|\cdot|\leq\delta} + D(x)_{\frac{1}{n}} + J(x)_{\frac{1}{n}} \in B(-x,r)\Big) \\ &\leq n\mathbb{P}\Big(D(x)_{\frac{1}{n}} + J(x)_{\frac{1}{n}} \in B(-x, |x|\delta + \underbrace{r+\delta}_{=\tilde{r}})\Big).\end{aligned}$$

As $B(-x - \sum_{i=1}^m x_i\beta^i\frac{1}{n} + b\frac{1}{n}, r) \subseteq B(-x, |x|\left|\sum_{i=1}^m \frac{\beta^i}{n}\right| + \left|\frac{b}{n}\right| + r) \subseteq B(-x, |x|\delta + \delta + r)$, we expanded the radius instead of shifting the ball $B(-x,r)$.
We note that the jump process $J(x)_t$ is spectrally positive. In other words, it has only $\mathbb{R}_+^m$-valued jumps, i.e. $\mu(\mathbb{R}_-^m\setminus\{0\}) = 0$ and $\mu^i(\mathbb{R}_-^m\setminus\{0\}) = 0$ for $i = 1,\ldots,m$. Exploiting this property, leads to

$$\begin{aligned}n\mathbb{P}\Big(L_{\frac{1}{n}}^{(x)} \in B(0,r)\Big) &\leq n\mathbb{P}\Big(D(x)_{\frac{1}{n}} + \underbrace{J(x)_{\frac{1}{n}}}_{\geq 0} \in B(-x, |x|\delta + \tilde{r})\Big) \\ &\leq n\mathbb{P}\Big(D(x)_{\frac{1}{n}} \in \bigtimes_{i=1}^{m}(-\infty, |x|\delta + \tilde{r} - x_i)\Big).\end{aligned}$$

Observe that $|x|\delta + \tilde{r} - x_i$ is negative for at least one $i \in \{1, \dots, m\}$ by a suitable choice of K and δ. Since $\max_{i=1,\dots,m} x_i \geq 1/\sqrt{m}|x|$, the hypercube $\times_{i=1}^{m}(-\infty, |x|\delta + \tilde{r} - x_i)$ is covered by the complement of the ball $B\left(0, |x|(1/\sqrt{m} - \delta) - \tilde{r}\right)$, see Figure 6.3. Using this together with the maximal inequality, see Theorem 2.1, we obtain

$$\begin{aligned} n\mathbb{P}\left(L_{\frac{1}{n}}^{(x)} \in B(0,r)\right) &\leq n\mathbb{P}\left(D(x)_{\frac{1}{n}} \in B^c(0, |x|(1/\sqrt{m} - \delta) - \tilde{r})\right) \\ &\leq n\mathbb{P}\left(\tau^{D(x)}_{B(0,|x|(1/\sqrt{m}-\delta)-\tilde{r})} \leq \frac{1}{n}\right) \\ &\leq n\frac{c}{n} \sup_{|\xi| \leq \frac{1}{|x|(1/\sqrt{m}-\delta)-\tilde{r}}} \left|\frac{1}{2}\alpha x|\xi|^2\right| \\ &\leq \tilde{c}\frac{|x|}{(|x|(1/\sqrt{m}-\delta)-\tilde{r})^2} \leq \frac{\varepsilon}{3}, \end{aligned}$$

where $\varepsilon > 0$ is arbitrary and all $x > K(\varepsilon, r, \delta)$. Now the remaining part is identical to

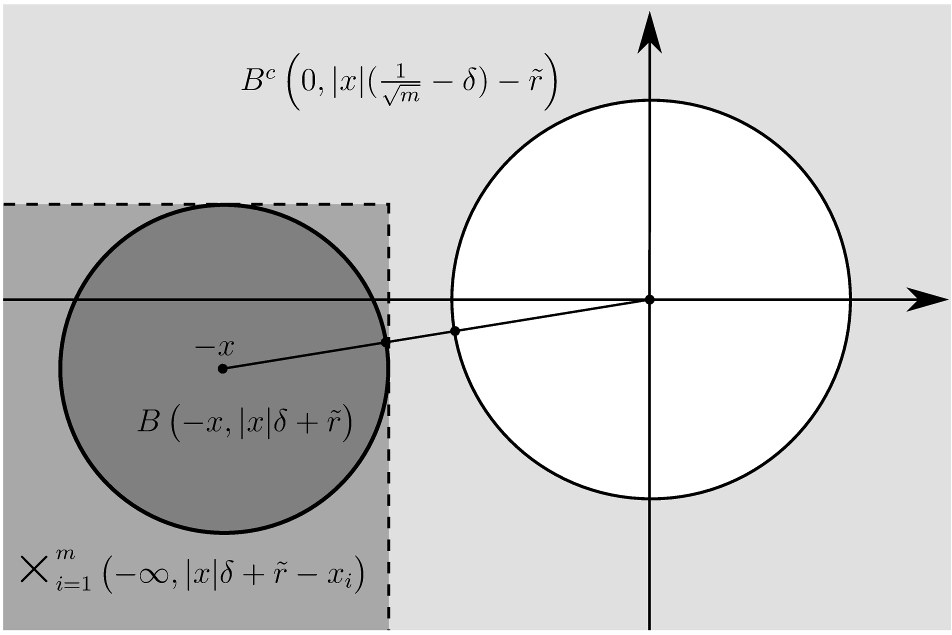

Figure 6.3: The complement of the ball $B^c\left(0, |x|(\frac{1}{\sqrt{m}} - \delta) - \tilde{r}\right)$ contains the hypercupe $\times_{i=1}^{m}\left(-\infty, |x|\delta + \tilde{r} - x_i\right)$ which covers the ball $B\left(-x, |x|\delta + \tilde{r}\right)$.

the proof of Theorem 6.1.
For the general case, $D = \mathbb{R}_+^m \times \mathbb{R}^n$, we distinguish between a multidimensional Ornstein-Uhlenbeck process, i.e. $m = 0$, and the mixed case $m, n > 0$. The first is already covered by Theorem 6.1. Therefore, let $m, n > 0$, so that we have x-dependent jumps on $\mathbb{R}^n$. Again denote by $(L_t^{(x)})_{t\geq 0}$ the Lévy process with characteristic exponent $\xi \mapsto q(x,\xi)$ and starting point $L_0^{(x)} = x$. Our aim is to show that the probability of $L_t^{(x)}$ lying in the ball $B(0,r)$ is sufficiently small by reducing it to the previous special case $D = \mathbb{R}_+^m$. Therefore

we note that the ball $B(0,r)$ is contained in the strip $S(B_{\mathbb{R}_+^m}(0,r)) := \{x = (x_I, x_{II}) \in \mathbb{R}_+^m \times \mathbb{R}^n;\ x_I \in B_{\mathbb{R}_+^m}(0,r)\} = \pi_{\mathbb{R}_+^m}^{-1} B_{\mathbb{R}_+^m}(0,r)$, where $\pi_{\mathbb{R}_+^m}$ is the projection onto the first m coordinates, i.e. onto $\mathbb{R}_+^m$, see Figure 6.4. Consequently, we use the probability that the projection $\pi_{\mathbb{R}_+^m} L_t^{(x)}$ hits $B_{\mathbb{R}_+^m}(0,r)$,

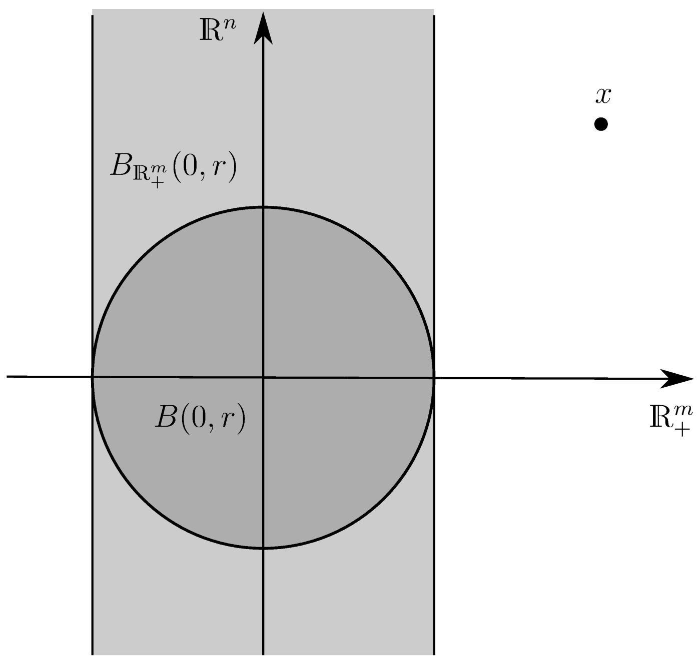

Figure 6.4: The strip $S(B_{\mathbb{R}_+^m}(0,r)) := \{x = (x_I, x_{II}) \in \mathbb{R}_+^m \times \mathbb{R}^n;\ x_I \in B_{\mathbb{R}_+^m}(0,r)\}$ contains the ball $B(0,r)$

$$n\mathbb{P}\left(L_{\frac{1}{n}}^{(x)} \in B(0,r)\right) \leq n\mathbb{P}\left(\pi_{\mathbb{R}_+^m} L_{\frac{1}{n}}^{(x)} \in B_{\mathbb{R}_+^m}(0,r)\right),$$

as an upper estimate. The structure of the affine symbol shows that the $\mathbb{R}_+^m$ projection is independent of $x_{II} \in \mathbb{R}^n$ as the x_{II} term only appears in the linear drift term and its coefficient β^j is subject to the condition $\beta_i^j = 0$ for $i = 1, \ldots, m$. In other words, $\pi_{\mathbb{R}_+^m} L_{\frac{1}{n}}^{(x)}$ is a Lévy process with state space $\mathbb{R}_+^m$ independent of $x_{II} \in \mathbb{R}^n$. Hence the above estimate for CBI processes also holds for $\pi_{\mathbb{R}_+^m} L_{\frac{1}{n}}^{(x)}$ and shows that the Markov chain approximation is valid for all affine processes without killing. $\square$

It is remarkable that due to the spectrally positive jump part the Markov chain approach allows to approximate even non-conservative affine processes. Theorem 6.3 does not show that the Markov chain approximation of an affine process coincides with the Euler scheme. If the affine process is conservative this statement is true. In order to show the consistency of these approaches in the general case we need the representation of the affine process as a stochastic differential equation and the existence of a solution.

For examples of path simulations using the Markov chain approximation we refer to Chapter 3, Figures 3.1, 3.2 and 3.3.

This approach can easily be extended to affine processes on the state space of symmetric positive semidefinite matrices. Since the jump measures allow only jumps into the space S_d^+, we can use the same estimation as for CBI processes in the proof of Theorem 6.4.

Corollary 6.5. *Let X be a stochastically continuous affine process with state space S_d^+. Assume that the corresponding symbol, given by*

$$\begin{aligned} q(x,\xi) = \langle b, \xi\rangle + \int_{S_d^+\setminus\{0\}} (1-\mathrm{e}^{\langle y,\xi\rangle})\mu(\,\mathrm{d}y) \\ -\xi\alpha\xi + B^\top(\xi) + \gamma + \int_{S_d^+\setminus\{0\}} \left(1-\mathrm{e}^{\langle \xi,y\rangle} + \langle \chi(y),\xi\rangle\right)\mu(\,\mathrm{d}y), \end{aligned}$$

has admissible parameters, cf. Theorem 4.7, and no killing, i.e. $q(x,0)=0$. For each $n \geq 1$ define a Markov chain $(Y^n(k))_{k\geq 1}$ with $Y^n(0) := x_0$ and transition kernel $\mu_{x,\frac{1}{n}}(\,\mathrm{d}y)$ given by

$$\int \mathrm{e}^{\mathbf{i}\langle y,\xi\rangle}\mu_{x,\frac{1}{n}}(\,\mathrm{d}y) = \mathrm{e}^{\mathbf{i}\langle x,\xi\rangle - \frac{1}{n}q(x,\xi)}, \quad x\in S_d^+, \xi \in S_d, n\geq 1.$$

Then

$$Y^n(\lfloor \cdot n\rfloor) \xrightarrow{d} X \quad (n\to\infty).$$

Here $\lfloor x\rfloor = \max\{k\in\mathbb{Z};\ k\leq x\}$ and $\xrightarrow{d}$ denotes convergence in distribution in the space of right continuous functions with left limits equipped with the Skorohod J_1 topology.

This result shows that the Markov chain approximation using the symbol of the process does not depend on the state space. Therefore this approach can easily be extended to more general state spaces as already indicated by the above corollary.

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Affirmation

Hereby I affirm that I wrote the present thesis without any inadmissible help by a third party and without using any other means than indicated. Thoughts that were taken directly or indirectly from other sources are indicated as such. This thesis has not been presented to any other examination board in this or a similar form, neither in this nor in any other country.

I have written this dissertation at Technischen Universität Dresden under the scientific supervision of Prof. Dr. René L. Schilling.

There have been no prior attempts to obtain a PhD at any other university.

I accept the requirements for obtaining a PhD (Promotionsordnung) of the Faculty of Science of the TU Dresden, issued February 23, 2011.

Versicherung

Hiermit versichere ich, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

Die vorliegende Dissertation habe ich an der Technischen Universität Dresden unter der wissenschaftlichen Betreuung von Prof. Dr. René L. Schilling angefertigt.

Es wurden zuvor keine Promotionsvorhaben unternommen.

Ich erkenne die Promotionsordnung der Fakultät Mathematik und Naturwissenschaften der TU Dresden vom 23. Februar 2011 an.

Dresden, den 14. März 2016