

Löb's axiom and cut-elimination theorem¹Katsumi Sasaki²**Abstract**

We consider Löb's axiom in modal logics. By adding it to the smallest normal modal logic \mathbf{K} , we obtain provability logic \mathbf{GL} , which is complete for the formal provability interpretation in Peano arithmetic \mathbf{PA} (see Solovay [Sol76]). So, \mathbf{GL} and Löb's axiom has been considered as one of the most important modal logics and axioms.

A cut-elimination theorem for \mathbf{GL} was proved in Valentini [Val83]. Valentini uses an induction on *degree*, *rank* and *width*. The first two parameters are used in the standard proof of cut-elimination theorem presented in Gentzen [Gen35], but the proof for \mathbf{GL} needs the third one *width*. The theorem was also proved semantically in Avron [Avr84]. Avron proved it by using completeness of \mathbf{GL} . However, the completeness cannot be obtained by the standard method, i.e., the canonical model. Here we can see the difficulty to deal with \mathbf{GL} .

The normal modal logic $\mathbf{K4}$ is a sublogic of \mathbf{GL} , which is obtained from \mathbf{K} by adding the transitivity axiom. $\mathbf{K4}$ is much easier to deal with than \mathbf{GL} . A cut-elimination theorem and completeness for $\mathbf{K4}$ are given by the standard method mentioned above.

\mathbf{GL} is also obtained by adding Löb's axiom to $\mathbf{K4}$. So, the knowledge of $\mathbf{K4}$ is useful for the discussion of \mathbf{GL} . In this paper, we give another proof of the cut-elimination theorem for \mathbf{GL} using a cut-free system for $\mathbf{K4}$ and a property of Löb's axiom.

1 Introduction

We use lower case Latin letters for propositional variables. Formulas are defined, as usual, from the propositional variables and the logical constant \perp (contradiction) by using logical connectives \wedge (conjunction), \vee (disjunction) \supset (implication) and \Box (necessity). We use upper case Latin letters, possibly with suffixes, for formulas. We use Greek letters for finite sets of formulas. By $\Box\Gamma$, we mean the set $\{\Box A \mid A \in \Gamma\}$.

Definition 1.1. The degree $d(A)$ of a formula A is defined inductively as follows:

- (1) $d(p) = 1$, for each propositional variable p ,
- (2) $d(\perp) = 1$,
- (3) $d(A \wedge B) = d(A \vee B) = d(A \supset B) = d(A) + d(B) + 1$,
- (4) $d(\Box A) = d(A) + 1$.

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By **K4**, we mean the smallest set of formulas containing all the tautologies and axioms

$$\Box(p \supset q) \supset (\Box p \supset \Box q) \text{ and } \Box p \supset \Box \Box p$$

and closed under modus ponens, substitution and necessitation, i.e., $A \in \mathbf{K4}$ implies $\Box A \in \mathbf{K4}$. By **GL**, we mean the smallest set of formulas containing all the theorems in **K4** and Löb's axiom

$$L(p) = \Box(\Box p \supset p) \supset \Box p.$$

and closed under modus ponens, substitution and necessitation.

By a sequent, we mean the expression

$$\Gamma \rightarrow \Delta.$$

For brevity's sake, we write

$$A_1, \dots, A_k, \Gamma_1, \dots, \Gamma_\ell \rightarrow \Delta_1, \dots, \Delta_m, B_1, \dots, B_n$$

instead of

$$\{A_1, \dots, A_k\} \cup \Gamma_1 \cup \dots \cup \Gamma_\ell \rightarrow \Delta_1 \cup \dots \cup \Delta_m \cup \{B_1, \dots, B_n\}.$$

By $\text{Sub}(\Gamma \rightarrow \Delta)$, we mean the set of subformulas of each formula in $\Gamma \cup \Delta$.

The sequent style system **GK4** for **K4** is defined from the following axioms and rules in the usual way.

Axioms of GK4

$$A \rightarrow A$$

$$\perp \rightarrow$$

Inference rules of GK4

$$\frac{\Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta} (T \rightarrow) \qquad \frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, A} (\rightarrow T)$$

$$\frac{\Gamma \rightarrow \Delta, A \quad A, \Pi \rightarrow \Lambda}{\Gamma, \Pi - \{A\} \rightarrow \Delta - \{A\}, \Lambda} (\text{cut})$$

$$\frac{A_i, \Gamma \rightarrow \Delta}{A_1 \wedge A_2, \Gamma \rightarrow \Delta} (\wedge \rightarrow_i) \qquad \frac{\Gamma \rightarrow \Delta, A \quad \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \wedge B} (\rightarrow \wedge)$$

$$\frac{A, \Gamma \rightarrow \Delta \quad B, \Gamma \rightarrow \Delta}{A \vee B, \Gamma \rightarrow \Delta} (\vee \rightarrow) \qquad \frac{\Gamma \rightarrow \Delta, A_i}{\Gamma \rightarrow \Delta, A_1 \vee A_2} (\rightarrow \vee_i)$$

$$\frac{\Gamma \rightarrow \Delta, A \quad B, \Gamma \rightarrow \Delta}{A \supset B, \Gamma \rightarrow \Delta} (\supset \rightarrow) \qquad \frac{A, \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \supset B} (\rightarrow \supset)$$

$$\frac{\Gamma, \Box \Gamma \rightarrow A}{\Box \Gamma \rightarrow \Box A} (\Box_{K4})$$

The following two lemmas can be proved in the usual way.

Lemma 1.2. $\rightarrow A \in \mathbf{GK4}$ if and only if $A \in \mathbf{K4}$.

Lemma 1.3. If $\Gamma \rightarrow \Delta \in \mathbf{GK4}$, then there exists a cut-free proof figure for $\Gamma \rightarrow \Delta$ in $\mathbf{GK4}$.

The sequent style system \mathbf{GGL} for \mathbf{GL} is the system obtained from $\mathbf{GK4}$ by replacing (\Box_{K4}) by the following inference rule:

$$\frac{\Box A, \Gamma, \Box \Gamma \rightarrow A}{\Box \Gamma \rightarrow \Box A} (\Box_{GL}).$$

The following lemma can also be proved in the usual way.

Lemma 1.4. $\rightarrow A \in \mathbf{GGL}$ if and only if $A \in \mathbf{GL}$.

Definition 1.5. A subfigure of a proof figure P is defined as follows:

- (1) P is a subfigure of P ,
- (2) if $P = \frac{P_1}{\Gamma \rightarrow \Delta}$, then each subfigure of P_1 is a subfigure of P ,
- (3) if $P = \frac{P_1 \quad P_2}{\Gamma \rightarrow \Delta}$, then each subfigure of P_1 or P_2 is a subfigure of P .

A proof figure Q is called a proper subfigure of a proof figure P if Q is a subfigure of P and $Q \neq P$.

2 A property of Löb's axiom

In this section, we show a property of Löb's axiom.

Definition 2.1. The expression $\Box^n A$ is defined inductively as follows:

- (1) $\Box^0 A = A$,
- (2) $\Box^{k+1} A = \Box(\Box^k A)$.

The following lemma is important for the proof of cut-elimination theorem.

Lemma 2.2. $\Box^n L(A) \rightarrow L(A) \in \mathbf{GK4}$, for any $n \geq 0$.

Proof. By an induction on k , we can show

$$\Box^{k+1} L(A) \rightarrow \Box^k L(A) \in \mathbf{GK4}$$

for any $k \geq 0$. Using cut, possibly several times, we obtain the lemma. \dashv

Corollary 2.3. *For any $n \geq 0$,*

$$\Gamma \rightarrow \Delta \in \mathbf{GGL} \text{ if and only if } \Gamma \rightarrow \Delta \in \mathbf{GK4} + \Box^n L(p),$$

where $\mathbf{GK4} + \Box^n L(p)$ is the system obtained by adding $\rightarrow \Box^n L(A)$ to $\mathbf{GK4}$ as an axiom.

Lemma 2.4. *Let P be a proof figure for $\Gamma \rightarrow \Delta$ in $\mathbf{GK4} + \Box^{n+1} L(p)$. Then there exist formulas A_1, \dots, A_m such that*

$$\Box^{n+1} L(A_1), \dots, \Box^{n+1} L(A_m), \Gamma \rightarrow \Delta \in \mathbf{GK4}.$$

Proof. We use an induction on the number $\#(P)$ of axioms of the form $\rightarrow \Box^{n+1} L(A)$ in P . If $\#(P) = 0$, then P is a proof figure for $\Gamma \rightarrow \Delta$ in $\mathbf{GK4}$. Suppose that $\#(P) > 0$ and the lemma holds for any P^* such that $\#(P^*) < \#(P)$. Then there exists an axiom $\rightarrow \Box^{n+1} L(A_1)$ in P for some A_1 . For a subfigure Q of P , we define $h(Q)$ as follows:

$$(1) h(A \rightarrow A) = \frac{A \rightarrow A}{\Box^{n+1} L(A_1), A \rightarrow A},$$

$$(2) h(\perp \rightarrow) = \frac{\perp \rightarrow}{\Box^{n+1} L(A_1), \perp \rightarrow},$$

$$(3) h(\rightarrow \Box^{n+1} L(A)) = \frac{\rightarrow \Box^{n+1} L(A)}{\Box^{n+1} L(A_1) \rightarrow \Box^{n+1} L(A)}, \text{ where } A \neq A_1,$$

$$(4) h(\rightarrow \Box^{n+1} L(A_1)) = \Box^{n+1} L(A_1) \rightarrow \Box^{n+1} L(A_1),$$

$$(5) h\left(\frac{P_1 \cdots P_k}{\Gamma \rightarrow \Delta}\right) = \frac{h(P_1) \cdots h(P_k)}{\Box^{n+1} L(A_1), \Gamma \rightarrow \Delta}$$

if the inference rule that introduces $\Gamma \rightarrow \Delta$ is not (\Box_{K4}) ,

$$(6) h\left(\frac{P_1}{\Box \Gamma \rightarrow \Box A}\right) = \frac{\frac{h(P_1)}{\text{using } (T \rightarrow), \text{ possibly several times}}}{\frac{L(A_1), \Gamma, \Box^{n+1} L(A_1), \Box \Gamma \rightarrow A}{\Box^{n+1} L(A_1), \Box \Gamma \rightarrow \Box A}}$$

if the inference rule that introduces $\Box \Gamma \rightarrow \Box A$ is (\Box_{K4}) .

Note that $h(P)$ is a proof figure for

$$\Box^{n+1} L(A_1), \Gamma \rightarrow \Delta$$

satisfying $\#(h(P)) < \#(P)$. Using the induction hypothesis, we obtain the lemma. \dashv

3 Cut-elimination

In this section, we prove the following theorem using the cut-free system $\mathbf{GK4}$ and Lemma 2.4.

Lemma 3.1. *If $\Gamma \rightarrow \Delta \in \mathbf{GGL}$, then there exists a cut-free proof figure for $\Gamma \rightarrow \Delta$ in \mathbf{GGL} .*

To prove the theorem above, we provide some preparations.

Definition 3.2. By \mathbf{GGL}^* , we mean the system obtained from \mathbf{GGL} by adding the inference rule (\Box_{K4}) in $\mathbf{GK4}$.

Definition 3.3. Let P be a cut-free proof figure in \mathbf{GGL}^* . We define $dep_{\Box}(P)$ as follows:

$$\begin{aligned} (1) \quad & dep_{\Box}(A \rightarrow A) = dep_{\Box}(\perp \rightarrow) = 0, \\ (2) \quad & dep_{\Box}\left(\frac{P_1 \cdots P_n}{\Gamma \rightarrow \Delta}\right) \\ &= \begin{cases} dep_{\Box}(P_1) + 1 & \text{if } I \text{ is } (\Box_{K4}) \text{ or } (\Box_{GL}) \\ \max\{dep_{\Box}(P_1), \dots, dep_{\Box}(P_n)\} & \text{otherwise} \end{cases} \end{aligned}$$

where I is the inference rule that introduces $\Gamma \rightarrow \Delta$ in $\frac{P_1 \cdots P_n}{\Gamma \rightarrow \Delta}$.

Lemma 3.4. *Let P be a cut-free proof figure for*

$$\Box^n \Pi, \Gamma \rightarrow \Delta$$

in \mathbf{GGL}^ . If $dep_{\Box}(P) < n$ and $\Pi \cap \text{Sub}(\Gamma \rightarrow \Delta) = \emptyset$, then there exists a cut-free proof figure for $\Gamma \rightarrow \Delta$ in \mathbf{GGL}^* .*

Proof. We use an induction on P . If $\Pi = \emptyset$, the lemma is obvious. Suppose that $\Pi \neq \emptyset$ and the lemma holds for any proper subfigure of P . Then P is not axiom, and hence, there exists an inference rule I that introduces the end sequent of P . Here we only show the case that I is either (\Box_{K4}) or (\Box_{GL}) since the other case can be shown by the induction hypothesis. The inference rule I is of the form:

$$\frac{\Lambda, \Box^{n-1}\Pi, \Box^n, \Pi, \Gamma', \Box\Gamma' \rightarrow A}{\Box^n \Pi, \Box\Gamma' \rightarrow \Box A}$$

where $\Lambda \in \{\{\Box A\}, \emptyset\}$ and $\Box\Gamma' \rightarrow \Box A$ is $\Gamma \rightarrow \Delta$. Let P_1 be the proof figure that introduces the upper sequent of I . Since I is either (\Box_{K4}) or (\Box_{GL}) , we have $dep_{\Box}(P_1) < dep_{\Box}(P) < n$, and thereby, $dep_{\Box}(P_1) < n - 1$. Also we have $(\Pi \cup \Box\Pi) \cap \text{Sub}(\Gamma \rightarrow \Delta) = \emptyset$. Since $\text{Sub}(\Lambda, \Gamma', \Box\Gamma' \rightarrow A) \subseteq \text{Sub}(\Gamma \rightarrow \Delta)$, we have $(\Pi \cup \Box\Pi) \cap \text{Sub}(\Lambda, \Gamma', \Box\Gamma' \rightarrow A) = \emptyset$. Using the induction hypothesis, there exists a cut-free proof figure for $\Lambda, \Gamma', \Box\Gamma' \rightarrow A$ in \mathbf{GGL}^* . Using (\Box_{K4}) or (\Box_{GL}) , we obtain the lemma. \dashv

By $\mathcal{P}(\Box A)$, we mean the set of each cut-free proof figure P in \mathbf{GGL}^* such that the inference rule introducing the end sequent of P is either (\Box_{K4}) or (\Box_{GL}) and its principal formula in the succedent is $\Box A$.

Definition 3.5. We define a mapping $h_{\Box C}$ on the set of cut-free proof figures in \mathbf{GGL}^* as follows:

- (1) $h_{\Box C}(A \rightarrow A) = \frac{A \rightarrow A}{\Box C, A \rightarrow A}$,
- (2) $h_{\Box C}(\perp \rightarrow) = \frac{\perp \rightarrow}{\Box C, \perp \rightarrow}$,
- (3) $h_{\Box C}\left(\frac{P_1 \cdots P_n}{\Gamma \rightarrow \Delta}\right) = \frac{h_{\Box C}(P_1) \cdots h_{\Box C}(P_n)}{\Box C, \Gamma \rightarrow \Delta}$ if $\frac{P_1 \cdots P_n}{\Gamma \rightarrow \Delta} \notin \mathcal{P}(\Box D)$

for any $\Box D$,

- (4) $h_{\Box C}\left(\frac{P_1}{\Box \Gamma \rightarrow \Box A}\right) = \frac{\frac{h_{\Box C}(P_1)}{\text{using } (T \rightarrow), \text{ possibly several times}}}{\frac{\Box A, C, \Gamma, \Box C, \Gamma \rightarrow A}{\Box C, \Gamma \rightarrow \Box A}}$ if $\frac{P_1}{\Box \Gamma \rightarrow \Box A} \in \mathcal{P}(\Box A)$ for each $A \neq C$,

- (5) $h_{\Box C}\left(\frac{P_1}{\Box \Gamma \rightarrow \Box C}\right) = \frac{\frac{\text{using } (T \rightarrow), \text{ possibly several times}}{\Box C, \Gamma \rightarrow \Box C}}{\Box C \rightarrow \Box C}$ if $\frac{P_1}{\Box \Gamma \rightarrow \Box C} \in \mathcal{P}(\Box C)$.

By $\#_{\Box}(P)$, we mean the sum of the number of inference rule (\Box_{K4}) in P and the number of inference rule (\Box_{GL}) in P .

Lemma 3.6. *Let P be a cut-free proof figure for $\Gamma \rightarrow \Delta$ in \mathbf{GGL}^* . If there exists a subfigure $Q \in \mathcal{P}(\Box C)$ of P , then $h_{\Box C}(P)$ is a cut-free proof figure for $\Box C, \Gamma \rightarrow \Delta$ such that $\#_{\Box}(P) > \#_{\Box}(h_{\Box C}(P))$ and $dep_{\Box}(P) \geq dep_{\Box}(h_{\Box C}(P))$.*

Proof. By an induction on P .

Lemma 3.7. *Let P be a cut-free proof figure for*

$$\Box^{2n+2}\Pi, \Gamma \rightarrow \Delta$$

in \mathbf{GGL}^ , where n is the number of elements in $\{C \mid \Box C \in \text{Sub}(\Gamma \rightarrow \Delta)\}$. Then there exists a cut-free proof figure for $\Gamma \rightarrow \Delta$ in \mathbf{GGL}^* .*

Proof. We use an induction on $\#_{\Box}(P) + \omega(dep_{\Box}(P))$. We note that

$$\Box^{n+1}\Pi \cap \text{Sub}(\Gamma \rightarrow \Delta) = \emptyset$$

and the end sequent of P can be expressed as

$$\Box^{n+1}(\Box^{n+1}\Pi), \Gamma \rightarrow \Delta.$$

If $dep_{\Box}(P) < n + 1$, then by Lemma 3.4, we obtain the lemma. Suppose that $dep_{\Box}(P) \geq n + 1$ and the lemma holds for any proper subfigure of P . Then there exists a sequence

$$P_1, \dots, P_{n+1}, \dots, P_{dep_{\Box}(P)}$$

of subfigures of P satisfying

- (1) $P_i \in \mathcal{P}(\Box C_i)$ for some C_i ,
- (2) P_{i+1} is a proper subfigure of P_i .

We note $C_i \in \text{Sub}(\Gamma \rightarrow \Delta)$ for $i \leq n+1$. So, there exist i and j such that $C_i = C_j$ and $1 \leq i < j \leq n+1$. On the other hand, P_i is of the form

$$\frac{P'_i}{\Box \Gamma'_i \rightarrow \Box C_i}$$

Using Lemma 3.6, $h_{\Box C_i}(P'_i)$ is a cut-free proof figure for $\Box C_i, \Gamma'_i, \Box \Gamma'_i \rightarrow C_i$ such that $\#_{\Box}(P'_i) > \#_{\Box}(h_{\Box C_i}(P'_i))$ and $dep_{\Box}(P'_i) \geq dep_{\Box}(h_{\Box C_i}(P'_i))$.

By P' , we mean the figure obtained from P by replacing P'_i by $h_{\Box C_i}(P'_i)$. Since

$$\frac{\Box C_i, \Gamma'_i, \Box \Gamma'_i \rightarrow C_i}{\Box \Gamma'_i \rightarrow \Box C_i}$$

is an inference rule in \mathbf{GGL}^* , P' is a cut-free proof figure for the end sequent of P such that $\#_{\Box}(P) > \#_{\Box}(P')$ and $dep_{\Box}(P) \geq dep_{\Box}(P')$. Using the induction hypothesis, we obtain the lemma. \dashv

Lemma 3.8. *Let P be a cut-free proof figure for $\Gamma \rightarrow \Delta$ in \mathbf{GGL}^* . Then there exists a cut-free proof figure for $\Gamma \rightarrow \Delta$ in \mathbf{GGL} .*

Proof. By replacing each inference rule (\Box_{K4}) in P by

$$\frac{\Gamma, \Box \Gamma \rightarrow A}{\frac{\Box A, \Gamma, \Box \Gamma \rightarrow A}{\Box \Gamma \rightarrow \Box A}}$$

we obtain a cut-free proof figure in \mathbf{GGL} . \dashv

Proof of Theorem 3.1. Let n be the number of elements in $\{C \mid \Box C \in \text{Sub}(\Gamma \rightarrow \Delta)\}$. By Corollary 2.3 and Lemma 2.4, there exist formulas A_1, \dots, A_m such that

$$\Box^{2n+2}L(A_1), \dots, \Box^{2n+2}L(A_m), \Gamma \rightarrow \Delta \in \mathbf{GK4}.$$

Using Lemma 1.3, there exists a cut-free proof figure for the above sequent in $\mathbf{GK4}$, and hence, in \mathbf{GGL}^* . Using Lemma 3.7, there exists a cut-free proof figure for $\Gamma \rightarrow \Delta$ in \mathbf{GGL}^* . Using Lemma 3.8, we obtain the theorem. \dashv

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