On p-duo Semimodules

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Abstract
The concept of p-duo semimodule is introduced as a generalization of duo semimodule, where a semimodule M is said to be a p-duo if every pure subsemimodule of M is fully invariant. Many results about this concept are given.

Keywords: p-duo semimodule, duo semimodule, weak duo semimodule, pure semimodule.

1-Introduction
Throughout all semirings are commutative have identity and all semimodules are unitital. R is a semiring and Ma left R-semimodule. A subsemimodule N of a semimodule M is called fully invariant if f(N) \subseteq N, for every R-endomorphism f of M. It is clear that 0 and M are fully invariant subsemimodules of M. The R-semimodule M is called duo if every subsemimodule of M is fully invariant. The semiring R is a duo if it is duo as R-semimodule. It is clear that every semiring is a duo semiring. Also we introduced the concept of weak duo semimodules, where an R-semimodule M is called weak duo if every direct summand subsemimodule of M is fully invariant.

Also, the concept of purely duo(shortly p-duo) semimodule is introduced where an R-semimodule M is called a p-duo if each pure subsemimodule of M is fully invariant where a subsemimodule N of M is said to be pure if IM \cap N = IN for every ideal I of R. Also, p-duo semimodule, and some conditions under which p-duo and weak duo are equivalent is studied.

2-Preliminaries
Some definitions that needed in this paper, will be introduced.

Definition 2.1:[Chaudhari & Bonde, 20105]
Let R be a semiring, a left R-semimodule is a commutative monoid(M,+) with additive identity 0_M for which we have a function R \times M \to M, defined by (r,x) \mapsto rx (scalar multiplication), which satisfies the following conditions for all elements r and s of R and all elements x and y of M:

(i) \ (rs)x = r(sx)
(ii) \ r(x + y) = rx + ry
(iii) \ (r + s)x = rx + sy
(iv) \ 0Rx = 0 = r0 for all r \in R and x \in M
If $1_R x = x$ hold for each $x \in M$ then the semimodule $M$ is called unitary.

**Definition 2.2:** [Chaudhari & Bonde, 2010]

A non-empty subset $N$ of a left $R$-semimodule $M$ is called subsemimodule of $M$ if $N$ is closed under addition and scalar multiplication, that is $N$ is itself a semimodule over $R$, (denoted by $N \triangleleft M$).

**Definition 2.3:** [Golan, 2013]

Let $R$ be a semiring and $L \triangleleft M$ ($R$-semimodule). Then $L$ is said to be a direct summand of $M$ if there exists $R$-subsemimodule $K$ such that $M = L \oplus K$ and $M$ is called a direct sum of $L$ and $K$.

**Definition 2.4:** [Abdulameer, 2017]

A left $R$-semimodule is said to be semisimple if it's a direct sum of its simple subsemimodule.

**Definition 2.5:** [Ebrahimi & Shajari, 2010]

An $R$-semimodule $M$ is called multiplication if for each subsemimodule $N$ of $M$ there exist some ideal $I$ of $R$ such that $IM = N$.

**Definition 2.6:** [Katsov et al., 2009]

If $M$ is an $R$-semimodule then its left annihilator is $\text{ann}_R(M) = \{ r \in R: rm = 0 \text{ for every element } m \in M \}$.

**Definition 2.7:** [Abdulameer, 2017]

A semimodule $M$ is called quasi-injective if for any $R$-semimodule $A$, any $R$-monomorphism $f: A \to M$ any $R$-homomorphism $\alpha: A \to M$, there exists $R$-homomorphism $\varphi: M \to M$ (endomorphism) such that $f = \alpha \varphi$.

\[
\begin{array}{c}
A \xrightarrow{\alpha} M \\
\downarrow f \\
M \xrightarrow{\varphi}
\end{array}
\]

**Definition 2.8:** [Althani, 2011]

A semimodule $M$ is called quasi-projective if for any $R$-semimodule $A$, any $R$-epimorphism $\alpha: M \to A$ any $R$-homomorphism $\varphi: M \to A$, there exists $R$-homomorphism $\psi: M \to M$ (endomorphism) such that $\alpha = \varphi \psi$.

\[
\begin{array}{c}
M \xrightarrow{\psi} M \\
\downarrow f \\
A \xleftarrow{\varphi}
\end{array}
\]
Definition 2.9:[Abdulameer, 2017]
A subsemimodule $N$ of $M$ is said to be fully invariant if $f(N) \subseteq N$ for each $R$-endomorphism $f$ on $M$.

Definition 2.10:[Abdulameer, 2017]
A semimodule $M$ is said to be duo if each subsemimodule of $M$ is fully invariant.

3- p-duo semimodules
In [Özcan & Harmanci, 2006; Anderson & Fuller, 1974] weak duo and p-duo modules were introduced respectively. Analogously, the similar concepts for semimodules is introduced.

Definition 3.1:
A semimodule $M$ is called weak duo if every direct summand subsemimodule of $M$ is fully invariant.

Definition 3.2:
A subsemimodule $N$ of a semimodule $M$ is called pure if $IM \cap N = IN$ for each ideal $I$ of $R$.

Definition 3.3:
A semimodule $M$ is called a p-duo if each pure subsemimodule of $M$ is fully invariant.

Remark 3.4:
1-Every duo semimodule is p-duo and every p-duo is weakly duo.
2-Every multiplication semimodule is a duo semimodule, hence a p-duo semimodule and a weakly duo semimodule.
3-Every pure simple semimodule $M$ is a p-duo semimodule, hence a weak duo semimodule.

Proposition 3.5:
A direct summand of p-duo semimodule is a p-duo.

Proof:
Let $L$ be a direct summand of a p-duo $R$-semimodule. That is $M = L \oplus K$ for some $K \hookrightarrow M$. Let $N$ be a pure subsemimodule of $L$ and let $f:L \rightarrow L$ be an $R$-homomorphism semimodule. Since $L$ is a direct summand, then $L$ is pure subsemimodule in $M$, hence $N$ is a pure subsemimodule in $M$.

Defined $h = f \pi_L: M \rightarrow M$ by $h(x) = f(x)$
$h$ is a well-defined $R$-homomorphism. It follows that $h(N) \subseteq N$, since $M$ is a p-duo semimodule and $N$ is a pure subsemimodule in $M$. But $h(N) = f(N), (N \hookrightarrow L)$. Hence $f(N) \subseteq N$; that is $N$ is fully invariant subsemimodule of $L$. Thus $L$ is a p-duo semimodule.
Lemma 3.6:
If $N$ is a fully invariant subsemimodule of $M$ and if $M = K \oplus H$, then $N = (N \cap K) \oplus (N \cap H)$.

Proof:
Let $n \in N$, since $M = K \oplus H \Rightarrow n = k + h$ and $\pi_K : M \to M \Rightarrow \pi_K(N) \subseteq N$ (fully invariant)

$$\pi_K(n) = k \Rightarrow k \in N \Rightarrow k \in N \cap K$$

Similarly, $h \in N \cap H$

So $\oplus = (N \cap K) + (N \cap H)$ and $(N \cap K) \cap + (N \cap H) = N \cap (K \cap H) = N \cap (0) = 0$, so $N = (N \cap K) \oplus (N \cap H)$.  

In [9] the purely quasi-injective of modules was introduced. Analogously, the similar concept for semimodules is introduced.

Definition 3.7:
An $R$-semimodule $M$ is called purely quasi-injective if every pure subsemimodule $N$ of $M$ and every $f : N \to M$, there exists an $R$-homomorphism $h : M \to M$ such that $h \circ i = f$ where $i$ is the inclusion mapping.

Proposition 3.8:
Let $M$ be an $R$-semimodule such that every cyclic subsemimodule is pure. Then $M$ is a P-duo semimodule if and only if for each $f \in \text{End}(M)$ and for each $m \in M$, there exists $r \in R$ such that $f(m) = rm$.

Proof:
$\Rightarrow$ Let $f \in \text{End}(M), m \in M$. Since $< m >$ is pure (where $< m >$ denotes the cyclic subsemimodule generated by $m$), then $f(< m >) \subseteq < m >$. Hence the result is obtained.

$\Leftarrow$ The stated condition implies $f(N) \subseteq N$ for every $f \in \text{End}(M)$. It follows that $M$ is a duo semimodule. Hence it is a P-duo semimodule.

Remark 3.9:
If $M$ is a semisimple semimodule. Then the following statements are equivalent:
1-$M$ is a duo semimodule.
2-$M$ is a p-duo semimodule.
3-$M$ is a weak duo semimodule.

Proposition 3.10:
Let $M$ be a P-duo $R$-semimodule. Then
1-If $M$ is purely quasi-injective, then every pure subsemimodule of $M$ is a P-duo semimodule.
2-If $M$ is quasi-projective, then for any pure subsemimodule $N$ of $M$, $M/N$ is a P-duo $R$-semimodule.
Proof: 
1-Let $N$ be a pure subsemimodule and $K$ be a pure subsemimodule of $N$. Let $f: N \rightarrow N$ be a homomorphism. Since $N$ is a pure subsemimodule in $M$ and $M$ is a purely quasi-injective semimodule, there exists $h: M \rightarrow M$ such that $h \circ i = i \circ f$ where $i$ is the inclusion mapping of $N$ into $M$.

Thus $h \circ i(K) = h(K)$. But $K$ is a pure subsemimodule in $N$ and $N$ is a pure submodule in $M$, implies $K$ is a pure subsemimodule in $M$. Hence $h(K) \subseteq K$.

Also $h \circ i(K) = i \circ f(K) = f(K)$. Thus $h(K) = f(K)$ and so $f(K) \subseteq K$. Therefore $N$ is a P-duo semimodule.

2-Let $L/K$ be a pure subsemimodule of $M/K$. and let $h: M/K \rightarrow M/K$ be an $R$-homomorphism. Let $\pi: M \rightarrow M/K$ be the natural epimorphism. Since $M/K$ is quasi-projective, there exists $h^*: M \rightarrow M$ such that $\pi \circ h^* = h \circ \pi$. Hence $h^*(m) + K = h(m + K)$ for each $m \in M$. But $L/K$ is a pure subsemimodule in $M/K$ and $K$ is a pure subsemimodule in $M$, so that $L$ is a pure subsemimodule in $M$.

It follows that $h^*(L) \subseteq L$, since $M$ is a P-duo semimodule. Hence $h^*(L/K) = h^*(L)/K = \pi(h^*(L)/K) \subseteq L/K$, Thus $L/K$ is a P-duo semimodule. 

Remark 3.11: 
Let a semimodule $M = L_1 \oplus L_2$ be a direct sum of subsemimodules $L_1, L_2$. Then $L_1$ is fully invariant subsemimodule of $M$ if and only if $Hom(L_1, L_2) = 0$

Proposition 3.12: 
Let a semimodule $M = L \oplus K$ be a direct sum of subsemimodules $L, K$ such that $M$ is a p-duo semimodule. Then $Hom(L, K) = 0$. 

31
Proof:
Since $L$ is a direct sum of $M$, $L$ is a pure subsemimodule in $M$. But $M$ is a $p$-duo semimodule, so $L$ is fully invariant subsemimodule in $M$. Hence $Hom(L, K) = 0$ by note (3.11).

Lemma 3.13:
Let $M$ be a semimodule. If $annM_1 + annM_2 = R$, with $M_1, M_2$ are two semimodule, then $N = I_1N \oplus I_2N$.

Proof:
Let $I_1 = annM_1, I_2 = annM_2$

$I_1M \cap N = I_1N \subseteq I_1M = I_1(M_1 + M_2) = M_2$

Similarly, $I_2M \cap N = I_2N \subseteq I_2M_1 \cap M_2 = (0)$

Now, let $n \in N \Rightarrow n = r_1n + r_2n, r_1 \in I_1, r_2 \in I_2 \Rightarrow n \in I_1N + I_2N$

So $N = I_1N \oplus I_2N$.

Theorem 3.14:
Let an $R$-semimodule $M = L_1 \oplus L_2$ be a direct sum of subsemimodules $L_1, L_2$ such that $annL_1 + annL_2 = R$. Then $M$ is a $p$-duo semimodule if and only if $L_1$ and $L_2$ are $p$-duo semimodule and $Hom(L_i, L_j) = 0$ for $i \neq j, i, j \in \{1, 2\}$.

Proof:
$\Rightarrow$ By proposition(3.5) and Proposition(3.12).

$\Leftarrow$ Let $N$ be a pure subsemimodule of $M$. since $annL + annK = R$, then by lemma(3.13)$N = N_1 \oplus N_2$ for some $N_1 \leftarrow L, N_2 \leftarrow K$. Hence $N_1$ is a pure subsemimodule in $L_1$ and $N_2$ is a pure subsemimodule in $L_2$. Let $f: M \rightarrow M$ be an $R$-homomorphism.

Then $\rho_jf_i: L_j \rightarrow L_j, j = 1, 2$, where $\rho_j$ is the canonical projection and $i_j$ is the inclusion map. Hence $\rho_jf_i(N_j) \subseteq N_j, j = 1, 2$, since $L_j(j = 1, 2)$ is a $p$-duo semimodule. Moreover by hypothesis $\rho_k f_{i_j}(N_j)(N_2) = 0$ for $k \neq j(k, j \in \{1, 2\})$. Then $f(N) = f(N_1) + f = f(i_1(N_1)) + f(i_2(N_2)) = (\rho_1 + \rho_2) \left( f(i_1(N_1)) + f(i_2(N_2)) \right) = \rho_1 \left( f(i_1(N_1)) \right) + \rho_2 \left( f(i_2(N_2)) \right) + \rho_1 \left( f(i_2(N_2)) \right) + \rho_2 \left( f(i_2(N_2)) \right) = \rho_1 \left( f(i_1(N_1)) \right) + \rho_2 \left( f(i_2(N_2)) \right) \subseteq N_1 + N_2 = N$. Thus $M$ is a $p$-duo semimodule.

Lemma 3.15:
Let $M$ be an $R$-semimodule such that $M = \oplus_{i \in I} M_i$. If $N$ is fully invariant subsemimodule of $M$, then $N = \oplus_{i \in I}(N \cap M_i)$.

Proof: As in Lemma (3.6).

Theorem 3.16:
Let a semimodule $M = \oplus_{i \in I} M_i$. Then $M$ is a $p$-duo semimodule if and only if $1-M_i$ is a $p$-duo semimodule for all $i \in I$.

1-$Hom(M_i, M_j) = 0$ for all $i \neq j, j \in I$.

2-$N = \oplus_{i \in I}(N \cap M_i)$ for every pure subsemimodule $N$ of $M$. 

32
Proof:

⇒ By proposition (3.5), proposition (3.12) and Lemma (3.15).

⇐ let \( N \) be a pure subsemimodule of \( M \). By (3), \( N = \bigoplus_{i \in I} (N \cap M_i) \). Thus \( N \cap M_i \) is a pure subsemimodule in \( M_i \). Let \( f: M \rightarrow M \). For any \( j \in I \). Consider the following

\[ M_j \xrightarrow{i_j} M \xrightarrow{f} M \xrightarrow{\rho_j} M_j \]

Where \( i_j \) is the inclusion map and \( \rho_j \) is the canonical projection. Hence \( \rho_j f i_j: M_j \rightarrow M_j \) and so \( \rho_j f i_j (N \cap M_i) \subseteq N \cap M_i \) for each \( j \in I \). By (2), \( \text{Hom}(M_i, M_j) = 0 \) for all \( i \neq j, j \in I \). Hence \( f(\bigoplus_{j \in I} (N \cap M_j)) \subseteq \bigoplus_{j \in I} (\rho_j f i_j (N \cap M_i)) = N \). Thus \( M \) is a p-duo semimodule.

In [Al-Bahraany, 2000] the pure intersection property of modules was introduced. Analogously, the similar concept for semimodules is introduced.

Definition 3.17:

An \( R \)-semimodule \( M \) is said to satisfy pure intersection property (shortly \( \text{PIP} \)) if the intersection of any two pure subsemimodule is pure too.

Corollary 3.18:

Let \( M = \bigoplus_{i \in I} M_i \). Then \( M \) is a p-duo semimodule if the following conditions hold:

1- \( \bigoplus_{i \in I} M_i \) is a p-duo for every finite subset \( \hat{I} \) of \( I \).

2- \( M \) satisfies PIP.

Proof:

By (1), \( M_i \) is a p-duo semimodule for every \( i \in I \). Also \( M_i \bigoplus M_j \) is a p-duo semimodule for each \( i \neq j, i, j \in I \). Let \( x \in N \), hence \( x \in \bigoplus_{i \in \hat{I}} M_i = L \), for some finite subset \( \hat{I} \) of \( I \).

Thus \( x \in N \cap L \). By (2), \( N \cap L \) is a pure subsemimodule in \( M \). But \( N \cap L \subseteq L \), so \( N \cap L \) is a pure subsemimodule in \( L \). Since \( L \) is a p-duo semimodule by (1), \( N \cap L \) is a fully invariant subsemimodule in \( L \). Thus \( N \cap L = \bigoplus_{i \in \hat{I}} (N \cap M_i) \). It follows that \( x \in \bigoplus_{i \in \hat{I}} (N \cap M_i) \) and so \( x \in \bigoplus_{i \in \hat{I}} (N \cap M_i) \). Thus \( N = \bigoplus_{i \in \hat{I}} (N \cap M_i) \) and hence \( M \) is a p-duo semimodule by Theorem (3.16).

In [Saad et al., 1990] the summand sum property and summand intersection property of modules were introduced respectively. Analogously, the similar concepts for semimodules is introduced.

Definition 3.19:

An \( R \)-semimodule is said to satisfy summand sum property if \( K + L \) is a direct summand of \( M \) whenever \( K \) and \( L \) are direct summands of \( M \).

Definition 3.20:

An \( R \)-semimodule is said to satisfy summand intersection property if \( K \cap L \) is a direct summand of \( M \) whenever \( K \) and \( L \) are direct summands of \( M \).
Proposition 3.21:

Let $M$ be a $P$-duo semimodule. If $L$ is a direct summand of $M$ and $N$ is a pure subsemimodule of $M$, then $L \cap N$ is a pure subsemimodule of $M$.

Proof:

Since $L$ is a direct summand of $M$, $M = L \oplus H$ for some $H \hookrightarrow M$. Since $M$ is a $P$-duo semimodule and $K$ is a pure subsemimodule, by (Lemma ) then $K$ is a fully invariant. Hence $K = (K \cap L) \oplus (K \cap H)$. Thus $K \cap L$ is a direct summand of $K$, so $K \cap L$ is a puresubsemimodule in $K$. But $K$ is a pure subsemimodule in $M$, hence $K \cap L$ is a pure subsemimodule in $M$.

Proposition 3.22:

Let $M$ be an $R$-semimodule, then the following two statements are equivalent:

1. $M$ is a $p$-duo semimodule.
2. For each two pure subsemimodule of $M$ with zero intersection, then their sum is fully invariant in $M$.

Proof:

(1$\rightarrow$2) It is clear.

(2$\rightarrow$1) Let $N$ be a pure subsemimodule of $M$. Let $H = (0)$, then $H$ is a pure subsemimodule in $M$ and $N \cap H = (0)$. Hence by (2), $N = N + H$ is a fully invariant. Thus $M$ is a $p$-duo semimodule. ◊

Lemma 3.23:

An $R$-semimodule $M$ satisfies PIP if $I(N \cap L) = IN \cap IL$, for each ideal $I$ of $R$ and for each pure subsemimodules $N, L$ of $M$.

Proof:

Let $N, L$ be two pure subsemimodules of $M$. Then $IM \cap N = IN$ and $IM \cap L = IL$. Hence $IM \cap (N \cap L) = (IM \cap N) \cap L = IN \cap L$, also $IM \cap (N \cap L) = (IM \cap L) \cap N = IL \cap N$. Hence $IN \cap L = IL \cap N$. On the other hand, $I(N \cap L) = IN \cap IL$. Claim that $IN \cap L = IN \cap IL$. Let $x \in IN \cap L = IL \cap N$. Hence $x \in IN \cap IL$ so $IN \cap L \subseteq IN \cap IL$ and $IN \cap IL \subseteq IN \cap L$. So $IN \cap IL = IN \cap L$. Thus $IM \cap (N \cap L) = IN \cap L = IN \cap IL = I(N \cap L)$. Therefore $M$ is satisfies PIP.

Reference


