SUBORDINATION PROPERTIES FOR A CERTAIN CLASS OF ANALYTIC FUNCTIONS WITH COMPLEX ORDER

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In this paper, we derive several subordination results for a certain class of analytic functions defined by the generalized Al-Oboudi differential operator. Relevant connections of some of the results obtained with those in earlier works are also provided.

1. Introduction

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

Further, by $\mathcal{S}$ we will denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

Also let $\mathcal{S}^*(b)$ and $\mathcal{K}(b)$ denote, respectively, the subclasses of $\mathcal{A}$ consisting of functions that are starlike of complex order $b \ (b \in \mathbb{C} \setminus \{0\})$ and convex of complex order $b \ (b \in \mathbb{C} \setminus \{0\})$ in $\mathbb{U}$.

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In particular, the classes $S^*: = S^*(1)$ and $K: = K(1)$ are the familiar classes of starlike and convex functions in $U$, respectively.

For two functions $f$ and $g$, analytic in $U$, we say that the function $f$ is subordinate to $g$ in $U$, and write

$$f(z) \prec g(z) \quad (z \in U),$$

if there exists a Schwarz function $\omega$, which is analytic in $U$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in U)$ such that

$$f(z) = g(\omega(z)) \quad (z \in U).$$

Indeed, it is known that

$$f(z) \prec g(z) \quad (z \in U) \Rightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

Furthermore, if the function $g$ is univalent in $U$, then we have the following equivalence

$$f(z) \prec g(z) \quad (z \in U) \iff f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

The following definition of fractional derivative by Owa [10] (also by Srivastava and Owa [15]) will be required in our investigation.

The fractional derivative of order $\gamma$ is defined, for a function $f$, by

$$D^{\gamma}_zf(z) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\gamma}} dt \quad (0 \leq \gamma < 1), \quad (2)$$

where the function $f$ is analytic in a simply connected region of the complex $z$-plane containing the origin, and the multiplicity of $(z-t)^{-\gamma}$ is removed by requiring $\log(z-t)$ to be real when $z-t > 0$.

It readily follows from (2) that

$$D^{\gamma}_zf^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\gamma)} z^{k-\gamma} \quad (0 \leq \gamma < 1, k \in \mathbb{N} = \{1, 2, \ldots\}).$$

Using the operator $D^{\gamma}_zf$, Owa and Srivastava [11] introduced the operator $\Omega^\gamma : A \rightarrow A$, which is known as an extension of fractional derivative and fractional integral, as follows:

$$\Omega^\gamma f(z) = \Gamma(2-\gamma) z^{\gamma} D^{\gamma}_zf(z)$$

$$= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)} a_kz^k \quad \gamma \neq 2, 3, 4, \ldots \quad (3)$$
Note that
\[ \Omega^0 f(z) = f(z). \]

In [2], Al-Oboudi and Al-Amoudi defined the linear multiplier fractional differential operator \( D_{\lambda}^{n,\gamma} \) (which is known as the generalized Al-Oboudi differential operator) as follows:
\[
D_0^0 f(z) = f(z),
\]
\[
D_\lambda^{1,\gamma} f(z) = (1 - \lambda) \Omega^\gamma f(z) + \lambda z (\Omega^\gamma f(z))',
\]
\[
= D_\lambda^\gamma (f(z)), \quad \lambda \geq 0, \quad 0 \leq \gamma < 1,
\]
\[
D_\lambda^{2,\gamma} f(z) = D_\lambda^\gamma \left( D_\lambda^{1,\gamma} f(z) \right),
\]
\[
\vdots
\]
\[
D_\lambda^{n,\gamma} f(z) = D_\lambda^\gamma \left( D_\lambda^{n-1,\gamma} f(z) \right), \quad n \in \mathbb{N}.
\]

If \( f \) is given by (1), then by (3), (4) and (5), we see that
\[
D_\lambda^{n,\gamma} f(z) = z + \sum_{k=2}^{\infty} \Psi_{k,n}(\gamma, \lambda) a_k z^k, \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},
\]
where
\[
\Psi_{k,n}(\gamma, \lambda) = \left[ \frac{\Gamma(k+1) \Gamma(2-\gamma)}{\Gamma(k+1-\gamma)} \left( 1 + \lambda (k-1) \right) \right]^n.
\]

**Remark 1.1.** (i) When \( \gamma = 0 \), we get Al-Oboudi differential operator [1].
(ii) When \( \gamma = 0 \) and \( \lambda = 1 \), we get Salagean differential operator [12].
(iii) When \( n = 1 \) and \( \lambda = 0 \), we get Owa-Srivastava fractional differential operator [11].

Let \( G_{\gamma,\lambda}^{n}(\delta, b, A, B) \) denote the class of functions \( f \in \mathcal{A} \) satisfying
\[
1 + \frac{1}{b} \left( 1 - \delta \frac{D_{\lambda}^{n,\gamma} f(z)}{z} + \delta (D_{\lambda}^{n,\gamma} f(z))' - 1 \right) < \frac{1 + A z}{1 + B z}
\]
or satisfying
\[
\left| \frac{1 - \delta \frac{D_{\lambda}^{n,\gamma} f(z)}{z} + \delta (D_{\lambda}^{n,\gamma} f(z))' - 1}{(A - B) b - B \left[ 1 - \delta \frac{D_{\lambda}^{n,\gamma} f(z)}{z} + \delta (D_{\lambda}^{n,\gamma} f(z))' - 1 \right]} \right| < 1,
\]
where \( z \in \mathbb{U}, \ b \in \mathbb{C} \setminus \{0\}, \delta \geq 0, \ -1 \leq B < A \leq 1 \) and \( D_{\lambda}^{n,\gamma} \) is the generalized Al-Oboudi differential operator.
In [13], by using the Salagean differential operator $D^n$, Sivasubramanian et al. defined the class

$$G^n_{0,1} (\delta, b, A, B) = G_n(\delta, b, A, B)$$

$$= \left\{ f \in A : 1 + \frac{1}{b} \left( (1 - \delta) \frac{D^n f(z)}{z} + \delta (D^n f(z))' - 1 \right) < \frac{1 + Az}{1 + Bz} \right\}$$

which generalizes the class

$$G^n_{0,1} (\delta, b, 1, -1) = G_n(\delta, b)$$

$$= \left\{ f \in A : \Re \left\{ 1 + \frac{1}{b} \left( (1 - \delta) \frac{D^n f(z)}{z} + \delta (D^n f(z))' - 1 \right) \right\} > 0 \right\}$$

introduced by Aouf [3].

We note that, for $z \in \mathbb{U}$,

(i) $G^n_{\gamma, \lambda} (\delta, b, 1, -1) = G^n_{\gamma, \lambda} (\delta, b)$

(ii) $G^n_{\gamma, \lambda} (0, b, 1, -1) = G^n_{\gamma, \lambda} (b)$

(iii) $G^n_{\gamma, \lambda} (1, b, 1, -1) = R^n_{\gamma, \lambda} (b)$

(iv) $G^0_{\gamma, \lambda} (0, b, 1, -1) = G(b)$

(v) $G^0_{\gamma, \lambda} (1, b, 1, -1) = R(b)$

(vi) $G^0_{\gamma, \lambda} (0, 1 - \alpha, 1, -1) = G_{\alpha}$

(vii) $G^0_{\gamma, \lambda} (1, 1 - \alpha, 1, -1) = R_{\alpha}$

(viii) $G^0_{\gamma, \lambda} (\delta, b, 1, -1) = G(\delta, b)$

The class $R(b)$ was studied by Halim [8], the class $G_{\alpha}$ was studied by Chen [5, 6] and the class $R_{\alpha}$ was studied by Ezrohi [7].

**Definition 1.2** (Hadamard product or Convolution). Given two functions $f$ and $g$ in the class $A$, where $f$ is given by (1) and $g$ is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

the Hadamard product (or convolution) $f \ast g$ is defined by

$$(f \ast g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g \ast f)(z) \quad (z \in \mathbb{U}).$$
Definition 1.3 (Subordinating Factor Sequence). A sequence \( \{b_k\}_{k=1}^\infty \) of complex numbers is said to be a subordinating factor sequence if, whenever \( f \) of the form (1) is analytic, univalent and convex in \( \mathbb{U} \), we have the subordination given by
\[
\sum_{k=1}^\infty a_k b_k z^k < f(z) \quad (z \in \mathbb{U}; \quad a_1 = 1).
\]

Lemma 1.4 ([16]). The sequence \( \{b_k\}_{k=1}^\infty \) is a subordinating factor sequence if and only if
\[
\Re \left\{ 1 + 2 \sum_{k=1}^\infty b_k z^k \right\} > 0 \quad (z \in \mathbb{U}).
\]

2. Main Result

Now, we prove the following theorem which gives a sufficient condition for functions belonging to the class \( \mathcal{G}_n^{\gamma, \lambda} (\delta, b, A, B) \).

Theorem 2.1. Let the function \( f \) which is defined by (1) satisfy the following condition:
\[
\sum_{k=2}^\infty (1 + |B|) (1 + \delta (k - 1)) \Psi_{k,n} (\gamma, \lambda) |a_k| \leq (A - B) |b|, \quad (10)
\]
then \( f \in \mathcal{G}_n^{\gamma, \lambda} (\delta, b, A, B) \).

Proof. Suppose that the inequality (10) holds. Then we have for \( z \in \mathbb{U} \),
\[
\begin{align*}
&\left| (1 - \delta) \frac{D_{\lambda}^{n, \gamma} f(z)}{z} + \delta \left( D_{\lambda}^{n, \gamma} f(z) \right)' - 1 \right| \\
&\quad - (A - B) b - B \left| (1 - \delta) \frac{D_{\lambda}^{n, \gamma} f(z)}{z} + \delta \left( D_{\lambda}^{n, \gamma} f(z) \right)' - 1 \right| \\
&\quad = \sum_{k=2}^\infty (1 + \delta (k - 1)) \Psi_{k,n} (\gamma, \lambda) a_k z^{k-1} \\
&\quad - (A - B) b - B \sum_{k=2}^\infty (1 + \delta (k - 1)) \Psi_{k,n} (\gamma, \lambda) a_k z^{k-1} \\
&\quad \leq \sum_{k=2}^\infty (1 + \delta (k - 1)) \Psi_{k,n} (\gamma, \lambda) |a_k| |z|^{k-1} \\
&\quad - (A - B) |b| - |B| \sum_{k=2}^\infty (1 + \delta (k - 1)) \Psi_{k,n} (\gamma, \lambda) |a_k| |z|^{k-1} < 
\end{align*}
\]
Let in the subordination result (function class $G$) the inequality which consists of functions $f$, $g$, $h$, and suppose that $g \in A$ and let $f$ defined by (12) and Srivastava and Attiya (14). Then we have

$$< \sum_{k=2}^{\infty} (1 + |B|) (1 + \delta (k - 1)) \Psi_{2,n}(\gamma, \lambda) |a_k| - (A - B) |b| \leq 0,$$

which shows that $f$ belongs to the class $G_{\gamma, \lambda} (\delta, b, A, B)$. 

In view of Theorem 2.1, we now introduce the subclass $G_{\gamma, \lambda}^n (\delta, b, A, B)$ which consists of functions $f \in A$ whose Taylor-Maclaurin coefficients satisfy the inequality (10). We note that

$$G_{\gamma, \lambda} (\delta, b, A, B) \subset G_{\gamma, \lambda}^n (\delta, b, A, B).$$

In this work, we prove several subordination relationships involving the function class $G_{\gamma, \lambda}^n (\delta, b, A, B)$ employing the technique used earlier by Attiya [4] and Srivastava and Attiya [14].

**Theorem 2.2.** Let the function $f$ defined by (1) be in the class $G_{\gamma, \lambda}^n (\delta, b, A, B)$ and suppose that $g \in K$. Then

$$\frac{(1 + |B|) (1 + \delta) \Psi_{2,n}(\gamma, \lambda)}{2 [(1 + |B|) (1 + \delta) \Psi_{2,n}(\gamma, \lambda) + (A - B) |b|]} (f * g)(z) < g(z) \quad (z \in U) \quad (11)$$

and

$$\Re \{ f(z) \} > - \frac{(1 + |B|) (1 + \delta) \Psi_{2,n}(\gamma, \lambda) + (A - B) |b|}{(1 + |B|) (1 + \delta) \Psi_{2,n}(\gamma, \lambda)} \quad (z \in U). \quad (12)$$

The constant factor

$$\frac{(1 + |B|) (1 + \delta) \Psi_{2,n}(\gamma, \lambda)}{2 [(1 + |B|) (1 + \delta) \Psi_{2,n}(\gamma, \lambda) + (A - B) |b|]}$$

in the subordination result (11) cannot be replaced by a larger one.

**Proof.** Let $f \in G_{\gamma, \lambda}^n (\delta, b, A, B)$ and let $g(z) = z + \sum_{k=2}^{\infty} c_k z^k \in K$. Then we have

$$\frac{(1 + |B|) (1 + \delta) \Psi_{2,n}(\gamma, \lambda)}{2 [(1 + |B|) (1 + \delta) \Psi_{2,n}(\gamma, \lambda) + (A - B) |b|]} (f * g)(z) = \frac{(1 + |B|) (1 + \delta) \Psi_{2,n}(\gamma, \lambda)}{2 [(1 + |B|) (1 + \delta) \Psi_{2,n}(\gamma, \lambda) + (A - B) |b|]} \left( z + \sum_{k=2}^{\infty} a_k c_k z^k \right). \quad (13)$$

Thus, by Definition 1.3, the subordination result (11) will hold true if the sequence

$$\left\{ \frac{(1 + |B|) (1 + \delta) \Psi_{2,n}(\gamma, \lambda)}{2 [(1 + |B|) (1 + \delta) \Psi_{2,n}(\gamma, \lambda) + (A - B) |b|]} a_k \right\}_{k=1}^{\infty} \quad (14)$$
is a subordinating factor sequence, with \( a_1 = 1 \). In view of Lemma 1.4, this is equivalent to the following inequality:

\[
\Re \left\{ 1 + \sum_{k=1}^{\infty} \frac{(1 + |B|) (1 + \delta) \Psi_{2,n}(\gamma, \lambda)}{[(1 + |B|) (1 + \delta) \Psi_{2,n}(\gamma, \lambda) + (A - B) |b|]} a_k z^k \right\} > 0 \quad (z \in \mathbb{U}).
\]  

(15)

Since 

\[(1 + \delta (k - 1)) \Psi_{k,n}(\gamma, \lambda)\]

is an increasing function of \( k \) \((k \geq 2)\), when \(|z| = r\) \((0 < r < 1)\), we have

\[
\Re \left\{ 1 + \sum_{k=1}^{\infty} \frac{(1 + |B|) (1 + \delta) \Psi_{2,n}(\gamma, \lambda)}{[(1 + |B|) (1 + \delta) \Psi_{2,n}(\gamma, \lambda) + (A - B) |b|]} a_k z^k \right\} \]

\[
= \Re \left\{ 1 + \frac{1 + |B|}{(1 + |B|) (1 + \delta) \Psi_{2,n}(\gamma, \lambda) + (A - B) |b|} \sum_{k=2}^{\infty} (1 + \delta) \Psi_{2,n}(\gamma, \lambda) a_k |r^k - 1| (1 + |B|) (1 + \delta) \Psi_{2,n}(\gamma, \lambda) + (A - B) |b| \right\} \]

\[
\geq 1 - \frac{1 + |B|}{(1 + |B|) (1 + \delta) \Psi_{2,n}(\gamma, \lambda) + (A - B) |b|} \sum_{k=2}^{\infty} (1 + \delta (k - 1)) \Psi_{k,n}(\gamma, \lambda) a_k |r^k - 1| (1 + |B|) (1 + \delta) \Psi_{2,n}(\gamma, \lambda) + (A - B) |b| \]

\[
> 1 - \frac{(A - B) |b|}{(1 + |B|) (1 + \delta) \Psi_{2,n}(\gamma, \lambda) + (A - B) |b|} \]

\[
= 1 - r > 0,
\]

where we have also made use of the assertion \((10)\) of Theorem 2.1. Then \((15)\) holds true in \( \mathbb{U} \). This proves the inequality \((11)\). The inequality \((12)\) follows from \((11)\) by taking the convex function

\[
g(z) = \frac{z}{1 - z} = z + \sum_{k=2}^{\infty} z^k.
\]

To prove the sharpness of the constant

\[
\frac{(1 + |B|) (1 + \delta) \Psi_{2,n}(\gamma, \lambda)}{2 [(1 + |B|) (1 + \delta) \Psi_{2,n}(\gamma, \lambda) + (A - B) |b|]},
\]

we consider the function \( f_0 \in \mathcal{G}_{\gamma,\lambda}^{n+}(\delta, b, A, B) \) given by

\[
f_0(z) = z - \frac{(A - B) |b|}{(1 + |B|) (1 + \delta) \Psi_{2,n}(\gamma, \lambda)} z^2.
\]  

(16)
Thus from (11), we have
\[
\frac{(1 + |B|)(1 + \delta)\Psi_{2,n}(\gamma, \lambda)}{2[(1 + |B|)(1 + \delta)\Psi_{2,n}(\gamma, \lambda) + (A - B)|b|]} f_0(z) < \frac{z}{1 - z} \quad (z \in \mathbb{U}). \tag{17}
\]

It can easily be verified for the function \(f_0\) given by (16) that
\[
\min_{|z| \leq r} \left\{ \mathcal{R} \left( \frac{(1 + |B|)(1 + \delta)\Psi_{2,n}(\gamma, \lambda)}{2[(1 + |B|)(1 + \delta)\Psi_{2,n}(\gamma, \lambda) + (A - B)|b|]} f_0(z) \right) \right\} = -\frac{1}{2}. \tag{18}
\]

This shows that the constant
\[
\frac{(1 + |B|)(1 + \delta)\Psi_{2,n}(\gamma, \lambda)}{2[(1 + |B|)(1 + \delta)\Psi_{2,n}(\gamma, \lambda) + (A - B)|b|]
\]
is the best possible, which completes the proof of Theorem 2.2. \(\square\)

For the choices \(\gamma = 0\) and \(\lambda = 1\) in Theorem 2.2, we get the following corollary.

**Corollary 2.3** ([13, Theorem 2.2]). *Let the function \(f\) defined by (1) be in the class \(G_n^* (\delta, b, A, B)\) and suppose that \(g \in \mathcal{K}\). Then*
\[
\frac{(1 + |B|)(1 + \delta)2^n}{2[(1 + |B|)(1 + \delta)2^n + (A - B)|b|]} (f * g)(z) < g(z) \quad (z \in \mathbb{U}) \tag{19}
\]

*and*
\[
\mathcal{R} \{f(z)\} > -\frac{(1 + |B|)(1 + \delta)2^n + (A - B)|b|}{(1 + |B|)(1 + \delta)2^n} \quad (z \in \mathbb{U}).
\]

*The constant factor*
\[
\frac{(1 + |B|)(1 + \delta)2^n}{2[(1 + |B|)(1 + \delta)2^n + (A - B)|b|]
\]
in the subordination result (19) cannot be replaced by a larger one.

For the choices of \(\gamma = 0\), \(\lambda = 1\) and \(A = 1, B = -1\) in Theorem 2.2, we get the following corollary.

**Corollary 2.4** ([3, Theorem 1]). *Let the function \(f\) defined by (1) be in the class \(G_n^* (\delta, b)\) and suppose that \(g \in \mathcal{K}\). Then*
\[
\frac{(1 + \delta)2^n}{2[(1 + \delta)2^n + |b|]} (f * g)(z) < g(z), \quad (z \in \mathbb{U}) \tag{20}
\]

*and*
\[
\mathcal{R} \{f(z)\} > -\frac{(1 + \delta)2^n + |b|}{(1 + \delta)2^n}, \quad (z \in \mathbb{U}).
\]
The constant factor
\[
\frac{(1 + \delta) 2^n}{2 [(1 + \delta) 2^n + |b|]}
\]
in the subordination result (20) cannot be replaced by a larger one.

For the choices of \( n = 0, \gamma = 0, \lambda = 1 \) and \( A = 1, B = -1 \) in Theorem 2.2, we get the following corollary.

**Corollary 2.5.** Let the function \( f \) defined by (1) be in the class \( G^*(\delta, b) \) and suppose that \( g \in \mathcal{K} \). Then
\[
\frac{1 + \delta}{2 (1 + \delta + |b|)} (f * g)(z) < g(z), \quad (z \in \mathbb{U}) \tag{21}
\]
and
\[
\Re \{f(z)\} > -\frac{1 + \delta + |b|}{1 + \delta}, \quad (z \in \mathbb{U}).
\]
The constant factor
\[
\frac{1 + \delta}{2 (1 + \delta + |b|)}
\]
in the subordination result (21) cannot be replaced by a larger one.

For the choices of \( \delta = 0, n = 0, \gamma = 0, \lambda = 1 \) and \( A = 1, B = -1 \) in Theorem 2.2, we get the following corollary.

**Corollary 2.6.** Let the function \( f \) defined by (1) be in the class \( G^*(b) \) and suppose that \( g \in \mathcal{K} \). Then
\[
\frac{1}{2 (1 + |b|)} (f * g)(z) < g(z), \quad (z \in \mathbb{U}) \tag{22}
\]
and
\[
\Re \{f(z)\} > -(1 + |b|), \quad (z \in \mathbb{U}).
\]
The constant factor
\[
\frac{1}{2 (1 + |b|)}
\]
in the subordination result (22) cannot be replaced by a larger one.

For the choices of \( b = 1 - \alpha \) (\( 0 \leq \alpha < 1 \)), \( \delta = 0, n = 0, \gamma = 0, \lambda = 1 \) and \( A = 1, B = -1 \) in Theorem 2.2, we get the following corollary.
Corollary 2.7. Let the function $f$ defined by (1) be in the class $G_\alpha^*$ and suppose that $g \in \mathcal{K}$. Then
\[
\frac{1}{2(2-\alpha)} (f \ast g)(z) < g(z), \quad (z \in \mathbb{U})
\] (23)
and
\[
\Re \{f(z)\} > -(2-\alpha), \quad (z \in \mathbb{U}).
\]
The constant factor
\[
\frac{1}{2(2-\alpha)}
\]
in the subordination result (23) cannot be replaced by a larger one.

For the choices of $b = 1 - \alpha$ ($0 \leq \alpha < 1$), $\delta = 1$, $n = 0$, $\gamma = 0$, $\lambda = 1$ and $A = 1, B = -1$ in Theorem 2.2, we get the following corollary.

Corollary 2.8. Let the function $f$ defined by (1) be in the class $R_\alpha^*$ and suppose that $g \in \mathcal{K}$. Then
\[
\frac{1}{3-\alpha} (f \ast g)(z) < g(z), \quad (z \in \mathbb{U})
\] (24)
and
\[
\Re \{f(z)\} > -\frac{3-\alpha}{2}, \quad (z \in \mathbb{U}).
\]
The constant factor
\[
\frac{1}{3-\alpha}
\]
in the subordination result (24) cannot be replaced by a larger one.

REFERENCES


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