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Majorization for certain classes of meromorphic functions defined by integral operator

Abstract. Here we investigate a majorization problem involving starlike meromorphic functions of complex order belonging to a certain subclass of meromorphic univalent functions defined by an integral operator introduced recently by Lashin.

1. Introduction and preliminaries. Let $f(z)$ and $g(z)$ be analytic in the open unit disk

$$\Delta = \{ z \in \mathbb{C} \text{ and } |z| < 1 \}.$$

For analytic functions $f(z)$ and $g(z)$ in $\Delta$, we say that $f(z)$ is majorized by $g(z)$ in $\Delta$ (see [9]) and write

$$f(z) \ll g(z) \quad (z \in \Delta),$$

if there exists a function $\phi(z)$, analytic in $\Delta$ such that $|\phi(z)| \leq 1$, and

$$f(z) = \phi(z)g(z) \quad (z \in \Delta).$$

Let $\Sigma$ denote the class of meromorphic functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k,$$

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which are analytic and univalent in the punctured unit disk
\[ \Delta^* := \{ z \in \mathbb{C} : 0 < |z| < 1 \} := \Delta \setminus \{0\} \]
with a simple pole at the origin.

For functions \( f_j \in \Sigma \) given by
\[ f_j(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,j} z^k \quad (j = 1, 2; z \in \Delta^*) \]
we define the Hadamard product (or convolution) of \( f_1 \) and \( f_2 \) by
\[ (f_1 * f_2)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z) \]

Analogously to the operators defined by Jung, Kim and Srivastava [7] on the normalized analytic functions, Lashin [8] introduced the following integral operators
\[ \mathcal{P}^\alpha_\beta : \Sigma \longrightarrow \Sigma \]
defined by
\[ \mathcal{P}^\alpha_\beta f(z) = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{z^{\beta+1}} \int_0^z \left( \log \frac{z}{t} \right)^{\alpha-1} f(t) dt \]
\( \alpha > 0, \beta > 0; z \in \Delta^* \), where \( \Gamma(\alpha) \) is the familiar Gamma function.

Using the integral representation of the Gamma function and (1.4), it can be easily shown that
\[ \mathcal{P}^\alpha_\beta f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left( \frac{\beta}{\beta + k + 1} \right)^\alpha a_k z^k \quad (\alpha > 0, \beta > 0; z \in \Delta^*) \]

Obviously
\[ \mathcal{P}^1_\beta f(z) := J_\beta. \]

The operator
\[ J_\beta : \Sigma \longrightarrow \Sigma \]
has also been studied by Lashin [8].

It is easy to verify that (see [8]),
\[ z(\mathcal{P}^\alpha_\beta f(z))' = \beta \mathcal{P}^{\alpha-1}_\beta f(z) - (\beta + 1) \mathcal{P}^\alpha_\beta f(z). \]

**Definition 1.1.** A function \( f(z) \in \Sigma \) is said to be in the class \( \mathcal{S}^{\alpha,j}_{\beta}(\gamma) \) of meromorphic functions of complex order \( \gamma \neq 0 \) in \( \Delta \) if and only if
\[ \Re \left\{ 1 - \frac{1}{\gamma} \left( \frac{z(\mathcal{P}^\alpha_\beta f(z))^{(j+1)}}{(\mathcal{P}^\alpha_\beta f(z))^{(j)}} + j + 1 \right) \right\} > 0 \]
\( (z \in \Delta, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \alpha > 0, \beta > 0, \gamma \in \mathbb{C} \setminus \{0\} \)
Clearly, we have the following relationships:

(i) $S_{0,0}^0(\gamma) = S(\gamma)$ \hspace{1em} (\gamma \in \mathbb{C} \setminus \{0\}),

(ii) $S_{0,0}^0(1 - \eta) = S^*(\eta)$ \hspace{1em} (0 \leq \eta < 1).

The classes $S(\gamma)$ and $S^*(\eta)$ are said to be classes of meromorphic starlike univalent functions of complex order $\gamma \neq 0$ and meromorphic starlike univalent functions of order $\eta$ ($\eta \in \mathbb{R}$ such that $0 \leq \eta < 1$) in $\Delta^*$.

A majorization problem for the normalized classes of starlike functions has been investigated by Altinas et al. [1] and MacGregor [9]. In the recent paper Goyal and Goswami [2] generalized these results for the class of multivalent functions, using fractional derivatives operators. Further, Goyal et al. [3], Goswami and Wang [4], Goswami [5], Goswami et al. [6] studied majorization property for different classes. In this paper, we will study majorization properties for the class of meromorphic functions using integral operator $P_{\beta}^j$.

2. Majorization problems for the class $S_{\beta}^{\alpha,j}(\gamma)$.

**Theorem 2.1.** Let the function $f \in \Sigma$ and suppose that $g \in S_{\beta}^{\alpha,j}(\gamma)$. If $(P_{\beta}^j f(z))^{(j)}$ is majorized by $(P_{\beta}^j g(z))^{(j)}$ in $\Delta^*$, then

\begin{equation}
|P_{\beta}^{-1} f(z)|^{(j)} \leq |P_{\beta}^{-1} g(z)|^{(j)} \text{ for } |z| \leq r_1(\beta,\gamma),
\end{equation}

where

\begin{equation}
r_1(\beta,\gamma) = \frac{k_1 - \sqrt{k_1^2 - 4\beta(\beta + 2\gamma)}}{2|\beta + 2\gamma|},
\end{equation}

and

\begin{equation}
k_1 = \beta + 2 + |\beta + 2\gamma|, (\beta > 0, j \in \mathbb{N}_0, \gamma \in \mathbb{C} \setminus \{0\}).
\end{equation}

**Proof.** Since $g \in S_{\beta}^{\alpha,j}(\gamma)$, we find from (2.1) that if

\begin{equation}
h_1(z) = 1 - \frac{1}{\gamma} \left( \frac{z(P_{\beta}^j g(z))^{(j+1)}}{(P_{\beta}^j g(z))^{(j)}} + j + 1 \right)
\end{equation}

(\alpha, \beta > 0, \gamma \in \mathbb{C} \setminus \{0\}, j \in \mathbb{N}_0), \text{ then } \Re\{h_1(z)\} > 0 \text{ (} z \in \Delta \text{) and}

\begin{equation}
h_1(z) = \frac{1 + w(z)}{1 - w(z)} \quad (w \in \mathcal{P}),
\end{equation}

where $\mathcal{P}$ denotes the well-known class of bounded analytic functions in $\Delta$ and $w(z) = c_1 z + c_2 z^2 + \ldots$ satisfies the conditions

$w(0) = 0$ and $|w(z)| \leq |z|$ (\(z \in \Delta\)).
Making use of (2.3) and (2.4), we get

\[
\frac{z(\mathcal{P}_\beta^{\alpha-1}g(z))^{(j+1)}}{(\mathcal{P}_\beta^{\alpha}g(z))^{(j)}} = \frac{1 + j - 2\gamma w(z) - (1 + j)}{1 - w(z)}.
\]

By the principle of mathematical induction, and (1.11), we easily get

\[
(\mathcal{P}_\beta^{\alpha}g(z))^{(j)} = \beta(\mathcal{P}_\beta^{\alpha-1}g(z))^{(j)} - (\beta + j + 1)(\mathcal{P}_\beta^{\alpha}g(z))^{(j)},
\]

(\alpha > 1, \beta > 0; z \in \Delta^*). Now using (2.6) in (2.5), we find that

\[
\frac{\beta(\mathcal{P}_\beta^{\alpha-1}g(z))^{(j)}}{(\mathcal{P}_\beta^{\alpha}g(z))^{(j)}} = \frac{(\beta + j + 1) + (1 + j - 2\gamma w(z) - (1 + j)}{1 - w(z)}
\]

or

\[
(\mathcal{P}_\beta^{\alpha}g(z))^{(j)} = \frac{\beta(1 - w(z))}{\beta - (\beta + 2\gamma)w(z)}(\mathcal{P}_\beta^{\alpha-1}g(z))^{(j)}.
\]

Since \( |w(z)| \leq |z| \) (\( z \in \Delta \)), the formula (2.6) yields

\[
\left| (\mathcal{P}_\beta^{\alpha}g(z))^{(j)} \right| \leq \frac{\beta[1 + |z|]}{\beta - |\beta + 2\gamma||z|} \left| (\mathcal{P}_\beta^{\alpha-1}g(z))^{(j)} \right|.
\]

Next since \((\mathcal{P}_\beta^{\alpha}f(z))^{(j)}\) is majorized by \((\mathcal{P}_\beta^{\alpha}g(z))^{(j)}\) in the unit disk \( \Delta^* \), from (1.3), we have

\[
(\mathcal{P}_\beta^{\alpha}f(z))^{(j)} = \phi(z)(\mathcal{P}_\beta^{\alpha}g(z))^{(j)}.
\]

Differentiating it with respect to \( z \) and multiplying by \( z \), we get

\[
z(\mathcal{P}_\beta^{\alpha}f(z))^{(j+1)} = z\phi'(z)(\mathcal{P}_\beta^{\alpha}g(z))^{(j)} + z\phi(z)(\mathcal{P}_\beta^{\alpha}g(z))^{(j+1)}.
\]

Using (2.7), in the above equality, it yields

\[
(\mathcal{P}_\beta^{\alpha-1}f(z))^{(j)} = \frac{z\phi'(z)}{\beta}(\mathcal{P}_\beta^{\alpha}g(z))^{(j)} + \phi(z)(\mathcal{P}_\beta^{\alpha-1}g(z))^{(j)}.
\]

Thus, nothing that \( \phi \in \mathcal{P} \) satisfies the inequality (see, e.g. Nehari [6])

\[
|\phi'(z)| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2}
\]

and making use of (2.8) and (2.10) in (2.9), we get

\[
\left| (\mathcal{P}_\beta^{\alpha-1}f(z))^{(j)} \right| \leq \left( |\phi(z)| + \frac{1 - |\phi(z)|^2}{1 - |z|} \frac{|z|}{|\beta - |2\gamma + \beta||z||} \right) \left| (\mathcal{P}_\beta^{\alpha-1}g(z))^{(j)} \right|,
\]

which upon setting

\[
|z| = r \quad \text{and} \quad |\phi(z)| = \rho \quad (0 \leq \rho \leq 1),
\]
Corollary 2.2. Let the function $f \in \Sigma$ and suppose that $g \in S^{1,0}_\beta(\gamma)$. If $(J_\beta f(z))^{(j)}$ is majorized by $(J_\beta g(z))^{(j)}$ in $\Delta^*$, then
\begin{equation}
|f(z)|^{(j)} \leq |g(z)|^{(j)} \quad \text{for } |z| \leq r_2(\beta, \gamma),
\end{equation}
where
\begin{equation}
r_2(\beta, \gamma) = \frac{k_2 - \sqrt{k_2^2 - 4\beta(\beta + 2\gamma)}}{2(\beta + 2\gamma)}
\end{equation}
and
\begin{equation}
k_2 = \beta + 2 + |\beta + 2\gamma|, \quad (\beta > 0, \ j \in \mathbb{N}_0, \ \gamma \in \mathbb{C}\setminus\{0\}).
\end{equation}
Further putting $\beta = 1$ and $\gamma = 1 - \eta$, $j = 0$ in Corollary 2.1, we get
Corollary 2.2. Let the function $f \in \Sigma$ and suppose that $g \in S^{1,0}_1(1 - \eta)$. If $(J_1 f(z))$ is majorized by $(J_1 g(z))$ in $\Delta^*$, then
\begin{equation}
|f(z)| \leq |g(z)| \quad \text{for } |z| \leq r_3,
\end{equation}
where
\begin{equation}
r_3 = \frac{3 - \eta - \sqrt{\eta^2 - 4\eta + 6}}{3 - \eta}.
\end{equation}
For $\eta = 0$, the above corollary reduces to the following result:
Corollary 2.3. Let the function $f(z) \in \Sigma$ and suppose that $g \in S^{1,0}_1(1) := S^{1,0}_1$. If $(J_1 f(z))$ is majorized by $(J_1 g(z))$ in $\Delta^*$, then
\begin{equation}
|f(z)| \leq |g(z)| \quad \text{for } |z| \leq \frac{3 - \sqrt{6}}{3}.
\end{equation}
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References