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Multiple-interaction kinetic modeling of a virtual-item gambling economy

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In recent years, there has been a proliferation of online gambling sites, which has made gambling more accessible with a consequent rise in related problems, such as addiction. Hence, the analysis of the gambling behavior at both the individual and the aggregate levels has become the object of several investigations. In this paper, resorting to classical methods of the kinetic theory, we describe the behavior of a multiagent system of gamblers participating in lottery-type games on a virtual-item gambling market. The comparison with previous, often empirical, results highlights the ability of the kinetic approach to explain how the simple microscopic rules of a gambling-type game produce complex collective trends, which might be difficult to interpret precisely by looking only at the available data.

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I. INTRODUCTION

Gambling is usually perceived as a complex multidimensional activity fostered by several different motivations [1]. Due to the rapid technological developments, in the past decade the possibility of online gambling has enormously increased [2], leading to the simultaneous rise of related behavioral problems. As remarked in [3], structural characteristics of online gambling, such as the speed and the availability, have led to the conclusion that online gambling has a high potential risk of addiction.

A nonsecondary aspect of the impressive increase in online gambling sites is related to economic interests. Indeed, the expansion of the video-gaming industry has resulted in the formation of a new market, in which gamblers are the actors, that has reached the level of billions of dollars. The continuous expansion of this market depends on many well-established reasons, which include its easy accessibility, low entry barriers, and immediate outcome.

As documented in [4], mathematical modeling of these relatively new phenomena has attracted the interest of current research, with the aim of understanding the aggregate behavior of a system of gamblers. In the aforementioned work, the behavior of online gamblers has been studied by methods of statistical physics. In particular, the analysis has been focused on a popular type of virtual-item gambling, the jackpot, i.e., a lottery-type game which occupies a big portion of the gambling market on the web. As pointed out in [4], to be able to model the complex online gambling behavior at both the individual and aggregate levels is quickly becoming a pressing need for adolescent gambling prevention and possibly for virtual gambling regulation.

The gambling data sets used in [4] have been extracted from the publicly available history page of a gambling site. The huge number of gambling rounds, as well as the time period (more than seven months) taken into account, allowed for a consistent fitting. In particular, the collected data highlight that the number of rounds played by each gambler follows a log-normal distribution and, moreover, that the distribution function of the winnings exhibits fat tails.

The huge number of gamblers and the well-defined rules of the game allow us to treat the system of gamblers as a particular multiagent economic system in which the agents invest (risk) part of their personal wealth to obtain a marked improvement of their economic conditions. Unlike classical models of the trading activity [5], in this gambling economy particular attention needs to be paid to the behavioral reasons pushing people to gamble even in the presence of high risks. By looking at the jackpot game from this perspective and resorting to the classical modeling of multiagent systems via kinetic equations of Boltzmann and Fokker-Planck type [5], we will be able to obtain a detailed interpretation of the data sets collected in [4].

Our forthcoming analysis will be split in two parts. In the first part we will discuss the kinetic modeling of gambling and we will study in particular the distribution in time of the tickets played and won by the gamblers. Our modeling approach is largely inspired by the similarities of the jackpot game with the so-called winner take all game described in detail in [5] and furthermore by the results obtained in [6,7] about generalized Maxwellian kinetic equations with multiple microscopic interactions. Nevertheless, the high number of gamblers taking part in the game, the presence of a percentage cut on the winnings operated by the site, and the continuous refilling of tickets to play introduce essential differences. In

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the second part we will deal with the behavioral aspects linked to the online gambling. This is a phenomenon that may be fruitfully described by resorting to a skewed distribution and consequently may be modeled along the lines of recent papers [8,9]. The behavioral aspects of the gambling and their relationships with other economically relevant phenomena have been discussed in a number of papers (cf., e.g., [10] and the references therein). Also, the emergence of the skewed log-normal distribution was noticed before. The novelty of the present approach is that we enlighten the principal behavioral aspects at the basis of a reasonable kinetic description.

Returning to the kinetic description of the jackpot game, it is interesting to remark that some related problems have been studied before. The presence of the site cut, which can be regarded as a sort of dissipation, suggests that the time evolution of the distribution function of the tickets played and won by the gamblers may be described in a way similar to other well-known dissipative kinetic models, such as that of the Maxwell-type granular gas studied by Ernst and Brito [11] or that of the Pareto tail formation in self-similar solutions of an economy undergoing recession [12]. However, essential differences remain. Unlike the situations described in [11,12], where the loss of the energy or of the mean value, respectively, was artificially restored by a suitable scaling of the variables, in the present case the percentage cut on each wager, leading to an exponential loss of the mean value of the winnings, is refilled randomly because of the persistent activity of the gamblers even in the presence of losses. A second difference concerns the necessity to take into account a high number of participants in the jackpot game. In [4] it was conjectured that the shape of the steady-state distribution emerging from the game rules does not change as the number of participants increases. Consequently, all models studied there were limited to the description of the evolution of winnings in a game with a very small number of gamblers. Here we adopt instead a different strategy, inspired by the model introduced by Bobylev and Windfall [7]. It is worth mentioning that multiple interaction models of multiagent systems have also been introduced, in the economic context, to describe the trading behavior of a group of individuals playing the so-called minority game [13,14]. We will duly compare the results of our analysis with this setting in the Conclusion of the paper.

From the detailed kinetic description of the online jackpot game, and unlike the analysis proposed in [4], we conclude that the game mechanism does not actually give rise to a power-law-type steady distribution of the tickets played and won by the gamblers. The formation of such a fat tail may, however, be obtained by resorting to a different linearization of the game, which, while apparently close to the actual nonlinear version, may be shown numerically to produce quite different trends.

In more detail, the paper is organized as follows. In Sec. II we introduce the microscopic model of the jackpot game with N gamblers and its nonlinear Boltzmann-type kinetic description with multiple interactions (Sec. II A). Next, in the limit of N large, we replace the N-interaction dynamics with a sort of mean-field individual interaction, which gives rise to a linear Boltzmann-type model (Sec. II B). We study the large-time trend of the linear model by means of a Fokker-Planck asymptotic analysis, which shows that no fat tails are produced at equilibrium (Sec. II C). Finally, by resorting to a different linearization of the multiple-interaction model based on the preservation of the first two statistical moments of the distribution function, we produce an alternative kinetic model whose equilibrium distribution has indeed a power-law-type fat tail (Sec. II D). Nevertheless, we argue that such a linear model does not provide a description of the gambling dynamics completely equivalent to the original multiple-interaction model and hence it does not describe exactly the original jackpot game. In Sec. III we discuss a model for the distribution of the tickets which the gamblers purchase to participate in successive rounds of the jackpot game. This study complements previous work on the gambling dynamics, as it provides the basis to model the refilling of tickets mentioned above. In Sec. IV we illustrate the evolution of the real game predicted by the multiple-interaction kinetic model and that of the various linearized models by means of several numerical experiments, which confirm the theoretical findings of the previous sections. Finally, in Sec. V we summarize the main results of the work.

II. KINETIC MODELS OF JACKPOT GAMES

A. Maxwell-type models

The jackpot game we are going to study is very simple to describe. At given intervals of time, which may last from a few seconds to several minutes, the site opens a new round of the game that the gamblers may attend. The gamblers participate in the game by placing a bet with a certain number of lottery tickets purchased with one or several skins deposited to the gambling site. There is only one winning ticket in each round of the game. The winning ticket is drawn when the total number of skins deposited as wagers in that round exceeds a certain threshold. The draw is based on a uniformly distributed random number with a range equal to the total number of tickets purchased in that round. The gambler who holds the winning ticket wins all the wagers, i.e., the deposited skins in that round, after a site cut (percentage cut) has been subtracted.

As usual in the kinetic description, we assume that the gamblers are indistinguishable [5]. This means that, at any time $t \geqslant 0$, the state of a gambler is completely characterized by their wealth, expressed by the number $x \geqslant 0$ of owned tickets. Consequently, the microscopic state of the gamblers is fully characterized by the density, or distribution function, $f = f(x, t)$.

The precise meaning of the density f is the following. Given a subdomain $D \subseteq \mathbb{R}_+$, the integral

$$\int_D f(x,t)dx$$

represents the number of individuals possessing a number $x \in D$ of tickets at time $t \geqslant 0$. We assume that the density function is normalized to one, i.e.,

$$\int_{\mathbb{R}_+} f(x,t)dx = 1,$$

so that f may be understood as a probability density.

The time evolution of the density f is due to the fact that rounds are programmed at regular time intervals and gamblers continuously upgrade their number of tickets x at each new round. In analogy with the classical kinetic theory of rarefied gases, we refer to a single upgrade of the quantity x as an interaction.

The game has evident similarities with the winner take all game described in detail in [5], Chap. 5. The main differences are the presence of a high number of participants and of the site cut. Indeed, while the microscopic interactions in the winner take all game are pointwise conservative, any round of the online jackpot game leads to a loss of the value returned to the gamblers.

Let us consider a number N of gamblers, with $N \gg 1$, who participate in a sequence of rounds. At the initial time, the gamblers (indexed by $k = 1, \dots, N$) buy certain numbers $x_k = x_k(0)$ of tickets, with the intention to play for a while. While it is clear that actually $x_k \in \mathbb{N}_+$, in order to avoid inessential difficulties, and without loss of generality, we will consider $x_k \in \mathbb{R}_+$. Moreover, we may fix a unitary price for the tickets, so as to identify straightforwardly the number of tickets with the amount of money owned by the gamblers. We assume that each gambler participates in a round by using only a small fraction of their tickets, say, $\epsilon\alpha_k x_k$, where $0 < \epsilon \ll 1$ while the α_k's may be either constant or random coefficients. In the simplest case, i.e., $\alpha_k = 1$ for all k, the total number of tickets played by the gamblers in a single round is $\epsilon \sum_{k=1}^N x_k$.

At fixed time intervals of length $\Delta t > 0$, a ticket is chosen randomly. The owner of that ticket wins an amount of money corresponding to the value of the total number of tickets played in that round, minus a certain fixed cut operated by the site. Let us denote by $x_k(t-1)$ the number of tickets possessed by the kth gambler right before the next round. If $\delta > 0$ denotes the percentage cut operated by the site, after the new round the quantities $x_k(t-1)$ update to

$$x_k(t) = (1-\epsilon)x_k(t-1) + \epsilon(1-\delta)\sum_{j=1}^N x_j(t-1)I(A(t-1)-k) \qquad (1)$$

for $k = 1, 2, \dots, N$. In (1), $A(t-1) \in \{1, \dots, N\}$ is a discrete random variable giving the index of the winner in the forthcoming round. Since the winner is chosen by extracting uniformly one of the played tickets, the random variable $A(t-1)$ may be characterized by the law

$$\mathbb{P}(A(t-1) = k) = \frac{x_k(t-1)}{\sum_{j=1}^N x_j(t-1)}, \qquad (2)$$

with $k = 1, 2, \dots, N$. Furthermore, in (1) the function $I(n)$, for $n \in \mathbb{Z}$, is defined by

$$I(0) = 1, \quad I(n) = 0\, \forall\, n \neq 0.$$

Because of the fixed cut operated by the site, the total number of tickets, viz., the amount of money, in the hands of the gamblers diminishes at each round so that, in the long run, the gamblers remain without tickets to play. On the other hand, as noticed in the recent analysis in [4], the data published by the jackpot site certify that this never happens. One may easily identify at least two explanations. First, gamblers with high losses are continuously replaced by new gamblers entering the game. Second, in the presence of repeated losses the gamblers continuously refill the amount of money available to their wagers by drawing on their personal reserves of wealth. Notice that we may easily identify the new gamblers entering the game with those leaving it by simply assuming that the number N of gamblers remains constant in time. Taking this nonsecondary aspect into account, we modify the upgrade rule (1) as

$$x_k(t) = (1-\epsilon)x_k(t-1) + \epsilon\beta Y_k(t-1) + \epsilon(1-\delta)\sum_{j=1}^N x_j(t-1)I(A(t-1)-k), \qquad (3)$$

with $k = 1, \dots, N$. In (3), $\beta \geqslant 0$ is a fixed constant, which identifies the rate of refilling of the tickets. Moreover, the Y_k's are non-negative, independent and identically distributed random variables giving the number of refilled tickets. In agreement with [4], and as explained in full detail in Sec. III, one can reasonably assume that the random variables Y_k are log-normally distributed.

The upgrade rules (2) and (3) lead straightforwardly to a Boltzmann-type kinetic model describing the time evolution of the density $f(x,t)$ of a population of gamblers who play an N-player jackpot game, independently and repeatedly, according to the interaction

$$x_k' = (1-\epsilon)x_k + \epsilon\beta Y_k + \epsilon(1-\delta)\sum_{j=1}^N x_j I(A-k) \qquad (4)$$

for $k = 1, 2, \dots, N$ and where $A \in \{1, \dots, N\}$ is a discrete random variable with the law

$$\mathbb{P}(A = k) = \frac{x_k}{\sum_{j=1}^N x_j}, \quad k = 1, \dots, N.$$

In (4), the quantity x_k represents the number of tickets, hence the amount of money, put into the game by the kth gambler, while the quantity x_k' is the new number of tickets owned by the kth gambler after the draw of the winning ticket.

Starting from the microscopic interaction (4), the study of the time evolution of the distribution function f may be obtained by resorting to kinetic collisionlike models [5]. Specifically, the evolution of any observable quantity φ, i.e., any quantity which may be expressed as a function of the microscopic state x, is given by the Boltzmann-type equation

$$\frac{d}{dt}\int_{\mathbb{R}_+}\varphi(x)f(x,t)dx = \frac{1}{\tau N}\int_{\mathbb{R}_+^N}\sum_{k=1}^N\langle\varphi(x_k') - \varphi(x_k)\rangle\prod_{j=1}^N f(x_j,t)dx_j, \qquad (5)$$

where τ denotes a relaxation time and $\langle\cdot\rangle$ is the average with respect to the distributions of the random variables Y_k and A contained in (4). Note that the interaction term on the right-hand side of (5) takes into account the whole set of gamblers, and consequently it depends on the N-product of the density functions $f(x_j,t)$, $j = 1, \dots, N$. Thus, the evolution of f obeys a highly nonlinear Boltzmann-type equation.

Remark 1. In the classical kinetic theory of rarefied gases, the binary collision integral depends on a nonconstant collision kernel, which selects the collisions according to the relative velocities of the colliding particles. Conversely, the interaction integral in (5) has a constant kernel, chosen equal to 1 without loss of generality. This corresponds, in the jargon of the classical kinetic theory, to consideration of Maxwellian interactions. Remarkably, in the case of the jackpot game, this assumption corresponds perfectly to the description of the game under investigation, since one may realistically assume that the numbers of tickets played by different gamblers are uncorrelated.

Choosing $\varphi(x) = 1$ in (5) yields

$$\frac{d}{dt}\int_{\mathbb{R}_+} f(x,t)dx = 0,$$

meaning that the total mass of the system is conserved in time. It is worth pointing out that, as a matter of fact, this is the only conserved quantity in (5).

In order to better understand the time evolution of f, as well as the role of the site cut, we begin by considering the situation in which the gamblers do not refill their tickets, which corresponds to letting $\beta = 0$. In this case, the interactions (4) being linear in the x_k's, we can compute explicitly the evolution in time of the mean number of tickets

$$m(t) := \int_{\mathbb{R}_+} xf(x,t)dx$$

owned by the gamblers. Indeed, since

$$\left\langle \sum_{k=1}^N x'_k \right\rangle = (1-\epsilon)\sum_{k=1}^N x_k + \epsilon(1-\delta)\sum_{j=1}^N x_j \sum_{k=1}^N \mathbb{P}(A=k)$$
$$= (1-\epsilon\delta)\sum_{k=1}^N x_k, \tag{6}$$

choosing $\varphi(x) = x$ in (5), we obtain

$$\frac{dm}{dt} = -\frac{\epsilon\delta}{\tau}m. \tag{7}$$

As expected, the presence of a percentage cut $\delta > 0$ in the jackpot game leads to an exponential decay to zero of the mean number of tickets at a rate proportional to $\frac{\epsilon\delta}{\tau}$.

As far as higher-order moments of the distribution function f are concerned, analytic results may be obtained at the cost of more complicated computations, due to the nonlinearity of the Boltzmann-type equation (5). This unpleasant fact is evident by computing, e.g., the second-order moment, i.e., the energy of the system, which amounts to choosing $\varphi(x) = x^2$ in (5). In this case, we have

$$\left\langle \sum_{k=1}^N (x'_k)^2 \right\rangle = [(1-\epsilon)^2 + 2\epsilon(1-\epsilon)(1-\delta)]\sum_{k=1}^N x_k^2$$
$$+ \epsilon^2(1-\delta)^2\left(\sum_{k=1}^N x_k\right)^2. \tag{8}$$

Note that the term $(\sum_{k=1}^N x_k)^2$, once integrated against the N product of the distribution functions, produces a dependence on both the second moment and the square of the first moment, whose decay law has been established in (7).

It is now clear that, while giving a precise picture of the evolution of the jackpot game, the highly nonlinear Boltzmann-type equation (5) may essentially be treated only numerically.

B. Linearized model

A considerable simplification occurs in the presence of a large number N of participants in the game. In this situation, at any time $t > 0$ we have

$$\sum_{k=1}^N x_k = N\frac{1}{N}\sum_{k=1}^N x_k \approx Nm(t). \tag{9}$$

In practice, if N is large enough we may approximate the empirical mean number of tickets $\frac{1}{N}\sum_{k=1}^N x_k$ of the gamblers participating in a round of the game with the theoretical mean number of tickets m owned by the entire population of potential gamblers. Hence, still considering for the moment the case $\beta = 0$, the interaction (4) may be restated as

$$x'_k = (1-\epsilon)x_k + \epsilon N(1-\delta)m(t)I(\tilde{A}-k) \tag{10}$$

for $k = 1, 2, \dots, N$ and where $\tilde{A} \in \{1, \dots, N\}$ is the discrete random variable with the (approximate) law

$$\mathbb{P}(\tilde{A}=k) \approx \frac{x_k}{Nm(t)}, \quad k = 1, \dots, N.$$

Remark 2. Owing to the approximation (9), the usual properties $\mathbb{P}(\tilde{A}=k) \leqslant 1$ and $\sum_{k=1}^N \mathbb{P}(\tilde{A}=k) = 1$ may be fulfilled, in general, only in a mild sense, which however becomes tighter and tighter as N grows. We refrain from investigating precisely the proper order of magnitude of N because, as we will see in a moment, we will be mostly interested in the asymptotic regime $N \to \infty$.

Before proceeding further, we observe that in a recent paper [7] the linearization resulting from considering a large number of gamblers was proposed in an economic context. The same type of approximation was also used in [15] to linearize a Boltzmann-type equation describing the exchange of goods according to microeconomic principles.

The main consequence of the interaction rule (10) is that the each postinteraction number of tickets x'_k depends linearly only on the preinteraction number x_k and on the (theoretical) mean number of tickets $m(t)$. Plugging (10) into (5) leads then to a linear Boltzmann-type equation. In particular, the time evolution of the observable quantities $\varphi = \varphi(x)$ is now given by

$$\frac{d}{dt}\int_{\mathbb{R}_+} \varphi(x)f(x,t)dx = \frac{1}{\tau}\int_{\mathbb{R}_+} \langle \varphi(x') - \varphi(x)\rangle f(x,t)dx, \tag{11}$$

where

$$x' = (1-\epsilon)x + \epsilon N(1-\delta)m(t)I(\bar{A}-1) \tag{12}$$

and the random variable $\bar{A} \in \{0, 1\}$ is such that

$$\mathbb{P}(\bar{A}=1) = \frac{x}{Nm(t)}. \tag{13}$$

In practice, since it is no longer necessary to label the single gamblers participating in a round of the jackpot game, we use

$\bar{A}$ simply to decide whether the randomly chosen gambler x wins ($\bar{A}=1$) or not ($\bar{A}=0$) in that round.

Equation (11) allows for a simplified and explicit computation of the statistical moments of the distribution function f. In particular, it gives the right evolution of the first moment like in (7). We remark, however, that the simplified interaction rules (12) and (13) have two main weak points. First, since the mean value $m(t)$ follows the decay given by (7), and thus it is in particular nonconstant in time, the interaction (12) features an explicit dependence on time. Second, if ϵ is fixed independently of N, the number ϵN of tickets played in a single game tends to blow up as N increases. At that point, the kinetic model does not represent the target jackpot game anymore. Therefore, while maintaining the fundamental linear characteristics, which make the model amenable to analytical investigations, it is essential to combine the large number of gamblers in each round with a simultaneously small value of ϵ. Indeed, it is realistic to assume that the product ϵN, which characterizes the percent of the number of tickets played in each game, remains finite for every $N \gg 1$ and $\epsilon \ll 1$. We express this assumption by letting $\epsilon \sim \kappa N^{-1}$, where $\kappa > 0$ is a constant, so that

$$\lim_{N\to\infty} \epsilon N = \kappa. \tag{14}$$

Remark 3. Notice that the rate of decay of the mean value m in the linear model (11), which, as already observed, equals that of the nonlinear model given by (7), is bounded away from zero for any value of ϵ if and only if $\tau \sim \epsilon$. Therefore, in order to maintain the correct decay of the mean value for any value of ϵ and N in the linearized model, we will assume, without loss of generality, that $\tau = \epsilon$.

We are now ready to reinclude in the dynamics also the refilling of money operated by the gamblers drawing on their personal reserves of wealth. Assuming a very large number $N \gg 1$ of gamblers together with (14) and taking also Remark 3 into account, the jackpot game with refilling is well described by the linear kinetic equation

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}_+}&\varphi(x)f(x,t)dx\\ &=\frac{1}{\epsilon}\int_{\mathbb{R}_+^2}\langle\varphi(x')-\varphi(x)\rangle f(x,t)\Phi(y)dx\,dy,\end{aligned} \tag{15}$$

where

$$x' = (1-\epsilon)x + \epsilon\beta Y + \kappa(1-\delta)m(t)I(\bar{A}-1), \tag{16}$$

with $\bar{A} \in \{0,1\}$ and, recalling (13),

$$\mathbb{P}(\bar{A}=1) = \epsilon\frac{x}{\kappa m(t)}.$$

From (15) we define by $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ the probability density function of the random variable Y describing the refilling or money operated by the gamblers. Motivated by the discussion contained in Sec. III, we assume that Φ is a log-normal probability density function. This agrees with the behavior of the gamblers observed in [4] and ensures that the moments of Y are all finite. In particular,

$$M := \int_{\mathbb{R}_+} y\Phi(y)dy < +\infty. \tag{17}$$

Taking $\varphi(x) = x$ in (15), we obtain that the mean number of tickets owned by the gamblers obeys now the equation

$$\frac{dm}{dt} = -\delta m + \beta M,$$

whence

$$m(t) = m_0 e^{-\delta t} + \frac{\beta M}{\delta}(1-e^{-\delta t}), \tag{18}$$

with $m_0 := m(0) \geqslant 0$. Remarkably, m does not depend on ϵ. Moreover, in the presence of refilling, m is uniformly bounded in time from above and from below:

$$\min\left\{m_0, \frac{\beta M}{\delta}\right\} \leqslant m(t) \leqslant \max\left\{m_0, \frac{\beta M}{\delta}\right\}.$$

Note that, for $\beta, M > 0$, the mean number of tickets m no longer decays to zero but tends asymptotically to the value $\frac{\beta M}{\delta}$.

Choosing now $\varphi(x) = e^{-i\xi x}$, where $\xi \in \mathbb{R}$ and i is the imaginary unit, we obtain the Fourier-transformed version of the kinetic equation (15),

$$\partial_t \hat{f}(\xi,t) = \frac{1}{\epsilon}\int_{\mathbb{R}_+^2}\langle e^{-i\xi x'} - e^{-i\xi x}\rangle f(x,t)\Phi(y)dx\,dy, \tag{19}$$

where, as usual, $\hat{f}$ denotes the Fourier transform of the distribution function f:

$$\hat{f}(\xi,t) := \int_{\mathbb{R}_+} f(x,t)e^{-i\xi x}dx.$$

Taking (13) into account, the right-hand side of (19) can be written as the sum of two contributions

$$A_\epsilon(\xi,t) = \frac{1}{\epsilon}\int_{\mathbb{R}_+}(e^{-i\epsilon\beta\xi y}-1)\int_{\mathbb{R}_+}\left[e^{-i\xi[(1-\epsilon)x+\kappa(1-\delta)m(t)]}\frac{\epsilon x}{\kappa m(t)} + e^{-i(1-\epsilon)\xi x}\left(1-\frac{\epsilon x}{\kappa m(t)}\right)\right]f(x,t)dx\,\Phi(y)dy,$$

$$B_\epsilon(\xi,t) = \frac{1}{\epsilon}\int_{\mathbb{R}_+}\left[(e^{-i\xi[(1-\epsilon)x+\kappa(1-\delta)m(t)]} - e^{-i\xi x})\frac{\epsilon x}{\kappa m(t)} + (e^{-i(1-\epsilon)\xi x} - e^{-i\xi x})\left(1-\frac{\epsilon x}{\kappa m(t)}\right)\right]f(x,t)dx.$$

In the limit $\epsilon \to 0^+$, viz., $N \to \infty$, we obtain

$$\lim_{\epsilon\to 0^+} A_\epsilon(\xi,t) = -i\beta M\xi\hat{f}(\xi,t)$$

and

$$\lim_{\epsilon\to 0^+} B_\epsilon(\xi,t) = \left[\frac{i}{\kappa m(t)}(e^{-i\kappa m(t)(1-\delta)\xi}-1)-\xi\right]\partial_\xi\hat{f}(\xi,t),$$

which shows that, for $N \gg 1$ and in the regime (14), the nonlinear kinetic model (5) with the scaling $\tau = \epsilon$ (cf. Remark 3) is well approximated by the Fourier-transformed linear equation

$$\partial_t \hat{f} = \left[\frac{i}{\kappa m(t)}(e^{-i\kappa m(t)(1-\delta)\xi} - 1) - \xi\right]\partial_\xi \hat{f} - i\beta M\xi \hat{f}. \tag{20}$$

This equation may be used to compute recursively the time evolution of the statistical moments of f, upon recalling the relationship

$$m_n(t) := \int_{\mathbb{R}_+} x^n f(x,t)dx = i^n \partial_\xi^n \hat{f}(0,t), \quad n \in \mathbb{N}, \tag{21}$$

and to check their possible blowup indicating the formation of fat tails in f.

C. Explicit steady states and boundedness of moments

To gain further information on (20) in the physical variable x, let us consider at first the case in which the constant κ is small, say, $\kappa \ll 1$. Expanding the exponential function appearing in (20) in a Taylor series up to the second order, we obtain

$$\left[\frac{i}{\kappa m(t)}(e^{-i\kappa m(t)(1-\delta)\xi} - 1) - \xi\right]\partial_\xi \hat{f} \approx \left[-\delta\xi - \frac{i\kappa m(t)}{2}(1-\delta)^2\xi^2\right]\partial_\xi \hat{f}. \tag{22}$$

Within this approximation, we can return from (20) to the physical variable x by the inverse Fourier transform. In particular, we get

$$\partial_t f(x,t) = \frac{\kappa(1-\delta)^2 m(t)}{2}\partial_x^2[xf(x,t)] + \partial_x[(\delta x - \beta M)f(x,t)], \tag{23}$$

which is a Fokker-Planck-type equation with variable diffusion coefficient. Notice that the mean value of the solution to (23) coincides with (18). In particular, if $m_0 = \frac{\beta M}{\delta}$ then the mean value remains constant in time:

$$m(t) \equiv \frac{\beta M}{\delta}\ \forall t > 0.$$

In this simple case, (23) has a stationary solution, say, $f_\infty = f_\infty(x)$, which is easily found by solving the differential equation

$$\frac{\kappa(1-\delta)^2}{2}\frac{\beta M}{\delta}\partial_x(xf_\infty) + (\delta x - \beta M)f_\infty = 0$$

and which turns out to be a Γ probability density function

$$f_\infty(x) = \frac{\left(\frac{2\delta^2}{\kappa(1-\delta)^2\beta M}\right)^{2\delta/\kappa(1-\delta)^2}}{\Gamma\left(\frac{2\delta}{\kappa(1-\delta)^2}\right)} x^{2\delta/\kappa(1-\delta)^2-1}e^{-2\delta^2/\kappa(1-\delta)^2\beta M x}. \tag{24}$$

Since f_∞ has moments bounded of any order, we conclude that no fat tail is produced in this case.

In the general case, i.e., without invoking the approximation (22), we may check that the same qualitative asymptotic trend emerges by resorting to the following argument. Let us define

$$D(\xi,t) := \frac{i}{\kappa m(t)}(e^{-i\kappa m(t)(1-\delta)\xi} - 1) - (1-\delta)\xi$$

so that (20) may be rewritten as

$$\partial_t \hat{f} = D(\xi,t)\partial_\xi \hat{f} - \delta\xi\partial_\xi \hat{f} - i\beta M\xi \hat{f}. \tag{25}$$

The function $D(\xi,t)$ satisfies

$$D(0,t) = \partial_\xi D(0,t) = 0,$$

while, for $n \geqslant 2$,

$$\partial_\xi^n D(0,t) = [i\kappa m(t)]^{n-1}(1-\delta)^n,$$

and further, owing to the Leibniz rule,

$$\partial_\xi^n[D(\xi,t)\partial_\xi \hat{f}(\xi,t)]|_{\xi=0} = \sum_{k=2}^n \binom{n}{k}\partial_\xi^k D(0,t)\partial_\xi^{n-k+1}\hat{f}(0,t). \tag{26}$$

Notice that the highest-order derivative of $\hat{f}$ appearing on the right-hand side of (26) is of order $n-1$. Therefore, taking the nth ξ derivative of (25) and computing in $\xi = 0$, while recalling (21), yields, for $n \geqslant 2$,

$$\frac{dm_n}{dt} = -n\delta m_n + \mathcal{E}(m_1,\dots,m_{n-1}), \tag{27}$$

where $\mathcal{E}$ is a term containing only moments of order equal to at most $n-1$. The exact expression of $\mathcal{E}$ may be obtained from (21)–(26) but, in any case, (27) shows recursively that the statistical moments of f of any order are uniformly bounded in time if they are bounded at the initial time. Therefore, we conclude that fat tails do not form also in the general case described by (20).

Remark 4. The uniform boundedness of all moments of f has been actually proved only for the linearized kinetic model (11) and (12) in the limit regime $\epsilon \to 0^+$, viz., $N \to \infty$. Nevertheless, the result so obtained suggests that also the "real" kinetic model, described by the highly nonlinear Boltzmann-type equation (5), may behave in the same way. This is in contrast to the conclusions drawn in [4], where, resorting to some simplified models, the authors justify the formation of power-law tails in the distribution of the gambler winnings.

D. Are power-law tails correct?

As briefly outlined in Remark 4, the solution to the linearized kinetic model of the jackpot game does not possess fat tails. In order to investigate the possible reasons behind the fat tails apparently observed in [4], in the following we introduce an alternative linear kinetic model of the jackpot game, still derived from the microscopic interaction (4), whose equilibrium density exhibits indeed power-law-type fat tails. This model may be obtained by resorting to a different linearization of (5). Nevertheless, as observed via numerical experiments in Sec. IV, such a linearized equation, while apparently very close to the original nonlinear model, produces a quite different large-time trend compared to the one described by (20).

Let us fix $\beta = 0$ in (4) and let us assume, without loss of generality, that the extracted winner is the gambler $k = 1$.

Then

$$x_1' = (1-\epsilon)x_1 + \epsilon(1-\delta)\sum_{j=1}^{N} x_j,$$

$$x_k' = (1-\epsilon)x_k, \quad k = 2, 3, \dots, N,$$

which implies [cf. also (8)]

$$\sum_{k=1}^{N}(x_k')^2 = (1-\epsilon)^2\sum_{k=1}^{N}x_k^2 + 2\epsilon(1-\epsilon)(1-\delta)x_1\sum_{k=1}^{N}x_k + \epsilon^2(1-\delta)^2\left(\sum_{k=1}^{N}x_k\right)^2.$$

Taking into account the expression (6) of the mean value, we obtain

$$\frac{N\sum_{k=1}^{N}(x_k')^2}{\left(\sum_{k=1}^{N}x_k'\right)^2} = \frac{(1-\epsilon)^2}{(1-\epsilon\delta)^2}\frac{N\sum_{k=1}^{N}x_k^2}{\left(\sum_{k=1}^{N}x_k\right)^2} + N\epsilon^2(1-\delta)^2 + 2N\epsilon(1-\epsilon)(1-\delta)\frac{x_1}{\sum_{k=1}^{N}x_k} \approx \frac{N\sum_{k=1}^{N}x_k^2}{\left(\sum_{k=1}^{N}x_k\right)^2}$$

for $N \gg 1$ large and consequently $\epsilon \ll 1$ small. Indeed,

$$\frac{x_1}{\sum_{k=1}^{N}x_k} = \frac{\frac{1}{N}x_1}{\frac{1}{N}\sum_{k=1}^{N}x_k} \approx \frac{x_1}{Nm(t)} \xrightarrow{N\to\infty} 0.$$

In other words, for a large number N of gamblers and a correspondingly small percentage ϵ of tickets played in a single game, the relationship (14) implies that the quantity

$$\chi := \frac{N\sum_{k=1}^{N}x_k^2}{\left(\sum_{k=1}^{N}x_k\right)^2} \tag{28}$$

may be regarded approximately as a collision invariant of the interaction (4). Since

$$\left(\sum_{k=1}^{N}x_k\right)^2 \leqslant N\sum_{k=1}^{N}x_k^2,$$

it follows that $\chi \geqslant 1$. Note that this result does not depend on the choice of the winner in each round of the jackpot game.

Using (14) and (28) in (8), in this asymptotic approximation we obtain

$$\left\langle\sum_{k=1}^{N}(x_k')^2\right\rangle = [(1-\epsilon)^2 + 2\epsilon(1-\epsilon)(1-\delta)]\sum_{k=1}^{N}x_k^2 + \epsilon(1-\delta)^2\frac{\kappa}{\chi}\sum_{k=1}^{N}x_k^2 = \left[(1-\epsilon\delta)^2 + \epsilon(1-\delta)^2\left(\frac{\kappa}{\chi}-\epsilon\right)\right]\sum_{k=1}^{N}x_k^2, \tag{29}$$

whence, choosing $\varphi(x) = x^2$ in (5),

$$\frac{dm_2}{dt} = \left[\epsilon\left((1-\delta)^2\frac{\kappa}{\chi} - 2\right) - \epsilon^2(1-2\delta)\right]m_2. \tag{30}$$

This equation shows that the ratio κ/χ is of paramount importance to classify the large-time trend of the energy of the distribution f, hence also of f itself. Indeed, the sign of the coefficient

$$c(\kappa, \chi, \epsilon) := \epsilon\left((1-\delta)^2\frac{\kappa}{\chi} - 2\right) - \epsilon^2(1-2\delta)$$

determines if f converges asymptotically in time to a Dirac δ centered in $x = 0$ [when $c(\kappa, \chi, \epsilon) < 0$] or if it spreads over the whole positive real line [when $c(\kappa, \chi, \epsilon) > 0$].

This discussion suggests a consistent way to eliminate the time dependence in the interaction (12) while preserving the main macroscopic properties of the jackpot game, such as the right time evolutions of the mean [cf. (7)] and of the energy [cf. (30)]. Specifically, we proceed as follows. For all observable quantities $\varphi = \varphi(x)$, we consider the linear kinetic model (11) with the linear interaction rule

$$x' = (1-\epsilon\delta)x + \sqrt{\epsilon}x\eta_\epsilon, \tag{31}$$

where $\epsilon > 0$. In (31), η_ϵ is a discrete random variable taking only the two values $-\sqrt{\epsilon}(1-\delta)$ and $M_\epsilon/\sqrt{\epsilon}$ with probabilities

$$\mathbb{P}(\eta_\epsilon = -\sqrt{\epsilon}(1-\delta)) = 1 - p_\epsilon, \quad \mathbb{P}\left(\eta_\epsilon = \frac{M_\epsilon}{\sqrt{\epsilon}}\right) = p_\epsilon,$$

where $p_\epsilon \in [0, 1]$ and $M_\epsilon > 0$ are two constants to be properly fixed.

We interpret the rule (31), together with the prescribed values of η_ϵ, as follows: A gambler, who enters the game with a number of tickets (viz., an amount of money) equal to ϵx, may win a jackpot equal to $(M_\epsilon - \epsilon\delta)x$ with probability p_ϵ. The gambler may also lose the amount ϵx put into the game with probability $1 - p_\epsilon$. In particular, we determine p_ϵ by imposing $\langle\eta_\epsilon\rangle = 0$, which guarantees that (31) reproduces the correct evolution of the mean provided by (12) [indeed, in such a case we have $\langle x'\rangle = (1-\epsilon\delta)x$]. We find then

$$p_\epsilon = \frac{\epsilon(1-\delta)}{M_\epsilon + \epsilon(1-\delta)}.$$

Using this, we discover $\langle\eta_\epsilon^2\rangle = M_\epsilon(1-\delta)$, whence

$$\langle(x')^2\rangle = [(1-\epsilon\delta)^2 + \epsilon(1-\delta)M_\epsilon]x^2. \tag{32}$$

A comparison between formulas (29) and (32) allows us to conclude that the choice

$$M_\epsilon = (1-\delta)\left(\frac{\kappa}{\chi} - \epsilon\right)$$

further implies a time evolution of the energy identical to (30). Notice that the positivity of M_ϵ is guaranteed by choosing $\epsilon \ll 1$ small enough.

After deriving the linearized model for $\beta = 0$, we may reinclude the refilling of tickets or money in the interaction rule

$$x' = (1-\epsilon\delta)x + \epsilon\beta Y + \sqrt{\epsilon}x\eta_\epsilon, \tag{33}$$

where, as stated in Sec. II B, the random variable $Y \in \mathbb{R}_+$ is described by a prescribed log-normal probability density function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$. Within this approximation of the dynamics, the evolution of the distribution function $g = g(x,t)$ of the tickets (viz., the money) played and won by a large number of gamblers participating in the jackpot game is then described by the linear kinetic equation [cf. also (15)]

$$\frac{d}{dt}\int_{\mathbb{R}_+}\varphi(x)g(x,t)dx = \frac{1}{\tau}\int_{\mathbb{R}_+^2}\langle\varphi(x')-\varphi(x)\rangle g(x,t)\Phi(y)dx\,dy, \tag{34}$$

with x' given by (33).

1. Fokker-Planck description of the jackpot game

The linear kinetic equation (34) describes the evolution of the distribution function due to interactions of the type (33). As discussed in Sec. II B, for large values of the number N of gamblers participating in a round, and therefore, in view of (14), a small value of ϵ, the interaction (33) produces a small variation in the number of tickets owned by a gambler. We say then that, in such a regime, the interaction (33) is quasi-invariant or grazing. Consequently, a finite (i.e., non-infinitesimal) evolution of the distribution function g may be observed only if each gambler participates in a huge number of interactions (33) during a fixed period of time. This is achieved by means of the scaling $\tau \sim \epsilon$ like in Sec. II B (cf. Remark 3).

In this scaling, the kinetic model (34) is shown to approach its continuous counterpart given by a Fokker-Planck-type equation [5,16–19]. In the present case, (34) is well approximated by the weak form of a different linear Fokker-Planck equation with variable coefficients

$$\frac{d}{dt}\int_{\mathbb{R}_+}\varphi(x)g(x,t)dx = \int_{\mathbb{R}_+}\left(-\varphi'(x)(\delta x-\beta M)+\frac{\tilde{\sigma}}{2}\varphi''(x)x^2\right)g(x,t)dx, \tag{35}$$

where M is the mean refilling of tickets [cf. (17)] and where we have defined

$$\tilde{\sigma} := \lim_{\epsilon\to 0^+} M_\epsilon = (1-\delta)\frac{\kappa}{\chi}.$$

Then, provided the boundary terms produced by the integration by parts vanish, (35) may be recast in a strong form as

$$\partial_t g(x,t) = \frac{\tilde{\sigma}}{2}\partial_x^2[x^2 g(x,t)] + \partial_x[(\delta x-\beta M)g(x,t)]. \tag{36}$$

This equation describes the evolution of the distribution function g of the number of tickets $x \in \mathbb{R}_+$ played by the gamblers at time $t > 0$ in the limit of the grazing interactions. The advantage of this equation over (34) is that its unique steady state g_∞ with unitary mass may be explicitly computed:

$$g_\infty(x) = \frac{\left(\frac{2\beta M}{\tilde{\sigma}}\right)^{1+2\delta/\tilde{\sigma}}}{\Gamma\left(1+\frac{2\delta}{\tilde{\sigma}}\right)}\frac{e^{-2\beta M/\tilde{\sigma}x}}{x^{2+2\delta/\tilde{\sigma}}}. \tag{37}$$

We observe that this is an inverse Γ probability density function with parameters linked to the details of the microscopic interaction (33).

Remark 5. A comparison between (23) and the Fokker-Planck equation (36) shows that, while the drift term is the same, the coefficient of the diffusion term is proportional to x in (23) and to x^2 in (36). This difference determines, in the latter case, the formation of fat tails, which is consistent with the claim made in [4]. Nevertheless, as briefly explained before, the approach based on the interaction (33) leading to (36) in the quasi-invariant regime does *not* actually describe exactly the jackpot game. Indeed, it admits that all gamblers may win simultaneously, although with a very small probability.

2. The case $\beta = 0$

Further explicit computations on the Fokker-Planck equation (36) may be done in the case $\beta = 0$, which corresponds to the situation in which gamblers enter the game with a certain number of tickets, viz., amount of money, and use only those tickets, viz., money, to play. Then the distribution function $g = g(x,t)$ solves the equation

$$\partial_t g(x,t) = \frac{\tilde{\sigma}}{2}\partial_x^2[x^2 g(x,t)] + \delta\partial_x[xg(x,t)]. \tag{38}$$

Setting

$$\tilde{g}(x,t) := e^{-\delta t}g(e^{-\delta t}x,t),$$

which is easily checked to be in turn a distribution function with unitary mass at each time $t > 0$, we see that $\tilde{g}$ solves the diffusion equation

$$\partial_t\tilde{g}(x,t) = \frac{\tilde{\sigma}}{2}\partial_x^2[x^2\tilde{g}(x,t)], \tag{39}$$

with the same initial datum as that prescribed to (38), because $\tilde{g}(x,0) = g(x,0)$.

The unique solution to (39) corresponding to an initial datum $g_0(x)$ is given by the expression

$$\tilde{g}(x,t) = \int_{\mathbb{R}_+}\frac{1}{z}g_0\left(\frac{x}{z}\right)L_t(z)dz, \tag{40}$$

where

$$L_t(x) := \frac{1}{\sqrt{2\pi\tilde{\sigma}tx}}\exp\left(-\frac{\left(\log x+\frac{\tilde{\sigma}}{2}t\right)^2}{2\tilde{\sigma}t}\right)$$

is a log-normal probability density. Indeed, (39) possesses a unique source-type solution given by a log-normal density with unit mean, which at time $t = 0$ coincides with a Dirac δ centered in $x = 1$ (cf. [20]).

Both the mass and the mean of (40) are conserved in time, while initially bounded moments of order $n \geqslant 2$ grow exponentially at rate $n(n-1)$. Moreover, (40) can be shown to converge in time to $L_t(x)$ in various norms (see [20]).

Starting from (40), we easily obtain that the unique solution to the original Fokker-Planck equation (38) is given by

$$g(x,t) = \int_{\mathbb{R}_+}\frac{1}{z}g_0\left(\frac{x}{z}\right)\tilde{L}_t(z)dz, \tag{41}$$

where

$$\tilde{L}_t(x) = \frac{1}{\sqrt{2\pi\tilde{\sigma}tx}} \exp\left(-\frac{\left[\log x + \left(\delta + \frac{\tilde{\sigma}}{2}\right)t\right]^2}{2\tilde{\sigma}t}\right). \quad (42)$$

Note that, as expected, the mean value of the log-normal density (42) decays exponentially in time:

$$\int_{\mathbb{R}_+} x\tilde{L}_t(x)dx = e^{-\delta t}.$$

Consequently, if $X_t \sim g(x,t)$ is a stochastic process with probability density equal to the solution of (38), the mean of X_t decays exponentially to zero at the same rate and

$$\langle X_t \rangle = \int_{\mathbb{R}_+} xg(x,t)dx = e^{-\delta t}\langle X_0 \rangle.$$

Taking advantage of the representation formula (41), we can easily compute also higher-order moments of the solution. In particular, the variance of X_t is equal to

$$\langle X_t^2 \rangle - \langle X_t \rangle^2 = \langle X_0^2 \rangle e^{(\tilde{\sigma}-2\delta)t} - \langle X_0 \rangle^2 e^{-2\delta t}.$$

From here we see that the large-time trend of the variance depends on the sign of the quantity $\tilde{\sigma} - 2\delta$. If $\tilde{\sigma} < 2\delta$, the variance converges exponentially to zero, and thus all gamblers tend, in the long run, to lose all their tickets (viz., money). Conversely, if $\tilde{\sigma} > 2\delta$, the variance blows up for large times. This situation is analogous to the winner take all behavior [5], where the asymptotic steady state is a Dirac δ centered in zero but at any finite time a small decreasing number of gamblers possesses a huge number of tickets, sufficient to sustain the growth of the variance.

III. AGENT BEHAVIOR ON GAMBLING

A nonsecondary aspect of the online gambling is related to the behavioral trends of the gamblers. The data analysis in [4] focuses in particular on two characteristics of the gambling activity: first, the waiting time, defined as the time, measured in seconds, between successive bets by the same gambler, and second, the number of rounds played by individual gamblers. The study of this second aspect may shed light on the reasons behind a high gambling frequency and therefore also on possible addiction problems caused by gambling.

The fitting of the number of rounds played by individual gamblers during the period covered by gambling logs allowed the authors of [4] to conclude that the number of rounds is well described by a log-normal distribution. This result is in agreement with other studies (cf., e.g., [10] and references therein) where the mean gambling frequency is put in close relation with the alcohol consumption. Starting from the pioneering contribution [21], it has long been acknowledged that there exists a positive correlation between the level of alcohol consumption in a population and the proportion of heavy drinkers in the society. This relationship is known under several names, such as the total consumption model or the single distribution theory. Previous research has also found that its validity is not limited to the alcohol consumption but extends to different human phenomena.

In some recent papers [8,9] we introduced a kinetic description of a number of human behavioral phenomena, which recently has been applied also to the study of alcohol consumption [22]. The modeling assumptions in [22] allowed us to classify the alcohol consumption distribution as a generalized Γ probability density, which includes the log-normal distribution as a particular case. Recalling that, as discussed above, alcohol consumption shows many similarities to the gambling activity and taking inspiration from [9,22], we may explain exhaustively two main phenomena linked to the gambler behavior: on the one hand, the distribution of the number of tickets which individual gamblers play (including the refilling) in a single round of the jackpot game, and on the other hand, the distribution of the number of rounds played by individual gamblers in time.

A. Kinetic modeling and value functions

The evolution of the number density of tickets which the gamblers purchase to participate in successive rounds of the jackpot game may still be treated by resorting to the principles of statistical mechanics. Specifically, one can think of the population of gamblers as a multiagent system: Each gambler undergoes a sequence of microscopic interactions, through which the individual updates the personal number of tickets. In order to keep the connection with the classical kinetic theory of rarefied gases, these interactions obey suitable and universal rules, which, in the absence of well-defined physical laws, are designed so as to take into account at best some of the psychological aspects related to gambling.

Due to the nature of the game, the players know that there is a high probability to lose and a small one to win. For this reason, they are usually prepared to participate in a sequence of rounds, hoping to win in at least one of them. The involvement in the game pushes the gamblers to participate in successive rounds by purchasing an increasing number of tickets, so as to increase the probability to win. On the other hand, the attempt to safeguard the personal wealth suggests that they fix an *a priori* upper bound to the number of tickets purchased. These two aspects, clearly in conflict, are characteristic of a typical human behavior, which has been recently modeled in similar situations [8,9,22]. There the microscopic interactions have been built by taking inspiration from the pioneering analysis by Kahneman and Twersky [23] about decisional processes under risk.

In the present case, the aforementioned safeguarding tendency may be modeled by assuming that the gamblers have in mind an ideal number $\bar{w} > 0$ of tickets to buy in each round and simultaneously a threshold $\bar{w}_L > \bar{w}$ which they had better not exceed in order to avoid a (highly probable) excessive loss of money. Hence, the natural tendency of the gamblers to increase their number of tickets $w > 0$ bought for the forthcoming rounds has to be coupled with the limit value $\bar{w}_L$, which it would be wise not to exceed. Following [8,9], we may realize a gambler update via the following rule:

$$w' = w - \Psi\left(\frac{w}{\bar{w}_L}\right)w + w\eta. \quad (43)$$

In (43), w and w' denote the numbers of tickets played in the last round and in the forthcoming one, respectively. The function Ψ plays the role of the so-called value function in the prospect theory by Kahneman and Twersky [23]. Specifically,

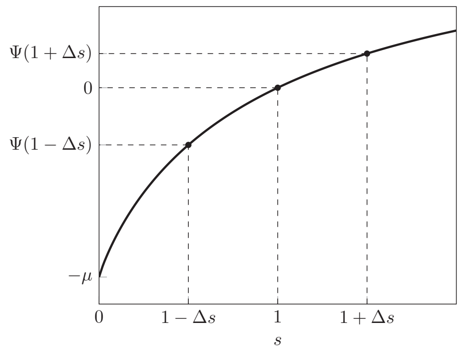

FIG. 1. Function Ψ given in (44).

it determines the update of the number of tickets in a skewed way, so as to reproduce the behavioral aspects discussed above. Analogously to [8], we let

$$\Psi(s) := \mu\frac{s^\alpha - 1}{s^\alpha + 1}, \quad s \geqslant 0, \tag{44}$$

where $\mu, \alpha \in (0, 1)$ are suitable constants characterizing the agent behavior. In particular, μ denotes the maximum variation in the number of tickets allowed in a single interaction (43), indeed

$$|\Psi(s)| \leqslant \mu \, \forall \, s \geqslant 0. \tag{45}$$

Hence, a small value of μ describes gamblers who buy a regular number of tickets in each round.

The function Ψ given in (44) maintains most of the physical properties required for the value function in the prospect theory [23] and is particularly suited to the present situation. In the microscopic interaction (43), the minus sign in front of Ψ is related to the fact that the desire to increase the probability to win pushes a gambler to increase the number w of purchased tickets when $w < \bar{w}_L$. At the same time, the tendency to safeguard the personal wealth induces the gambler to reduce the number of purchased tickets when $w > \bar{w}_L$. Moreover, the function Ψ is such that

$$-\Psi(1 - \Delta s) > \Psi(1 + \Delta s) \, \forall \, \Delta s \in (0, 1)$$

(cf. Fig. 1). This inequality means that if two gamblers are at the same distance from the limit value $\bar{w}_L$ from below and from above, respectively, the gambler starting from below will move closer to the optimal value $\bar{w}_L$ than the gambler starting from above. In other words, it is typically easier for gamblers to allow themselves to buy more tickets when the optimal threshold has not been exceeded than to limit themselves when the optimal threshold has already been exceeded.

Finally, in order to take into account a certain amount of human unpredictability in buying tickets in a new round, it is reasonable to assume that the new number of tickets may be affected by random fluctuations, expressed by the term $w\eta$ in (43). Specifically, η is a centered random variable

$$\langle \eta \rangle = 0, \quad \langle \eta^2 \rangle = \lambda > 0,$$

meaning that the random fluctuations are negligible on average. Moreover, to be consistent with the necessary nonnegativity of w', we assume that $\eta > -1 + \mu$, i.e., that the support of η is bounded from the left.

Remark 6. The behavior modeled by (43), which in principle concerns only the losers, may actually be applied also to the unique winner. Indeed, if the winner remains into the game, the pleasure to play will be dominant, so it is reasonable to imagine that the future behavior will not depend too much on the number of tickets gained in the last round.

Remark 7. The discussion set forth applies also to the modeling of the number of rounds played by individual gamblers in a fixed period of time, which has been considered in [4]. In particular, we may assume that the gamblers establish *a priori* to play for a limited number of times, in order to spend only a certain total amount of money; however then, as it happens in the single game, it is more difficult to stop than to continue. This can be well described by the rule (43) and by the value function (44), where now w represents the number of rounds played in the time period.

Let now $h = h(w, t)$ be the distribution function of the number of tickets purchased by a gambler in a certain round of the jackpot game. As anticipated at the beginning of this section, its time evolution may be obtained by resorting to kinetic collisionlike models [5] based on (43). In particular, since the interaction (43) depends only on the behavior of a single gambler, h obeys a linear Boltzmann-type equation of the form

$$\frac{d}{dt}\int_{\mathbb{R}_+} \varphi(w)h(w,t)dw = \frac{1}{\tau}\int_{\mathbb{R}_+} \langle \varphi(w') - \varphi(w)\rangle h(w,t)dw \tag{46}$$

[cf. (11)], where the constant $\tau > 0$ measures the interaction frequency and φ is any observable quantity.

Since the elementary interaction (43) is nonlinear with respect to w, the only conserved quantity in (46) is obtained from $\varphi(w) = 1$,

$$\frac{d}{dt}\int_{\mathbb{R}_+} h(w,t)dw = 0,$$

which implies that the solution to (46) remains a probability density at all times $t > 0$ if it is so at the initial time $t = 0$. The evolution of higher-order moments is difficult to compute explicitly. As a representative example let us take $\varphi(w) = w$, which provides the evolution of the mean number of tickets purchased by the gamblers over time:

$$m(t) := \int_{\mathbb{R}_+} wh(w,t)dw.$$

Since

$$\langle w' - w \rangle = \mu\frac{w^\alpha - \bar{w}_L^\alpha}{w^\alpha + \bar{w}_L^\alpha} w,$$

we obtain

$$\frac{dm}{dt} = \frac{\mu}{\tau}\int_{\mathbb{R}_+} \frac{w^\alpha - \bar{w}_L^\alpha}{w^\alpha + \bar{w}_L^\alpha} wh(w,t)dw. \tag{47}$$

This equation is not explicitly solvable. However, in view of (45), m remains bounded at any time $t > 0$ provided it is so initially, with the explicit upper bound (cf. [9])

$$m(t) \leqslant m_0 e^{(\mu/\tau)t},$$

where $m_0 := m(0)$. From (47) it is however not possible to deduce whether the time variation of m is or is not monotone.

Taking now $\varphi(w) = w^2$ in (46) and considering that

$$\langle (w')^2 - w^2 \rangle = \left[\Psi^2\left(\frac{w}{\bar{w}_L}\right) - 2\Psi\left(\frac{w}{\bar{w}_L}\right) + \lambda \right] w^2 \leqslant (3\mu + \lambda) w^2$$

because of (45) together with $0 < \mu < 1$, we see that the boundedness of the energy at the initial time implies that of the energy at any subsequent time $t > 0$, with the explicit upper bound

$$m_2(t) \leqslant m_{2,0} e^{/[(3\mu+\lambda)\tau]t},$$

where $m_{2,0} := m_2(0)$.

B. Fokker-Planck description and equilibria

The linear kinetic equation (46) is valid for every choice of the parameters α, μ, and λ, which characterize the microscopic interaction (43). In real situations, however, a single interaction, namely, a participation in a new round of the jackpot game, does not induce a marked change in the value of w. This situation is close to that discussed in Sec. II D 1, where we called these interactions grazing collisions [5,19].

Similarly to Sec. II D 1, we may easily take such a smallness into account by scaling the microscopic parameters in (43) and (46) as

$$\alpha \to \epsilon\alpha, \quad \lambda \to \epsilon\lambda, \quad \tau = \epsilon, \tag{48}$$

where $0 < \epsilon \ll 1$. A thorough discussion of these scaling assumptions may be found in [8,17]. In particular, here we mention that the rationale behind the coupled scaling of the parameters α and λ and of the frequency of the interactions τ is the following: Since the scaled interactions are grazing, and consequently produce a very small change in w, a finite (i.e., noninfinitesimal) variation of the distribution function g may be observed only if each gambler participates in a very large number of interactions within a fixed period of time.

As already observed in Sec. II D 1, when grazing interactions dominate, the kinetic model (46) is well approximated by a Fokker-Planck type equation [5,19]. Exhaustive details on such an approximation in the kinetic theory of socioeconomic systems may be found in [17]. In short, the mathematical idea is the following: If φ is sufficiently smooth and $w' \approx w$ because interactions are grazing, one may expand $\varphi(w')$ in a Taylor series about w. Plugging such an expansion into (46) with the value function (44) and taking the scaling (48) into account, one obtains

$$\frac{d}{dt}\int_{\mathbb{R}_+} \varphi(w)h(w,t)dw = \int_{\mathbb{R}_+} \left(-\frac{\alpha\mu}{2}\varphi'(w)w\log\frac{w}{\bar{w}_L} + \frac{\lambda}{2}\varphi''(w)w^2 \right) \times h(w,t)dw + \frac{1}{\epsilon}\mathcal{R}_\epsilon(w,t),$$

where $\mathcal{R}_\epsilon$ is a remainder such that $\frac{1}{\epsilon}\mathcal{R}_\epsilon \to 0$ as $\epsilon \to 0^+$ (cf. [17]). Therefore, under the scaling (48), the kinetic equation (46) is well approximated by the equation

$$\frac{d}{dt}\int_{\mathbb{R}_+} \varphi(w)h(w,t)dw = \int_{\mathbb{R}_+} \left(-\frac{\alpha\mu}{2}\varphi'(w)w\log\frac{w}{\bar{w}_L} + \frac{\lambda}{2}\varphi''(w)w^2 \right) h(w,t)dw.$$

This equation may be recognized as the weak form of the Fokker-Planck equation with variable coefficients

$$\partial_t h(w,t) = \frac{\lambda}{2}\partial_w^2[w^2 h(w,t)] + \frac{\alpha\mu}{2}\partial_w\left(w\log\frac{w}{\bar{w}_L} h(w,t) \right), \tag{49}$$

upon assuming that the boundary terms produced by the integration by parts vanish. Like in Sec. II D 1, the Fokker-Planck description (49) is advantageous over the original Boltzmann-type equation (46) because it allows for an explicit computation of the steady-state distribution function, say, $h_\infty = h_\infty(w)$. The latter solves the first-order ordinary differential equation

$$\frac{\lambda}{2}\frac{d}{dw}[w^2 h_\infty(w)] + \frac{\alpha\mu}{2} w\log\frac{w}{\bar{w}_L} h_\infty(w) = 0,$$

whose unique solution with unitary mass is

$$h_\infty(w) = \frac{1}{\sqrt{2\pi\sigma}\,w}\exp\left(-\frac{(\log w - \theta)^2}{2\sigma} \right), \tag{50}$$

where

$$\sigma := \frac{\lambda}{\alpha\mu}, \quad \theta := \log\bar{w}_L - \sigma.$$

Therefore, in very good agreement with the observations made in [4], the equilibrium distribution function predicted by the microscopic rule (43) with the value function (44) in the grazing interaction regime is a log-normal probability density, whose mean and variance are easily computed from the known formulas for log-normal distributions:

$$m_\infty := \bar{w}_L e^{-\sigma/2}, \quad \mathrm{Var}(g_\infty) := \bar{w}_L^2(1 - e^{-\sigma}).$$

In particular, these quantities are fractions of $\bar{w}_L$ and $\bar{w}_L^2$, respectively, depending only on the ratio $\sigma = \frac{\lambda}{\alpha\mu}$ between the variance λ of the random fluctuation η and the portion $\alpha\mu$ of the maximum rate μ of variation in the number of tickets purchased by a gambler in a single round. If

$$\sigma > 2\log\frac{\bar{w}_L}{\bar{w}},$$

then the asymptotic mean m_∞ is lower than the fixed ideal number $\bar{w}$ of tickets to be purchased in each round. This identifies a population of gamblers capable of not being too deeply involved in the jackpot game.

Figure 2 shows that the asymptotic profile (50) describes excellently the large-time distribution of the Boltzmann-type equation (46) in the quasi-invariant regime, i.e., ϵ small in

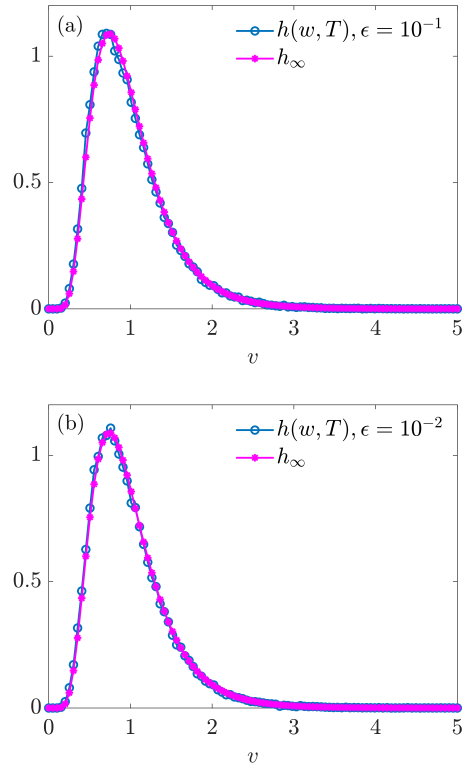

FIG. 2. Comparison of (50) with the numerically computed large-time solution (at the computational time $T = 10$) to the Boltzmann-type equation (46). The value function is (44) with $\mu = 0.5$ and $\alpha = 1$. Moreover, the binary interaction is (43) with $\lambda = \frac{1}{10}$ and $w_L = e^{\lambda/2\mu}$. We considered the quasi-invariant scaling (48) with (a) $\epsilon = 10^{-1}$ and (b) $\epsilon = 10^{-2}$.

(48). The solution to (46) has been obtained numerically via a standard Monte Carlo method.

IV. NUMERICAL TESTS

In this section we provide numerical insights into the various models discussed before, resorting to direct Monte Carlo methods for collisional kinetic equations and to the recent structure preserving methods for Fokker-Planck equations. For a comprehensive presentation of these numerical methods, the interested reader is referred to [5,24–26].

We begin by integrating the multiple-interaction Boltzmann-type model (5) so as to assess its equivalence with the linearized model (15) in the case $N \gg 1$ with $\epsilon N = \kappa > 0$, as predicted theoretically in Sec. II B. Next we also test numerically the consistency of the Fokker-Planck equation (23) with the linearized Boltzmann-type equation (15). Subsequently, we investigate the kinetic model with fat tails discussed in Sec. II D 1. In particular, we evaluate numerically some discrepancies that it presents with the other models.

A. Test 1: Multiple-interaction Boltzmann-type model and its linearized version

The multiple-interaction Boltzmann-type equation (5) can be fruitfully written in strong form, to put in evidence the gain and loss parts of the integral operator

$$\begin{aligned}\partial_t f(x,t) &= \frac{1}{\epsilon}\left\langle \int_{\mathbb{R}^{N-1}} \left(\frac{1}{J} \prod_{k=1}^{N} f({}'x_k, t) - \prod_{k=1}^{N} f(x_k,t) \right) dx_2 \cdots dx_N \right\rangle \\ &= \frac{1}{\epsilon} Q^+(f,\dots,f)(x,t) - \frac{1}{\epsilon} f(x,t),\end{aligned} \tag{51}$$

where Q^+ is the gain operator

$$Q^+(f,\dots,f)(x,\,t) := \frac{1}{\epsilon}\left\langle \int_{\mathbb{R}^{N-1}} \prod_{k=1}^{N} \frac{1}{J} f({}'x_k,\,t)\,dx_2\dots dx_N \right\rangle$$

and J is the Jacobian of the transformation (4) from the preinteraction variables $\{{}'x_k\}_{k=1}^N$ to the postinteraction variables $\{x_k\}_{k=1}^N$. We discretize (51) in time through a forward scheme on the mesh $t^n := n\Delta t$, $\Delta t > 0$. With the notation $f^n(x) := f(x, t^n)$, we obtain the following semidiscrete formulation:

$$f^{n+1}(x) = \left(1 - \frac{\Delta t}{\epsilon}\right) f^n(x) + \frac{\Delta t}{\epsilon} Q^+(f^n,\dots,f^n)(x).$$

By choosing $\Delta t = \epsilon$, the loss part disappears and at each time step only the gain operator Q^+ needs to be computed.

We recall that the multiple-interaction microscopic dynamics are given by (4). In particular, motivated by the results of Sec. III, we choose the Y_k's as independent and identically distributed random variables with log-normal probability density:

$$\Phi(y) = \frac{1}{\sqrt{4\pi}y} \exp\left(-\frac{(\log y + 1)^2}{2}\right). \tag{52}$$

A comparison with (50) shows that this corresponds to $\sigma = 2$ and $\bar{w}_L = e$, so $M = \langle Y_k \rangle = 1$ for all k.

In parallel, we consider the linearized Boltzmann-type equation (15), which we have shown to be formally equivalent to the multiple-interaction model for a large number of gamblers N. The semidiscrete-in-time formulation of the linearized model reads

$$f^{n+1}(x) = \left(1 - \frac{\Delta t}{\epsilon}\right) f^n(x) + \frac{\Delta t}{\epsilon}\left\langle \int_{\mathbb{R}_+} \frac{1}{J} f^n({}'x)\,dx \right\rangle,$$

where now the microscopic dynamics are given by (16) with $\kappa = \epsilon N$ and $Y \sim \Phi(y)$ like before [cf. (52)].

In both cases, we solve the interaction dynamics by a Monte Carlo scheme, considering a random sample of 10^6 particles with initial uniform distribution in the interval [0,2], thus $f_0(x) := f(x,0) = \frac{1}{2}\mathbb{1}_{[0,2]}(x)$, where $\mathbb{1}$ denotes the characteristic function.

In Fig. 3 we compare the evolutions of the two models in the time interval $t \in [0, 2]$ for $\delta = 0.2$ and $\beta = 0$ in (4) and (16) [cf. also (10)], i.e., in particular, with no refilling. In Fig. 4 we perform the same test in the larger-time interval $t \in [0, 25]$ for $\delta = \beta = 0.2$, i.e., by including also the refilling. In both cases, we see clearly that if N is sufficiently large, the linearized model is able to catch the multiple-interaction dynamics at each time, whereas differences can be observed if N is relatively small.

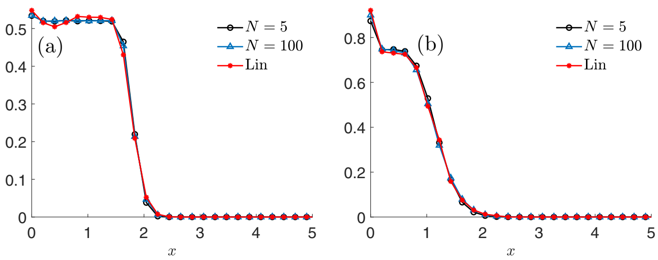

FIG. 3. Test 1: Without refilling. Evolution at different times (a) $t = 0.1$ and (b) $t = 2$ of the multiple-interaction Boltzmann-type model with either $N = 5$ gamblers (open circles) or $N = 100$ gamblers (open triangles) and of its linearized version (closed circles) in the time interval [0,2] for $\delta = 0.2$ and $\beta = 0$. We considered $\kappa = 0.1$.

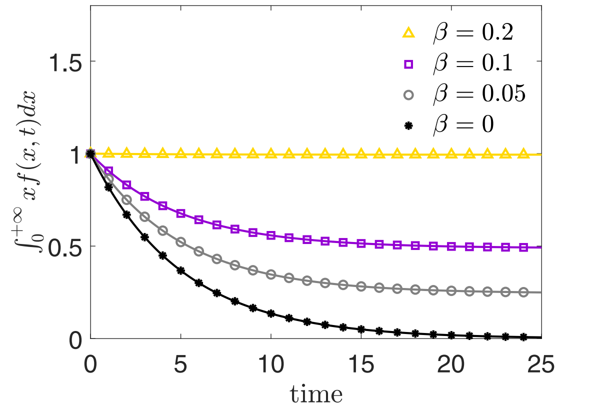

FIG. 5. Evolution of the mean number of tickets $m(t)$ in the time interval [0,25] for $\delta = 0.2$, $\kappa = 0.1$, and several choices of β.

Moreover, in the linearized model, we know that the mean number of tickets owned by the gamblers during the jackpot game is given by (18). In Fig. 5 we show instead the time evolution of the mean of the solution to the multiple-interaction Boltzmann-type model for several choices of the refilling parameter β. We observe good agreement with the theoretical results and in particular we see that the mean value tends indeed asymptotically to $\frac{\beta M}{\delta}$, as expected.

B. Test 2: Fokker-Planck approximation for large *N*

In the case $\epsilon, \kappa \ll 1$, the interactions (16) are quasi-invariant, and hence the linearized Boltzmann-type model (15) is well described by the Fokker-Planck equation (23). In the case of a constant mean value $m(t) \equiv m_0 = \frac{\beta M}{\delta}$ of the number of tickets owned by the gamblers, the steady distribution is the Γ probability density function (24). In this section we compare numerically the large-time distributions produced by either the multiple-interaction Boltzmann-type model (5) or the linearized Boltzmann-type model (15) with (24).

Like before, we consider a uniform initial distribution $f_0(x)$ in the interval [0,2] and moreover a random variable Y log-normally distributed according to (52), thus in particular with mean $M = 1$. We also set $\beta = \delta = 0.2$ in the microscopic interactions (4) and (16), so the mean value of the ticket distribution is always $m_0 = \frac{\beta M}{\delta} = 1$, consistently with the Fokker-Planck regime in which we are able to compute explicitly the steady distribution (24).

In Fig. 6(a) we compare the large-time distribution of the multiple-interaction Boltzmann-type model for an increasing number of gamblers participating in each round of the jackpot game ($N = 10^2$ and $N = 10^3$, respectively) with the asymptotic Γ probability density (24) computed from the Fokker-

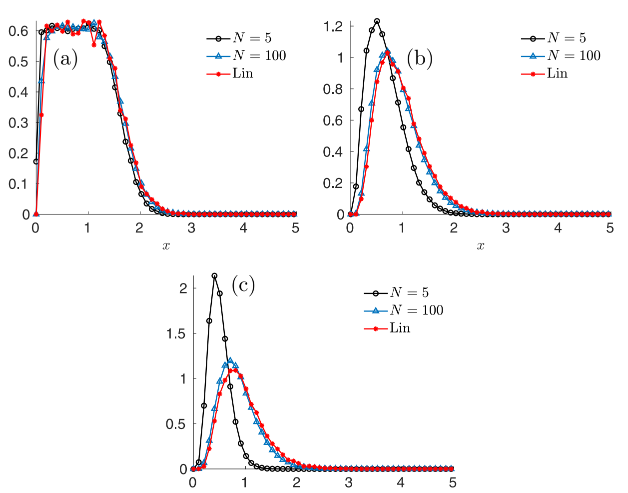

FIG. 4. Test 1: With refilling. Evolution at different times (a) $t = 1$, (b) $t = 5$, and (c) $t = 25$ of the multiple-interaction Boltzmann-type model with either $N = 5$ gamblers (open circles) or $N = 100$ gamblers (open triangles) and of its linearized version (closed circles) in the time interval [0,25] for $\delta = \beta = 0.2$ [log-normal refilling sampled from (52)]. We considered $\kappa = 0.1$.

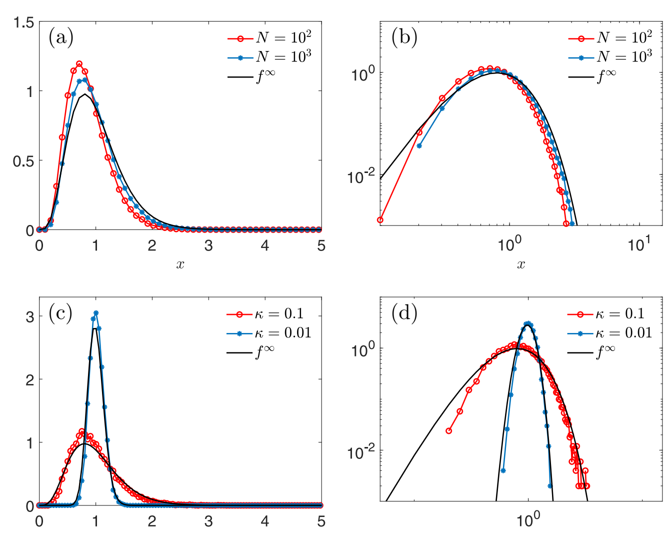

FIG. 6. Test 2. (a) and (b) Comparison of the steady distribution of the multiple-interaction Boltzmann-type model (5) with the Fokker-Planck asymptotic distribution (24) (solid line) and its log-log plot [in (b)] for $N = 10^2$ (open circles), $N = 10^3$ (closed circles), and fixed $\kappa = 0.1$. (c) and (d) Comparison of the steady distribution of the linearized Boltzmann-type model (15) with the Fokker-Planck asymptotic distribution (24) (solid line) and its log-log plot for $\kappa = 0.1$ (open circles) and $\kappa = 0.01$ (closed circles).

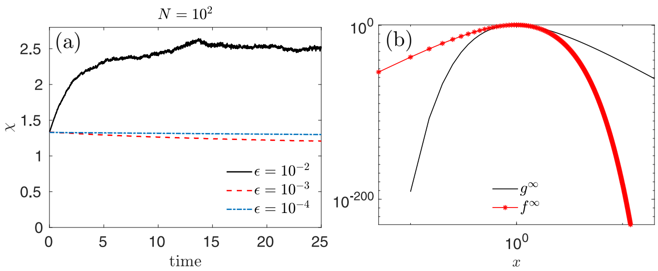

FIG. 7. Test 3. (a) Estimate of the approximate collision invariant χ [cf. (28)]. (b) A log-log plot of the distributions (24) and (37).

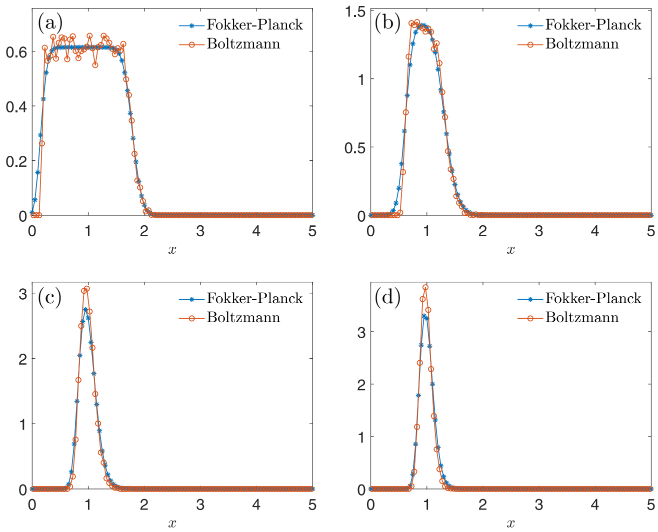

FIG. 8. Test 3. Comparison between the time evolutions of the Boltzmann-type model (33) and (34) (open circles) and of its Fokker-Planck approximation (36) (stars) in the quasi-invariant regime at times (a) $t = 1$, (b) $t = 5$, (c) $t = 10$, and (d) $t = 25$. The following parameters have been used: $\beta = \delta = 0.2$, $M = 1$, and $\kappa = 10^{-2}$. The value of the approximate collision invariant χ is estimated from the multiple-interaction Boltzmann-type model like in Fig. 7.

Planck equation. We clearly see that, for N large enough, the Fokker-Planck steady solution provides a good approximation of the equilibrium distribution of the real multiple-interaction model. In Fig. 6(b) we show the log-log plot of the same distributions, which allows us to appreciate that in particular the Fokker-Planck solution reproduces correctly the tail of the equilibrium distribution of the multiple-interaction model, thereby confirming that no fat tails have to be expected in the distribution of the tickets owned by the gamblers.

In Fig. 6(c) we compare instead the large-time distribution of the linearized Boltzmann-type model with the asymptotic Γ probability density (24) for decreasing values of κ ($\kappa = 0.1$ and $\kappa = 0.01$, respectively). In Fig. 6(d) we show the log-log plot of the same distributions to stress, in particular, the goodness of the approximation of the tail provided by (24).

C. Test 3: Fat-tail case

In Sec. II D 1 we derived the alternative linear Boltzmann-type model (33)–(34), which preserves some of the main macroscopic properties of the original multiple-interaction model (4) and (5). In particular, it accounts for the right evolution of the first and second moments of the distribution function.

We grounded such a derivation on the consideration that, for N large and ϵ small, the quantity χ defined in (28) may be treated approximately as a collision invariant of the N-gambler dynamics. In Fig. 7(a) we test numerically this assumption by taking $N = 10^2$ and some values of the scaling parameter ϵ decreasing from 10^{-2} to 10^{-4}. In particular, since χ depends actually on the evolving microscopic states $x_1, \dots, x_N$ of the agents, we plot the time evolution of χ for $t \in [0, 25]$. Such a time evolution is computed with the Monte Carlo method described in Sec. IV A, starting from an initial sample of $S = 10^6$ particles. Therefore, we get $N = 10^2$ subsamples of $S/N = 10^4$ particles, each of which produces a Monte Carlo estimate of the time trend of χ. From these samples we compute finally the average time trend of χ, namely, each of the curves plotted in Fig. 7(a). Consistently with our theoretical findings, we observe that, for ϵ small enough, χ may be actually regarded as a collision invariant.

In the quasi-invariant limit, the solution to the linear Boltzmann-type model (33)–(34) has been shown to approach that of the Fokker-Planck equation (36). Its explicitly computable steady state is the inverse Γ probability density (37), which, unlike the equilibrium distribution (24) approximating the trend of the multiple-interaction model for large N, exhibits a fat tail. In Fig. 7(b) we show the log-log plot of the distributions (24) and (37), which stresses the difference in their tails.

In order to check the consistency of the Fokker-Planck regime described, in the quasi-invariant limit, by (36) with the Boltzmann-type model (33) and (34), in Fig. 8 we show the time evolution of the distribution function g computed with both models for $t \in [0, 25]$, starting from an initial uniform distribution for $x \in [0, 2]$. In both cases, we treat χ as a collision invariant of the N-gambler model. Thus, we first computed the value of χ from the model (4) and (5) (with $N = 10^4$) and then we used it in the binary rules (33), where χ determines the values that η_ϵ can take, and in the diffusion coefficient $\tilde{\sigma}$ of (36). From Fig. 8 we see that the two models remain close to each other at every time and approach the same steady distribution for large times, as expected.

Finally, we quantify the distance between the solution f to the Fokker-Planck equation (23), which reproduces the large-time trend of the multiple-interaction Boltzmann-type model (4) and (5) [cf. Test 2 in Sec. IV B and the solution g to Fokker-Planck equation (36), which describes instead the large-time trend of the linear diffusive Boltzmann-type model (33) and (34)]. We consider in particular the relative L^1 error

$$E_\kappa(t) := \int_{\mathbb{R}_+} \frac{|g(x,t) - f(x,t)|}{f(x,t)} dx \tag{53}$$

for several values of the constant κ [cf. (14)], which appears as a coefficient in both Fokker-Planck equations. In particular, we consider $\kappa = 10^{-1}, 10^{-2}, 10^{-3}$ and we take $f(x,0) = g(x,0) = \frac{1}{2}\mathbb{1}_{[0,2]}(x)$ as the initial (uniform) distribution. By means of semi-implicit structure preserving (SP) methods,

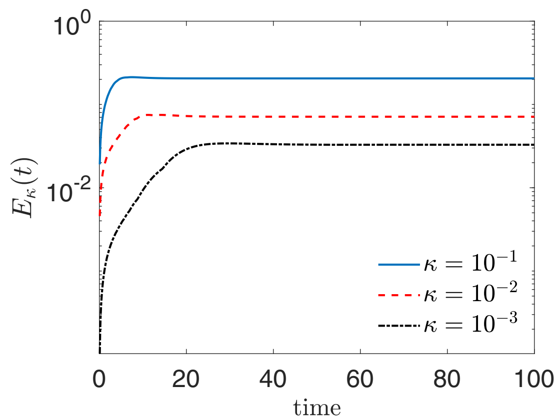

FIG. 9. Test 3. Relative L^1 error (53) between the solutions to the Fokker-Planck equations (23) and (36). The numerical solution of both models is obtained by means of semi-implicit SP methods over the computational domain [0,10] in the x variable, with $\Delta t = \Delta x = 10/N_x$ and $N_x = 401$ nodes.

we guarantee the positivity and the large-time accuracy of the numerical solution to both models. (The interested reader is referred to [26] for further details on this numerical technique.) From Fig. 9 we see that E_κ decreases with κ, although its order of magnitude remains non-negligible. Hence, the diffusive model with fat tails may approach, in a sense, the nondiffusive one with slim tails, but visible differences remain between them as a consequence of the fact that the diffusive model describes a jackpot game which is not completely equivalent to the real one caught by the nondiffusive model.

V. CONCLUSION

In this paper we introduced and discussed kinetic models of online jackpot games, i.e., lottery-type games which occupy a big portion of the web gambling market. Unlike the classical kinetic theory of rarefied gases, where binary collisions are dominant, in this case the game is characterized by simultaneous interactions among a large number $N \gg 1$ of gamblers, which leads to a highly nonlinear Boltzmann-type equation for the evolution of the density of the gambler's winnings. When participating in repeated rounds of the jackpot game, the gamblers continuously refill the number of tickets available to play and at the same time their winnings undergo a percentage cut operated by the site which administers the game. Hence, through the study of the evolution of the mean number of tickets and its variance, one realizes that the solution of the model should approach in time a nontrivial steady state describing the equilibrium distribution of the gambler's winnings.

In the limit $N \to \infty$, we showed that the multiple-interaction kinetic model can be suitably linearized so as to get access to analytical information about the large-time trend of its solution. We proposed two different linearizations, which, while apparently both consistent with the original nonlinear model, exhibit marked differences for large times. The solution to the linear model presented in Sec. II B converges towards a steady state with all moments bounded. In some cases, such a steady state can be written explicitly in the form of a Γ probability density function. Conversely, the solution to the linear model considered in Sec. II D converges towards a steady state in the form of an inverse Γ probability density function, hence with Pareto-type fat tails. We explained the different trend of the second model as a consequence of a too strong loss of correlation among the gamblers, which is instead present in the original nonlinear multiple-interaction model and also in its linear approximation proposed in Sec. II B. Numerical results showed indeed that the solution to this linear model is in perfect agreement with that to the fully nonlinear kinetic model.

The main conclusion which can be drawn from the present analysis is that the wealth economy of a multiagent system in which the trading activity relies on the rules of the jackpot game does not lead to a stationary distribution exhibiting Pareto-type fat tails, as it happens instead in a real economy. Unlike the real trading economy, where the small richest part of the population owns a relevant percentage of the total wealth, in the economy of the jackpot game the class of rich people is still very small but it does not own a consistent percentage of the total wealth (measured in terms of tickets played and won in time). In other words, it is exceptional to become rich by just playing the jackpot game and in such a case it is further exceptional to become very rich.

A nonsecondary conclusion of the present analysis is that the rules of the jackpot game imply a strong correlation among the gamblers participating in the game. Indeed, in each round of the game there is just one gambler who wins while all the other gamblers lose. Any approximation of the fully nonlinear model needs to take into account this aspect. This is clearly in contrast with a real trading economy, where the agents may instead take advantage simultaneously of their trading activity.

It is further interesting to compare the online gambling dynamics studied here with the market ecology among different types of traders depicted by the so-called minority game [13,14]. Connections with the so-called Kolkata restaurant problem have been discussed in [27]. The latter is a stylized model of a financial market, in which a group of $N \in \mathbb{N}$ traders is divided into two categories: the producers, who use the market to exchange goods, and the speculators, who aim to gain from the fluctuations of the market. Both types of users operate by either selling or buying. However, the producers exhibit an essentially deterministic behavior, while the speculators may choose among different strategies, taking advantage of past trading experiences. Furthermore, they may decide not to trade if they estimate that the market is not offering convenient opportunities. The revenue of each trade is modeled in such a way that the minority of the agents obtains the greatest reward. Such an idea is motivated, e.g., by the fact that, in the presence of a large number of buyers, the sellers, who are the minority, may raise the price of their goods. Such dynamics may be rephrased, in the language of a game, by saying that in each round (viz., trade) at most $\lceil \frac{N}{2} - 1 \rceil$ players (viz., traders) may win (viz., earn). With this interpretation, we notice that, unlike the online gambling, the minority game allows several players to win in a single round. Not by chance, then, in [13] the authors found, by means of numerical simulations, that the statistical distribution of the revenues exhibits a fat tail. A theoretical explanation of this fact may be based on the conclusions above, in particular on the fact that a small probability of a few players winning simultaneously in a single round is enough to generate fat-tailed distributions (cf. Sec. II D). Furthermore, in the case of the minority game, some players are assumed to learn from previous rounds, which implies a refinement of their game strategy towards the maximization of their incomes. This is a further element which may enhance the formation of a fat tail in the revenue distribution. In the case of online gambling, learning processes are actually not reported in the behavioral analysis [4]. On the other hand, it can be easily argued that addiction issues strongly prevent effective forms of learning by the gamblers.

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