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P-persistent homology of finite topological spaces

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A. Patania, G. Petri, F. Vaccarino

P-PERSISTENT HOMOLOGY OF FINITE TOPOLOGICAL SPACES

Abstract. Let *P* be a finite partially ordered set. We show that for any *P*-persistent object *X* in the category of finite topological spaces, there is a *P*- weighted graph, whose clique complex has the same *P*-persistent homology as *X*. We apply this to the study of weighted graphs vs persistence of their embedding into metric spaces.

1. Introduction

The study of topological spaces and the related computational methods are receiving an unprecedented attention from fields as diverse as biology and social sciences [3], [10], [13], [16] and [17].

The original motivation of this work is to provide a firm mathematical background for the results obtained in [16, 17], where the authors defined a filtration of a weighted network directly in terms of edge weights. The rationale behind this was the observation that embedding a network into a metric space generally obfuscates most of its interesting structures [15], which become instead evident when one focuses on the weighted connectivity structures without enforcing a metric. To achieve this, we use the abstract framework of category theory to get a closer look at the key ideas behind *P*-persistent homology expanding on the work by [2, 5, 14] in order to obtain a clear understanding of the mathematical structure behind the observation done in [16] that embedding a weighted network into a metric space generally obfuscates most of its interesting structures, which become evident when one focuses on the weighted connectivity structures without enforcing a metric.

Figure 1 illustrates this through the H_1 and H_2 persistent diagrams for two different filtrations obtained from a dataset of face-to-face contacts among children in an elementary school (see the Sociopatterns project [18] for details).

The *metrical filtration* is obtained in the standard way: given a metric (weighted shortest path in this case), one constructs a sequence of Rips-Vietoris complexes by studying the change in the overlap of ε-neighbourhoods of vertices while varying their radius ε (Figure 1 right). The non metrical one relies instead on associating clique complexes to a series of binary networks obtained from a progressively descending thresholding on edge weights (Figure 1 left). The difference between the diagrams of the two filtrations is evident: in the first case, most of the generators have short persistence and are thus distributed along the diagonal; in the second generators display a range of persistence, including some very large ones that signal the presence of interesting heterogeneities in the network structure.
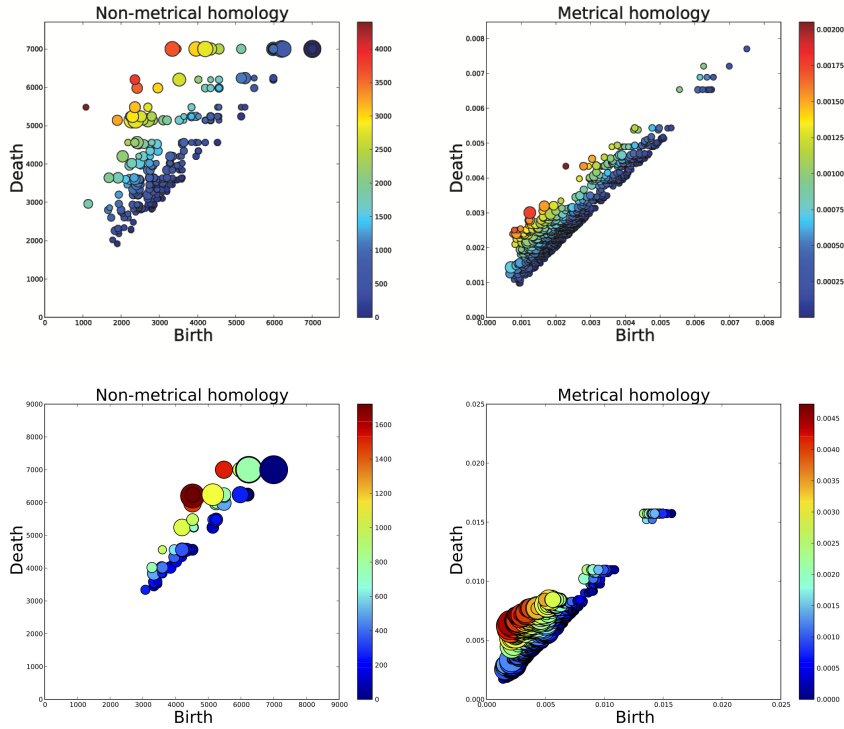
Figure 1: Comparison of results obtained for H_1 (*top row*) and H_2 (*bottom row*) on the Sociopatterns network with weighted graph filtration (*left*) and metrical filtration (*right* as described in the main text. The colour of dots represents the corresponding generator's persistence, while the size is proportional to number of simplices composing a chain representative of the generator's class (we choose here as representative the standard output of javaplex [8]).

More interestingly, we will see that the attempt at providing a mathematical formalisation of the approach used in [16] yields a much more general result regarding *P*-persistent homology.

In order to obtain this result, we need to introduce a number of notions. In the next section (Sec. 2) we briefly define the notion of *P*-persistence, which is our main object of study. Subsequently we lay down the vocabulary needed, introducing finite topological spaces and their equivalence to partially ordered sets, the relation between simplicial complexes, their associated order complexes and graphs. Then, in Section 3, we focus on our main contributions: we then prove the equivalence between the sub-categories of weighted graphs whose morphisms preserve the poset structure of the

weights $\overline{\mathcal{G}}_P$, and that of one-critical P-persistent graphs $\mathcal{G}_1^P$ (Theorem 2). Furthermore, we show that there exist adjoint functors that describe the relation between the sub-categories involved in the equivalence and the other sup-categories (Theorems 3,4). Finally, in Subsection 4 we recall the concept of homology and use the categorical equivalence to prove Theorem 7 providing a theoretical explanation of the observations done in [16].

2. *P*-persistence

In this section we recall a series of standard facts on finite topological spaces and simplicial complexes. We will also introduce the relative categories and functors, which will be the object of our study.

Let us firstly recall the definition of poset.

DEFINITION 1. *A partially ordered set, briefly a poset, is a pair $P = (P, \leqslant)$, where P is a set and $\leqslant$ is an order relation on it, i.e. a reflexive, antisymmetric, and transitive relation on P.*
Posets form a category, denoted by $\mathscr{P}$, where morphisms are the order preserving functions. In this paper we will consider only finite posets.

REMARK 1. Every poset P is a category on its own, where the objects are the elements of P, and there is a (unique) morphism $x \to y$ if and only if $x \leqslant y$, for all $x, y \in P$.

Following [3, Sec.2.3], a P-persistent object is given by the following definition.

DEFINITION 2. *Let P be a poset and $\mathscr{A}$ an arbitrary category. A P-persistent object in $\mathscr{A}$ is a functor $\varphi : P \to \mathscr{A}$. P-persistent objects in $\mathscr{A}$ with their natural transformations form a category, which we will denote, as usual, by $\mathscr{A}^P$. Given two categories $\mathscr{A}$ and $\mathscr{B}$, to any functor $\phi : \mathscr{A} \to \mathscr{B}$ corresponds a functor $\mathscr{A}^P \to \mathscr{B}^P$. It is given by $\varphi \in \mathscr{A}^P \mapsto \phi \circ \varphi$. It will be denoted by ϕ^P.*

As said in the introduction, we are interested in studying persistent objects in the category of finite topological spaces and in a suitable category of graphs. Let us then introduce the reader to the categories we are interested in and some functors relating them.

2.1. Basic categories and functors

We want to start with a few considerations on finite topological spaces i.e. topological spaces with a finite number of elements, which we can imagine to be given as some sampling taken from a dataset. Finiteness is not a constraint for our purposes, since every application will have a finite data space, although possibly very large.

Finite topological spaces form a subcategory, denoted by $\mathscr{T}_f$, of the category $\mathscr{T}$ of topological spaces and continuous maps.

A T_0 space is a topological space such that for any two different points x and y there is an open set which contains one of these points and not the other. Two such points will be called topologically distinguishable. It is clear that this property is highly desirable in order to be able to extract meaningful information from a topological space.

In this paper we will denote by $\mathscr{T}_f^0$ the category of finite T_0−spaces.

From here on when we write topological space we will intend finite topological space if not otherwise stated.

Kolmogorov quotient

It may happen that a space we are working with is not T_0, but this difficulty is easy to overcome as it is shown by the following, which is well known, see e.g. [1, Prop.1.3.1].

PROPOSITION 1. *Let X be a finite space not* T_0. *Let* $X/\sim$ *be the Kolmogorov quotient of X defined by* $x \sim y$ *if there does not exist an open set which contains one of these points and not the other. Then,* $X/\sim$ *is* T_0 *and the quotient map* $q : X \to x_0$ *is a homotopy equivalence.*
The Kolmogorov quotient $X \to X/\sim$ *induces a functor from the category of topological spaces to the category of* T_0−*spaces.*

Since homology is defined up to weak homotopy equivalence (Definition 17), the Kolmogorov quotient allows us to restrict our analysis from general topological spaces to T_0−spaces without any loss of information.

Finite T_0−spaces are posets

THEOREM 1. *There is an isomorphism of categories:*
$$\mathscr{T}_f^0 \cong \mathscr{P}$$

Proof. Let $X \in \mathscr{T}_f^0$, for $x \in X$ let U_x be the intersection of all the closed sets in X that contain *x*. Then we can give in *X* an order relation in the following way:

(1) $x \leqslant y \leftrightarrow U_x \subseteq U_y$

Since *X* is T_0 this relation is a partial order. In this way we have a correspondence $X \mapsto (X, \leqslant)$ which induces a functor $\mathscr{T}_f^0 \to \mathscr{P}$.

On the other end, a poset $P \in \mathscr{P}$ is also a topological space via the *Alexandrov topology*. In this topology the closed sets are the *lower sets*: $\Gamma \subset P$ such that $\forall x, y \in P$ with $x \in \Gamma$ and $y \leqslant x$ implies that $y \in \Gamma$. A poset endowed with this topology satisfies the T_0 condition. The assignment of this topology on *P* induces a functor $\mathscr{P} \to \mathscr{T}_f^0$ which is left and right inverse of the previous one, that is:

(2) $\mathscr{T}_f^0 \cong \mathscr{P}$

We refer the reader to [1, Chap.1] for details.   □

REMARK 2. (i) A preordered set is a set endowed with a binary relation which is reflexive and transitive. A poset is a special kind of preorder obtained by requiring the relation to be antisymmetric. Preordered sets form a category *Preord* the same way as posets. The isomorphism in Theorem 1 can be extended to an isomorphism $\mathscr{T}_f \cong Preord_f$, where the latter is the category of finite preordered sets.
(ii) Let $X \in \mathscr{T}_f$ be a T_1-space, i.e. for all $x, y \in X$ there exist two open sets $A, B \subseteq X$ such that $x \in A$, $y \in B$ and $y \notin A$, $x \notin B$. Then X has the discrete topology and the process just described is not very informative because in this case the poset $(X, \leqslant)$ is discrete. We will see how to deal with finite metric spaces, that are T_1, in the last section of this paper.

From now on we will identify any $X \in \mathscr{T}_f$ with the poset associated to its Kolmogorov quotient i.e., by abuse of notation, we will write

$$X = (X, \leqslant) = (X/\sim, \leqslant)$$

where $\leqslant$ is the order relation given in (1).

Simplicial Complexes

The basic idea of a simplicial complex is that of gluing together, in a coherent way, points, lines, triangles, tetrahedra, and higher dimensional equivalents. We will give now a more formal definition.

DEFINITION 3. *An (abstract)* simplicial complex *is a non empty family Σ of finite subsets, called faces, of a vertex set V such that $\sigma \subset \tau \in \Sigma$ implies that $\sigma \in \Sigma$.*

We assume that the vertex set is finite and totally ordered. A face of $n+1$ vertices is called an $n-$face, denoted by $[p_0, \ldots, p_n]$, and n is its dimension. We set, as usual, the dimension of the empty set as -1, following [9, Sec.2.1].
A $0-$face is a vertex, a $1-$face is an edge, a $2-$face is a full triangle, a $3-$face is a full tetrahedron, etc.
The *dimension* of a simplicial complex the highest dimension of the faces in the complex. We call *vertex set V* of Σ the union of the one point elements of Σ.
For a simplicial complex Σ and a non negative integer k we denote by Σ_k its $k-skeleton$ which is defined as

$$(3) \qquad \Sigma_k := \{\sigma \in \Sigma : \dim \sigma \leqslant k\}$$

The $k-$skeleton of Σ is a simplicial complex in the obvious way.
Simplicial complexes form a category, $\mathscr{S}$, where a morphism of simplicial complex is called *simplicial map* and is given by a map on vertices such that the image of a face is again a face.

We are going to recall some well known relations between simplicial complexes, topological spaces and posets.

PROPOSITION 2. *There exists a functor $\mathscr{O} : \mathscr{P} \to \mathscr{S}$ which associates to every poset P a simplicial complex, called the order complex.*

Proof. For every $P \in \mathscr{P}$ we can construct a simplicial complex as follows:

$$[x_0, \ldots, x_k] \in \mathscr{O}(P) \text{ if and only if } x_0 < x_1 < \cdots < x_k, \text{ for all } x_j \in P.$$

$\mathscr{O}(P)$ is called the *order complex of P*. □

DEFINITION 4. *Every simplicial complex can be made into a topological space by considering it a poset, i.e. $\Gamma \subseteq \Sigma$ is closed if and only if Γ is a simplicial complex. This gives a functor $\pi : \mathscr{S} \to \mathscr{P} \simeq \mathscr{T}_f^0$ by $\pi(\Sigma) = (\Sigma, \subseteq)$ the poset with elements the simplices in Σ and as partial order the inclusion of simplices.*
Given a simplicial complex Σ we write $\mathscr{O}(\Sigma) := \mathscr{O}(\pi(\Sigma))$. The simplicial complex $\mathscr{O}(\Sigma)$ is called the barycentric subdivision of Σ.
By abuse of notation we will also write $\mathscr{O}(X) := \mathscr{O}(X/\sim, \leqslant)$ for all $X \in \mathscr{T}_f$ via the isomorphism in Theorem 1.

It is well known that Σ and $\mathscr{O}(\Sigma)$, endowed with the Alexandrov topology, are weakly homotopy equivalent (we refer the reader to Definition 17). We refer the interested reader to [1, Section 1.4] for further details.

Graphs

DEFINITION 5. *A reflexive graph is a pair $G = (V, E)$, where V is a finite set whose elements are called vertices and $E = \Delta_{V \times V} \cup \hat{E}'$ with $\hat{E} \subseteq \binom{V}{2}$, i.e. is a graph which has an edge (called self-loop) (v, v) for every vertex $v \in V$. Equivalently, reflexive graphs can be seen as one dimensional simplicial complexes, identifying self-loops and vertices with 0-simplices and edges in $\hat{E}$ with 1-simplices. We will denote by $\mathscr{G}$ the category with objects reflexive graphs and morphisms the simplicial maps defined via the given identification with one dimensional simplicial complexes.*

It should be clear that $\mathscr{G}$ is isomorphic to the full subcategory of $\mathscr{S}$ whose objects are the one dimensional simplicial complexes.

REMARK 3. It is useful to notice that, since graphs in $\mathscr{G}$ are defined as $G = (V, E)$, then the null graph $G_{\varnothing} = (\varnothing, \varnothing)$ is an object in $\mathscr{G}$.

DEFINITION 6. *A clique in a graph $G = (V, E)$ is a complete subgraph of G i.e. is a subgraph $G' = (V', E')$ with $V' \subset V$ and $E' \subset E$ such that $E' = \binom{V'}{2}$.*
Given a graph $G \in \mathscr{G}$ there is a covariant functor, $\mathrm{Cl} : \mathscr{G} \to \mathscr{S}$, called the clique functor given by $[v_0, \ldots, v_k] \in \mathrm{Cl}(G)$ if and only if $(v_i, v_j) \in E$ for all $0 \leqslant i \neq j \leqslant k$..

Note that this is well defined because $(v, v) \mapsto [v]$, for all $v \in V$.

DEFINITION 7. *Given a simplicial complex Σ there is a functor $k_1 : \mathscr{S} \to \mathscr{G}$ where $k_1(\Sigma)$ is the (reflexive) graph corresponding to the $1-$skeleton of Σ.*

REMARK 4. In general $\Sigma \neq \mathrm{Cl}(k_1(\Sigma))$. For example if we consider $\Sigma = \{[a],[b],[c],[a,b],[a,c],[b,c]\}$, this is a simplicial complex but $\mathrm{Cl}(k_1(\Sigma)) = \Sigma \cup \{[a,b,c]\}$.

DEFINITION 8. *A simplicial complex Σ is called a flag (or a clique complex) if $\Sigma = \mathrm{Cl}(k_1(\Sigma))$. Flag complexes form a subcategory of $\mathscr{S}$ denoted by $\mathscr{F}$.*

REMARK 5. It is easy to see that $\mathscr{O}(X)$ is a flag complex for all $X \in \mathscr{T}_f$. In particular this implies that, for all $\Sigma \in \mathscr{S}$, the barycentric subdivision $\mathscr{O}(\Sigma)$ is a flag complex.

PROPOSITION 3. *The functors $\mathrm{Cl} : \mathscr{G} \to \mathscr{F}$ and $k_{1|\mathscr{F}} : \mathscr{F} \to \mathscr{G}$ give an isomorphism $\mathscr{G} \simeq \mathscr{F}$.*

Proof. Obvious. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

To summarize:

$$(4) \qquad \mathscr{S} \underset{\mathscr{O} \circ \pi}{\overset{\pi}{\longrightarrow}} \mathscr{P} \simeq \mathscr{T}_f^0 \overset{\mathscr{O}}{\longrightarrow} \mathscr{F} \underset{k_1}{\overset{\mathrm{Cl}}{\underset{\cong}{\rightleftarrows}}} \mathscr{G}$$

2.2. *P*-weighted graphs

DEFINITION 9. *Let $P \in \mathscr{P}$ be a poset and $G = (V,E) \in \mathscr{G}$ a graph, let us denote by $G \in \mathscr{S}$ the corresponding one dimensional simplicial complex. A $P-$weighted graph is a pair (G,ω), where $\omega : (G,\subseteq) \to P$ is a morphism of posets (i.e. a function $G \to P$) that is continuous in the Alexandrov topology.*

EXAMPLE 1. *Let $G = (V,E) \in \mathscr{G}$ the graph with $V = \{1,2,3,4\}$ and $E = \{(1,2),(2,3),(3,4),(1,3)\}$, then (G,ω) is a weighted graph where $\omega(\{1,2,3,(1,2),(2,3)\}) \mapsto 1$, $\omega(\{4,(3,4)\}) \mapsto 2$, $\omega(\{(1,3)\}) \mapsto 3$.*

DEFINITION 10. *We define as $\mathscr{G}_P$ the category of P-weighted graphs, having objects P-weighted graphs and whose morphisms $\alpha : (G,\omega) \to (H,\theta)$ are induced by a simplicial map $\rho : G \to H$, such that $\alpha(G_v) \subseteq H_v$, where for any $v \in P$, $G_v = \{x \in G \,|\, \omega(x) \leqslant v\}$.*

3. Main results: equivalences and adjunctions

In this section we introduce our main contributions, exposing the relationships between the category of weighted graphs $\mathscr{G}_P$, the category of *P*-persistent graphs $\mathscr{G}^P$, and some of their particular subcategories.
We have the following.

PROPOSITION 4. *For all $P \in \mathscr{P}$, there is a functor $\Phi_P : \mathscr{G}_P \to \mathscr{G}^P$.*

Proof. Let $(G, \omega) \in \mathscr{G}_P$. From the definition of G_v we have that $G_u \subseteq G_v$ for every $u \leqslant v$.
We can associate to $(G, \omega) \in \mathscr{G}_P$ a *P*-persistent object in $\varphi_G \in \mathscr{G}^P$, namely $\varphi_G(v) = G_v$ with the inclusions maps $\varphi_G(u \leqslant v) : G_u \hookrightarrow G_v$ for all $v \in P$, $u \in P_v$. It is easy to check that the correspondence $(G, \omega) \to \varphi_G$ is natural in G. Therefore Φ_P is a functor between the two categories.   □

PROPOSITION 5. *For all $P \in \mathscr{P}$ there exists a functor $\Psi_P : \mathscr{G}^P \to \mathscr{G}_P$.*

Proof. Choose $\varphi \in \mathscr{G}^P$, and, for every $v \in P$, set $\varphi_v := \varphi(v) \in \mathscr{G}$.
Let $\omega^\varphi : \coprod_{v \in P} \varphi_v \to P$ be given by $\omega^\varphi|_{\varphi_v} = v$. It is easy to check that the correspondence $\varphi \mapsto (\coprod_{v \in P} \varphi_v, \omega^\varphi)$ is natural in φ, thus giving a functor $\Psi_P : \mathscr{G}^P \to \mathscr{G}_P$.   □

DEFINITION 11. *Let $\bar{\mathscr{G}}_P$ be the subcategory of $\mathscr{G}_P$ with the same objects, and morphisms the maps $\alpha : (C, \omega) \to (D, \omega')$ such that for every $x \in G$, $\omega'(\alpha(x)) = \omega(x)$. We set $\bar{\Phi}_P$ as the restriction of Φ_P to $\bar{\mathscr{G}}_P$.*

REMARK 6. It is useful to notice that, actually, $\Psi_P : \mathscr{G}^P \to \bar{\mathscr{G}}_P$. Since $\Psi_P(\varphi) \in Ob(\mathscr{G}_P) = Ob(\bar{\mathscr{G}}_P)$ for all $\varphi \in \mathscr{G}^P$, we just show that $\Psi_P(\mu)$ preserves weights for every $\mu : \varphi \to \tau \in \mathscr{G}^P$. Indeed from the definition of the weights $\omega^\varphi, \omega^\tau$ we have that $(\omega^\varphi)^{-1}(u) = \varphi_u$ then $\Psi_P(\mu)(\varphi_u) \subseteq \tau_u = (\omega^\tau)^{-1}(u)$.

DEFINITION 12. *Let $\mathscr{G}^P_{inc}$ be the subcategory of $\mathscr{G}^P$ whose objects are $\varphi \in \mathscr{G}^P$ such that the morphisms $\varphi(u \leqslant v) : \varphi(u) \to \varphi(v)$ are inclusions. We set Ψ^ι_P as the restriction of Ψ_P to $\mathscr{G}^P_{inc}$.*

Following Carlsson [4, Definition 1] we introduce the concept of *one critical P*-persistent object.

DEFINITION 13. *Let $\varphi \in \mathscr{G}^P_{inc}$. φ is said to be one-critical if for all $v \in P$, for all $(x, y) \in E_{\varphi(a)}$*

$$(5) \qquad\qquad \exists! \, m_{xy} = \min\{u \in P | \varphi_{uv}(x, y) = (x, y)\}$$

The one-critical P-persistent objects form a subcategory of $\mathscr{G}^P_{inc}$, which will be denoted by $\mathscr{G}^P_1$. We set Ψ^P_1 as the restriction of Ψ_P to $\mathcal{G}^P_1$.

REMARK 7. It is useful to notice that $\Phi_P : \mathscr{G}_P \to \mathscr{G}^P_1$. Indeed consider $(G, \omega) \in$

$\mathscr{G}_P$. By definition of Φ_P, it is clear that $\varphi_G \in \mathscr{G}_1^P$, with $m_{xy} = \omega(x, y)$.

THEOREM 2. *The categories $\mathscr{G}_1^P$ and $\bar{\mathscr{G}}_P$ are equivalent.*

Proof. It is a well known fact in category theory that a functor is an equivalence if and only if it is full, faithful and essentially surjective. Then to prove the equivalence of category we need to verify that Φ_P has these three properties.

Consider $(G, \omega), (H, \theta) \in \bar{\mathscr{G}}_P$, and $\alpha \in \hom_{\bar{\mathscr{G}}_P}((G, \omega), (H, \theta))$. The functor Φ_P is essentially surjective if it is surjective on objects up to isomorphism. Let $\varphi \in \mathscr{G}_1^P$, then we can construct $(G, \omega) \in \bar{\mathscr{G}}_P$ by $G := \bigcup_{a \in P} \varphi(a)$ and $\omega((x, y)) = m_{xy}$ (see 5). It follows that $\Phi_P((G, \omega))$ is such that, for all $u \in P$, one has $\varphi_G(u) = \{x \in G \mid \omega(x) \leqslant u\} = \bigcup_{a \in P; a \leqslant u} \varphi(a) \cong \varphi(u)$ by definition of φ.

The functor Φ_P is full if the map

$$\Phi_P((G, \omega), (H, \theta)) : \hom_{\bar{\mathscr{G}}_P}((G, \omega), (H, \theta)) \to \hom_{\mathscr{G}_1^P}(\varphi_G, \varphi_H)$$

is surjective for all $(G, \omega), (H, \theta) \in \bar{\mathscr{G}}_P$.
Consider a morphism $\rho : \varphi_G \to \varphi_H$ in $\mathscr{G}_1^P$. Let $\alpha : E_G \to E_H$ be given by $\alpha((x, y)) = \rho_{\omega(x, y)}(x, y)$, for every $(x, y) \in E_G$. Then α is a morphism $(G, \omega) \to (H, \theta)$ in $\bar{\mathscr{G}}_P$, because, since $\rho_{\omega(x, y)} : \varphi_G(\omega(x, y)) \to \varphi_H(\omega(x, y))$, we have that $\theta(\alpha((x, y))) = \omega(x, y)$. It is clear that $\Phi_P(\alpha)$ is ρ by the definitions of Φ_P and α.

As last step, we prove that Φ_P is faithful, i.e. that the map

$$\Phi_P((G, \omega), (H, \theta)) : \hom_{\bar{\mathscr{G}}_P}((G, \omega), (H, \theta)) \to \hom_{\mathscr{G}_1^P}(\varphi_G, \varphi_H)$$

is injective for all $(G, \omega), (H, \theta) \in \bar{\mathscr{G}}_P$.
Consider $\alpha, \beta \in \hom_{\bar{\mathscr{G}}_P}((G, \omega), (H, \theta))$ such that $\Phi(\alpha) = \Phi(\beta)$. This means that $\Phi(\alpha)_v = \Phi(\beta)_v$ for all $v \in P$, but this implies that $\alpha|_{G_v} = \beta|_{G_v}$ for all $v \in P$, then $\alpha = \beta$. $\square$

3.1. Adjunctions

Beside the equivalence in Theorem 2, there are also some results on the relationships between the other categories involved.

THEOREM 3. $\bar{\Phi}_P$ *is left adjoint of* Ψ_P, *that is*

$$\hom_{\bar{\mathscr{G}}_P}((X, \omega), \Psi_P(\varphi)) \cong \hom_{\mathscr{G}^P}(\bar{\Phi}_P((X, \omega)), \varphi)$$

Proof. Let $\pi : (X, \omega) \to (\coprod_P X_u, \omega^{\varphi_X})$ given by $\pi(x) = (x, \omega(x)) \in X_{\omega(x)}$, for every $x \in (X, \omega)$. This map is well defined and is actually a morphism in the category $\bar{\mathscr{G}}_P$ since $\omega^{\varphi_X}(\pi(x)) = \omega^{\varphi_X}((x, \omega(x))) = \omega(x)$.
To prove the assumption we will show that, for every $\alpha \in \hom_{\bar{\mathscr{G}}_P}((X, \omega), \Psi_P(\varphi))$, there is a unique morphism in $\mathscr{G}^P$, $\bar{\alpha} : \bar{\Phi}_P((X, \omega)) \to \varphi$ such that the following diagram

commutes

$$(6) \qquad (X,\omega) \xrightarrow{\ \pi\ } \Psi_P(\bar{\Phi}_P((X,\omega)))$$

$$\alpha \downarrow \qquad \swarrow \Psi_P(\bar{\alpha})$$

$$\Psi_P(\varphi)$$

Let $\alpha \in \hom_{\mathscr{G}_P}((X,\omega),(\coprod_P \varphi(u),\omega^\varphi))$, then α will be such that $\omega^\varphi(\alpha(x)) = \omega(x)$, for every $x \in (X,\omega)$. By construction of ω^φ we will have that $\alpha(x) \in \varphi(\omega(x))$, and with a little abuse of notation we will write $\alpha : x \mapsto (\alpha(x),\omega(x))$.

Let $\bar{\alpha} : \bar{\Phi}_P((X,\omega)) \to \varphi$ be the morphism in $\mathscr{G}^P$ defined through $\bar{\alpha}|_{X_u} : X_u \to \varphi(u)$ with $\bar{\alpha}|_{X_u}(x) = \varphi_{\omega(x)u}(\alpha(x),\omega(x))$, where $\varphi_{\omega(x)u} = \varphi(\omega(x) \leqslant u) : \varphi(\omega(x)) \to \varphi(u)$.

We still have to show that diagram 6 commutes, i.e. $\Psi_P(\bar{\alpha}) \circ \pi = \alpha$. Let x be an element of (X,ω) with weight $\omega(x)$, then $\pi(x) = (x,\omega(x)) \in X_{\omega(x)}$, so $\bar{\alpha}(\pi(x)) = \varphi_{\omega(x)\omega(x)}(\alpha(x),\omega(x)) = (\alpha(x),\omega(x))$. Follows that the diagram commutes and this proves the adjuction.

$\square$

There is another adjunction.

THEOREM 4. Ψ_P^ι *is left adjoint of* Φ_P.

In order to prove this theorem we need some technical lemmata.

LEMMA 1. *There is a natural transformation* $\varepsilon : \Psi_P^\iota \Phi_P \longrightarrow 1_{\mathscr{G}_P}$.

Proof. Consider $(G,\omega) \in \mathscr{G}_P$, then

$$(7) \qquad \mathscr{G}_P \xrightarrow{\ \Phi_P\ } \mathscr{G}_{inc}^P \xrightarrow{\ \Psi_P^\iota\ } \mathscr{G}_P$$

$$(G,\omega) \longmapsto \{G_v\} \longmapsto \left(\coprod_{v \in P} G_v, \omega^{\varphi_G}\right)$$

Define now $\varepsilon_{(G,\omega)}$ as follows:

$$(8) \qquad \varepsilon_{(G,\omega)} : \left(\coprod_{v \in P} G_v, \omega^{\varphi_G}\right) \longrightarrow (G,\omega)$$
$$(x,u) \mapsto x$$

This map is well defined since $\omega^{\varphi_G}((x,u)) = u \geqslant \omega(x) = \omega(\varepsilon_{(G,\omega)}(x,u))$.
Consider now (F,τ), and $\alpha : (G,\omega) \to (F,\tau)$ in $\mathscr{G}_P$, trivially the following diagram

commutes:

$$(9) \quad \begin{array}{ccc} (G,\omega) & \xrightarrow{\Psi_P^\iota \circ \Phi_P} (\coprod_{v \in P} G_v, \omega^{\varphi_G}) \xrightarrow{\varepsilon_{(G,\omega)}} (G,\omega) \\ \downarrow \alpha & \Psi_P^\iota(\Phi_P(\alpha)) \downarrow & \downarrow \alpha \\ (F,\tau) & \xrightarrow{\Psi_P^\iota \circ \Phi_P} (\coprod_{v \in P} F_v, \omega^{\varphi_F}) \xrightarrow{\varepsilon_{(F,\tau)}} (F,\tau) \end{array}$$

ε is the natural transformation we were searching for. $\qquad\square$

LEMMA 2. *There is a natural trasformation* $\eta : \Psi_P^\iota \Phi_P \longrightarrow 1_{\mathscr{G}_{inc}^P}$.

Proof. Consider $\varphi \in \mathscr{G}_{inc}^P$, $\Phi \circ \Psi(\varphi) = (\coprod_{u \leqslant v} \phi(u), \subseteq)$.

$$(10) \quad \mathscr{G}_{inc}^P \xrightarrow{\quad \Psi_P^\iota \quad} \mathscr{G}_P \xrightarrow{\quad \Phi_P \quad} \mathscr{G}_{inc}^P$$

$$\{\varphi(v), \subseteq\}_{v \in P} \longmapsto (\coprod_{v \in P} \varphi(v), \omega^\varphi) \longmapsto \{\coprod_{u \leqslant v} \varphi(u), \subseteq\}$$

Define now η_φ as follows:

$$(11) \quad \begin{array}{rcl} \eta_\varphi : \varphi(v) & \to & \coprod_{u \leqslant v} \varphi(u) \\ x & \mapsto & (x,v) \end{array}$$

Consider now θ, and $\alpha : \varphi \to \theta$ in $\mathscr{G}_{inc}^P$, the following diagram commutes:

$$(12) \quad \begin{array}{ccc} \varphi(v) & \xrightarrow{\eta_\varphi} & \coprod_{u \leqslant v} \varphi(u) \\ \downarrow \alpha_v & & \downarrow \coprod \alpha_u \\ \theta(v) & \xrightarrow{\eta_\theta} & \coprod_{u \leqslant v} \theta(u) \end{array}$$

where for every $(x,w) \in \coprod_{u \leqslant v} \varphi(u)$, $\coprod \alpha_u((x,w)) = (\alpha(x),w)$.
η is the natural transformation we were searching for. $\qquad\square$

Proof of Theorem 4. We will prove the unit-counit adjunction, with ε and η the natural transformations defined in Lemma 1, and 2.

To prove the adjunction we verify that the following compositions are the identity transformation of the respective categories.

$$(13) \quad \Phi_P \xrightarrow{\eta\Phi} \Phi_P \Psi_P^\iota \Phi_P \xrightarrow{\Phi\varepsilon} \Phi_P \qquad \Psi_P^\iota \xrightarrow{\Psi\eta} \Psi_P^\iota \Phi_P \Psi_P^\iota \xrightarrow{\varepsilon\Psi} \Psi_P^\iota$$
$$\qquad\qquad id_{\mathscr{G}_P} \qquad\qquad\qquad\qquad id_{\mathscr{G}_{inc}^P}$$

which means that for each (G, ω) in $\mathscr{G}_P$ and each φ in $\mathscr{G}^P$,

$$1_{\Psi_P^\iota(\varphi)} = \varepsilon_{\Psi_P^\iota(\varphi)} \circ \Psi_P^\iota(\eta_\varphi) \tag{14}$$

$$1_{\Phi_P((G,\omega))} = \Phi_P(\varepsilon_{(G,\omega)}) \circ \eta_{\Phi_P((G,\omega))} \tag{15}$$

We will start by verifying equation 14. Let $\varphi \in \mathscr{G}_{inc}^P$, we know that $\eta_\varphi : \varphi \longrightarrow \Phi_P \circ \Psi_P^\iota(\varphi)$ is a natural transformation defined for every $v \in P$ by

$$\eta_\varphi(v): \quad \begin{array}{ccc} \varphi(v) & \longrightarrow & \Phi_P(\Psi_P^\iota(\varphi))(v) = \coprod_{u \leqslant v} \varphi(u) \\ x & \mapsto & (x,v) \end{array} \tag{16}$$

Then

$$\Psi_P^\iota(\eta_\varphi) : \Psi_P^\iota(\varphi) \to (\coprod_{v \in P} \Phi_P(\Psi_P^\iota(\varphi))(v), \omega^{\Phi_P \Psi_P^\iota(\varphi)}) = (\coprod_{v \in P} \coprod_{u \leqslant v} \varphi(u), \omega^{\Phi_P \Psi_P^\iota(\varphi)}),$$

where $\omega^{\Phi_P \Psi_P^\iota(\varphi)}|_{\coprod_{u \leqslant v} \varphi(u)} = v$. From the definition of ε we gave in Lemma 1 we deduce that

$$\varepsilon_{\Psi_P^\iota(\varphi)} : \Psi_P^\iota \circ \Phi_P(\Psi_P^\iota(\varphi)) \longrightarrow \Psi_P^\iota(\varphi).$$

One has that $\Psi_P^\iota \circ \Phi_P(\Psi_P^\iota(\varphi))$ is the weighted graph $(\coprod_{v \in P} \Psi_P^\iota(\varphi)_v, \omega^{\Phi_P \Psi_P^\iota(\varphi)})$, where $\Psi_P^\iota(\varphi)_v = \{x \in \Psi_P^\iota(\varphi) | \omega^\varphi(x) \leqslant v\} = \coprod_{u \leqslant v} \varphi(u)$, and $\omega^{\Phi_P \Psi_P^\iota(\varphi)}|_{\coprod_{u \leqslant v} \varphi(u)} = v$.

$$\varepsilon_{\Psi_P^\iota(\varphi)}: \quad \begin{array}{ccc} (\coprod_{v \in P} \coprod_{u \leqslant v} \varphi(u), \omega^{\Phi_P \Psi_P^\iota(\varphi)}) & \longrightarrow & \Psi_P^\iota(\varphi) \\ ((x,u),v) & \mapsto & (x,u) \end{array} \tag{17}$$

where $\Psi_P^\iota(\varphi) = (\coprod_{v \in P} \varphi(v), \omega^\varphi)$, with $\omega^\varphi|_{\varphi(v)} = v$.

Then $\varepsilon_{\Psi_P^\iota(\varphi)} \circ \Psi_P^\iota(\eta_\varphi) = 1_{\Psi_P^\iota(\varphi)}$ as the following shows:

$$\begin{array}{ccccc} (\coprod_{v \in P} \varphi(v), \omega^\varphi) & \xrightarrow{\Psi(\eta_\varphi)} & (\coprod_{v \in P} \coprod_{u \leqslant v} \varphi(u), \omega^{\Phi_P \Psi_P^\iota(\varphi)}) & \xrightarrow{\varepsilon_{\Psi_P^\iota(\varphi)}} & (\coprod_{v \in P} \varphi(v), \omega^\varphi) \\ (x,v) & \mapsto & ((x,v),v) & \mapsto & (x,v) \end{array} \tag{18}$$

We verify now identity 15. Consider $(G, \omega) \in \mathscr{G}_P$, we have that

$$\eta_{\Phi_P((G,\omega))} : \Phi_P((G,\omega)) \to \Phi_P \circ \Psi_P^\iota(\Phi((G,\omega)))$$

where $\Phi_P((G,\omega))(v) = G_v$, with $G_v = \{x \in G | \omega(x) \leqslant v\}$. For every $v \in P$, we find that $\eta_{\Phi_P((G,\omega))}$ is determined by:

$$\eta_{\Phi_P((G,\omega))}(v): \quad \begin{array}{ccc} G_v & \longrightarrow & \coprod_{u \leqslant v} G_u \\ x & \mapsto & (x,v) \end{array} \tag{19}$$

Considering that $\varepsilon_{(G,\omega)} : (\coprod_{v \in P} G_v, \omega^{\varphi_G}) \mapsto (G,\omega)$, where $\omega^{\varphi_G}|_{G_v} = v$. We have that $\Phi_P(\varepsilon_{(G,\omega)})$ is defined for every $v \in P$:

$$(20) \qquad \Phi_P(\varepsilon_{(G,\omega)})(v) : \quad \Phi_P(\Psi^t_P \circ \Phi_P((G,\omega)))(v) \quad \longrightarrow \quad \Phi_P((G,\omega))(v)$$

where

$$\Phi_P(\Psi^t_P \circ \Phi_P((G,\omega)))(v) = \{(x,u) \in \coprod_{v \in P} G_v \text{ s.t. } \omega^{\varphi_G}((x,u)) = u \leqslant v\} = \coprod_{u \leqslant v} G_u$$

and $\Phi_P((G,\omega))(v) = G_v$. This gives the following natural transformation:

$$(21) \qquad \begin{array}{ccccc} G_v & \xrightarrow{\eta_{\Phi_P((G,\omega))}(v)} & \coprod_{u \leqslant v} G_u & \xrightarrow{\Phi_P(\varepsilon_{(G,\omega)})(v)} & G_v \\[2mm] x & \longmapsto & (x,v) & \longmapsto & x \end{array}$$

which proves that $\Phi_P(\varepsilon_{(G,\omega)}) \circ \eta_{\Phi_P((G,\omega))} = 1_{\Phi_P((G,\omega))}$. $\qquad\square$

4. Applications to homology

Homology

The basic idea behind algebraic topology is to functorially attach algebraic objects to topological spaces in order to discern their properties. Homology theory does so by introducing functors from the category of topological spaces (or some related category) and continuous maps to the category of modules over a commutative base ring, such that these modules are topological invariants.
We will first introduce homology over simplicial complexes, which are our main setting, and then we will proceed to define it over general topological spaces.

4.1. Simplicial homology

Fixed a field k, in the following, by vector space we intend a k—vector space.
Given a simplicial complex Σ of dimension d, for $0 \leqslant n \leqslant d$ consider the vector spaces $C_n := C_n(\Sigma)$ with basis the set of n-faces in Σ. Elements in C_n are called *n-chains*.

The linear maps sending a n-face to the alternate sum of it's $(n-1)$-faces are called *boundaries* and share the property $\partial_{n-1} \circ \partial_n = 0$.

$$\begin{array}{ccc} \partial_n : C_n & \longrightarrow & C_{n-1} \\[2mm] [p_0, \ldots, p_n] & \rightarrow & \sum_{i=0}^{n} (-1)^i [p_0, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n]. \end{array}$$

The subspace $\ker \partial_n$ of C_n is called the vector space of *n-cycles* and denoted by $Z_n := Z_n(\Sigma)$. The subspace $\operatorname{Im} \partial_{n+1}$ of C_n, is called the vector space of *n-boundaries* and denoted by $B_n := B_n(\Sigma)$.

REMARK 8. From $\partial_{n-1} \circ \partial_n = 0$ it follows that $B_n \subseteq Z_n$ for all n.

DEFINITION 14. *The $n-$th simplicial homology space of Σ, with coefficients in k, is the vector space* $H_n := H_n(\Sigma) := Z_n/B_n$. *We denote by* $\beta_n := \beta_n(\Sigma)$ *the rank of* H_n : *it is usually called the n-th Betti number of Σ.*

The first Betti numbers of Σ have an easy intuitive meaning: the 0-th Betti number is the number of connected components of Σ, the first Betti number is the number of two dimensional (poligonal) holes, the third Betti number is the number of three dimensional holes (convex polyhedron).

REMARK 9. It easy to check that C_n, Z_n, B_n and, therefore, H_n are all functors $\mathscr{S} \to \operatorname{Vect}_k$, where Vect_k denotes the category of vector spaces and linear mappings.

There is plenty of literature on homology and in particular on simplicial homology, we refer the interested reader to [12, Chapter 3]. In particular, one can find thereby, the proof of the following.

PROPOSITION 6. *The functors H_i are invariants by homeomorphism and homotopy type.*

DEFINITION 15. *Let $G \in \mathscr{G}$ be a graph. We now define as the homology space of G the vector space*

$$(22) \qquad\qquad H_i(G) := H_i(\operatorname{Cl}(G)).$$

PROPOSITION 7. *Let Σ be a simplicial complex. Then, there exists a graph $G \in \mathscr{G}$ such that $H_i(\Sigma) = H_i(G)$.*

Proof. The proof is a consequence of Remark 4 and 5. It is sufficient to consider as $G = k_1(\mathscr{O}(\pi(\Sigma)))$, the 1-skeleton of the barycentric subdivision of Σ, which is a flag complex. $\qquad\square$

4.2. Singular homology

Simplicial homology has an analogue for general topological spaces, namely *singular homology*, whose definition and properties we briefly recall now. Although we confine ourself into the category of finite topological spaces, the following definition remains valid for arbitrary topological spaces. We address the interested reader to [12, Chapter 4], [7, Chapter 2] for a thorough treatise on these topics.

Let $X \in \mathscr{T}_f$ be a topological space, the chain spaces C_n are in this case replaced by the vector spaces C_n^S freely generated by the set of all continuous functions from the

geometric realization of the standard n-simplex Δ^n to X.
The boundaries are then defined in the following way thus making (C_n^S, ∂_n^S) a chain complex.

DEFINITION 16. *Let σ be a generator of C_n, i.e. a continuous function from $\Delta^n \to X$. Then the boundary homomorphism ∂_n^S can be constructed in the following way:*

$$\partial_n^S(\sigma) = \sum_i^n \sigma|_{[v_0,\ldots,v_{i-1},v_{i+1},\ldots,v_n]}$$

where $\sigma|_{[v_0,\ldots,v_{i-1},v_{i+1},\ldots,v_n]}$ is the restriction of σ to $[v_0,\ldots,v_{i-1},v_{i+1},\ldots,v_n]$.

It is easy to verify that $\partial_n^S \circ \partial_{n+1}^S = 0$, thus we can define the homology spaces as we did for simplicial homology. We will denote the ith *singular homology space* by $H_i^S(X)$. By general nonsense it is easy to check that H_i^S gives a functor $\mathscr{T}_f \to \text{Vect}_k$.

THEOREM 5. *For any simplicial complex Σ, the singular homology groups are isomorphic to the simplicial homology groups.*

(23) $$\forall i \in \mathbb{N} \qquad H_i^S(\Sigma) \cong H_i(\Sigma)$$

The proof for this theorem is non-trivial and should the reader be interested, [7, Theorem 2.27] has clear proofs of this theorem.

DEFINITION 17. *Let $X, Y \in \mathscr{T}_f$, and let $\pi_n(X, x)$ denote the homotopy group of the space X at base point $x \in X$.*
A map $f : X \to Y$ is a weak homotopy equivalence if the following condition are both verified:

1. *f induces an isomorphism of the connected components of X and Y*

$$\Pi_0(f) : \Pi_0(X) \to \Pi_0(Y)$$

2. *for all $x \in X$, and $n \geqslant 1$ is an isomorphism on the homotopy groups*

$$\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, f(x))$$

There is the following result.

THEOREM 6 (McCord, [11]). *Let $X \in \mathscr{T}_f$ with $X/\sim$ its Kolmogorov quotient, then $\mathscr{O}(X/\sim) \in \mathscr{F}$ is weak homotopy equivalent to X.*

We refer the interested reader to [1, Chapter 1.4]. In view of this result it makes sense to set $\mathscr{O}(X) := \mathscr{O}(X/\sim)$ for all $X \in \mathscr{T}_f$.
Theorem 5 and 6 together imply that $H_i^S(X) \cong H_i(\mathscr{O}(X))$ for all $X \in \mathscr{T}_f$.

Moreover, since $\mathscr{O}(X)$ is a flag complex $\mathrm{H}_i(\mathscr{O}(X)) = \mathrm{H}_i(\mathrm{Cl}(k_1(\mathscr{O}(X))))$, that is the homology of the graph which is the 1-skeleton of $\mathscr{O}(X)$. Thus we can restrict ourselves to the study of the homology of $k_1(\mathscr{O}(X))$.

In conclusion, the following diagram commutes:

(24)
$$
\begin{array}{ccc}
\mathscr{T}_f \longrightarrow & \mathscr{T}_f^0 & \xrightarrow{\ \mathrm{H}_i^S\ } \mathrm{Vect}_k \\
 & {\scriptstyle\mathscr{O}}\Big\downarrow & \nearrow^{\mathrm{H}_i} \ \ \Big\uparrow{\scriptstyle\mathrm{H}_i} \\
 & \mathscr{F} & \underset{\mathrm{Cl}}{\overset{k_1}{\rightleftarrows}} \mathscr{G}
\end{array}
$$

Although these observations are interesting per se, they become much more significant if we consider not only the homological structure of a data space but also its *P*-persistent properties.

DEFINITION 18. *Let* $\tau \in \mathscr{T}_f^P$, *the composition* $\mathrm{H}_i^S \circ \tau \in \mathrm{Vect}_k^P$ *will be called the* i^{th} *P-persistent homology of* $\tau \in \mathscr{T}_f^P$.

Taking in consideration the concepts defined in the previous section, we have the following result which states that for any *P*-persistent finite topological space, there is a one-critical *P*-persistent graph having the same *P*-persistent homology.

PROPOSITION 8. *Let* $\tau \in \mathscr{T}_f^P$, *then there is* $\theta \in \mathscr{G}_{\mathbf{1}}^P$ *such that*

(25)
$$
\mathrm{H}_i^S \circ \tau \cong \mathrm{H}_i \circ \mathrm{Cl} \circ \theta
$$

as functors.

Proof. The commutativity of diagram 24 implies that the following diagram is commutative:

(26)
$$
\begin{array}{ccc}
\mathscr{T}_f^P \longrightarrow & (\mathscr{T}_f^0)^P & \xrightarrow{\ (\mathrm{H}_i^S)^P\ } \mathrm{Vect}_k^P \\
 & {\scriptstyle(k_1 \circ \mathscr{O})^P}\Big\downarrow & \nearrow_{(\mathrm{H}_i \circ \mathrm{Cl})^P} \\
 & \mathscr{G}^P &
\end{array}
$$

Therefore the statement holds with $\theta = k_1 \circ \mathscr{O} \circ \tau$. $\qquad\square$

The above result implies that *P*-persistent singular homology of finite spaces can be computed as *P*-persistent homology of graphs, and translates into the following existence theorem.

THEOREM 7. *Let* $\tau \in \mathscr{T}_f^P$ *be a P-filtration of topological spaces such that* τ_{ab} : $X_a \to X_b$ *is injective for all* $a, b \in P$ *with* $a \leqslant b$, *then exists a weighted graph* $(G, \omega) \in \bar{\mathscr{G}}_P$

such that

$$\mathrm{H}_i^S \circ \tau \cong \mathrm{H}_i(\mathrm{Cl}(\Phi_P(G, \omega))).$$

Proof. It follows from Proposition 8 and Theorem 2. □

REMARK 10. Let us consider a metric space (X, d), since X is T_1, then it has discrete topology. Therefore H_0^S is isomorphic to k^X, and $\mathrm{H}_i^S = \{0\}$ for all $i > 0$.
It is customary to associate to (X, d) a nice simplicial complex, namely the Vietoris-Rips complex X_ε. which is explicitly defined as $\mathrm{Cl}(D_\varepsilon(X))$ where D_ε is the graph with $V = X$, and $(x, y) \in E$ if and only if $d(x, y) \leqslant \varepsilon$. It is, therefore, clear that this approach is already included in our analysis.

4.3. Considerations on topological strata

In [16] the authors adopted different techniques to build filtrations of simplicial complexes from weighted networks. We are going to focus on two of these that are qualitatively different: a metrical, and a non-metrical filtration. The metrical filtration was obtained constructing a sequence of Vietoris-Rips complexes by studying the change in the overlap of ε-neighbourhoods of vertices while varying their radius ε, considering as metric of the underlying space the inverse-weighted shortest path. The non-metrical one relied instead on associating clique complexes to a series of binary networks obtained from a progressively less restrictive thresholding on the edge weights. The comparison highlighted a clear difference between the diagrams of the two filtrations: in the metric case, most generators had short persistence and were thus distributed along the diagonal; in the non-metric, generators displayed a range of persistences, including some very large ones, and thus poitned to the presence of interesting heterogeneities in the network structure which were not noticeable via the metric filtration.
This result can be reinterpreted in the light of Theorem 7. To do so, we need to define the two filtrations in terms of category theory.
Let $\mathbb{R}$ denote, as usual, the set of real numbers, which in this setting is considered as a (totally) ordered set via its usual ordering. Let us consider a (finite) metric space $X = (X, d)$ where $d : X \times X \to \mathbb{R}$. There is a weighted graph (G_X, ω_d) associated to X, namely its distance graph, which is the complete graph on the (vertex) set X, whose edges are weighted by the distance between their extrema. The Vietoris-Rips filtration associated to X is then $\mathrm{Cl}(\Phi_{\mathbb{R}}(G_X, \omega_d))$. It should be clear that $\Phi_{\mathbb{R}}(G_X, \omega_d) \in \mathscr{G}_{inc}^{\mathbb{R}}$.
Let us consider now a weighted graph (G, w) having weights in $\mathbb{R}$. We can bestow the set of vertices V of G with various distances induced by the weights and graph structure, see [6]. One classic example is the weighted shortest path metric, that is $\tilde{d}(x, y) = \min\{\sum_{(u,v) \in p} w(u, v)\}_p$ where p is a path between the vertices x and y. As we have just explained above, associated to $(V, \tilde{d})$ there is then an element of $\mathscr{G}_{inc}^{\mathbb{R}}$, namely $(G_V, \omega_{\tilde{d}})$. After this step one can compose with the clique functor and with the homology functor to obtain persistent homology. The metric filtration introduced in [16] was computed in this way and can be written in categorical form as $\mathrm{Cl}(\Phi_{\mathbb{R}}(G_V, \omega_{\tilde{d}}))$, while the non-metric filtration is $\mathrm{Cl}(\Phi_{\mathbb{R}}(G, w))$. Theorem 7 tells us that, while it is possible to reconstruct the original data structure from the non-metrical filtration, this

is not possible when we obfuscate the data by adopting a metrical lens, confirming the empirical results found in [15] and the validity of that approach, which also finds further empirical support in [16] and [17].

4.4. Conclusions

In this article we proved the categorical equivalence between a subcategory of weighted graphs and that of corresponding *P*-persistent objects. Moreover, we showed how these results influence topological data analysis in the construction of suitable filtrations of simplicial complexes, which are able to encode the correct persistent homology. Finally we also showed how these results give a formal window into why it can be unwise to use metric tools to construct simplicial complexes when datasets are not necessary sampled from a metric space.

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Alice PATANIA, Francesco VACCARINO,
Dipartimento di Scienze Matematiche, Politecnico di Torino
C.so Duca degli Abruzzi n.24, 10129 Torino, ITALIA
e-mail: `alice.patania@polito.it, francesco.vaccarino@polito.it`

Alice PATANIA, Giovanni PETRI, Francesco VACCARINO,
ISI Foundation
Via Alassio 11/c, 10126 Torino, ITALIA
e-mail: `alice.patania@isi.it, giovanni.petri@isi.it, francesco.vaccarino@isi.it`