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Original
Ulrich bundles on non-special surfaces with $p_g=0$ and $q=1$ / Casnati, Gianfranco. - In: REVISTA MATEMATICA COMPLUTENSE. - ISSN 1139-1138. - STAMPA. - 32:2(2019), pp. 559-574.

Availability:
This version is available at: 11583/2707351 since: 2020-02-13T09:07:55Z

Publisher:
Springer

Published
DOI:10.1007/s13163-019-00295-1

Rev Mat Complut
https://doi.org/10.1007/s13163-017-0248-z

Ulrich bundles on non-special surfaces with $p_g = 0$ and $q = 1$

Gianfranco Casnati[1]

Received: 8 June 2017 / Accepted: 21 October 2017

Abstract Let S be a surface with $p_g(S) = 0$, $q(S) = 1$ and endowed with a very ample line bundle $\mathcal{O}_S(h)$ such that $h^1\big(S, \mathcal{O}_S(h)\big) = 0$. We show that such an S supports families of dimension p of pairwise non-isomorphic, indecomposable, Ulrich bundles for arbitrary large p. Moreover, we show that S supports stable Ulrich bundles of rank 2 if the genus of the general element in $|h|$ is at least 2.

Keywords Vector bundle · Ulrich bundle

Mathematics Subject Classification Primary 14J60; Secondary 14J26 · 14J27 · 14J28

1 Introduction and notation

Throughout the whole paper we will work on an algebraically closed field k of characteristic 0 and $\mathbb{P}^N$ will denote the projective space over k of dimension N. The word surface will always denote a projective smooth connected surface.

If X is a smooth variety, then the study of vector bundles supported on X is an important tool for understanding its geometric properties. If $X \subseteq \mathbb{P}^N$, then X is naturally polarised by the very ample line bundle $\mathcal{O}_X(h) := \mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{O}_X$: in this

The author is a member of GNSAGA Group of INdAM and is supported by the framework of PRIN 2015 'Geometry of Algebraic Varieties', cofinanced by MIUR.

✉ Gianfranco Casnati
gianfranco.casnati@polito.it

1 Dipartimento di Scienze Matematiche, Politecnico di Torino, c.so Duca degli Abruzzi 24, 10129 Turin, Italy

Published online: 17 November 2017

case, at least from a cohomological point of view, the simplest bundles $\mathcal{F}$ on X are the ones which are *Ulrich with respect to* $\mathcal{O}_X(h)$, i.e. such that

$$h^i\big(X, \mathcal{F}(-ih)\big) = h^j\big(X, \mathcal{F}(-(j+1)h)\big) = 0$$

for each $i > 0$ and $j < \dim(X)$.

The existence of Ulrich bundles on each variety is a problem raised by D. Eisenbud and F.O. Schreyer in [19] (see [10] for a survey on Ulrich bundles). There are many partial results (e.g. see [2,3,7–9,11–13,15,17,18,26–28,31]). Nevertheless, all such results and those ones proved in [20] seem to suggest that Ulrich bundles exist at least when X satisfies an extra technical condition, namely that X is *arithmetically Cohen–Macaulay*, i.e. projectively normal and such that

$$h^i\big(X, \mathcal{O}_S(th)\big) = 0$$

for each $i = 1, \dots, \dim(X) - 1$ and $t \in \mathbb{Z}$. When X is not arithmetically Cohen–Macaulay, the literature is very limited (e.g. see [9] and [14]).

Now let $S \subseteq \mathbb{P}^N$ be a surface and set $p_g(S) := h^2\big(S, \mathcal{O}_S\big)$, $q(S) := h^1\big(S, \mathcal{O}_S\big)$, whence $\chi(\mathcal{O}_S) := 1 - q(S) + p_g(S) = 0$. Thanks to the Enriques–Kodaira classification of surfaces, we know that $\kappa(S) \leq 1$ and $K_S^2 \leq 0$ (see [6], Theorem X.4 and Lemma VI.1). In what follows we will denote by $\mathrm{Pic}(S)$ the Picard group of S: it is a group scheme and the connected component $\mathrm{Pic}^0(S) \subseteq \mathrm{Pic}(S)$ of the identity is an abelian variety of dimension $q(S)$ parameterising the line bundles algebraically equivalent to $\mathcal{O}_S$.

In this paper we first rewrite the proof of Proposition 6 of [10], in order to be able to extend its statement to a slightly wider class of surfaces.

Our modified statement is as follows: recall that $\mathcal{O}_S(h)$ is called *special* if $h^1\big(S, \mathcal{O}_S(h)\big) \neq 0$, *non-special* otherwise.

Theorem 1.1 *Let S be a surface with $p_g(S) = 0$, $q(S) = 1$ and endowed with a very ample non-special line bundle $\mathcal{O}_S(h)$.*

If $\mathcal{O}_S(\eta) \in \mathrm{Pic}^0(S) \setminus \{\, \mathcal{O}_S \,\}$ is such that $h^0\big(S, \mathcal{O}_S(K_S \pm \eta)\big) = h^1\big(S, \mathcal{O}_S(h \pm \eta)\big) = 0$, then for each general $C \in |\mathcal{O}_S(h)|$ and each general set $Z \subseteq C$ of $h^0\big(S, \mathcal{O}_S(h)\big)$ points, there is a rank 2 Ulrich bundle $\mathcal{E}$ with respect to $\mathcal{O}_S(h)$ fitting into the exact sequence

$$0 \longrightarrow \mathcal{O}_S(h + K_S + \eta) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{Z|S}(2h + \eta) \longrightarrow 0. \tag{1}$$

As pointed out in [10], Proposition 6, when S is a bielliptic surface then each very ample line bundle $\mathcal{O}_S(h)$ is automatically non-special and there always exists a non-trivial $\mathcal{O}_S(\eta) \in \mathrm{Pic}^0(S)$ of order 2 satisfying the above vanishings: thus the bundle $\mathcal{E}$ defined in Theorem 1.1 is actually *special*, i.e. $c_1(\mathcal{E}) = 3h + K_S$. We can argue similarly if S is either *anticanonical*, i.e. $|-K_S| \neq \emptyset$, or geometrically ruled.

A condition forcing the indecomposability of a coherent sheaf $\mathcal{F}$ on an n–dimensional variety X is its stability. Recall that the *slope* $\mu(\mathcal{F})$ and the *reduced Hilbert polynomial* $p_{\mathcal{F}}(t)$ of $\mathcal{F}$ with respect to the very ample polarisation $\mathcal{O}_X(h)$ are

$$\mu(\mathcal{F}) = c_1(\mathcal{F})h^{n-1}/\mathrm{rk}(\mathcal{F}), \qquad p_{\mathcal{F}}(t) = \chi(\mathcal{F}(th))/\mathrm{rk}(\mathcal{F}).$$

The coherent sheaf $\mathcal{F}$ is called μ–semistable (resp. μ–stable) if for all subsheaves $\mathcal{G}$ with $0 < \mathrm{rk}(\mathcal{G}) < \mathrm{rk}(\mathcal{F})$ we have $\mu(\mathcal{G}) \le \mu(\mathcal{F})$ (resp. $\mu(\mathcal{G}) < \mu(\mathcal{F})$).

The coherent sheaf $\mathcal{F}$ is called semistable (resp. stable) if for all $\mathcal{G}$ as above $p_{\mathcal{G}}(t) \le p_{\mathcal{F}}(t)$ (resp. $p_{\mathcal{G}}(t) < p_{\mathcal{F}}(t)$) for $t \gg 0$.

On an arbitrary variety we have the following chain of implications

$$\mathcal{F} \text{ is } \mu-\text{stable} \Rightarrow \mathcal{F} \text{ is stable} \Rightarrow \mathcal{F} \text{ is semistable} \Rightarrow \mathcal{F} \text{ is } \mu\text{-semistable}.$$

Nevertheless, when we restrict our attention to Ulrich bundles, the two notions of (semi)stability and μ–(semi)stability actually coincide.

A priori, it is not clear whether the bundles constructed in Theorem 1.1 are stable. In Sect. 4 we deal with their stability as follows. The *sectional genus* of S with respect to $\mathcal{O}_S(h)$ is defined as the genus of a general element of $|h|$. By the adjunction formula

$$\pi(\mathcal{O}_S(h)) := \frac{h^2 + hK_S}{2} + 1.$$

Notice that the equality $\pi(\mathcal{O}_S(h)) = 0$ would imply the rationality of S (e.g. see [1] and the references therein), contradicting $q(S) = 1$. Thus $\pi(\mathcal{O}_S(h)) \ge 1$ in our setup.

Theorem 1.2 *Let S be a surface with $p_g(S) = 0$, $q(S) = 1$ and endowed with a very ample non-special line bundle $\mathcal{O}_S(h)$.*

If $\pi(\mathcal{O}_S(h)) \ge 2$, then the bundle $\mathcal{E}$ constructed in Theorem 1.1 from a general set $Z \subseteq C \subseteq S$ of $h^0\big(S, \mathcal{O}_S(h)\big)$ points is stable.

Once that the existence of Ulrich bundles of low rank is proved, one could be interested in understanding how large a family of Ulrich bundles supported on S can actually be. In particular we say that a smooth variety $X \subseteq \mathbb{P}^N$ is *Ulrich–wild* if it supports families of dimension p of pairwise non-isomorphic, indecomposable, Ulrich bundles for arbitrary large p.

The last result proved in this paper concerns the Ulrich–wildness of the surfaces we are dealing with.

Theorem 1.3 *Let S be a surface with $p_g(S) = 0$, $q(S) = 1$ and endowed with a very ample non-special line bundle $\mathcal{O}_S(h)$. Then S is Ulrich–wild.*

In Sect. 2 we list some general results on Ulrich bundles on polarised surfaces. In Sect. 3 we prove Theorem 1.1. In Sect. 4 we first recall some easy facts about the stability of Ulrich bundles, giving finally the proof of Theorem 1.2. In Sect. 5 we prove Theorem 1.3.

Finally, the author would like to thank the referee for her/his comments which have allowed us to improve the whole exposition.

2 General results

In general, an Ulrich bundle $\mathcal{F}$ on $X \subseteq \mathbb{P}^N$ collects many interesting properties (see Sect. 2 of [19]). The following ones are particularly important.

- $\mathcal{F}$ is globally generated and its direct summands are Ulrich as well.
- $\mathcal{F}$ is *initialized*, i.e. $h^0\big(X, \mathcal{F}(-h)\big) = 0$ and $h^0\big(X, \mathcal{F}\big) \neq 0$.
- $\mathcal{F}$ is *aCM*, i.e. $h^i\big(X, \mathcal{F}(th)\big) = 0$ for each $i = 1, \dots, \dim(X) - 1$ and $t \in \mathbb{Z}$.

Let S be a surface. The Serre duality for $\mathcal{F}$ is

$$h^i\big(S, \mathcal{F}\big) = h^{2-i}\big(S, \mathcal{F}^\vee(K_S)\big), \qquad i = 0, 1, 2,$$

and the Riemann–Roch theorem is

$$\begin{aligned} h^0\big(S, \mathcal{F}\big) + h^2\big(S, \mathcal{F}\big) = h^1\big(S, \mathcal{F}\big) \\ + \operatorname{rk}(\mathcal{F})\chi(\mathcal{O}_S) + \frac{c_1(\mathcal{F})(c_1(\mathcal{F}) - K_S)}{2} - c_2(\mathcal{F}). \end{aligned} \tag{2}$$

Proposition 2.1 *Let S be a surface endowed with a very ample line bundle $\mathcal{O}_S(h)$. If $\mathcal{E}$ is a vector bundle on S, then the following assertions are equivalent:*

1. *$\mathcal{E}$ is an Ulrich bundle with respect to $\mathcal{O}_S(h)$;*
2. *$\mathcal{E}^\vee(3h + K_S)$ is an Ulrich bundle with respect to $\mathcal{O}_S(h)$;*
3. *$\mathcal{E}$ is an aCM bundle and*

$$\begin{aligned} c_1(\mathcal{E})h &= \operatorname{rk}(\mathcal{E})\frac{3h^2 + hK_S}{2}, \\ c_2(\mathcal{E}) &= \frac{c_1(\mathcal{E})^2 - c_1(\mathcal{E})K_S}{2} - \operatorname{rk}(\mathcal{E})(h^2 - \chi(\mathcal{O}_S)); \end{aligned} \tag{3}$$

4. *$h^0\big(S, \mathcal{E}(-h)\big) = h^0\big(S, \mathcal{E}^\vee(2h + K_S)\big) = 0$ and Equalities (3) hold.*

Proof See [14], Proposition 2.1. □

The following corollaries are immediate consequences of the above characterization.

Corollary 2.2 *Let S be a surface endowed with a very ample line bundle $\mathcal{O}_S(h)$. If $\mathcal{O}_S(D)$ is a line bundle on S, then the following assertions are equivalent:*

1. *$\mathcal{O}_S(D)$ is an Ulrich bundle with respect to $\mathcal{O}_S(h)$;*
2. *$\mathcal{O}_S(3h + K_S - D)$ is an Ulrich bundle with respect to $\mathcal{O}_S(h)$;*
3. *$\mathcal{O}_S(D)$ is an aCM bundle and*

$$D^2 = 2(h^2 - \chi(\mathcal{O}_S)) + DK_S, \qquad Dh = \frac{1}{2}(3h^2 + hK_S); \tag{4}$$

4. *$h^0\big(S, \mathcal{O}_S(D - h)\big) = h^0\big(S, \mathcal{O}_S(2h + K_S - D)\big) = 0$ and Equalities (4) hold.*

Proof See [14], Corollary 2.2. □

Recall that a rank 2 Ulrich bundle $\mathcal{E}$ on S is special if $c_1(\mathcal{E}) = 3h + K_S$.

Corollary 2.3 *Let S be a surface endowed with a very ample line bundle $\mathcal{O}_S(h)$.*

If $\mathcal{E}$ is a vector bundle of rank 2 on S, then the following assertions are equivalent:

1. *$\mathcal{E}$ is a special Ulrich bundle with respect to $\mathcal{O}_S(h)$;*
2. *$\mathcal{E}$ is initialized and*

$$c_1(\mathcal{E}) = 3h + K_S, \qquad c_2(\mathcal{E}) = \frac{5h^2 + 3hK_S}{2} + 2\chi(\mathcal{O}_S).$$

Proof See [14], Corollary 2.4. □

3 Existence of rank 2 Ulrich bundles

We start this section by recalling some facts on the classical Picard variety of a surface.

Lemma 3.1 *Let S be a surface endowed with a very ample line bundle $\mathcal{O}_S(h)$.*

Let $C \in |h|$ be general and let $i : C \to S$ be the inclusion map. Then the morphism $i^* \colon \operatorname{Pic}^0(S) \to \operatorname{Pic}^0(C)$ is injective.

Proof Let $\mathcal{O}_S(\eta) \in \operatorname{Pic}^0(S)\setminus\{\ \mathcal{O}_S\ \}$. The cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_S(\eta - h) \longrightarrow \mathcal{O}_S(\eta) \longrightarrow i^*\mathcal{O}_S(\eta) \longrightarrow 0$$

yields the exact sequence

$$H^0\big(S, \mathcal{O}_S(\eta)\big) \longrightarrow H^0\big(C, i^*\mathcal{O}_S(\eta)\big) \longrightarrow H^1\big(S, \mathcal{O}_S(\eta - h)\big).$$

Since $\mathcal{O}_S(\eta) \in \operatorname{Pic}^0(S)\setminus\{\ \mathcal{O}_S\ \}$, it follows that $h^0\big(S, \mathcal{O}_S(\eta)\big) = 0$. The Kodaira vanishing theorem implies $h^1\big(S, \mathcal{O}_S(\eta - h)\big) = 0$. We deduce $h^0\big(C, i^*\mathcal{O}_S(\eta)\big) = 0$, hence $i^*\mathcal{O}_S(\eta) \not\cong \mathcal{O}_C$. □

Now, let S be a surface with $p_g(S) = 0$ and $q(S) = 1$. Then $\operatorname{Pic}^0(S)$ is an elliptic curve: in particular $\operatorname{Pic}^0(S)$ contains three pairwise distinct non-trivial divisors of order 2 whose restrictions to C are still non-trivial and pairwise non-isomorphic, thanks to Lemma 3.1 above.

In order to prove Theorem 1.1 we will make use of the Hartshorne–Serre correspondence on surfaces. We recall that a locally complete intersection subscheme Z of dimension zero on a surface S is Cayley–Bacharach (CB for short) with respect to a line bundle $\mathcal{O}_S(A)$ if, for each $Z' \subseteq Z$ of degree $\deg(Z) - 1$, the natural morphism $H^0\big(S, \mathcal{I}_{Z|S}(A)\big) \to H^0\big(S, \mathcal{I}_{Z'|S}(A)\big)$ is an isomorphism.

Theorem 3.2 *Let S be a surface and $Z \subseteq S$ a locally complete intersection subscheme of dimension* 0.

Then there exists a vector bundle $\mathcal{F}$ of rank 2 *on S fitting into an exact sequence of the form*

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_{Z|S}(A) \longrightarrow 0, \tag{5}$$

if and only if Z is CB with respect to $\mathcal{O}_S(A + K_S)$.

Proof See Theorem 5.1.1 in [23]. □

We now prove Theorem 1.1 stated in the introduction. As we already noticed therein, its proof for $hK_S = 0$ coincides with the one of Proposition 6 in [10] because in this case the vanishing $h^1\big(S, \mathcal{O}_S(h \pm \eta)\big) = 0$ follows immediately from the Kodaira vanishing theorem as we will show below in Corollary 3.4.

Proof of Theorem 1.1 Recall that by hypothesis $p_g(S) = h^1\big(S, \mathcal{O}_S(h)\big) = 0$ and $q(S) = 1$. It follows that $\chi(\mathcal{O}_S) = 0$ and

$$h^2\big(S, \mathcal{O}_S(h)\big) = h^0\big(S, \mathcal{O}_S(K_S - h)\big) \le h^0\big(S, \mathcal{O}_S(K_S)\big) = 0,$$

thus $S \subseteq \mathbb{P}^N$, where

$$N := h^0\big(S, \mathcal{O}_S(h)\big) - 1 = \frac{h^2 - hK_S}{2} - 1 \ge 4, \tag{6}$$

because $q(S) = 0$ for each surface $S \subseteq \mathbb{P}^3$.

Let $C := S \cap H \in |h|$ be a general hyperplane section and let $i : C \to S$ be the inclusion morphism. The curve C is non-degenerate in $\mathbb{P}^{N-1} \cong H \subseteq \mathbb{P}^N$. Indeed the exact sequence

$$0 \longrightarrow \mathcal{I}_{S|\mathbb{P}^N}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^N}(1) \longrightarrow \mathcal{O}_S(h) \longrightarrow 0$$

implies $h^0\big(\mathbb{P}^N, \mathcal{I}_{S|\mathbb{P}^N}(1)\big) = h^1\big(\mathbb{P}^N, \mathcal{I}_{S|\mathbb{P}^N}(1)\big) = 0$. Thus, the exact sequence

$$0 \longrightarrow \mathcal{I}_{S|\mathbb{P}^N}(1) \longrightarrow \mathcal{I}_{C|\mathbb{P}^N}(1) \longrightarrow \mathcal{I}_{C|S}(h) \longrightarrow 0$$

implies $h^0\big(\mathbb{P}^N, \mathcal{I}_{C|\mathbb{P}^N}(1)\big) = 1$, because $\mathcal{I}_{C|S}(h) \cong \mathcal{O}_S$. Finally the exact sequence

$$0 \longrightarrow \mathcal{I}_{H|\mathbb{P}^N}(1) \longrightarrow \mathcal{I}_{C|\mathbb{P}^N}(1) \longrightarrow \mathcal{I}_{C|H}(1) \longrightarrow 0$$

and the isomorphism $\mathcal{I}_{H|\mathbb{P}^N}(1) \cong \mathcal{O}_{\mathbb{P}^N}$ yields $h^0\big(C, \mathcal{I}_{C|H}(1)\big) = 0$.

It follows the existence of a reduced subscheme $Z \subseteq C \subseteq S$ of degree $N+1$ whose points are in general position inside $H \cong \mathbb{P}^{N-1}$. Thus Z is CB with respect to $\mathcal{O}_S(h)$, hence there exists Sequence (5) with $\mathcal{O}_S(A) \cong \mathcal{O}_S(h - K_S)$, thanks to Theorem 3.2.

Let $\mathcal{O}_S(\eta) \in \mathrm{Pic}^0(S) \setminus \{\mathcal{O}_S\}$ be such that $h^0\big(S, \mathcal{O}_S(K_S \pm \eta)\big) = h^1\big(S, \mathcal{O}_S(h \pm \eta)\big) = 0$ and set $\mathcal{E} := \mathcal{F}(h + K_S + \eta)$. The bundle $\mathcal{E}$ fits into Sequence (1) and satisfies Equalities (3). If we show that $h^0\big(S, \mathcal{E}(-h)\big) = h^0\big(S, \mathcal{E}^\vee(2h + K_S)\big) = 0$, then we conclude that $\mathcal{E}$ is Ulrich thanks to Proposition 2.1 above. Notice that the second vanishing is equivalent to $h^0\big(S, \mathcal{E}(-h - 2\eta)\big) = 0$ because $c_1(\mathcal{E}) = 3h + K_S + 2\eta$.

The vanishing $h^0\big(S, \mathcal{O}_S(K_S \pm \eta)\big) = 0$ implies

$$h^0\big(S, \mathcal{E}(-h)\big) \le h^0\big(S, \mathcal{I}_{Z|S}(h + \eta)\big), \qquad h^0\big(S, \mathcal{E}(-h - 2\eta)\big) \le h^0\big(S, \mathcal{I}_{Z|S}(h - \eta)\big).$$

The exact sequence

$$0 \longrightarrow \mathcal{I}_{C|S} \longrightarrow \mathcal{I}_{Z|S} \longrightarrow \mathcal{I}_{Z|C} \longrightarrow 0 \tag{7}$$

and the isomorphisms $\mathcal{I}_{C|S} \cong \mathcal{O}_S(-h)$ and $\mathcal{I}_{Z|C} \cong \mathcal{O}_C(-Z)$ imply

$$h^0\big(S, \mathcal{I}_{Z|S}(h \pm \eta)\big) \le h^0\big(C, \mathcal{O}_C(-Z) \otimes \mathcal{O}_S(h \pm \eta)\big)$$

because $h^0\big(S, \mathcal{O}_S(\pm\eta)\big) = 0$. Thanks to the general choice of the points in Z, the Riemann–Roch theorem on C and the adjunction formula $\mathcal{O}_C(K_C) \cong i^*\mathcal{O}_S(h + K_S)$ on S give

$$\begin{aligned} h^0\big(C, \mathcal{O}_C(-Z) \otimes \mathcal{O}_S(h \pm \eta)\big) &= h^0\big(C, i^*\mathcal{O}_S(h \pm \eta)\big) - \deg(Z) \\ &= h^2+1-\pi(\mathcal{O}_S(h)) - \deg(Z)+h^1\big(C, i^*\mathcal{O}_S(h \pm \eta)\big)=h^0\big(C, i^*\mathcal{O}_S(K_S \mp \eta)\big). \end{aligned}$$

The exact sequence

$$0 \longrightarrow \mathcal{O}_S(-h) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_C \longrightarrow 0 \tag{8}$$

implies the existence of the exact sequence

$$\begin{aligned} H^0\big(S, \mathcal{O}_S(K_S \mp \eta)\big) &\longrightarrow H^0\big(C, i^*\mathcal{O}_S(K_S \mp \eta)\big) \\ &\longrightarrow H^1\big(S, \mathcal{O}_S(K_S - h \mp \eta)\big) \cong H^1\big(S, \mathcal{O}_S(h \pm \eta)\big). \end{aligned}$$

Thus the hypothesis on $\mathcal{O}_S(K_S \pm \eta)$ and $\mathcal{O}_S(h \pm \eta)$ forces $h^0\big(C, i^*\mathcal{O}_S(K_S \mp \eta)\big) = 0$. □

It is natural to ask when the vanishings $h^1\big(S, \mathcal{O}_S(K_S \pm \eta)\big) = h^1\big(S, \mathcal{O}_S(h \pm \eta)\big) = 0$ actually occur. We list below some related result.

Corollary 3.3 *Let S be a surface with $p_g(S) = 0$, $q(S) = 1$ and endowed with a very ample non-special line bundle $\mathcal{O}_S(h)$.*

Then S supports Ulrich bundles of rank $r \le 2$.

Proof Since each direct summand of an Ulrich bundle is Ulrich as well, it follows from Theorem 1.1 that it suffices to prove the existence of $\mathcal{O}_S(\eta) \in \mathrm{Pic}^0(S) \setminus \{ \mathcal{O}_S \}$ such that $h^0\big(S, \mathcal{O}_S(K_S \pm \eta)\big) = h^1\big(S, \mathcal{O}_S(h \pm \eta)\big) = 0$.

Let $\mathcal{P}$ be the Poincaré line bundle on $S \times \mathrm{Pic}(S)$. Recall that (e.g. see [29], Lecture 19), if $p\colon S \times \mathrm{Pic}(S) \to \mathrm{Pic}(S)$ is the projection on the second factor and $\mathcal{L} \in \mathrm{Pic}(S)$, then the restriction of $\mathcal{P}$ to the fibre $p^{-1}(\mathcal{L}) \cong S$ is isomorphic to the line bundle $\mathcal{L}$. The line bundle $\mathcal{P}$ is thus flat on $\mathrm{Pic}(S)$.

Let $\mathcal{P}_0$ be the restriction of $\mathcal{P}$ to $S \times \mathrm{Pic}^0(S)$, $A \subseteq S$ a divisor, $s\colon S \times \mathrm{Pic}(S) \to S$ the projection on the first factor. The line bundle $\mathcal{P}_0 \otimes s^*\mathcal{O}_S(A)$ is flat over $\mathrm{Pic}^0(S)$ and parameterizes the line bundles on S algebraically equivalent to $\mathcal{O}_S(A)$. Thus the semicontinuity theorem (e.g. see Theorem III.12.8 of [22]) applied to the sheaf

$\mathcal{P}_0 \otimes s^*\mathcal{O}_S(A)$ and the map $p_0\colon S \times \mathrm{Pic}^0(S) \to \mathrm{Pic}^0(S)$ imply that for each $i = 0, 1, 2$ and $c \in \mathbb{Z}$ the sets

$$\mathcal{V}^i_A(c) := \{\, \eta \in \mathrm{Pic}^0(S) \mid h^i\big(S, \mathcal{O}_S(A \pm \eta)\big) > c \,\},$$

are closed inside $\mathrm{Pic}^0(S)$. In particular $\mathcal{V} := \mathcal{V}^1_h(0) \cup \mathcal{V}^0_{K_S}(0)$ is closed.

By definition $\mathcal{O}_S \in \mathrm{Pic}^0(S)\backslash\mathcal{V} \neq \emptyset$. Thus for each general $\mathcal{O}_S(\eta) \in \mathrm{Pic}^0(S)$, the hypothesis $h^0\big(S, \mathcal{O}_S(K_S \pm \eta)\big) = h^1\big(S, \mathcal{O}_S(h \pm \eta)\big) = 0$ is satisfied and the statement is then completely proved. □

Notice that the above result guarantees the existence of an Ulrich bundle $\mathcal{E}$ with $c_1(\mathcal{E}) = 3h + K_S + 2\eta$ fitting into Sequence (1). Such bundle is special if and only if $\mathcal{O}_S(\eta)$ has order 2. It is not clear if such a choice can be done in general. Anyhow in some particular cases we can easily prove an existence result also for special Ulrich bundles: we start from Beauville's result for *bielliptic surfaces*, i.e. minimal surfaces S with $p_g(S) = 0$, $q(S) = 1$ and $\kappa(S) = 0$ (see Proposition 6 of [10]).

Corollary 3.4 *Let S be a bielliptic surface endowed with a very ample line bundle $\mathcal{O}_S(h)$.*

Then $\mathcal{O}_S(h)$ is non-special and S supports special Ulrich bundles of rank 2.

Proof If $\kappa(S) = 0$, then K_S is numerically trivial, hence $h - K_S \pm \eta$ is ample for each choice of $\mathcal{O}_S(\eta) \in \mathrm{Pic}^0(S)$, thanks to the Nakai criterion. Thus the vanishing $h^1\big(S, \mathcal{O}_S(h \pm \eta)\big) = 0$ follows from the Kodaira vanishing theorem: in particular $\mathcal{O}_S(h)$ is non-special.

We can find $\mathcal{O}_S(\eta) \in \mathrm{Pic}^0(S)\backslash\{\, \mathcal{O}_S, \mathcal{O}_S(\pm K_S) \,\}$ of order 2, because there are three non-trivial and pairwise non-isomorphic elements of order 2 in $\mathrm{Pic}^0(S)$. Thus $h^0\big(S, \mathcal{O}_S(K_S \pm \eta)\big) = 0$ because $K_S \pm \eta$ is not trivial by construction, hence the statement follows from Theorem 1.1. □

The surface S is *anticanonical* if $|-K_S| \neq \emptyset$: in particular $p_g(S) = 0$. The ampleness of $\mathcal{O}_S(h)$ implies $hK_S < 0$ in this case.

Corollary 3.5 *Let S be an anticanonical surface with $q(S) = 1$ and endowed with a very ample line bundle $\mathcal{O}_S(h)$.*

Then $\mathcal{O}_S(h)$ is non-special and S supports special Ulrich bundles of rank 2.

Proof If $A \in |-K_S|$, then $\omega_A \cong \mathcal{O}_A$ by the adjunction formula. We have $h^1\big(A, \mathcal{O}_S(h \pm \eta) \otimes \mathcal{O}_A\big) = h^0\big(A, \mathcal{O}_S(-h \mp \eta) \otimes \mathcal{O}_A\big)$, for each $\mathcal{O}_S(\eta) \in \mathrm{Pic}^0(S)$.

On the one hand, if $h^0\big(A, \mathcal{O}_S(-h \mp \eta) \otimes \mathcal{O}_A\big) > 0$, then $-hC \geq 0$ for some irreducible component $C \subseteq A$. On the other hand $\mathcal{O}_S(h)$ is ample, hence $hC > 0$.

The contradiction implies $h^0\big(A, \mathcal{O}_S(-h \mp \eta) \otimes \mathcal{O}_A\big) = 0$, hence the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_S(h + K_S \mp \eta) \longrightarrow \mathcal{O}_S(h \mp \eta) \longrightarrow \mathcal{O}_S(h \mp \eta) \otimes \mathcal{O}_A \longrightarrow 0$$

and the Kodaira vanishing theorem yield $h^1\big(S, \mathcal{O}_S(h \mp \eta)\big) = 0$. In particular $\mathcal{O}_S(h)$ is non-special. Finally $hK_S < 0$, hence $h^0\big(S, \mathcal{O}_S(K_S \pm \eta)\big) = 0$.

The statement then follows from Theorem 1.1 by taking any non-trivial $\mathcal{O}_S(\eta) \in \mathrm{Pic}^0(S)$ of order 2. □

Recall that a *geometrically ruled surface* is a surface S with a surjective morphism $p\colon S \to E$ onto a smooth curve such that every fibre of p is isomorphic to $\mathbb{P}^1$. If S is geometrically ruled, then $p_g(S) = 0$ and $q(S)$ is the genus of E (see [22], Chapter V.2 for further details).

Remark 3.6 Let S be a geometrically ruled surface on an elliptic curve E so that $p_g(S) = 0$ and $q(S) = 1$. Thanks to the results in [22], Chapter V.2, we know the existence of a vector bundle $\mathcal{H}$ of rank 2 on E such that $h^0\big(E, \mathcal{H}\big) \neq 0$ and $h^0\big(E, \mathcal{H}(-P)\big) = 0$ for each $P \in E$ and $S \cong \mathbb{P}(\mathcal{H})$. Then p can be identified with the natural projection map $\mathbb{P}(\mathcal{H}) \to E$. The group $\mathrm{Pic}(S)$ is generated by the class ξ of $\mathcal{O}_{\mathbb{P}(\mathcal{H})}(1)$ and by $p^*\,\mathrm{Pic}(E)$. If we set $\mathcal{O}_E(\mathfrak{h}) := \det(\mathcal{H})$ and $e := -\deg(\mathfrak{h})$, then $e \geq -1$ (see [30]). Moreover, $K_S = -2\xi + p^*\mathfrak{h}$.

There exists an exact sequence

$$0 \longrightarrow \mathcal{O}_E \longrightarrow \mathcal{H} \longrightarrow \mathcal{O}_E(\mathfrak{h}) \longrightarrow 0. \tag{9}$$

The symmetric product of Sequence (9) yields

$$0 \longrightarrow \mathcal{H}(-\mathfrak{h}) \longrightarrow S^2\mathcal{H}(-\mathfrak{h}) \longrightarrow \mathcal{O}_E(\mathfrak{h}) \longrightarrow 0. \tag{10}$$

Sequence (9) splits if and only if $\mathcal{H}$ is decomposable. Thus, if this occurs, then $S^2\mathcal{H}(-\mathfrak{h})$ contains $\mathcal{O}_E$ as direct summand, whence

$$h^0\big(S, \mathcal{O}_S(-K_S)\big) \geq h^0\big(E, \mathcal{O}_E\big) = 1. \tag{11}$$

because $h^0\big(S, \mathcal{O}_S(-K_S)\big) = h^0\big(E, S^2\mathcal{H}(-\mathfrak{h})\big)$, thanks to the projection formula.

Assume that $\mathcal{H}$ is indecomposable. Then either $\mathcal{O}_E(\mathfrak{h}) = \mathcal{O}_E$ or $\mathcal{O}_E(\mathfrak{h}) \neq \mathcal{O}_E$. In the first case the cohomology of Sequences (9) and (10) again implies Inequality (11).

If $\mathcal{O}_E(\mathfrak{h}) \neq \mathcal{O}_E$, then Lemma 22 of [4] implies that $S^2\mathcal{H}(-\mathfrak{h})$ is the direct sum of the three non-trivial elements of order 2 of $\mathrm{Pic}(E)$, hence $h^0\big(S, \mathcal{O}_S(-K_S)\big) = 0$.

We conclude that a geometrically ruled surface on an elliptic curve is anticanonical if and only if $e \geq 0$.

Thanks to the above remark and Corollary 3.5, we know that each geometrically ruled surface S with $q(S) = 1$ and $e \geq 0$ supports special Ulrich bundles of rank 2 with respect to each very ample line bundle $\mathcal{O}_S(h)$. We can extend the result also to the case $e = -1$.

Corollary 3.7 *Let S be a geometrically ruled surface with $q(S) = 1$ and endowed with a very ample line bundle $\mathcal{O}_S(h)$.*

Then $\mathcal{O}_S(h)$ is non-special and S supports special Ulrich bundles of rank 2.

Proof We have to prove the statement only for $e = -1$. If $\mathcal{O}_S(h) = \mathcal{O}_{\mathbb{P}(\mathcal{H})}(a\xi + p^*\mathfrak{b})$, then $\deg(\mathfrak{b}) > -a/2$ (see [22], Proposition V.2.21). Thus the Table in Proposition 3.1 of [21] implies that $h^1\big(S, \mathcal{O}_S(h \pm \eta)\big) = 0$ for each $\eta \in \mathrm{Pic}^0(S)$.

Again the statement follows from Theorem 1.1 by taking any non-trivial $\mathcal{O}_S(\eta)$ of order 2. □

Remark 3.8 When $g = 1$, the corollary above extend Propositions 3.1, 3.3 and Theorem 3.4 of [2] to the range $e \le 0$.

Recall that an embedded surface $S \subseteq \mathbb{P}^N$ is called *non-degenerate* if it is not contained in any hyperplane.

Corollary 3.9 *Let $S \subseteq \mathbb{P}^4$ be a non-degenerate non-special surface with $p_g(S) = 0$. Then S supports special Ulrich bundles of rank* 2.

Proof The cohomology of Sequence (8) tensored by $\mathcal{O}_S(h)$ implies $h^1\big(C, i^*\mathcal{O}_S(h)\big) = 0$. In particular such surfaces are sectionally non-special (see [24] for details). Non-special and sectionally non-special surfaces are completely classified in [24] and [25]. They satisfy $q(S) \le 1$ and, if equality holds, then they are either quintic scrolls over elliptic curves, or the Serrano surfaces (these are very special bielliptic surfaces of degree 10: see [32]). The results above and Sect. 4 of [14] yields the statement. □

Remark 3.10 Linearly normal non-special surface $S \subseteq \mathbb{P}^4$ with $p_g(S) = 0$ satisfy $3 \le h^2 \le 10$ (see [24] and [25]). If $h^2 \le 6$, such surfaces are known to support Ulrich line bundles: see [27] for the case $q(S) = 0$ and [10], Assertion 2) of Proposition 5 for the case $q(S) = 1$.

4 Stability of Ulrich bundles

We start this section by recalling the following result: see [13], Theorem 2.9 for its proof.

Theorem 4.1 *Let X be a smooth variety endowed with a very ample line bundle $\mathcal{O}_X(h)$.*

If $\mathcal{E}$ is an Ulrich bundle on X with respect to $\mathcal{O}_X(h)$, the following assertions hold:

1. *$\mathcal{E}$ is semistable and μ–semistable;*
2. *$\mathcal{E}$ is stable if and only if it is μ–stable;*
3. *if*
$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{M} \longrightarrow 0$$
is an exact sequence of coherent sheaves with $\mathcal{M}$ torsion free and $\mu(\mathcal{L}) = \mu(\mathcal{E})$, then both $\mathcal{L}$ and $\mathcal{M}$ are Ulrich bundles.

We now prove Theorem 1.2 stated in the introduction.

Proof of Theorem 1.2 Assume that $\mathcal{E}$ is not stable: then it is not μ–stable, thanks to Theorem 4.1. In particular there exists a line subbundle $\mathcal{O}_S(D) \subseteq \mathcal{E}$ such that $\mu(\mathcal{E}) = \mu(\mathcal{O}_S(D))$. Again Theorem 4.1 implies that $\mathcal{O}_S(D)$ is Ulrich.

On the one hand, if $\mathcal{O}_S(D)$ is contained in the kernel $\mathcal{K} \cong \mathcal{O}_S(h + K_S + \eta)$ of the map $\mathcal{E} \to \mathcal{I}_{Z|S}(2h + \eta)$ in Sequence (1), then $h^0\big(S, \mathcal{O}_S(h + K_S + \eta - D)\big) \ne 0$. On the other hand, Equality (4) and Inequality (6) imply

$$(h + K_S + \eta - D)h = -\frac{h^2 - hK_S}{2} = 1 - N \leq -3,$$

whence $h^0\big(S, \mathcal{O}_S(h + K_S + \eta - D)\big) = 0$.

We deduce that $\mathcal{O}_S(D) \not\subseteq \mathcal{K}$, hence the composite map $\mathcal{O}_S(D) \subseteq \mathcal{E} \to \mathcal{I}_{Z|S}(2h + \eta)$ is non-zero, i.e.

$$h^0\big(S, \mathcal{I}_{Z|S}(2h + \eta - D)\big) \neq 0. \tag{12}$$

Nevertheless, $\pi(\mathcal{O}_S(h)) \geq 2$ by hypothesis, then

$$(h + \eta - D)h = -\frac{h^2 + hK_S}{2} = 1 - \pi(\mathcal{O}_S(h)) \leq -1,$$

hence $h^0\big(S, \mathcal{I}_{C|S}(2h + \eta - D)\big) = h^0\big(S, \mathcal{O}_S(h + \eta - D)\big) = 0$.

Thus the cohomology of Sequence (7) tensored by $\mathcal{O}_S(2h + \eta - D)$ yields

$$h^0\big(S, \mathcal{I}_{Z|S}(2h + \eta - D)\big) \leq h^0\big(C, \mathcal{I}_{Z|C} \otimes \mathcal{O}_S(2h + \eta - D)\big),$$

hence

$$h^0\big(S, \mathcal{I}_{Z|S}(2h + \eta - D)\big) \leq \max\{\, 0, h^0\big(C, i^*\mathcal{O}_S(2h + \eta - D)\big) - N - 1 \,\}, \tag{13}$$

for a general choice of Z inside C. If $i^*\mathcal{O}_S(2h + \eta - D)$ is special, then the Clifford theorem and the second Equality (4) imply

$$h^0\big(C, i^*\mathcal{O}_S(2h + \eta - D)\big) \leq \frac{(2h + \eta - D)h}{2} + 1 = \frac{N + 3}{2} \leq N, \tag{14}$$

because $N \geq 4$ (see Inequality (6)). If $i^*\mathcal{O}_S(2h + \eta - D)$ is non-special, the Riemann–Roch theorem on C and the second Equality (4) return

$$h^0\big(C, i^*\mathcal{O}_S(2h + \eta - D)\big) = N + 2 - \pi(\mathcal{O}_S(h)) \leq N, \tag{15}$$

because $\pi(\mathcal{O}_S(h)) \geq 2$.

We obtain $h^0\big(S, \mathcal{I}_{Z|S}(2h + \eta - D)\big) = 0$ by combining Inequalities (13), (14) and (15). This equality contradicts Inequality (12), hence the bundle $\mathcal{E}$ is necessarily stable. □

Remark 4.2 If $\pi(\mathcal{O}_S(h)) = 1$, then S is a geometrically ruled surface embedded as a scroll by $\mathcal{O}_S(h) \cong \mathcal{O}_S(\xi + p^*\mathfrak{b})$ over an elliptic curve, thanks to [1], Theorem A (here we are using the notation introduced in Remark 3.6).

Moreover $(h + \eta - D)h = 0$, hence the argument in the above proof does not lead to any contradiction when $\mathcal{O}_S(D) \cong \mathcal{O}_S(h + \eta)$. Such a line bundle is actually Ulrich, because one easily checks that it satisfies all the conditions of Corollary 2.2.

In [16], via a slightly different but similar construction, we are able to show the existence special stable Ulrich bundles of rank 2 on elliptic scrolls.

Let S be a surface with $p_g(S) = 0$, $q(S) = 1$ and endowed with a very ample non-special line bundle $\mathcal{O}_S(h)$. Let

$$c_1 := 3h + K_S + 2\eta, \qquad c_2 := \frac{5h^2 + 3hK_S}{2},$$

where $\mathcal{O}_S(\eta) \in \mathrm{Pic}^0(S)\setminus\{\,\mathcal{O}_S\,\}$ satisfies

$$h^0\big(S, \mathcal{O}_S(K_S \pm \eta)\big) = h^1\big(S, \mathcal{O}_S(h \pm \eta)\big) = 0.$$

If $\pi(\mathcal{O}_S(h)) \geq 2$, then the coarse moduli space $\mathcal{M}_S^s(2; c_1, c_2)$ parameterizing isomorphism classes of stable rank 2 bundles on S with Chern classes c_1 and c_2 is non-empty (see Theorem 1.2). The locus $\mathcal{M}_S^{s,U}(2; c_1, c_2) \subseteq \mathcal{M}_S^s(2; c_1, c_2)$ parameterizing stable Ulrich bundles is open as pointed out in [13].

Proposition 4.3 *Let S be a surface with $p_g(S) = 0$, $q(S) = 1$ and endowed with a very ample non-special line bundle $\mathcal{O}_S(h)$.*

If $\pi(\mathcal{O}_S(h)) \geq 2$, then there is a component $\mathcal{U}_S(\eta)$ of dimension at least $h^2 - K_S^2$ in $\mathcal{M}_S^{s,U}(2; c_1, c_2)$ containing all the points representing the stable bundles $\mathcal{E}$ constructed in Theorem 1.1.

Proof Let us denote by $\mathcal{H}_S$ the Hilbert flag scheme of pairs (Z, C) where $C \in |\mathcal{O}_S(h)|$ and $Z \subseteq C$ is a 0–dimensional subscheme of degree $N+1$. The general $C \in |\mathcal{O}_S(h)|$ is smooth and its image via the map induced by $\mathcal{O}_S(h)$ generate a hyperplane inside $\mathbb{P}^N$. Thus the set $\mathcal{H}_S^U \subseteq \mathcal{H}_S$ of pairs (Z, C) corresponding to sets of points Z in a smooth curve $C \subseteq \mathbb{P}^N$ which are in general position in the linear space generated by C is open and non-empty.

We have a well–defined forgetful dominant morphism $\mathcal{H}_S \to |\mathcal{O}_S(h)|$ whose fibre over C is an open subset of the $(N+1)$–symmetric product of C. In particular $\mathcal{H}_S^U$ is irreducible of dimension $2N+1$. Let (Z, C) represent a point of $\mathcal{H}_S^U$: the Ulrich bundles associated to such a point via the construction described in Theorem 1.1 correspond to the sections of

$$\mathrm{Ext}_S^1\big(\mathcal{I}_{Z|S}(h - K_S), \mathcal{O}\big) \cong H^1\big(S, \mathcal{I}_{Z|S}(h)\big)^\vee.$$

By definition of $\mathcal{H}_S^U$, we have $h^0\big(C, \mathcal{I}_{Z|C}(h)\big) = 0$, hence the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{I}_{Z|C}(h) \longrightarrow \mathcal{O}_C(h) \longrightarrow \mathcal{O}_Z(h) \longrightarrow 0$$

and the Riemann–Roch theorem for $\mathcal{O}_C(h)$ yield $h^1\big(C, \mathcal{I}_{Z|C}(h)\big) = \deg(Z) - \chi(\mathcal{O}_C(h)) = 1$. Sequence (7), the isomorphism $\mathcal{I}_{C|S} \cong \mathcal{O}_S(-h)$ and the hypothesis $q(S) = 1$ and $p_g(S) = 0$ finally return $h^1\big(S, \mathcal{I}_{Z|S}(h)\big) = 2$, because $h^0\big(C, \mathcal{I}_{Z|C}(h)\big) = 0$ by definition.

Thus we have a family $\mathfrak{E}$ of Ulrich bundles of rank 2 with Chern classes c_1 and c_2 parameterised by a vector bundle on $\mathcal{H}_S^U$.

If $\pi(\mathcal{O}_S(h)) \geq 2$, then the bundles in the family are also stable for a general choice of Z. Since stability is an open property in a flat family (see [23], Proposition 2.3.1 and Corollary 1.5.11), it follows the existence of an irreducible open subset $\mathcal{H}_S^{s,U} \subseteq \mathcal{H}_S^U \subseteq \mathcal{H}_S$ of points corresponding to stable bundles.

Thus, we have a morphism $\mathcal{H}_S^{s,U} \to \mathcal{M}_S^{s,U}(2; c_1, c_2)$ whose image parameterizes the isomorphism classes of stable bundles constructed in Theorem 1.1. In particular such bundles, correspond to the points of a single irreducible component $\mathcal{U}_S(\eta) \subseteq \mathcal{M}_S^{s,U}(2; c_1, c_2)$.

Theorems 4.5.4 and 4.5.8 of [23] imply that $\dim(\mathcal{U}_S(\eta)) \geq 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S)$. Taking into account the definitions of c_1 and c_2, simple computations finally yield $\dim(\mathcal{U}_S(\eta)) \geq h^2 - K_S^2$. □

If we have some extra informations on the surface S, then we can describe $\mathcal{U}_S(\eta)$ as the following proposition shows.

Proposition 4.4 *Let S be an anticanonical surface with $p_g(S) = 0$, $q(S) = 1$ and endowed with a very ample line bundle $\mathcal{O}_S(h)$.*

If $\pi(\mathcal{O}_S(h)) \geq 2$, then $\mathcal{U}_S(\eta)$ is non-rational and generically smooth of dimension $h^2 - K_S^2$.

Proof Thanks to Corollary 3.5 we know that $\mathcal{O}_S(h)$ is non-special. Let $A \in |-K_S|$: the cohomology of

$$0 \longrightarrow \mathcal{O}_S(K_S) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_A \longrightarrow 0$$

tensored with $\mathcal{E} \otimes \mathcal{E}^\vee$ yields the exact sequence

$$0 \longrightarrow H^0\big(S, \mathcal{E} \otimes \mathcal{E}^\vee(K_S)\big) \longrightarrow H^0\big(S, \mathcal{E} \otimes \mathcal{E}^\vee\big) \longrightarrow H^0\big(A, \mathcal{E} \otimes \mathcal{E}^\vee \otimes \mathcal{O}_A\big).$$

Since $\mathcal{E}$ is stable (see Theorem 1.2), then it is simple, i.e. $h^0\big(S, \mathcal{E} \otimes \mathcal{E}^\vee\big) = 1$ (see [23], Corollary 1.2.8), hence the map

$$H^0\big(S, \mathcal{E} \otimes \mathcal{E}^\vee\big) \longrightarrow H^0\big(A, \mathcal{E} \otimes \mathcal{E}^\vee \otimes \mathcal{O}_A\big)$$

is injective. We deduce that $h^2\big(S, \mathcal{E} \otimes \mathcal{E}^\vee\big) = h^0\big(S, \mathcal{E} \otimes \mathcal{E}^\vee(K_S)\big) = 0$.

Thus $\mathcal{E}$ corresponds to a smooth point of $\mathcal{U}_S(\eta)$ and $\dim(\mathcal{U}_S(\eta)) = h^2 - K_S^2$, thanks to Corollary 4.5.2 of [23]. Finally, being $q(S) = 1$, then $\mathcal{U}_S(\eta)$ is irregular thanks to [5] as well. □

Remark 3.6 and the above proposition yield the following corollary.

Corollary 4.5 *Let S be a geometrically ruled surface with $q(S) = 1$, $e \geq 0$ and endowed with a very ample line bundle $\mathcal{O}_S(h)$.*

If $\pi(\mathcal{O}_S(h)) \geq 2$, then $\mathcal{U}_S(\eta)$ is non-rational and generically smooth of dimension h^2.

5 Ulrich–wildness

Let S be a surface with $p_g(S) = 0$ and $q(S) = 1$. Moreover $\pi(\mathcal{O}_S(h)) \ge 1$ because S is not rational, as pointed out in the introduction.

We will make use of the following result.

Theorem 5.1 *Let X be a smooth variety endowed with a very ample line bundle $\mathcal{O}_X(h)$.*

If $\mathcal{A}$ and $\mathcal{B}$ are simple Ulrich bundles on X such that $h^1\big(X, \mathcal{A} \otimes \mathcal{B}^\vee\big) \ge 3$ and $h^0\big(X, \mathcal{A} \otimes \mathcal{B}^\vee\big) = h^0\big(X, \mathcal{B} \otimes \mathcal{A}^\vee\big) = 0$, then X is Ulrich–wild.

Proof See [20], Theorem 1 and Corollary 1. □

An immediate consequence of the above Theorem is the proof of Theorem 1.3.

Proof of Theorem 1.3 Recall that S is a surface with $p_g(S) = 0$, $q(S) = 1$ and endowed with a very ample non-special line bundle $\mathcal{O}_S(h)$. We have $\pi(\mathcal{O}_S(h)) \ge 1$, $\chi(\mathcal{O}_S) = 0$ and $K_S^2 \le 0$.

If $\pi(\mathcal{O}_S(h)) \ge 2$, then Theorems 1.1 and 1.2 yield the existence of a stable special Ulrich bundle $\mathcal{E}$ of rank 2 on S.

The local dimension of $\mathcal{M}_S^s(2; c_1, c_2)$ at the point corresponding to $\mathcal{E}$ is at least $4c_2 - c_1^2 = h^2 - K_S^2 \ge 1$. Thus, there exists a second stable Ulrich bundle $\mathcal{G} \not\cong \mathcal{E}$ of rank 2 with $c_i(\mathcal{G}) = c_i$, for $i = 1, 2$. Both $\mathcal{E}$ and $\mathcal{G}$, being stable, are simple (see [23], Corollary 1.2.8).

Due to Proposition 1.2.7 of [23] we have $h^0\big(F, \mathcal{E} \otimes \mathcal{G}^\vee\big) = h^0\big(F, \mathcal{G} \otimes \mathcal{E}^\vee\big) = 0$, thus

$$h^1\big(F, \mathcal{E} \otimes \mathcal{G}^\vee\big) = h^2\big(F, \mathcal{E} \otimes \mathcal{G}^\vee\big) - \chi(\mathcal{E} \otimes \mathcal{G}^\vee) \ge -\chi(\mathcal{E} \otimes \mathcal{G}^\vee).$$

Equality (2) with $\mathcal{F} := \mathcal{E} \otimes \mathcal{G}^\vee$ and the equalities $\mathrm{rk}(\mathcal{E} \otimes \mathcal{G}^\vee) = 4$, $c_1(\mathcal{E} \otimes \mathcal{G}^\vee) = 0$ and $c_2(\mathcal{E} \otimes \mathcal{G}^\vee) = 4c_2 - c_1^2$ imply

$$h^1\big(F, \mathcal{E} \otimes \mathcal{G}^\vee\big) \ge 4c_2 - c_1^2 = h^2 - K_S^2 \ge 3.$$

because surfaces of degree up to 2 are rational. We conclude that S is Ulrich–wild, by Theorem 5.1.

Finally let $\pi(\mathcal{O}_S(h)) = 1$. In this case, S is a geometrically ruled surface on an elliptic curve E thanks to Theorem A of [1] embedded as a scroll bay $\mathcal{O}_S(h)$. Using the notations of Remark 3.6 we can thus assume that $\mathcal{O}_S(h) = \mathcal{O}_S(\xi + p^*\mathfrak{b})$, where $\deg(\mathfrak{b}) \ge e + 3$.

Assertion 2) of Proposition 5 in [10] yields that for each $\vartheta \in \mathrm{Pic}^0(E) \setminus \{\ \mathcal{O}_E\ \}$ the line bundle $\mathcal{L} := \mathcal{O}_S(h + p^*\vartheta) \cong \mathcal{O}_S(\xi + p^*\mathfrak{b} + p^*\vartheta)$ is Ulrich. It follows from Corollary 2.2 that $\mathcal{M} := \mathcal{O}_S(2h + K_S - p^*\vartheta) \cong p^*\mathcal{O}_E(2\mathfrak{b} + \mathfrak{h} - \vartheta)$ is Ulrich too.

Trivially, such bundles are simple and $h^0\big(S, \mathcal{L} \otimes \mathcal{M}^\vee\big) = h^0\big(S, \mathcal{M} \otimes \mathcal{L}^\vee\big) = 0$ because $\mathcal{L} \not\cong \mathcal{M}$. Since $\mathcal{L} \otimes \mathcal{M}^\vee \cong \mathcal{O}_S(\xi - p^*\mathfrak{b} - p^*\mathfrak{h} + 2\vartheta)$ and $e = -\deg(\mathfrak{h}) \ge -1$, it follows from Equality (2) that

$$h^1\big(S, \mathcal{L} \otimes \mathcal{M}^\vee\big) \ge -\chi(\mathcal{L} \otimes \mathcal{M}^\vee) = 2\deg(\mathfrak{b}) - e \ge e + 6 \ge 5.$$

The statement thus again follows from Theorem 5.1. □

The following consequence of the above theorem is immediate, thanks to Corollaries 3.4, 3.5, 3.7.

Corollary 5.2 *Let S be a surface endowed with a very ample line bundle $\mathcal{O}_S(h)$.*

If S is either bielliptic, or anticanonical with $q(S) = 1$, or geometrically ruled with $q(S) = 1$, then it is Ulrich–wild.

The following corollary strengthens the second part of the statements of Theorems 4.13 and 4.18 in [27].

Corollary 5.3 *Let $S \subseteq \mathbb{P}^4$ be a non-degenerate linearly normal non-special surface of degree at least 4 with $p_g(S) = 0$. Then S is Ulrich–wild.*

Proof As pointed out in the proof of Corollary 3.9 the surface S satisfies $q(S) \leq 1$ and if equality holds it is either an elliptic scroll or a bielliptic surface. Theorem 1.3 above and Sect. 5 of [14] yields that S is Ulrich–wild. □

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