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ON LIE ALGEBRAS RESPONSIBLE FOR ZERO-CURVATURE REPRESENTATIONS OF MULTICOMPONENT $(1+1)$-DIMENSIONAL EVOLUTION PDES

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Abstract. Zero-curvature representations (ZCRs) are one of the main tools in the theory of integrable $(1+1)$-dimensional PDEs. According to the preprint arXiv:1212.2199, for any given $(1+1)$-dimensional evolution PDE one can define a sequence of Lie algebras $\mathbb{F}^p$, $p=0,1,2,3,\dots$, such that representations of these algebras classify all ZCRs of the PDE up to local gauge equivalence. ZCRs depending on derivatives of arbitrary finite order are allowed. Furthermore, these algebras provide necessary conditions for existence of Bäcklund transformations between two given PDEs. The algebras $\mathbb{F}^p$ are defined in arXiv:1212.2199 in terms of generators and relations.

In the present paper, we describe some methods to study the structure of the algebras $\mathbb{F}^p$ for multicomponent $(1+1)$-dimensional evolution PDEs. Using these methods, we compute the explicit structure (up to non-essential nilpotent ideals) of the Lie algebras $\mathbb{F}^p$ for the Landau-Lifshitz, nonlinear Schrödinger equations, and for the n-component Landau-Lifshitz system of Golubchik and Sokolov for any $n>3$.

In particular, this means that for the n-component Landau-Lifshitz system we classify all ZCRs (depending on derivatives of arbitrary finite order), up to local gauge equivalence and up to killing nilpotent ideals in the corresponding Lie algebras. As a result of the classification, one obtains two main non-equivalent ZCRs: a well-known ZCR with values in the infinite-dimensional Lie algebra of certain $(n+1)\times(n+1)$ matrix-valued functions on some algebraic curve and a very different ZCR with values in the Lie algebra $\mathfrak{so}_{n-1}$. We prove that any other ZCR of the n-component Landau-Lifshitz system is equivalent to a reduction of these two ZCRs.

The presented methods to classify ZCRs can be applied also to other $(1+1)$-dimensional evolution PDEs. Furthermore, the obtained results can be used for proving non-existence of Bäcklund transformations between some PDEs, which will be described in forthcoming publications.

Contents

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2010 *Mathematics Subject Classification.* 37K30, 37K35.

1. Introduction

1.1. **The main ideas.** Let m be a positive integer. An m-component $(1+1)$-dimensional evolution PDE for functions $u^1(x,t),\dots,u^m(x,t)$ is a PDE of the form

$$\frac{\partial u^i}{\partial t}=F^i(x,t,u^1,\dots,u^m,\,u^1_1,\dots,u^m_1,\dots,u^1_d,\dots,u^m_d), \tag{1}$$

$$u^i=u^i(x,t),\qquad u^i_k=\frac{\partial^k u^i}{\partial x^k},\qquad i=1,\dots,m,\qquad k\in\mathbb{Z}_{>0}.$$

Here the number $d\geq 1$ is such that the functions F^i may depend only on the variables x, t, u^j, u^j_k for $k\leq d$.

A large part of the theory of integrable systems is devoted to the study of such PDEs. This class of PDEs includes many celebrated equations of mathematical physics (e.g., the KdV, Krichever-Novikov, Landau-Lifshitz, nonlinear Schrödinger equations). Many more PDEs can be written in the evolution form (1) after a suitable change of variables.

To simplify notation, we set $u^j_0=u^j$ and $u^j_t=\partial u^j/\partial t$ for $j=1,\dots,m$. Then the right-hand side of (1) can be written as $F^i(x,t,u^j_0,u^j_1,\dots,u^j_d)$, and the PDE (1) reads

$$u^i_t=F^i(x,t,u^j_0,u^j_1,\dots,u^j_d),\qquad i=1,\dots,m. \tag{2}$$

So (2) is a compact form of (1).

In this paper, integrability of PDEs is understood in the sense of soliton theory and the inverse scattering method. This is sometimes called S-integrability.

It is well known that, in order to investigate possible integrability properties of (1), one needs to consider zero-curvature representations (ZCRs).

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. For a PDE of the form (1), a *zero-curvature representation (ZCR) with values in* $\mathfrak{g}$ is given by $\mathfrak{g}$-valued functions

$$A=A(x,t,u^j_0,u^j_1,\dots,u^j_p),\qquad B=B(x,t,u^j_0,u^j_1,\dots,u^j_{p+d-1}) \tag{3}$$

satisfying

$$D_x(B)-D_t(A)+[A,B]=0. \tag{4}$$

The *total derivative operators* D_x, D_t in (4) are

$$D_x = \frac{\partial}{\partial x} + \sum_{\substack{i=1,\dots,m,\\ k\geq 0}} u^i_{k+1}\frac{\partial}{\partial u^i_k}, \qquad D_t = \frac{\partial}{\partial t} + \sum_{\substack{i=1,\dots,m,\\ k\geq 0}} D^k_x\big(F^i(x,t,u^j_0,u^j_1,\dots,u^j_d)\big)\frac{\partial}{\partial u^i_k}. \tag{5}$$

The number p in (3) is such that the function A may depend only on the variables x, t, u^j_k for $k \leq p$ and $j = 1,\dots,m$. Then equation (4) implies that the function B may depend only on x, t, $u^j_{k'}$ for $k' \leq p+d-1$.

Such ZCRs are said to be *of order* $\leq p$. In other words, a ZCR given by A, B is of order $\leq p$ iff $\dfrac{\partial A}{\partial u^i_l} = 0$ for all $l > p$.

Remark 1. The right-hand side $F^i(x,t,u^j_0,u^j_1,\dots,u^j_d)$ of (1) appears in condition (4), because F^i appears in the formula for the operator D_t in (5).

Note that (4) can be written as $[D_x + A,\, D_t + B] = 0$. Condition (4) is equivalent to the fact that the auxiliary linear system

$$\partial_x(W) = -AW, \qquad \partial_t(W) = -BW \tag{6}$$

is compatible modulo (1). Here $W = W(x,t)$ is an invertible $N \times N$ matrix-function.

Remark 2. When we consider a function $Q = Q(x,t,u^j_0,u^j_1,\dots,u^j_l)$ for some $l \in \mathbb{Z}_{\geq 0}$, we always assume that this function is analytic on an open subset of the manifold with the coordinates $x,t,u^j_0,u^j_1,\dots,u^j_l$ for $j = 1,\dots,m$. For example, Q may be a meromorphic function, because a meromorphic function is analytic on some open subset of the manifold. In particular, this applies to the functions (3).

We study the following problem. How to describe all ZCRs (3), (4) for a given PDE (1)?

In the case when $p = 0$ and the functions F^i, A, B do not depend on x, t, a partial answer to this question is provided by the Wahlquist-Estabrook prolongation method (WE method for short). Namely, for a given PDE of the form $u^i_t = F^i(u^j_0,u^j_1,\dots,u^j_d)$, $i = 1,\dots,m$, the WE method constructs a Lie algebra so that ZCRs of the form

$$A = A(u^j_0), \qquad B = B(u^j_0,u^j_1,\dots,u^j_{d-1}), \qquad D_x(B) - D_t(A) + [A,B] = 0 \tag{7}$$

correspond to representations of this algebra (see, e.g., [2, 25, 9]). It is called the *Wahlquist-Estabrook prolongation algebra*. Note that in (7) the function $A(u^j_0)$ depends only on u^j_0, $j = 1,\dots,m$.

To study the general case of ZCRs (3), (4) with arbitrary p for any equation (1), we need to consider gauge transformations.

Without loss of generality, one can assume that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}_N$ for some $N \in \mathbb{Z}_{>0}$, where $\mathfrak{gl}_N$ is the algebra of $N \times N$ matrices with entries from $\mathbb{R}$ or $\mathbb{C}$. So our considerations are applicable to both cases $\mathfrak{gl}_N = \mathfrak{gl}_N(\mathbb{R})$ and $\mathfrak{gl}_N = \mathfrak{gl}_N(\mathbb{C})$. And we denote by GL_N the group of invertible $N \times N$ matrices.

Let $\mathbb{K}$ be either $\mathbb{C}$ or $\mathbb{R}$. Then $\mathfrak{gl}_N = \mathfrak{gl}_N(\mathbb{K})$ and $\mathrm{GL}_N = \mathrm{GL}_N(\mathbb{K})$. In this paper, all algebras are supposed to be over the field $\mathbb{K}$.

Let $\mathcal{G} \subset \mathrm{GL}_N$ be the connected matrix Lie group corresponding to the Lie algebra $\mathfrak{g} \subset \mathfrak{gl}_N$. (That is, $\mathcal{G}$ is the connected immersed Lie subgroup of GL_N corresponding to the Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}_N$.) A *gauge transformation* is given by an invertible matrix-function $G = G(x,t,u^j_0,u^j_1,\dots,u^j_l)$ with values in $\mathcal{G}$.

For any ZCR (3), (4) and any gauge transformation $G = G(x,t,u^j_0,u^j_1,\dots,u^j_l)$, the functions

$$\tilde{A} = GAG^{-1} - D_x(G)\cdot G^{-1}, \qquad \tilde{B} = GBG^{-1} - D_t(G)\cdot G^{-1} \tag{8}$$

satisfy $D_x(\tilde{B}) - D_t(\tilde{A}) + [\tilde{A}, \tilde{B}] = 0$ and, therefore, form a ZCR. (This is explained in Remark 10 below.)

Moreover, since A, B take values in $\mathfrak{g}$ and G takes values in $\mathcal{G}$, the functions $\tilde{A}$, $\tilde{B}$ take values in $\mathfrak{g}$. (This is well known, but for completeness we prove this in Lemma 1 in Section 2.)

The ZCR (8) is said to be *gauge equivalent* to the ZCR (3), (4). For a given PDE (1), formulas (8) determine an action of the group of gauge transformations on the set of ZCRs of this PDE.

The WE method does not use gauge transformations in a systematic way. In the classification of ZCRs (7) this is acceptable, because the class of ZCRs (7) is relatively small.

The class of ZCRs (3), (4) is much larger than that of (7). As is shown in the present paper, gauge transformations play a very important role in the classification of ZCRs (3), (4). Because of this, the classical WE method does not produce satisfactory results for (3), (4), especially in the case $p > 0$.

To overcome this problem, we use the approach which we developed in [10, 11]. Namely, using some ideas from [10, 11], we find a normal form for ZCRs (3), (4) with respect to the action of the group of gauge transformations. Using the normal form of ZCRs, for any given equation (1), we define a Lie algebra $\mathbb{F}^p$ for each $p \in \mathbb{Z}_{\geq 0}$ so that the following property holds.

For every finite-dimensional Lie algebra $\mathfrak{g}$, any $\mathfrak{g}$-valued ZCR (3), (4) of order $\leq p$ is locally gauge equivalent to the ZCR arising from a homomorphism $\mathbb{F}^p \to \mathfrak{g}$.

More precisely, as is discussed below, we define a Lie algebra $\mathbb{F}^p$ for each $p \in \mathbb{Z}_{\geq 0}$ and each point a of the infinite prolongation $\mathcal{E}$ of the PDE (1). So the full notation for the algebra is $\mathbb{F}^p(\mathcal{E}, a)$.

Recall that the *infinite prolongation* $\mathcal{E}$ of (1) is the infinite-dimensional manifold with the coordinates

$$x, \qquad t, \qquad u^i_k, \qquad i = 1, \dots, m, \qquad k \in \mathbb{Z}_{\geq 0}. \tag{9}$$

(A more precise definition of the manifold $\mathcal{E}$ is given in Section 2.1.) The precise definition of $\mathbb{F}^p(\mathcal{E}, a)$ for any evolution PDE (1) is presented in Section 2. In this definition, the algebra $\mathbb{F}^p(\mathcal{E}, a)$ is given in terms of generators and relations.

For every finite-dimensional Lie algebra $\mathfrak{g}$, homomorphisms $\mathbb{F}^p(\mathcal{E}, a) \to \mathfrak{g}$ classify (up to gauge equivalence) all $\mathfrak{g}$-valued ZCRs (3), (4) of order $\leq p$, where functions A, B are defined on a neighborhood of the point $a \in \mathcal{E}$. See Section 2 for details.

The algebras $\mathbb{F}^p(\mathcal{E}, a)$ are responsible also for parameter-dependent ZCRs, see Remark 3 below.

For scalar evolution equations with $m = 1$ this approach was developed in [11]. In the present paper we study the algebras $\mathbb{F}^p(\mathcal{E}, a)$ for multicomponent evolution PDEs (1) with arbitrary m.

Note that the same Lie algebras $\mathbb{F}^p(\mathcal{E}, a)$ were used in [10] for a different purpose, see Remark 4 below.

Remark 3. In the theory of integrable $(1+1)$-dimensional PDEs, one is especially interested in ZCRs depending on a parameter λ. So consider a $\mathfrak{g}$-valued ZCR of the form

$$A = A(\lambda, x, t, u^j_0, u^j_1, \dots, u^j_p), \qquad B = B(\lambda, x, t, u^j_0, u^j_1, \dots, u^j_{p+d-1}),$$
$$D_x(B) - D_t(A) + [A, B] = 0, \tag{10}$$

where $\mathfrak{g}$-valued functions A, B depend on x, t, u^i_k and a parameter λ.

Let $\tilde{\mathfrak{g}}$ be the infinite-dimensional Lie algebra of functions $h(\lambda)$ with values in $\mathfrak{g}$. (Depending on the problem under study, one can consider analytic or meromorphic functions $h(\lambda)$. Or one can assume that λ runs through an open subset of some algebraic curve and consider $\mathfrak{g}$-valued functions $h(\lambda)$ on this algebraic curve.)

Then (10) can be regarded as a ZCR with values in $\tilde{\mathfrak{g}}$. After a suitable (parameter-dependent) gauge transformation, each ZCR of the form (10) corresponds to a homomorphism $\mathbb{F}^p(\mathcal{E}, a) \to \tilde{\mathfrak{g}}$. So the Lie algebras $\mathbb{F}^p(\mathcal{E}, a)$ are responsible also for parameter-dependent ZCRs.

Applications of $\mathbb{F}^p(\mathcal{E},a)$ to the theory of Bäcklund transformations are described in [10]. In the scalar case $m=1$, applications of $\mathbb{F}^p(\mathcal{E},a)$ to obtaining necessary conditions for integrability of scalar evolution equations are discussed in [11].

According to Section 2, the algebras $\mathbb{F}^p(\mathcal{E},a)$ for $p\in\mathbb{Z}_{\geq 0}$ are arranged in a sequence of surjective homomorphisms

$$\cdots\to\mathbb{F}^p(\mathcal{E},a)\to\mathbb{F}^{p-1}(\mathcal{E},a)\to\cdots\to\mathbb{F}^1(\mathcal{E},a)\to\mathbb{F}^0(\mathcal{E},a). \tag{11}$$

Recall that $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{R}$. We suppose that the variables x, t, u^i_k take values in $\mathbb{K}$. A point $a\in\mathcal{E}$ is determined by the values of the coordinates x, t, u^i_k at a. Let

$$a=(x=x_a,\,t=t_a,\,u^i_k=a^i_k)\,\in\,\mathcal{E},\qquad x_a,t_a,a^i_k\in\mathbb{K},\qquad i=1,\dots,m,\qquad k\in\mathbb{Z}_{\geq 0}, \tag{12}$$

be a point of $\mathcal{E}$. In other words, the constants x_a, t_a, a^i_k are the coordinates of the point $a\in\mathcal{E}$ in the coordinate system x, t, u^i_k.

Example 1. To clarify the definition of $\mathbb{F}^p(\mathcal{E},a)$, let us consider the case $m=p=1$. To this end, we fix an evolution PDE (1) with $m=1$ and study ZCRs of order ≤ 1 of this PDE.

Since we assume $m=1$, any ZCR of order ≤ 1 is written as follows

$$A=A(x,t,u^1_0,u^1_1),\qquad B=B(x,t,u^1_0,u^1_1,\dots,u^1_d),\qquad D_x(B)-D_t(A)+[A,B]=0. \tag{13}$$

According to Theorem 2 in Section 2, any such ZCR (13) on a neighborhood of $a\in\mathcal{E}$ is gauge equivalent to a ZCR of the form

$$\tilde A=\tilde A(x,t,u^1_0,u^1_1),\qquad \tilde B=\tilde B(x,t,u^1_0,u^1_1,\dots,u^1_d), \tag{14}$$

$$D_x(\tilde B)-D_t(\tilde A)+[\tilde A,\tilde B]=0, \tag{15}$$

$$\frac{\partial\tilde A}{\partial u^1_1}(x,t,u^1_0,a^1_1)=0,\qquad \tilde A(x,t,a^1_0,a^1_1)=0,\qquad \tilde B(x_a,t,a^1_0,a^1_1,\dots,a^1_d)=0, \tag{16}$$

where x_a, a^1_k are the constants determined by the point $a\in\mathcal{E}$ given by (12).

In other words, properties (16) determine a normal form for ZCRs (13) with respect to the action of the group of gauge transformations on a neighborhood of $a\in\mathcal{E}$.

A similar normal form for ZCRs (3), (4) with arbitrary m, p is described in Theorem 3 in Section 2.

Since the functions $\tilde A$, $\tilde B$ from (14), (16) are analytic on a neighborhood of $a\in\mathcal{E}$, these functions are represented as absolutely convergent power series

$$\tilde A=\sum_{l_1,l_2,i_0,i_1\geq 0}(x-x_a)^{l_1}(t-t_a)^{l_2}(u^1_0-a^1_0)^{i_0}(u^1_1-a^1_1)^{i_1}\cdot\tilde A^{l_1,l_2}_{i_0,i_1}, \tag{17}$$

$$\tilde B=\sum_{l_1,l_2,j_0,\dots,j_d\geq 0}(x-x_a)^{l_1}(t-t_a)^{l_2}(u^1_0-a^1_0)^{j_0}\dots(u^1_d-a^1_d)^{j_d}\cdot\tilde B^{l_1,l_2}_{j_0\dots j_d}. \tag{18}$$

Here $\tilde A^{l_1,l_2}_{i_0,i_1}$ and $\tilde B^{l_1,l_2}_{j_0\dots j_d}$ are elements of a Lie algebra, which we do not specify yet.

Using formulas (17), (18), we see that properties (16) are equivalent to

$$\tilde A^{l_1,l_2}_{i_0,1}=\tilde A^{l_1,l_2}_{0,0}=\tilde B^{0,l_2}_{0\dots 0}=0\qquad\qquad\forall\, l_1,l_2,i_0\in\mathbb{Z}_{\geq 0}. \tag{19}$$

To define $\mathbb{F}^1(\mathcal{E},a)$, we regard $\tilde A^{l_1,l_2}_{i_0,i_1}$, $\tilde B^{l_1,l_2}_{j_0\dots j_d}$ from (17), (18) as abstract symbols. By definition, the algebra $\mathbb{F}^1(\mathcal{E},a)$ is generated by the symbols $\tilde A^{l_1,l_2}_{i_0,i_1}$, $\tilde B^{l_1,l_2}_{j_0\dots j_d}$ for $l_1,l_2,i_0,i_1,j_0,\dots,j_d\in\mathbb{Z}_{\geq 0}$. Relations for these generators are provided by equations (15), (19). A more detailed description of this construction is given in Section 2 for arbitrary m, p. The scalar case $m=1$ is studied also in [11].

Remark 4. Let $q\in\mathbb{Z}_{>0}$. Consider a system of the form

$$w^l_x=\alpha^l(w^1,\dots,w^q,x,t,u^j_0,u^j_1,\dots,u^j_p), \tag{20}$$

$$w^l_t = \beta^l(w^1,\dots,w^q,x,t,u^j_0,u^j_1,\dots,u^j_{p+d-1}),\qquad w^l = w^l(x,t),\qquad l=1,\dots,q. \tag{21}$$

We assume that system (20), (21) is compatible modulo (1), which means the following. Differentiating equation (20) with respect to t and equation (21) with respect to x, one obtains some expressions for w^l_{xt} and w^l_{tx}. The expressions for w^l_{xt} and w^l_{tx} must coincide, modulo (1).

For example, the linear system (6) corresponding to the ZCR (3), (4) is compatible modulo (1). In general, in (20), (21) the functions α^l, β^l may depend on $w^1,\dots,w^q$ nonlinearly. It is well known that such compatible systems play an important role in the theory of Bäcklund transformations (see, e.g., [17]).

Recall that $\mathcal{E}$ is the manifold with the coordinates (9). We assume that the functions α^l, β^l from (20), (21) are defined on the manifold $\mathbf{W}\times\mathbf{U}$, where $\mathbf{W}$ is a manifold with the coordinates $w^1,\dots,w^q$ and $\mathbf{U}$ is an open subset of $\mathcal{E}$.

Let $\mathfrak{L}$ be the Lie algebra of vector fields on $\mathbf{W}$. Since $\mathbf{W}$ is a manifold with the coordinates $w^1,\dots,w^q$, the Lie algebra $\mathfrak{L}$ consists of vector fields of the form $\sum_{l=1}^q f^l(w^1,\dots,w^q)\dfrac{\partial}{\partial w^l}$.

Then

$$\mathcal{A} = \sum_{l=1}^q \alpha^l(w^1,\dots,w^q,x,t,u^j_0,u^j_1,\dots,u^j_p)\frac{\partial}{\partial w^l}, \tag{22}$$

$$\mathcal{B} = \sum_{l=1}^q \beta^l(w^1,\dots,w^q,x,t,u^j_0,u^j_1,\dots,u^j_{p+d-1})\frac{\partial}{\partial w^l} \tag{23}$$

can be regarded as functions on $\mathbf{U}\subset\mathcal{E}$ with values in $\mathfrak{L}$.

It is well known (and is explained in [10]) that system (20), (21) is compatible modulo (1) iff

$$D_x(\mathcal{B}) - D_t(\mathcal{A}) + [\mathcal{A},\mathcal{B}] = 0.$$

Therefore, (22), (23) can be viewed as a ZCR with values in $\mathfrak{L}$.

The preprint [10] shows that, up to gauge equivalence, compatible systems of the form (20), (21) can be described in terms of homomorphisms $\mathbb{F}^p(\mathcal{E},a)\to\mathfrak{L}$. (The notion of gauge equivalence for such systems is discussed in [10].)

The main goal of this paper is to demonstrate techniques for computation of the algebras $\mathbb{F}^p(\mathcal{E},a)$ for multicomponent evolution PDEs. Since the algebras $\mathbb{F}^p(\mathcal{E},a)$ are responsible for all ZCRs, computation of $\mathbb{F}^p(\mathcal{E},a)$ leads to classification of ZCRs up to gauge equivalence. The results of the paper are described in the next subsection.

1.2. **The main results.** As has been discussed above, for each $p\in\mathbb{Z}_{\ge0}$ the algebra $\mathbb{F}^p(\mathcal{E},a)$ is defined by a certain set of generators and relations arising from a normal form of ZCRs. In Theorem 7 in Section 2.5 we describe a smaller subset of generators for $\mathbb{F}^p(\mathcal{E},a)$. This result is very helpful in the computation of $\mathbb{F}^p(\mathcal{E},a)$ for concrete PDEs, which is demonstrated in Sections 5, 6.

In Theorem 8 in Section 3 we describe a relation between the algebra $\mathbb{F}^0(\mathcal{E},a)$ and the Wahlquist-Estabrook prolongation algebra. This is helpful in the computation of $\mathbb{F}^0(\mathcal{E},a)$ for PDEs whose Wahlquist-Estabrook prolongation algebra is known.

The main example of a PDE considered in this paper is the multicomponent generalization of the Landau-Lifshitz equation from [6, 23]. To describe this PDE, we need some notation.

For any $k\in\mathbb{Z}_{>0}$ and any k-dimensional vectors $V=(v^1,\dots,v^k)$ and $W=(w^1,\dots,w^k)$, we set $\langle V,W\rangle=\sum_{i=1}^k v^iw^i$.

Recall that $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{R}$. Fix an integer $n\ge2$. Let $r_1,\dots,r_n\in\mathbb{K}$ be such that $r_i\ne r_j$ for all $i\ne j$. Denote by $R=\operatorname{diag}(r_1,\dots,r_n)$ the diagonal $n\times n$ matrix with entries r_i. Consider

the PDE

$$S_t = \Big(S_{xx} + \frac{3}{2}\langle S_x, S_x\rangle S\Big)_x + \frac{3}{2}\langle S, RS\rangle S_x, \qquad \langle S, S\rangle = 1, \qquad R = \operatorname{diag}(r_1, \dots, r_n), \tag{24}$$

where $S = \big(s^1(x,t), \dots, s^n(x,t)\big)$ is an n-dimensional vector-function, and $s^i(x,t)$ take values in $\mathbb{K}$.

System (24) was introduced in [6]. According to [6], for $n = 3$ it coincides with the higher symmetry (the commuting flow) of third order for the Landau-Lifshitz equation. Thus (24) can be regarded as an n-component generalization of the Landau-Lifshitz equation. For this reason, we call (24) the n-component Landau-Lifshitz system.

The paper [6] considers also the following algebraic curve

$$\lambda_i^2 - \lambda_j^2 = r_j - r_i, \qquad i, j = 1, \dots, n, \tag{25}$$

in the space $\mathbb{K}^n$ with coordinates $\lambda_1, \dots, \lambda_n$. According to [6], this curve is of genus $1 + (n-3)2^{n-2}$, and system (24) possesses a ZCR parametrized by points of this curve. (The ZCR is described in Remark 5 below.)

System (24) has an infinite number of symmetries, conservation laws [6], and an auto-Bäcklund transformation with a parameter [1]. Soliton-like solutions of (24) can be found in [1]. In [23] system (24) and its symmetries are constructed by means of the Kostant–Adler scheme.

Remark 5. For $i, j = 1, \dots, n+1$, let $E_{i,j} \in \mathfrak{gl}_{n+1}$ be the $(n+1) \times (n+1)$ matrix with (i,j)-th entry equal to 1 and all other entries equal to zero.

From the results of [6, 23] one can obtain the following $\mathfrak{gl}_{n+1}$-valued ZCR for the PDE (24)

$$\mathcal{A} = \sum_{i=1}^{n} s^i \lambda_i (E_{i,n+1} + E_{n+1,i}), \tag{26}$$

$$\mathcal{B} = D_x^2(\mathcal{A}) + [D_x(\mathcal{A}), \mathcal{A}] + \Big(r_1 + \lambda_1^2 + \frac{1}{2}\langle S, RS\rangle + \frac{3}{2}\langle D_x(S), D_x(S)\rangle\Big)\mathcal{A}, \tag{27}$$

$$D_x(\mathcal{B}) - D_t(\mathcal{A}) + [\mathcal{A}, \mathcal{B}] = 0.$$

Here $\lambda_1, \dots, \lambda_n \in \mathbb{K}$ are parameters satisfying (25).

We regard $\lambda_i(E_{i,n+1} + E_{n+1,i})$ as $\mathfrak{gl}_{n+1}$-valued functions on the curve (25). Let $\mathfrak{L}$ be the infinite-dimensional Lie algebra of all polynomial $\mathfrak{gl}_{n+1}$-valued functions $M(\lambda_1, \dots, \lambda_n)$ on the curve (25).

Let $L(n) \subset \mathfrak{L}$ be the Lie subalgebra generated by the functions $\lambda_i(E_{i,n+1} + E_{n+1,i})$, $i = 1, \dots, n$. Using relations (25), one can easily show that $L(n)$ consists of linear combinations of the functions

$$((\lambda_1)^2 + r_1)^l \lambda_i (E_{i,n+1} + E_{n+1,i}), \qquad ((\lambda_1)^2 + r_1)^l \lambda_i \lambda_j (E_{i,j} - E_{j,i}), \tag{28}$$
$$i, j = 1, \dots, n, \qquad i < j, \qquad l \in \mathbb{Z}_{\geq 0}.$$

According to [9], the Lie algebra $L(n)$ is infinite-dimensional, and the functions (28) form a basis for it. We describe $L(n)$ in more detail in Section 4.

Note that the algebra $L(n)$ is very similar to infinite-dimensional Lie algebras that were studied in [22, 23]. According to [9], the Lie algebra $L(n)$ appears in the description of the Wahlquist-Estabrook prolongation algebra for the PDE (24). The paper [9] gives also a presentation for the algebra $L(n)$ in terms of a finite number of generators and relations.

Note that (26), (27) can be regarded as functions with values in $L(n)$. So (26), (27) can be viewed as a ZCR with values in $L(n)$.

Remark 6. Let $\mathfrak{so}_{n,1} \subset \mathfrak{gl}_{n+1}$ be the Lie algebra of the matrix Lie group $\mathrm{O}(n,1) \subset \mathrm{GL}_{n+1}$, which consists of invertible linear transformations that preserve the standard bilinear form of signature $(n,1)$.

The algebra $\mathfrak{so}_{n,1}$ has the following basis

$$E_{i,j} - E_{j,i}, \qquad i < j \leq n, \qquad E_{l,n+1} + E_{n+1,l}, \qquad l = 1, \dots, n.$$

Note that the functions (26), (27), (28) take values in $\mathfrak{so}_{n,1}$.

Remark 7. In order to describe the algebras $\mathbb{F}^p(\mathcal{E},a)$ for system (24), we need to resolve the constraint $\langle S,S\rangle=1$ for the vector-function $S=\big(s^1(x,t),\dots,s^n(x,t)\big)$. Following [6], we do this as follows

$$s^j=\frac{2u^j}{1+\langle u,u\rangle},\qquad j=1,\dots,n-1,\qquad s^n=\frac{1-\langle u,u\rangle}{1+\langle u,u\rangle},\tag{29}$$

where $u=\big(u^1(x,t),\dots,u^{n-1}(x,t)\big)$ is an $(n-1)$-dimensional vector-function.

Then system (24) can be written as a PDE of the form $u^i_t=u^i_3+G^i(u^j,u^j_1,u^j_2)$, $i=1,\dots,n-1$. The explicit formula for this PDE is (147) in Section 4. Substituting (29) in (26), (27), we see that (26), (27) is a ZCR of order ≤ 0 for this PDE. (That is, the function (26) does not depend on u^j_l for $l>0$.)

In Theorem 10 in Section 5.3 we construct for this PDE an $\mathfrak{so}_{n-1}$-valued ZCR of the form

$$A=A(u^j,u^j_1),\qquad B=B(u^j,u^j_1,u^j_2,u^j_3),\qquad D_x(B)-D_t(A)+[A,B]=0,\tag{30}$$

where $\mathfrak{so}_{n-1}$ is the Lie algebra of skew-symmetric $(n-1)\times(n-1)$ matrices with entries from $\mathbb{K}$. According to our notation, $u^j=u^j_0$, hence the ZCR (30) is of order ≤ 1. Note that the ZCR (30) does not depend on any parameters.

So we have two very different ZCRs for the same PDE (24), which can be transformed to the PDE (147) by the transformation (29). Namely, we have the $\mathfrak{gl}_{n+1}$-valued ZCR (26), (27) and the $\mathfrak{so}_{n-1}$-valued ZCR (30) described in Theorem 10.

One can embed the Lie algebras $\mathfrak{gl}_{n+1}$ and $\mathfrak{so}_{n-1}$ into the Lie algebra $\mathfrak{gl}_N$ for some $N\ge n+1$, and then one can regard these ZCRs as $\mathfrak{gl}_N$-valued ZCRs. One can ask whether these ZCRs can become gauge equivalent after suitable embeddings $\mathfrak{gl}_{n+1}\hookrightarrow\mathfrak{gl}_N$ and $\mathfrak{so}_{n-1}\hookrightarrow\mathfrak{gl}_N$. In Remark 29 in Section 5.5 we show that these ZCRs cannot become gauge equivalent.

Using the theory described in Sections 2, 3, we compute the algebras $\mathbb{F}^p(\mathcal{E},a)$ for the PDE (24) in Sections 4, 5, 6.

The PDE (24) is imposed on a vector-function $S=\big(s^1(x,t),\dots,s^n(x,t)\big)$ satisfying $\langle S,S\rangle=1$. We compute $\mathbb{F}^p(\mathcal{E},a)$ for this PDE in the case $n\ge 4$. (The cases $n=2,3$ are less interesting and will described elsewhere.)

In Section 4 we compute the algebra $\mathbb{F}^0(\mathcal{E},a)$ for this PDE, using its Wahlquist-Estabrook prolongation algebra described in [9].

In Section 5 we compute $\mathbb{F}^1(\mathcal{E},a)$, and in Section 6 the algebras $\mathbb{F}^k(\mathcal{E},a)$ for all $k\ge 2$ are computed for this PDE. (We describe the structure of these Lie algebras up to some non-essential nilpotent ideals.)

The results are summarized in the following theorem, which is proved in Section 6.4.

Theorem 1 (Section 6.4)**.** *Let $n\ge 4$. The Lie algebras $\mathbb{F}^p(\mathcal{E},a)$ for the PDE (24) have the following structure.*

The algebra $\mathbb{F}^0(\mathcal{E},a)$ is isomorphic to the algebra $L(n)$ defined in Remark 5.

There is an abelian ideal $\mathfrak{S}$ of $\mathbb{F}^1(\mathcal{E},a)$ such that $\mathbb{F}^1(\mathcal{E},a)/\mathfrak{S}\cong L(n)\oplus\mathfrak{so}_{n-1}$, where $\mathfrak{so}_{n-1}$ is the Lie algebra of skew-symmetric $(n-1)\times(n-1)$ matrices. The surjective homomorphism $\mathbb{F}^1(\mathcal{E},a)\to\mathbb{F}^0(\mathcal{E},a)$ from (11) coincides with the composition of the homomorphisms

$$\mathbb{F}^1(\mathcal{E},a)\to\mathbb{F}^1(\mathcal{E},a)/\mathfrak{S}\cong L(n)\oplus\mathfrak{so}_{n-1}\to L(n)\cong\mathbb{F}^0(\mathcal{E},a).$$

Let $\tau_k\colon\ \mathbb{F}^k(\mathcal{E},a)\to\mathbb{F}^{k-1}(\mathcal{E},a)$ be the surjective homomorphism from (11). Then for any $k\ge 2$ we have

$$[v_1,[v_2,v_3]]=0\qquad\forall\, v_1,v_2,v_3\in\ker\tau_k.$$

In particular, the kernel of τ_k is nilpotent.

For each $k \geq 1$, let $\varphi_k\colon \mathbb{F}^k(\mathcal{E}, a) \to L(n) \oplus \mathfrak{so}_{n-1}$ be the composition of the surjective homomorphisms

$$\mathbb{F}^k(\mathcal{E}, a) \to \mathbb{F}^1(\mathcal{E}, a) \to \mathbb{F}^1(\mathcal{E}, a)/\mathfrak{S} \cong L(n) \oplus \mathfrak{so}_{n-1},$$

where $\mathbb{F}^k(\mathcal{E}, a) \to \mathbb{F}^1(\mathcal{E}, a)$ arises from (11). Then

$$[h_1, [h_2, \dots, [h_{2k-2}, [h_{2k-1}, h_{2k}]] \dots]] = 0 \qquad \forall\, h_1, h_2, \dots, h_{2k} \in \ker \varphi_k.$$

In particular, the kernel of φ_k is nilpotent.

Remark 8. Nilpotent ideals of the Lie algebras $\mathbb{F}^p(\mathcal{E}, a)$ are not important for the main applications to the theory of Bäcklund transformations and integrability.

Theorem 1 says that, for the PDE (24) in the case $n \geq 4$, for each $k \geq 1$ there is a surjective homomorphism $\varphi_k\colon \mathbb{F}^k(\mathcal{E}, a) \to L(n) \oplus \mathfrak{so}_{n-1}$ such that the kernel of φ_k is nilpotent. Since nilpotent ideals of $\mathbb{F}^k(\mathcal{E}, a)$ are not important, one can say that "the main part" of $\mathbb{F}^k(\mathcal{E}, a)$ is $L(n) \oplus \mathfrak{so}_{n-1}$.

And one has also the isomorphism $\mathbb{F}^0(\mathcal{E}, a) \cong L(n)$ for this PDE, according to Theorem 1.

Consider a ZCR (3), a gauge transformation $G = G(x, t, u_0^j, u_1^j, \dots, u_l^j)$, and the corresponding gauge equivalent ZCR (8). Then, essentially, (3) and (8) carry the same information, because one can switch from (3) to (8) and vice versa, using G and G^{-1}.

Furthermore, the following fact about ZCRs with values in a Lie algebra $\mathfrak{g}$ is well known. If one is interested in applications to the theory of integrable systems, then one can ignore nilpotent ideals of the Lie algebra $\mathfrak{g}$.

Therefore, it makes sense to classify ZCRs up to gauge equivalence and up to killing nilpotent ideals in the corresponding Lie algebras. As we show below, the algebras $\mathbb{F}^p(\mathcal{E}, a)$ are helpful in this respect.

As has been said above, for every finite-dimensional Lie algebra $\mathfrak{g}$, homomorphisms $\mathbb{F}^p(\mathcal{E}, a) \to \mathfrak{g}$ classify (up to gauge equivalence) all $\mathfrak{g}$-valued ZCRs (3), (4) of order $\leq p$, where functions A, B are defined on a neighborhood of the point $a \in \mathcal{E}$.

According to Theorem 1, for the PDE (24) in the case $n \geq 4$, we have $\mathbb{F}^0(\mathcal{E}, a) \cong L(n)$, and for each $k \geq 1$ there is a surjective homomorphism $\varphi_k\colon \mathbb{F}^k(\mathcal{E}, a) \to L(n) \oplus \mathfrak{so}_{n-1}$ such that the kernel of φ_k is nilpotent.

This allows us to classify all ZCRs (up to gauge equivalence and up to killing nilpotent ideals) for the PDE (24) in the case $n \geq 4$ as follows. In Section 7 we prove that, after suitable gauge transformations and after killing some nilpotent ideals, any ZCR becomes isomorphic to a reduction of the direct sum of the $L(n)$-valued ZCR described in Remark 5 and the $\mathfrak{so}_{n-1}$-valued ZCR described in Remark 7. (The notions of direct sums and reductions of ZCRs are explained in Section 2.6.)

In other words, as a result of the classification of all ZCRs (depending on derivatives of arbitrary finite order) for the PDE (24) in the case $n \geq 4$, we obtain two main non-equivalent ZCRs: the $L(n)$-valued ZCR and the $\mathfrak{so}_{n-1}$-valued ZCR described above. In Section 7 we prove that any other ZCR for the considered PDE is essentially equivalent (up to killing nilpotent ideals) to a reduction of the direct sum of these two main ZCRs.

In our opinion, it is interesting to see that, for a given multicomponent evolution PDE, one can classify all ZCRs (depending on derivatives of arbitrary order), and, as a result of the classification, one obtains several non-equivalent ZCRs depending on different derivatives and taking values in different Lie algebras. As has been said above, we classify ZCRs up to gauge equivalence and up to killing nilpotent ideals in the corresponding Lie algebras.

In the present paper we do this for the multicomponent Landau-Lifshitz system, but the described computational techniques can be applied to many more evolution PDEs. (Although for some PDEs the computations may be very difficult.)

For some scalar evolution PDEs of orders 3, 5, 7 (including the KdV and Krichever-Novikov equations), a similar approach to classification of ZCRs was described in [8, 11]. Note that the multicomponent case considered in the present paper is much more sophisticated than the scalar case considered in [8, 11].

Using the methods developed in Sections 2–6, in Section 8 we describe the structure of the Lie algebras $\mathbb{F}^p(\mathcal{E},a)$, $p\in\mathbb{Z}_{\geq 0}$, for the classical Landau-Lifshitz and nonlinear Schrödinger equations in the case $\mathbb{K}=\mathbb{C}$.

As has been said above, in Theorem 8 in Section 3 we describe a relation between the algebra $\mathbb{F}^0(\mathcal{E},a)$ and the Wahlquist-Estabrook prolongation algebra. To compute $\mathbb{F}^0(\mathcal{E},a)$ for the classical Landau-Lifshitz and nonlinear Schrödinger equations in Section 8, we use this relation and the corresponding Wahlquist-Estabrook prolongation algebras computed in [16, 3].

For the classical Landau-Lifshitz equation, we show that $\mathbb{F}^0(\mathcal{E},a)$ is isomorphic to an infinite-dimensional Lie algebra of certain $\mathfrak{so}_3(\mathbb{C})$-valued functions on an elliptic curve. According to [16], this algebra arises from the well-known elliptic $\mathfrak{so}_3(\mathbb{C})$-valued ZCR of the Landau-Lifshitz equation [21, 5, 16]. For this equation, we show also that the Lie algebras $\mathbb{F}^q(\mathcal{E},a)$ with $q\in\mathbb{Z}_{>0}$ are obtained from the Lie algebra $\mathbb{F}^0(\mathcal{E},a)$ by applying several times the operation of central extension.

For the nonlinear Schrödinger equation, we show that $\mathbb{F}^0(\mathcal{E},a)$ is isomorphic to the direct sum of the infinite-dimensional Lie algebra

$$\mathfrak{sl}_2(\mathbb{C}[\lambda])=\mathfrak{sl}_2(\mathbb{C})\otimes_{\mathbb{C}}\mathbb{C}[\lambda]$$

and a one-dimensional abelian Lie algebra. Here $\mathbb{C}[\lambda]$ is the algebra of polynomials in the variable λ. The Lie algebra $\mathfrak{sl}_2(\mathbb{C}[\lambda])$ arises from the well-known parameter-dependent $\mathfrak{sl}_2(\mathbb{C})$-valued ZCR of the nonlinear Schrödinger equation, and λ corresponds to the parameter in the ZCR. For this equation, we show also that the Lie algebras $\mathbb{F}^q(\mathcal{E},a)$ with $q\in\mathbb{Z}_{>0}$ are obtained from the Lie algebra $\mathbb{F}^0(\mathcal{E},a)$ by applying several times the operation of central extension.

Using the algebras $\mathbb{F}^p(\mathcal{E},a)$ for $(1+1)$-dimensional evolution PDEs, the preprint [10] describes a necessary condition for existence of Bäcklund transformations between two given PDEs. This necessary condition is given in [10] in terms of the algebras $\mathbb{F}^p(\mathcal{E},a)$.

Using this necessary condition from [10] and knowing the structure of the algebras $\mathbb{F}^p(\mathcal{E},a)$, one can sometimes prove non-existence of Bäcklund transformations between two given PDEs. Examples of such results for scalar evolution equations are given in [10] and references therein. The methods to compute $\mathbb{F}^p(\mathcal{E},a)$ described in the present paper allow one to obtain similar results for some multicomponent PDEs, which will be discussed in forthcoming publications.

Remark 9. As has been said above, the ZCR (26), (27) for the PDE (24) is parametrized by points of the curve (25). Some other integrable PDEs with ZCRs parametrized by the curve (25) were introduced in [6, 7, 22]. It was noticed in [22] that the formulas $\lambda=\lambda_i^2+r_i,\ \ y=\prod_{i=1}^n\lambda_i$ provide a map from the curve (25) to the hyperelliptic curve $y^2=\prod_{i=1}^n(\lambda-r_i)$. According to [6], for $n>3$ the curve (25) itself is not hyperelliptic.

Remark 10. It is well known that equation (4) implies $D_x(\tilde{B})-D_t(\tilde{A})+[\tilde{A},\tilde{B}]=0$ for $\tilde{A}$, $\tilde{B}$ given by (8). Indeed, one has $D_x+\tilde{A}=G(D_x+A)G^{-1}$ and $D_x+\tilde{B}=G(D_t+B)G^{-1}$. Therefore,

$$\begin{aligned}D_x(\tilde{B})-D_t(\tilde{A})+[\tilde{A},\tilde{B}]=[D_x+\tilde{A},D_t+\tilde{B}]=[G(D_x+A)G^{-1},G(D_t+B)G^{-1}]=\\ =G[D_x+A,D_t+B]G^{-1}=G(D_x(B)-D_t(A)+[A,B])G^{-1}.\end{aligned}$$

Hence the equation $D_x(B)-D_t(A)+[A,B]=0$ implies $D_x(\tilde{B})-D_t(\tilde{A})+[\tilde{A},\tilde{B}]=0$.

Remark 11. Some other approaches to the study of the action of gauge transformations on ZCRs can be found in [13, 14, 15, 18, 19, 20] and references therein. For a given ZCR with values in a matrix Lie algebra $\mathfrak{g}$, the papers [13, 14, 18] define certain $\mathfrak{g}$-valued functions that transform by

conjugation when the ZCR transforms by gauge. Applications of these functions to construction and classification of some types of ZCRs are described in [13, 14, 15, 18, 19, 20].

To our knowledge, the theory of [13, 14, 15, 18, 19, 20] does not produce any infinite-dimensional Lie algebras responsible for ZCRs. So this theory does not contain the algebras $\mathbb{F}^p(\mathcal{E}, a)$.

1.3. **Abbreviations, conventions, and notation.** The following abbreviations, conventions, and notation are used in the paper.

ZCR = zero-curvature representation, WE = Wahlquist-Estabrook.

The symbols $\mathbb{Z}_{>0}$ and $\mathbb{Z}_{\geq 0}$ denote the sets of positive and nonnegative integers respectively.

$\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{R}$. All vector spaces and algebras are supposed to be over the field $\mathbb{K}$. We denote by $\mathfrak{gl}_N$ the algebra of $N \times N$ matrices with entries from $\mathbb{K}$ and by GL_N the group of invertible $N \times N$ matrices.

Also, we use the following notation for partial derivatives of functions $u^i = u^i(x, t)$, $i = 1, \dots, m$,

$$u^i_0 = u^i, \qquad u^i_k = \frac{\partial^k u^i}{\partial x^k}, \qquad k \in \mathbb{Z}_{\geq 0}.$$

2. Zero-curvature representations, gauge transformations, and the algebras $\mathbb{F}^p(\mathcal{E}, a)$

In this section we study the algebras $\mathbb{F}^p(\mathcal{E}, a)$ introduced in [10]. For completeness, we give detailed definitions of $\mathcal{E}$, $a \in \mathcal{E}$, and $\mathbb{F}^p(\mathcal{E}, a)$.

2.1. **The infinite prolongation of an evolution PDE.** As has been said in Section 1.1, we suppose that x, t, u^i_k take values in $\mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{R}$. Let $\mathbb{K}^\infty$ be the infinite-dimensional space with the coordinates

$$x, \qquad t, \qquad u^i_k, \qquad i = 1, \dots, m, \qquad k \in \mathbb{Z}_{\geq 0}. \tag{31}$$

The topology on $\mathbb{K}^\infty$ is defined as follows.

For each $l \in \mathbb{Z}_{\geq 0}$, consider the space $\mathbb{K}^{m(l+1)+2}$ with the coordinates x, t, u^i_k for $k = 0, 1, \dots, l$ and $i = 1, \dots, m$. One has the natural projection $\pi_l \colon \mathbb{K}^\infty \to \mathbb{K}^{m(l+1)+2}$ that "forgets" the coordinates $u^i_{k'}$ for $k' > l$.

Since $\mathbb{K}^{m(l+1)+2}$ is a finite-dimensional vector space, we have the standard topology on $\mathbb{K}^{m(l+1)+2}$. For any $l \in \mathbb{Z}_{\geq 0}$ and any open subset $V \subset \mathbb{K}^{m(l+1)+2}$, the subset $\pi_l^{-1}(V) \subset \mathbb{K}^\infty$ is, by definition, open in $\mathbb{K}^\infty$. Such subsets form a base of the topology on $\mathbb{K}^\infty$. In other words, we consider the smallest topology on $\mathbb{K}^\infty$ such that the maps $\pi_l \colon \mathbb{K}^\infty \to \mathbb{K}^{m(l+1)+2}$, $l \in \mathbb{Z}_{\geq 0}$, are continuous.

According to our notation, the PDE (1) can be written as (2). Let $\mathbb{K}^{m(d+1)+2}$ be the space with the coordinates x, t, u^i_k for $k = 0, 1, \dots, d$ and $i = 1, \dots, m$. Let $\mathbb{U} \subset \mathbb{K}^{m(d+1)+2}$ be an open subset such that the functions $F^i(x, t, u^j_0, u^j_1, \dots, u^j_d)$ from (2) are defined on $\mathbb{U}$.

The *infinite prolongation* $\mathcal{E}$ of the evolution PDE (1) can be defined as follows

$$\mathcal{E} = \pi_d^{-1}(\mathbb{U}) \subset \mathbb{K}^\infty.$$

So $\mathcal{E}$ is an open subset of the space $\mathbb{K}^\infty$ with the coordinates (31). The topology on $\mathcal{E}$ is induced by the embedding $\mathcal{E} \subset \mathbb{K}^\infty$.

Example 2. For any constants $e_1, e_2, e_3 \in \mathbb{K}$, consider the Krichever-Novikov equation [12, 24]

$$u_t = u_{xxx} - \frac{3}{2}\frac{(u_{xx})^2}{u_x} + \frac{(u - e_1)(u - e_2)(u - e_3)}{u_x}, \qquad u = u(x, t). \tag{32}$$

Since this is a scalar equation of order 3, we have here $m = 1$ and $d = 3$.

In our notation, we set $u^1 = u(x, t)$ and rewrite equation (32) as follows

$$u^1_t = F^1(x, t, u^1_0, u^1_1, u^1_2, u^1_3), \tag{33}$$

$$F^1(x,t,u^1_0,u^1_1,u^1_2,u^1_3)=u^1_3-\frac{3}{2}\frac{(u^1_2)^2}{u^1_1}+\frac{(u^1_0-e_1)(u^1_0-e_2)(u^1_0-e_3)}{u^1_1}, \tag{34}$$

where $u^1_0=u^1$ and $u^1_k=\dfrac{\partial^k u^1}{\partial x^k}$ for $k\in\mathbb{Z}_{\geq 0}$.

Let $\mathbb{K}^6$ be the space with the coordinates x, t, u^1_0, u^1_1, u^1_2, u^1_3. According to (34), the function F^1 is defined on the open subset $\mathbb{U}\subset\mathbb{K}^6$ determined by the condition $u^1_1\neq 0$.

In this example, $\mathbb{K}^\infty$ is the space with the coordinates x, t, u^1_k for $k\in\mathbb{Z}_{\geq 0}$. We have the map $\pi_3\colon\mathbb{K}^\infty\to\mathbb{K}^6$ that "forgets" the coordinates $u^1_{k'}$ for $k'>3$. The infinite prolongation $\mathcal{E}$ of equation (33) is the following open subset of $\mathbb{K}^\infty$

$$\mathcal{E}=\pi_3^{-1}(\mathbb{U})=\big\{(x,t,u^1_0,u^1_1,u^1_2,\dots)\in\mathbb{K}^\infty\,\big|\,u^1_1\neq 0\big\}.$$

2.2. **A normal form for ZCRs with respect to the action of gauge transformations.** Consider again an evolution PDE (1) with arbitrary $m,d\in\mathbb{Z}_{>0}$. As has been said above, the infinite prolongation $\mathcal{E}$ of (1) is an open subset of the space $\mathbb{K}^\infty$ with the coordinates (31).

A point $a\in\mathcal{E}$ is determined by the values of the coordinates x, t, u^i_k at a. Let

$$a=(x=x_a,\,t=t_a,\,u^i_k=a^i_k)\,\in\,\mathcal{E},\qquad x_a,t_a,a^i_k\in\mathbb{K},\qquad i=1,\dots,m,\qquad k\in\mathbb{Z}_{\geq 0}, \tag{35}$$

be a point of $\mathcal{E}$. In other words, the constants x_a, t_a, a^i_k are the coordinates of the point $a\in\mathcal{E}$ in the coordinate system x, t, u^i_k.

Recall that, for every $N\in\mathbb{Z}_{>0}$, we denote by $\mathfrak{gl}_N$ the algebra of $N\times N$ matrices with entries from $\mathbb{K}$ and by GL_N the group of invertible $N\times N$ matrices. Let $\mathrm{Id}\in\mathrm{GL}_N$ be the identity matrix.

By the standard Lie group – Lie algebra correspondence, for every Lie subalgebra $\mathfrak{g}\subset\mathfrak{gl}_N$ there is a unique connected immersed Lie subgroup $\mathcal{G}\subset\mathrm{GL}_N$ whose Lie algebra is $\mathfrak{g}$. We call $\mathcal{G}$ the connected matrix Lie group corresponding to the matrix Lie algebra $\mathfrak{g}\subset\mathfrak{gl}_N$.

For any $l\in\mathbb{Z}_{\geq 0}$, a matrix-function $G=G(x,t,u^j_0,u^j_1,\dots,u^j_l)$ with values in $\mathcal{G}$ is called a gauge transformation. Equivalently, one can say that a gauge transformation is given by a $\mathcal{G}$-valued function $G=G(x,t,u^j_0,u^j_1,\dots,u^j_l)$.

The following lemma is known, but for completeness we present a proof of it.

Lemma 1. *Let $N\in\mathbb{Z}_{>0}$ and $p\in\mathbb{Z}_{\geq 0}$. Let $\mathfrak{g}\subset\mathfrak{gl}_N$ be a matrix Lie algebra and $\mathcal{G}\subset\mathrm{GL}_N$ be the connected matrix Lie group corresponding to $\mathfrak{g}\subset\mathfrak{gl}_N$.*

Let

$$A=A(x,t,u^j_0,u^j_1,\dots,u^j_p),\qquad B=B(x,t,u^j_0,u^j_1,\dots,u^j_{p+d-1}),\qquad D_x(B)-D_t(A)+[A,B]=0 \tag{36}$$

be a ZCR of order $\leq p$ such that the functions A, B take values in $\mathfrak{g}$. Here D_x and D_t are given by (5).

Then for any $\mathcal{G}$-valued function

$$G=G(x,t,u^j_0,u^j_1,\dots,u^j_{p-1}) \tag{37}$$

depending on x, t, $u^j_0,\dots,u^j_{p-1}$, $j=1,\dots,m$, the functions

$$\tilde A=GAG^{-1}-D_x(G)\cdot G^{-1},\qquad\qquad \tilde B=GBG^{-1}-D_t(G)\cdot G^{-1} \tag{38}$$

form a $\mathfrak{g}$-valued ZCR of order $\leq p$. That is,

$$\tilde A=\tilde A(x,t,u^j_0,u^j_1,\dots,u^j_p),\quad \tilde B=\tilde B(x,t,u^j_0,u^j_1,\dots,u^j_{p+d-1}),\quad D_x(\tilde B)-D_t(\tilde A)+[\tilde A,\tilde B]=0, \tag{39}$$

and $\tilde A$, $\tilde B$ take values in $\mathfrak{g}$.

Formulas (38) *determine an action of the group of $\mathcal{G}$-valued gauge transformations* (37) *on the set of $\mathfrak{g}$-valued ZCRs of order $\leq p$.*

Proof. Since A, B take values in $\mathfrak{g}$ and G takes values in the connected matrix Lie group $\mathcal{G}$ corresponding to the Lie algebra $\mathfrak{g}\subset\mathfrak{gl}_N$, the functions

$$GAG^{-1},\quad GBG^{-1},\quad \frac{\partial}{\partial x}(G)\cdot G^{-1},\quad \frac{\partial}{\partial t}(G)\cdot G^{-1},\quad \frac{\partial}{\partial u^i_k}(G)\cdot G^{-1}\qquad \forall\, i,k \tag{40}$$

take values in $\mathfrak{g}$. Hence the functions $\tilde A$, $\tilde B$ given by (38) take values in $\mathfrak{g}$ as well.

Using formulas (5), (36), (38) and the fact that G may depend only on x, t, $u^j_0,\dots,u^j_{p-1}$, we easily get (39).

One has $D_x{+}\tilde A=G(D_x{+}A)G^{-1}$ and $D_x{+}\tilde B=G(D_t{+}B)G^{-1}$, which implies that formulas (38) determine an action of the group of $\mathcal{G}$-valued gauge transformations (37) on the set of $\mathfrak{g}$-valued ZCRs of order $\le p$. □

Remark 12. For any $l\in\mathbb{Z}_{\ge0}$ and $a\in\mathcal{E}$, when we consider a function $Q=Q(x,t,u^j_0,u^j_1,\dots,u^j_l)$ defined on a neighborhood of $a\in\mathcal{E}$, we always assume that the function is analytic on this neighborhood. For example, Q may be a meromorphic function defined on an open subset of $\mathcal{E}$ such that Q is analytic on a neighborhood of $a\in\mathcal{E}$.

In particular, this applies to the functions A, B, G, $\tilde A$, $\tilde B$ considered in Theorems 2, 3 below.

Theorems 2 and 3 below describe a normal form for ZCRs with respect to the action of the group of gauge transformations. To clarify the construction, we first consider the case of ZCRs of order ≤1 in Theorem 2. The general case of ZCRs of order $\le p$ for any $p\in\mathbb{Z}_{\ge0}$ is described in Theorem 3.

Theorem 2. *Let $m,N\in\mathbb{Z}_{>0}$. Let $\mathfrak{g}\subset\mathfrak{gl}_N$ be a matrix Lie algebra. Denote by $\mathcal{G}\subset\mathrm{GL}_N$ the connected matrix Lie group corresponding to $\mathfrak{g}\subset\mathfrak{gl}_N$.*

Let $\mathcal{E}$ be the infinite prolongation of an m-component evolution PDE (1)*. Consider a point $a\in\mathcal{E}$ given by* (35)*. According to* (35)*, the point a is determined by constants x_a, t_a, a^i_k.*

Let

$$A=A(x,t,u^j_0,u^j_1),\qquad B=B(x,t,u^j_0,u^j_1,\dots,u^j_d),\qquad D_x(B)-D_t(A)+[A,B]=0 \tag{41}$$

be a ZCR of order ≤1 such that the functions A, B are defined on a neighborhood of $a\in\mathcal{E}$ and take values in $\mathfrak{g}$.

Then, on a neighborhood of $a\in\mathcal{E}$, there is a $\mathcal{G}$-valued function

$$G=G(x,t,u^j_0) \tag{42}$$

depending on x, t, u^j_0, $j=1,\dots,m$, such that the functions

$$\tilde A=GAG^{-1}-D_x(G)\cdot G^{-1},\qquad \tilde B=GBG^{-1}-D_t(G)\cdot G^{-1} \tag{43}$$

satisfy

$$\forall\, i_0=1,\dots,m,\qquad \frac{\partial\tilde A}{\partial u^{i_0}_1}\,\Big|_{u^j_1=a^j_1\ \forall\, j,\ u^i_0=a^i_0\ \forall\, i>i_0}=0, \tag{44}$$

$$\tilde A\,\Big|_{u^j_0=a^j_0,\ u^j_1=a^j_1\ \forall\, j}=0, \tag{45}$$

$$\tilde B\,\Big|_{x=x_a,\ u^j_k=a^j_k\ \forall\, j,\ \forall\, k\ge0}=0, \tag{46}$$

and

$$G\,\Big|_{x=x_a,\ t=t_a,\ u^j_0=a^j_0\ \forall\, j}=\mathrm{Id}. \tag{47}$$

Note that, according to Lemma 1, the functions (43) *form a $\mathfrak{g}$-valued ZCR of order ≤1. That is,*

$$\tilde A=\tilde A(x,t,u^j_0,u^j_1),\qquad \tilde B=\tilde B(x,t,u^j_0,u^j_1,\dots,u^j_d),\qquad D_x(\tilde B)-D_t(\tilde A)+[\tilde A,\tilde B]=0, \tag{48}$$

and $\tilde{A}$, $\tilde{B}$ take values in $\mathfrak{g}$.

Remark 13. The notation $\tilde{B}\Big|_{x=x_a,\ u_k^j=a_k^j\ \forall j,\ \forall k\geq 0}$ in (46) means that we substitute $x=x_a$ and $u_k^j=a_k^j$ for all $j=1,\dots,m$ and all $k\geq 0$ in the function $\tilde{B}=\tilde{B}(x,t,u_0^j,u_1^j,\dots,u_d^j)$. That is,

$$\tilde{B}\Big|_{x=x_a,\ u_k^j=a_k^j\ \forall j,\ \forall k\geq 0}=\tilde{B}(x_a,t,a_0^j,a_1^j,\dots,a_d^j).$$

The notation in (44), (45), (47) can be understood in a similar way.

Proof. To clarify the main idea, let us consider first the case $m=2$. Then formulas (41), (48) become

$$A=A(x,t,u_0^1,u_0^2,u_1^1,u_1^2),\qquad B=B(x,t,u_0^1,u_0^2,u_1^1,u_1^2,\dots,u_d^1,u_d^2), \tag{49}$$

$$\tilde{A}=\tilde{A}(x,t,u_0^1,u_0^2,u_1^1,u_1^2),\qquad \tilde{B}=\tilde{B}(x,t,u_0^1,u_0^2,u_1^1,u_1^2,\dots,u_d^1,u_d^2). \tag{50}$$

For $m=2$ in (44) we have $i_0=1,2$, so condition (44) is equivalent to the following two equations

$$\frac{\partial\tilde{A}}{\partial u_1^1}(x,t,u_0^1,a_0^2,a_1^1,a_1^2)=0, \tag{51}$$

$$\frac{\partial\tilde{A}}{\partial u_1^2}(x,t,u_0^1,u_0^2,a_1^1,a_1^2)=0. \tag{52}$$

Conditions (45), (46) in the case $m=2$ can be written as

$$\tilde{A}(x,t,a_0^1,a_0^2,a_1^1,a_1^2)=0, \tag{53}$$

$$\tilde{B}(x_a,t,a_0^1,a_0^2,a_1^1,a_1^2,\dots,a_d^1,a_d^2)=0. \tag{54}$$

According to Lemma 1 in the case $p=1$, formulas (43) determine an action of the group of $\mathcal{G}$-valued gauge transformations (42) on the set of $\mathfrak{g}$-valued ZCRs of order ≤ 1.

To prove the statement of the theorem in the case $m=2$, we need to find a $\mathcal{G}$-valued gauge transformation $G=G(x,t,u_0^1,u_0^2)$ such that the transformed ZCR (43) satisfies (51), (52), (53), (54), and $G(x_a,t_a,a_0^1,a_0^2)=\mathrm{Id}$.

We are going to construct the required gauge transformation in several steps. First, we will construct a transformation to achieve property (52), then another transformation to get properties (51), (52), then another transformation to get properties (51), (52), (53), and finally another transformation to obtain all properties (51), (52), (53), (54).

Consider the ordinary differential equation (ODE)

$$\frac{\partial G_2}{\partial u_0^2}=G_2\cdot\left(\frac{\partial A}{\partial u_1^2}(x,t,u_0^1,u_0^2,a_1^1,a_1^2)\right) \tag{55}$$

with respect to the variable u_0^2 and an unknown function $G_2=G_2(x,t,u_0^1,u_0^2)$. The variables x, t, u_0^1 are regarded as parameters in this ODE.

Let $G_2(x,t,u_0^1,u_0^2)$ be a local solution of the ODE (55) with the initial condition $G_2(x,t,u_0^1,a_0^2)=\mathrm{Id}$. Since $\partial A/\partial u_1^2$ takes values in $\mathfrak{g}$, the function G_2 takes values in $\mathcal{G}$.

Set

$$\hat{A}=G_2AG_2^{-1}-D_x(G_2)\cdot G_2^{-1},\qquad \hat{B}=G_2BG_2^{-1}-D_t(G_2)\cdot G_2^{-1}. \tag{56}$$

Since G_2 takes values in $\mathcal{G}$, the functions $\hat{A}$, $\hat{B}$ take values in $\mathfrak{g}$. Using (56) and (55), we get

$$\begin{aligned}\frac{\partial\hat{A}}{\partial u_1^2}(x,t,u_0^1,u_0^2,a_1^1,a_1^2)&=G_2\left(\frac{\partial A}{\partial u_1^2}(x,t,u_0^1,u_0^2,a_1^1,a_1^2)\right)G_2^{-1}-\left(\frac{\partial}{\partial u_1^2}\big(D_x(G_2)\big)\right)G_2^{-1}=\\&=G_2\left(\frac{\partial A}{\partial u_1^2}(x,t,u_0^1,u_0^2,a_1^1,a_1^2)\right)G_2^{-1}-\frac{\partial G_2}{\partial u_0^2}G_2^{-1}=\end{aligned} \tag{57}$$

$$= G_2\left(\frac{\partial A}{\partial u_1^2}(x,t,u_0^1,u_0^2,a_1^1,a_1^2)\right)G_2^{-1} - G_2\left(\frac{\partial A}{\partial u_1^2}(x,t,u_0^1,u_0^2,a_1^1,a_1^2)\right)G_2^{-1} = 0.$$

Now consider the ODE

$$\frac{\partial G_1}{\partial u_0^1} = G_1 \cdot \left(\frac{\partial \hat{A}}{\partial u_1^1}(x,t,u_0^1,a_0^2,a_1^1,a_1^2)\right) \tag{58}$$

with respect to the variable u_0^1 and an unknown function $G_1 = G_1(x,t,u_0^1)$, where x, t are regarded as parameters.

Let $G_1(x,t,u_0^1)$ be a local solution of the ODE (58) with the initial condition $G_1(x,t,a_0^1) = \mathrm{Id}$. Since $\partial\hat{A}/\partial u_1^1$ takes values in $\mathfrak{g}$, the function G_1 takes values in $\mathcal{G}$.

Set

$$\bar{A} = G_1\hat{A}G_1^{-1} - D_x(G_1)\cdot G_1^{-1}, \qquad \bar{B} = G_1\hat{B}G_1^{-1} - D_t(G_1)\cdot G_1^{-1}. \tag{59}$$

Then (57), (58), (59) imply that $\bar{A}$ satisfies properties (51), (52), if we replace $\tilde{A}$ by $\bar{A}$ in (51), (52). Furthermore, since G_1 takes values in $\mathcal{G}$, the functions $\bar{A}$, $\bar{B}$ take values in $\mathfrak{g}$.

Let $\tilde{G} = \tilde{G}(x,t)$ be a local solution of the ODE

$$\frac{\partial \tilde{G}}{\partial x} = \tilde{G}\cdot\bar{A}(x,t,a_0^1,a_0^2,a_1^1,a_1^2)$$

with the initial condition $\tilde{G}(x_a,t) = \mathrm{Id}$, where t is viewed as a parameter. Set

$$\check{A} = \tilde{G}\bar{A}\tilde{G}^{-1} - D_x(\tilde{G})\cdot\tilde{G}^{-1}, \qquad \check{B} = \tilde{G}\bar{B}\tilde{G}^{-1} - D_t(\tilde{G})\cdot\tilde{G}^{-1}. \tag{60}$$

Then $\check{A}$ satisfies properties (51), (52), (53), if we replace $\tilde{A}$ by $\check{A}$ in (51), (52), (53).

Finally, let $\hat{G} = \hat{G}(t)$ be a local solution of the ODE

$$\frac{\partial \hat{G}}{\partial t} = \hat{G}\cdot\check{B}(x_a,t,a_0^1,a_0^2,a_1^1,a_1^2,\dots,a_d^1,a_d^2) \tag{61}$$

with the initial condition $\hat{G}(t_a) = \mathrm{Id}$. Set

$$\tilde{A} = \hat{G}\check{A}\hat{G}^{-1} - D_x(\hat{G})\cdot\hat{G}^{-1}, \qquad \tilde{B} = \hat{G}\check{B}\hat{G}^{-1} - D_t(\hat{G})\cdot\hat{G}^{-1}. \tag{62}$$

Then $\tilde{A}$, $\tilde{B}$ obey (51), (52), (53), (54).

Let $G = \hat{G}\cdot\tilde{G}\cdot G_1\cdot G_2$. Then equations (56), (59), (60), (62) imply

$$\tilde{A} = GAG^{-1} - D_x(G)\cdot G^{-1}, \qquad \tilde{B} = GBG^{-1} - D_t(G)\cdot G^{-1}.$$

Furthermore, since $G_2(x,t,u_0^1,a_0^2) = G_1(x,t,a_0^1) = \tilde{G}(x_a,t) = \hat{G}(t_a) = \mathrm{Id}$, we have $G(x_a,t_a,a_0^1,a_0^2) = \mathrm{Id}$. Thus $G = \hat{G}\cdot\tilde{G}\cdot G_1\cdot G_2$ satisfies all the required properties in the case $m = 2$.

This construction can be easily generalized to the case of arbitrary m. One can define G as the product $G = \hat{G}\cdot\tilde{G}\cdot G_1\cdot G_2\dots G_m$, where the $\mathcal{G}$-valued functions

$$G_q = G_q(x,t,u_0^1,\dots,u_0^q), \qquad q = 1,\dots,m, \qquad \tilde{G} = \tilde{G}(x,t), \qquad \hat{G} = \hat{G}(t)$$

are defined as solutions of certain ODEs similar to the ODEs considered above. □

The set $\{1,\dots,m\}\times\mathbb{Z}_{\geq 0}$ consists of pairs (i,k), where $i\in\{1,\dots,m\}$ and $k\in\mathbb{Z}_{\geq 0}$. Consider the following ordering $\preceq$ of the set $\{1,\dots,m\}\times\mathbb{Z}_{\geq 0}$

$$i,i'\in\{1,\dots,m\}, \qquad k,k'\in\mathbb{Z}_{\geq 0}, \qquad k\neq k',$$
$$(i,k)\prec(i',k') \text{ iff } k<k', \qquad (i,k)\prec(i',k) \text{ iff } i<i'. \tag{63}$$

That is, $(1,0)\prec(2,0)\prec\dots\prec(m,0)\prec(1,1)\prec(2,1)\prec\dots$.

As usual, the notation $(i_1,k_1)\succeq(i_2,k_2)$ means that either $(i_1,k_1)\succ(i_2,k_2)$ or $(i_1,k_1) = (i_2,k_2)$.

Remark 14. Let $F = F(x, t, u^i_k)$ be a function of the variables x, t, u^i_k. Let $i' \in \{1, \dots, m\}$ and $k' \in \mathbb{Z}_{\geq 0}$. Then the notation $F\Big|_{u^i_k = a^i_k \ \forall\, (i,k) \succ (i',k')}$ says that we substitute $u^i_k = a^i_k$ for all $(i, k) \succ (i', k')$ in the function F.

Similarly, the notation $F\Big|_{x = x_a,\ u^i_k = a^i_k \ \forall\, (i,k) \succeq (i',k')}$ means that we substitute $x = x_a$ and $u^i_k = a^i_k$ for all $(i, k) \succeq (i', k')$ in F.

Theorem 3. *Let $m, N \in \mathbb{Z}_{>0}$ and $p \in \mathbb{Z}_{\geq 0}$. Let $\mathfrak{g} \subset \mathfrak{gl}_N$ be a matrix Lie algebra. Denote by $\mathcal{G} \subset \mathrm{GL}_N$ the connected matrix Lie group corresponding to $\mathfrak{g} \subset \mathfrak{gl}_N$.*

Let $\mathcal{E}$ be the infinite prolongation of an m-component evolution PDE (1)*. Consider a point $a \in \mathcal{E}$ given by* (35)*. According to* (35)*, the point a is determined by constants x_a, t_a, a^i_k.*

Let

(64)
$$A = A(x, t, u^j_0, u^j_1, \dots, u^j_p), \qquad B = B(x, t, u^j_0, u^j_1, \dots, u^j_{p+d-1}), \qquad D_x(B) - D_t(A) + [A, B] = 0$$

be a ZCR of order $\leq p$ such that the functions A, B are defined on a neighborhood of $a \in \mathcal{E}$ and take values in $\mathfrak{g}$.

Then, on a neighborhood of $a \in \mathcal{E}$, there is a $\mathcal{G}$-valued function $G = G(x, t, u^j_0, u^j_1, \dots, u^j_{p-1})$ depending on x, t, $u^j_0, \dots, u^j_{p-1}$, $j = 1, \dots, m$, such that the functions

$$\tilde{A} = GAG^{-1} - D_x(G) \cdot G^{-1}, \qquad \tilde{B} = GBG^{-1} - D_t(G) \cdot G^{-1}$$

satisfy

$$\forall\, i_0 = 1, \dots, m, \qquad \forall\, k_0 \geq 1, \qquad \frac{\partial \tilde{A}}{\partial u^{i_0}_{k_0}}\Big|_{u^i_k = a^i_k \ \forall\, (i,k) \succ (i_0, k_0 - 1)} = 0, \tag{65}$$

$$\tilde{A}\Big|_{u^i_k = a^i_k \ \forall\, (i,k)} = 0, \tag{66}$$

$$\tilde{B}\Big|_{x = x_a,\ u^i_k = a^i_k \ \forall\, (i,k)} = 0, \tag{67}$$

and

$$G\Big|_{x = x_a,\ t = t_a,\ u^i_k = a^i_k \ \forall\, (i,k)} = \mathrm{Id}. \tag{68}$$

Note that, according to Lemma 1, the functions (43) *form a $\mathfrak{g}$-valued ZCR of order $\leq p$. That is, $\tilde{A}$, $\tilde{B}$ take values in $\mathfrak{g}$ and satisfy* (39)*.*

Proof. This theorem can be proved similarly to Theorem 2. One can define G as the product of several gauge transformations, which are defined as solutions of certain ODEs similar to the ODEs considered in the proof of Theorem 2. □

Fix a point $a \in \mathcal{E}$ given by (35), which is determined by constants x_a, t_a, a^i_k.

A ZCR

(69)
$$\mathsf{A} = \mathsf{A}(x, t, u^j_0, u^j_1, \dots, u^j_p), \qquad \mathsf{B} = \mathsf{B}(x, t, u^j_0, u^j_1, \dots, u^j_{p+d-1}), \qquad D_x(\mathsf{B}) - D_t(\mathsf{A}) + [\mathsf{A}, \mathsf{B}] = 0$$

is said to be *a-normal* if A, B satisfy the following equations

$$\forall\, i_0 = 1, \dots, m, \qquad \forall\, k_0 \geq 1, \qquad \frac{\partial \mathsf{A}}{\partial u^{i_0}_{k_0}}\Big|_{u^i_k = a^i_k \ \forall\, (i,k) \succ (i_0, k_0 - 1)} = 0, \tag{70}$$

$$\mathsf{A}\Big|_{u^i_k = a^i_k \ \forall\, (i,k)} = 0, \tag{71}$$

$$\mathsf{B}\Big|_{x = x_a,\ u^i_k = a^i_k \ \forall\, (i,k)} = 0. \tag{72}$$

Remark 15. For example, the ZCR $\tilde{A}$, $\tilde{B}$ described in Theorem 3 is a-normal, because $\tilde{A}$, $\tilde{B}$ obey (65), (66), (67). Theorem 3 implies that any ZCR on a neighborhood of $a \in \mathcal{E}$ is gauge equivalent to an a-normal ZCR.

Theorem 4. *Let $N \in \mathbb{Z}_{>0}$. Let*

$$\mathsf{A} = \mathsf{A}(x,t,u_0^j,u_1^j,\dots,u_p^j), \quad \mathsf{B} = \mathsf{B}(x,t,u_0^j,u_1^j,\dots,u_{p+d-1}^j), \quad D_x(\mathsf{B}) - D_t(\mathsf{A}) + [\mathsf{A},\mathsf{B}] = 0 \tag{73}$$

be an a-normal ZCR with values in $\mathfrak{gl}_N$. (So A, B take values in $\mathfrak{gl}_N$ and satisfy (70), (71), (72).)

Consider another a-normal ZCR with values in $\mathfrak{gl}_N$

$$\mathcal{A} = \mathcal{A}(x,t,u_0^j,u_1^j,\dots,u_q^j), \quad \mathcal{B} = \mathcal{B}(x,t,u_0^j,u_1^j,\dots,u_{q+d-1}^j), \quad D_x(\mathcal{B}) - D_t(\mathcal{A}) + [\mathcal{A},\mathcal{B}] = 0, \tag{74}$$

$$\forall\, i_0 = 1,\dots,m, \qquad \forall\, k_0 \geq 1, \qquad \frac{\partial \mathcal{A}}{\partial u_{k_0}^{i_0}}\,\Big|_{u_k^i = a_k^i \ \forall\,(i,k) \succ (i_0,k_0-1)} = 0, \tag{75}$$

$$\mathcal{A}\,\Big|_{u_k^i = a_k^i \ \forall\,(i,k)} = 0, \tag{76}$$

$$\mathcal{B}\,\Big|_{x = x_a,\ u_k^i = a_k^i \ \forall\,(i,k)} = 0. \tag{77}$$

Suppose that there is a function $\mathbf{G} = \mathbf{G}(x,t,u_0^j,u_1^j,\dots,u_l^j)$ with values in GL_N such that

$$\mathcal{A} = \mathbf{G}\mathsf{A}\mathbf{G}^{-1} - D_x(\mathbf{G})\cdot\mathbf{G}^{-1}, \tag{78}$$

$$\mathcal{B} = \mathbf{G}\mathsf{B}\mathbf{G}^{-1} - D_t(\mathbf{G})\cdot\mathbf{G}^{-1}. \tag{79}$$

In other words, we suppose that the a-normal ZCR A, B is gauge equivalent to the a-normal ZCR $\mathcal{A}$, $\mathcal{B}$ with respect to a gauge transformation $\mathbf{G} = \mathbf{G}(x,t,u_0^j,u_1^j,\dots,u_l^j)$.

Then the function $\mathbf{G}$ is actually a constant element of GL_N (that is, $\mathbf{G}$ does not depend on x, t, u_k^j), and we have

$$\mathcal{A} = \mathbf{G}\mathsf{A}\mathbf{G}^{-1}, \qquad \mathcal{B} = \mathbf{G}\mathsf{B}\mathbf{G}^{-1}. \tag{80}$$

Proof. Using (70), (75), (78), one can prove

$$\frac{\partial \mathbf{G}}{\partial u_k^j} = 0 \qquad \forall\, j,k \tag{81}$$

by descending induction on (j,k) with respect to the ordering $\prec$. Equation (81) means that the function $\mathbf{G}$ may depend only on x, t.

Now, taking into account (81) and (5), we can rewrite (78), (79) as

$$\mathcal{A} = \mathbf{G}\mathsf{A}\mathbf{G}^{-1} - \frac{\partial \mathbf{G}}{\partial x}\cdot\mathbf{G}^{-1}, \tag{82}$$

$$\mathcal{B} = \mathbf{G}\mathsf{B}\mathbf{G}^{-1} - \frac{\partial \mathbf{G}}{\partial t}\cdot\mathbf{G}^{-1}. \tag{83}$$

Substituting $u_k^i = a_k^i$ for all i, k in (82) and using (71), (76), we get $\partial\mathbf{G}/\partial x = 0$. Hence $\mathbf{G}$ may depend only on t. Substituting $x = x_a$ and $u_k^i = a_k^i$ for all i, k in (83) and using (72), (77), we get $\partial\mathbf{G}/\partial t = 0$.

Thus $\mathbf{G}$ does not depend on x, t, u_k^j, so $\mathbf{G}$ is a constant element of GL_N. Then $D_x(\mathbf{G}) = D_t(\mathbf{G}) = 0$, and relations (78), (79) imply (80). □

Remark 16. In the situation described in Theorem 4, since $\mathbf{G}$ is a constant element of GL_N, the equation $\mathcal{A} = \mathbf{G}\mathsf{A}\mathbf{G}^{-1}$ implies that the functions A and $\mathcal{A}$ depend on the same variables x, t, u_k^j.

2.3. **The algebras $\mathbb{F}^p(\mathcal{E},a)$.** Recall that $\mathcal{E}$ is the infinite prolongation of an m-component evolution PDE (1). The number $m\in\mathbb{Z}_{>0}$ is fixed throughout this section. Consider a point $a\in\mathcal{E}$ given by (35). According to (35), the point a is determined by constants x_a, t_a, a^i_k.

For each $q\in\mathbb{Z}_{\geq 0}$, let $\mathcal{M}_q$ be the set of matrices of size $m\times(q+1)$ with nonnegative integer entries. For a matrix $\gamma\in\mathcal{M}_q$, its entries are denoted by $\gamma_{i,k}\in\mathbb{Z}_{\geq 0}$, where $i=1,\dots,m$ and $k=0,\dots,q$. Let U_a^γ be the following product

$$U_a^\gamma=\prod_{\substack{i=1,\dots,m,\\ k=0,\dots,q}}\bigl(u^i_k-a^i_k\bigr)^{\gamma_{i,k}}. \tag{84}$$

Remark 17. For each $q\in\mathbb{Z}_{>0}$, $i_0\in\{1,\dots,m\}$, and $k_0\in\{1,\dots,q\}$, denote by $M^q_{i_0,k_0}\subset\mathcal{M}_q$ the subset of matrices α satisfying the following conditions

$$\alpha_{i_0,k_0}=1,\quad \forall\, k>k_0\quad \forall\, i\quad \alpha_{i,k}=0,\quad \forall\, i_1\neq i_0\quad \alpha_{i_1,k_0}=0,\quad \forall\, i_2>i_0\quad \alpha_{i_2,k_0-1}=0. \tag{85}$$

In other words, for each $k>k_0$ the k-th column of any matrix $\alpha\in M^q_{i_0,k_0}$ is zero, the k_0-th column contains only one nonzero entry $\alpha_{i_0,k_0}=1$, and in the (k_0-1)-th column one has $\alpha_{i_2,k_0-1}=0$ for all $i_2>i_0$.

Set also $M^0_{i_0,k_0}=\varnothing$ for all i_0, k_0. So the set $M^0_{i_0,k_0}$ is empty.

Let $N\in\mathbb{Z}_{>0}$ and $p\in\mathbb{Z}_{\geq 0}$. Consider again a matrix Lie algebra $\mathfrak{g}\subset\mathfrak{gl}_N$. According to Theorem 3, any $\mathfrak{g}$-valued ZCR (64) of order $\leq p$ defined on a neighborhood of $a\in\mathcal{E}$ is gauge equivalent to a $\mathfrak{g}$-valued ZCR

$$\tilde A=\tilde A(x,t,u^j_0,u^j_1,\dots,u^j_p),\qquad \tilde B=\tilde B(x,t,u^j_0,u^j_1,\dots,u^j_{p+d-1}), \tag{86}$$

$$D_x(\tilde B)-D_t(\tilde A)+[\tilde A,\tilde B]=0 \tag{87}$$

satisfying (66), (67), (68).

According to Remark 12, the $\mathfrak{g}$-valued functions $\tilde A$, $\tilde B$ are analytic on a neighborhood of $a\in\mathcal{E}$. Hence, in some neighborhood of $a\in\mathcal{E}$, the functions $\tilde A$, $\tilde B$ can be represented as absolutely convergent power series

$$\tilde A=\sum_{\alpha\in\mathcal{M}_p,\ l_1,l_2\in\mathbb{Z}_{\geq 0}}(x-x_a)^{l_1}(t-t_a)^{l_2}\cdot U_a^\alpha\cdot\tilde A^{l_1,l_2}_\alpha, \tag{88}$$

$$\tilde B=\sum_{\beta\in\mathcal{M}_{p+d-1},\ l_1,l_2\in\mathbb{Z}_{\geq 0}}(x-x_a)^{l_1}(t-t_a)^{l_2}\cdot U_a^\beta\cdot\tilde B^{l_1,l_2}_\beta, \tag{89}$$

$$\tilde A^{l_1,l_2}_\alpha,\tilde B^{l_1,l_2}_\beta\in\mathfrak{g}.$$

Remark 18. Using formulas (88), (89), we see that properties (66), (67), (68) are equivalent to

$$\tilde A^{l_1,l_2}_0=\tilde B^{0,l_2}_0=0,\quad \tilde A^{l_1,l_2}_{\hat\alpha}=0,\quad \hat\alpha\in M^p_{i_0,k_0},\quad i_0=1,\dots,m,\quad k_0=1,\dots,p,\quad l_1,l_2\in\mathbb{Z}_{\geq 0}, \tag{90}$$

where $M^p_{i_0,k_0}\subset\mathcal{M}_p$ is the set of matrices defined in Remark 17.

Remark 19. The main idea of the definition of the Lie algebra $\mathbb{F}^p(\mathcal{E},a)$ can be informally outlined as follows. According to Theorem 3 and Remark 18, any ZCR (64) of order $\leq p$ is gauge equivalent to a ZCR given by functions $\tilde A$, $\tilde B$ that are of the form (88), (89) and satisfy (87), (90).

To define $\mathbb{F}^p(\mathcal{E},a)$, we regard A^{l_1,l_2}_α, B^{l_1,l_2}_β from (88), (89) as abstract symbols. By definition, the Lie algebra $\mathbb{F}^p(\mathcal{E},a)$ is generated by the symbols A^{l_1,l_2}_α, B^{l_1,l_2}_β for $\alpha\in\mathcal{M}_p$, $\beta\in\mathcal{M}_{p+d-1}$, $l_1,l_2\in\mathbb{Z}_{\geq 0}$. Relations for these generators are provided by equations (87), (90). The details of this construction are presented below.

Let $\mathfrak{F}$ be the free Lie algebra generated by the symbols $\mathbf{A}^{l_1,l_2}_\alpha$, $\mathbf{B}^{l_1,l_2}_\beta$ for $\alpha\in\mathcal{M}_p$, $\beta\in\mathcal{M}_{p+d-1}$, $l_1,l_2\in\mathbb{Z}_{\geq 0}$. In particular, we have

$$\mathbf{A}^{l_1,l_2}_\alpha\in\mathfrak{F},\quad \mathbf{B}^{l_1,l_2}_\beta\in\mathfrak{F},\quad \bigl[\mathbf{A}^{l_1,l_2}_\alpha,\mathbf{B}^{l_1,l_2}_\beta\bigr]\in\mathfrak{F}\qquad\forall\,\alpha\in\mathcal{M}_p,\qquad\forall\,\beta\in\mathcal{M}_{p+d-1},\qquad\forall\,l_1,l_2\in\mathbb{Z}_{\geq 0}.$$

Consider the following formal power series with coefficients in $\mathfrak{F}$

$$\mathbf{A} = \sum_{\alpha\in\mathcal{M}_p,\ l_1,l_2\in\mathbb{Z}_{\geq 0}} (x-x_a)^{l_1}(t-t_a)^{l_2}\cdot U_a^\alpha\cdot \mathbf{A}_\alpha^{l_1,l_2},$$

$$\mathbf{B} = \sum_{\beta\in\mathcal{M}_{p+d-1},\ l_1,l_2\in\mathbb{Z}_{\geq 0}} (x-x_a)^{l_1}(t-t_a)^{l_2}\cdot U_a^\beta\cdot \mathbf{B}_\beta^{l_1,l_2}.$$

Set

$$D_x(\mathbf{B}) = \sum_{\beta\in\mathcal{M}_{p+d-1},\ l_1,l_2\in\mathbb{Z}_{\geq 0}} D_x\big((x-x_a)^{l_1}(t-t_a)^{l_2}U_a^\beta\big)\cdot \mathbf{B}_\beta^{l_1,l_2}, \tag{91}$$

$$D_t(\mathbf{A}) = \sum_{\alpha\in\mathcal{M}_p,\ l_1,l_2\in\mathbb{Z}_{\geq 0}} D_t\big((x-x_a)^{l_1}(t-t_a)^{l_2}U_a^\alpha\big)\cdot \mathbf{A}_\alpha^{l_1,l_2}, \tag{92}$$

$$[\mathbf{A},\mathbf{B}] = \sum_{\substack{\alpha\in\mathcal{M}_p,\ \beta\in\mathcal{M}_{p+d-1},\\ l_1,l_2,l_1',l_2'\in\mathbb{Z}_{\geq 0}}} (x-x_a)^{l_1+l_1'}(t-t_a)^{l_2+l_2'}\cdot U_a^\alpha\cdot U_a^\beta\cdot\big[\mathbf{A}_\alpha^{l_1,l_2},\mathbf{B}_\beta^{l_1',l_2'}\big]. \tag{93}$$

For any $\alpha\in\mathcal{M}_p$, $\beta\in\mathcal{M}_{p+d-1}$, $l_1,l_2\in\mathbb{Z}_{\geq 0}$, the expressions $D_x\big((x-x_a)^{l_1}(t-t_a)^{l_2}U_a^\beta\big)$ and $D_t\big((x-x_a)^{l_1}(t-t_a)^{l_2}U_a^\alpha\big)$ are functions of the variables x, t, u_k^i. Taking the corresponding Taylor series at the point (35), we regard these expressions as power series.

Then (91), (92), (93) are formal power series with coefficients in $\mathfrak{F}$, and we have

$$D_x(\mathbf{B}) - D_t(\mathbf{A}) + [\mathbf{A},\mathbf{B}] = \sum_{\gamma\in\mathcal{M}_{p+d},\ l_1,l_2\in\mathbb{Z}_{\geq 0}} (x-x_a)^{l_1}(t-t_a)^{l_2}\cdot U_a^\gamma\cdot \mathbf{Z}_\gamma^{l_1,l_2}$$

for some elements $\mathbf{Z}_\gamma^{l_1,l_2}\in\mathfrak{F}$.

Let $\mathfrak{I}\subset\mathfrak{F}$ be the ideal generated by the elements

$$\mathbf{Z}_\gamma^{l_1,l_2},\qquad \mathbf{A}_0^{l_1,l_2},\qquad \mathbf{B}_0^{0,l_2},\qquad \gamma\in\mathcal{M}_{p+d},\qquad l_1,l_2\in\mathbb{Z}_{\geq 0},$$
$$\mathbf{A}_{\hat\alpha}^{l_1,l_2},\qquad \hat\alpha\in M_{i_0,k_0}^p,\qquad i_0=1,\dots,m,\qquad k_0=1,\dots,p,\qquad l_1,l_2\in\mathbb{Z}_{\geq 0}.$$

Set $\mathbb{F}^p(\mathcal{E},a)=\mathfrak{F}/\mathfrak{I}$. Consider the natural homomorphism $\rho\colon\mathfrak{F}\to\mathfrak{F}/\mathfrak{I}=\mathbb{F}^p(\mathcal{E},a)$ and set

$$\mathbb{A}_\alpha^{l_1,l_2}=\rho\big(\mathbf{A}_\alpha^{l_1,l_2}\big),\qquad\qquad \mathbb{B}_\beta^{l_1,l_2}=\rho\big(\mathbf{B}_\beta^{l_1,l_2}\big).$$

The definition of $\mathfrak{I}$ implies that the power series

$$\mathbb{A} = \sum_{\alpha\in\mathcal{M}_p,\ l_1,l_2\in\mathbb{Z}_{\geq 0}} (x-x_a)^{l_1}(t-t_a)^{l_2}\cdot U_a^\alpha\cdot \mathbb{A}_\alpha^{l_1,l_2}, \tag{94}$$

$$\mathbb{B} = \sum_{\beta\in\mathcal{M}_{p+d-1},\ l_1,l_2\in\mathbb{Z}_{\geq 0}} (x-x_a)^{l_1}(t-t_a)^{l_2}\cdot U_a^\beta\cdot \mathbb{B}_\beta^{l_1,l_2} \tag{95}$$

satisfy

$$D_x(\mathbb{B}) - D_t(\mathbb{A}) + [\mathbb{A},\mathbb{B}] = 0. \tag{96}$$

Remark 20. The Lie algebra $\mathbb{F}^p(\mathcal{E},a)$ can be described in terms of generators and relations as follows.

Equation (96) is equivalent to some Lie algebraic relations for $\mathbb{A}_\alpha^{l_1,l_2}$, $\mathbb{B}_\beta^{l_1,l_2}$.

The algebra $\mathbb{F}^p(\mathcal{E},a)$ is given by the generators $\mathbb{A}_\alpha^{l_1,l_2}$, $\mathbb{B}_\beta^{l_1,l_2}$, the relations arising from (96), and the following relations

$$\mathbb{A}_0^{l_1,l_2}=\mathbb{B}_0^{0,l_2}=0,\quad \mathbb{A}_{\hat\alpha}^{l_1,l_2}=0,\quad \hat\alpha\in M_{i_0,k_0}^p,\quad i_0=1,\dots,m,\quad k_0=1,\dots,p,\quad l_1,l_2\in\mathbb{Z}_{\geq 0}. \tag{97}$$

Note that condition (97) is equivalent to the following equations

$$\forall\, i_0=1,\dots,m,\qquad \forall\, k_0\geq 1,\qquad \frac{\partial\mathbb{A}}{\partial u_{k_0}^{i_0}}\bigg|_{u_k^i=a_k^i\ \forall\,(i,k)\succ(i_0,k_0-1)}=0, \tag{98}$$

$$\mathbb{A}\Big|_{u^i_k=a^i_k\ \forall\,(i,k)}=0, \tag{99}$$

$$\mathbb{B}\Big|_{x=x_a,\ u^i_k=a^i_k\ \forall\,(i,k)}=0. \tag{100}$$

Remark 21. Let $\mathfrak{L}$ be a Lie algebra. If A, B are functions with values in $\mathfrak{L}$ and satisfy (69) then A, B form a ZCR of order $\le p$ with values in $\mathfrak{L}$.

Instead of functions with values in $\mathfrak{L}$, one can consider formal power series with coefficients in $\mathfrak{L}$. Then one gets the notion of *formal ZCRs with coefficients in* $\mathfrak{L}$.

More precisely, a *formal ZCR of order* $\le p$ *with coefficients in* $\mathfrak{L}$ is given by power series

$$\mathsf{A}=\sum_{\alpha\in\mathcal{M}_p,\ l_1,l_2\in\mathbb{Z}_{\ge 0}}(x-x_a)^{l_1}(t-t_a)^{l_2}\cdot U^\alpha_a\cdot\mathsf{A}^{l_1,l_2}_\alpha, \tag{101}$$

$$\mathsf{B}=\sum_{\beta\in\mathcal{M}_{p+d-1},\ l_1,l_2\in\mathbb{Z}_{\ge 0}}(x-x_a)^{l_1}(t-t_a)^{l_2}\cdot U^\beta_a\cdot\mathsf{B}^{l_1,l_2}_\beta \tag{102}$$

such that $\mathsf{A}^{l_1,l_2}_\alpha,\mathsf{B}^{l_1,l_2}_\beta\in\mathfrak{L}$ and $D_x(\mathsf{B})-D_t(\mathsf{A})+[\mathsf{A},\mathsf{B}]=0$.

If the power series (101), (102) satisfy (70), (71), (72) then this formal ZCR is said to be *a-normal*.

For example, since (94), (95) obey (96), (98), (99), (100) and $\mathbb{A}^{l_1,l_2}_\alpha,\mathbb{B}^{l_1,l_2}_\beta\in\mathbb{F}^p(\mathcal{E},a)$, the power series (94), (95) constitute an a-normal formal ZCR of order $\le p$ with coefficients in $\mathbb{F}^p(\mathcal{E},a)$.

Remark 22. Let $\mathfrak{L}$ be a Lie algebra. Consider an $\mathfrak{L}$-valued ZCR of order $\le p$ given by $\mathfrak{L}$-valued functions A, B satisfying (69). Then the Taylor series of the functions A, B are of the form (101), (102) and constitute a formal ZCR with coefficients in $\mathfrak{L}$.

Thus any $\mathfrak{L}$-valued ZCR can be regarded as a formal ZCR with coefficients in $\mathfrak{L}$, if we replace the $\mathfrak{L}$-valued functions by the corresponding Taylor series with coefficients in $\mathfrak{L}$.

The definition of the Lie algebra $\mathbb{F}^p(\mathcal{E},a)$ implies the following result.

Theorem 5. *Let* (101), (102) *be an a-normal formal ZCR of order* $\le p$ *with coefficients in a Lie algebra* $\mathfrak{L}$. *Then one has a homomorphism* $\mathbb{F}^p(\mathcal{E},a)\to\mathfrak{L}$ *given by the formulas* $\mathbb{A}^{l_1,l_2}_\alpha\mapsto\mathsf{A}^{l_1,l_2}_\alpha$, $\mathbb{B}^{l_1,l_2}_\beta\mapsto\mathsf{B}^{l_1,l_2}_\beta$ *for all* α, β, l_1, l_2.

Remark 23. Let $\hat{A}=\hat{A}(x,t,u^j_0,u^j_1,\dots,u^j_p)$, $\hat{B}=\hat{B}(x,t,u^j_0,u^j_1,\dots,u^j_{p+d-1})$ be an a-normal ZCR of order $\le p$ with values in a Lie algebra $\hat{\mathfrak{g}}$. We suppose that the $\hat{\mathfrak{g}}$-valued functions $\hat{A}$, $\hat{B}$ are defined on a neighborhood of $a\in\mathcal{E}$.

Then the Taylor series of the functions $\hat{A}$, $\hat{B}$ at the point $a\in\mathcal{E}$ constitute an a-normal formal ZCR of order $\le p$ with coefficients in $\hat{\mathfrak{g}}$. Therefore, by Theorem 5, we get a homomorphism $\mathbb{F}^p(\mathcal{E},a)\to\hat{\mathfrak{g}}$ which maps the coefficients of the power series (94), (95) to the corresponding coefficients of the Taylor series of the functions $\hat{A}$, $\hat{B}$.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. A homomorphism $\mu\colon\mathbb{F}^p(\mathcal{E},a)\to\mathfrak{g}$ is said to be *regular* if the power series

$$\tilde{A}=\sum_{\alpha\in\mathcal{M}_p,\ l_1,l_2\in\mathbb{Z}_{\ge 0}}(x-x_a)^{l_1}(t-t_a)^{l_2}\cdot U^\alpha_a\cdot\mu\big(\mathbb{A}^{l_1,l_2}_\alpha\big), \tag{103}$$

$$\tilde{B}=\sum_{\beta\in\mathcal{M}_{p+d-1},\ l_1,l_2\in\mathbb{Z}_{\ge 0}}(x-x_a)^{l_1}(t-t_a)^{l_2}\cdot U^\beta_a\cdot\mu\big(\mathbb{B}^{l_1,l_2}_\beta\big) \tag{104}$$

are absolutely convergent in a neighborhood of $a\in\mathcal{E}$. In other words, μ is regular iff (103), (104) are analytic functions on a neighborhood of $a\in\mathcal{E}$.

Since (94), (95) obey (96), the power series (103), (104) satisfy (87) for any homomorphism $\mu\colon\mathbb{F}^p(\mathcal{E},a)\to\mathfrak{g}$. Therefore, if μ is regular, the analytic functions (103), (104) form a ZCR with values in $\mathfrak{g}$. Denote this ZCR by $\mathbf{Z}(\mathcal{E},a,p,\mu)$.

Combining this construction with Theorem 3 and Remark 18, we obtain the following result.

Theorem 6. *Let $\mathfrak{g}$ be a finite-dimensional matrix Lie algebra. For any $\mathfrak{g}$-valued ZCR (64) of order $\le p$ on a neighborhood of $a \in \mathcal{E}$, there is a regular homomorphism $\mu\colon \mathbb{F}^p(\mathcal{E},a) \to \mathfrak{g}$ such that the ZCR (64) is gauge equivalent to the ZCR $\mathbf{Z}(\mathcal{E},a,p,\mu)$ given by (103), (104).*

The ZCR $\mathbf{Z}(\mathcal{E},a,p,\mu)$ takes values in the Lie algebra $\mu\big(\mathbb{F}^p(\mathcal{E},a)\big) \subset \mathfrak{g}$.

Recall that we have the power series $\mathbb{A}$, $\mathbb{B}$ with coefficients in $\mathbb{F}^p(\mathcal{E},a)$ given by formulas (94), (95). Formulas (103), (104) say that $\tilde{A} = \mu(\mathbb{A})$ and $\tilde{B} = \mu(\mathbb{B})$, in the sense that μ maps the coefficients of the power series $\mathbb{A}$, $\mathbb{B}$ to the corresponding coefficients of the Taylor series of the functions $\tilde{A}$, $\tilde{B}$.

So the ZCR (64) is gauge equivalent to the ZCR $\mathbf{Z}(\mathcal{E},a,p,\mu)$ given by the functions $\tilde{A} = \mu(\mathbb{A})$, $\tilde{B} = \mu(\mathbb{B})$.

2.4. **The homomorphism $\mathbb{F}^p(\mathcal{E},a) \to \mathbb{F}^{p-1}(\mathcal{E},a)$.** According to Remarks 20, 21, the Lie algebra $\mathbb{F}^p(\mathcal{E},a)$ is generated by $\mathbb{A}_\alpha^{l_1,l_2}$, $\mathbb{B}_\beta^{l_1,l_2}$, and the power series (94), (95) constitute an a-normal formal ZCR of order $\le p$ with coefficients in $\mathbb{F}^p(\mathcal{E},a)$.

In this subsection we suppose that $p \ge 1$. Constructing the Lie algebra $\mathbb{F}^{p-1}(\mathcal{E},a)$ in the same way, we get power series

$$\hat{\mathbb{A}} = \sum_{\hat\alpha\in\mathcal{M}_{p-1},\ l_1,l_2\in\mathbb{Z}_{\ge0}} (x-x_a)^{l_1}(t-t_a)^{l_2}\cdot U_a^{\hat\alpha}\cdot \hat{\mathbb{A}}_{\hat\alpha}^{l_1,l_2}, \tag{105}$$

$$\hat{\mathbb{B}} = \sum_{\hat\beta\in\mathcal{M}_{p+d-2},\ l_1,l_2\in\mathbb{Z}_{\ge0}} (x-x_a)^{l_1}(t-t_a)^{l_2}\cdot U_a^{\hat\beta}\cdot \hat{\mathbb{B}}_{\hat\beta}^{l_1,l_2} \tag{106}$$

such that the Lie algebra $\mathbb{F}^{p-1}(\mathcal{E},a)$ is generated by $\hat{\mathbb{A}}_{\hat\alpha}^{l_1,l_2}$, $\hat{\mathbb{B}}_{\hat\beta}^{l_1,l_2}$ for all $\hat\alpha \in \mathcal{M}_{p-1}$, $\hat\beta \in \mathcal{M}_{p+d-2}$, $l_1,l_2 \in \mathbb{Z}_{\ge0}$, and the power series (105), (106) constitute an a-normal formal ZCR of order $\le p-1$ with coefficients in $\mathbb{F}^{p-1}(\mathcal{E},a)$.

We are going to construct a surjective homomorphism $\tau_p\colon \mathbb{F}^p(\mathcal{E},a) \to \mathbb{F}^{p-1}(\mathcal{E},a)$. Since the algebra $\mathbb{F}^p(\mathcal{E},a)$ is generated by $\mathbb{A}_\alpha^{l_1,l_2}$, $\mathbb{B}_\beta^{l_1,l_2}$, it is sufficient to define $\tau_p(\mathbb{A}_\alpha^{l_1,l_2})$, $\tau_p(\mathbb{B}_\beta^{l_1,l_2})$.

To do this, we need to introduce some extra notation. Recall that, for each $q \in \mathbb{Z}_{\ge0}$, we denote by $\mathcal{M}_q$ the set of matrices of size $m \times (q+1)$ with nonnegative integer entries. For a matrix $\gamma \in \mathcal{M}_q$, its entries are denoted by $\gamma_{i,k} \in \mathbb{Z}_{\ge0}$, where $i = 1,\dots,m$ and $k = 0,\dots,q$.

For each $q \ge 1$ and each $m \times (q+1)$ matrix $\gamma \in \mathcal{M}_q$, we denote by $\mathbf{r}(\gamma)$ the $m \times q$ matrix with the entries $\mathbf{r}(\gamma)_{i,k} = \gamma_{i,k}$ for $i = 1,\dots,m$ and $k = 0,\dots,q-1$. In other words, the matrix $\mathbf{r}(\gamma) \in \mathcal{M}_{q-1}$ is obtained from the matrix γ by erasing the last column.

Let $l_1,l_2 \in \mathbb{Z}_{\ge0}$. For $\alpha \in \mathcal{M}_p$, $\beta \in \mathcal{M}_{p+d-1}$, we can consider the matrices $\mathbf{r}(\alpha) \in \mathcal{M}_{p-1}$, $\mathbf{r}(\beta) \in \mathcal{M}_{p+d-2}$ and the elements $\hat{\mathbb{A}}_{\mathbf{r}(\alpha)}^{l_1,l_2}, \hat{\mathbb{B}}_{\mathbf{r}(\beta)}^{l_1,l_2} \in \mathbb{F}^{p-1}(\mathcal{E},a)$.

For all $l_1,l_2 \in \mathbb{Z}_{\ge0}$, $\alpha \in \mathcal{M}_p$, $\beta \in \mathcal{M}_{p+d-1}$, we set

$$\tau_p(\mathbb{A}_\alpha^{l_1,l_2}) = \begin{cases} 0, \text{ if there is } i \in \{1,\dots,m\} \text{ such that } \alpha_{i,p} \ne 0,\\ \hat{\mathbb{A}}_{\mathbf{r}(\alpha)}^{l_1,l_2}, \text{ if } \alpha_{i,p} = 0 \text{ for all } i,\end{cases} \tag{107}$$

$$\tau_p(\mathbb{B}_\beta^{l_1,l_2}) = \begin{cases} 0, \text{ if there is } i \in \{1,\dots,m\} \text{ such that } \beta_{i,p+d-1} \ne 0,\\ \hat{\mathbb{B}}_{\mathbf{r}(\beta)}^{l_1,l_2}, \text{ if } \beta_{i,p+d-1} = 0 \text{ for all } i.\end{cases} \tag{108}$$

The definition of $\mathbb{F}^p(\mathcal{E},a)$ and $\mathbb{F}^{p-1}(\mathcal{E},a)$ implies that $\tau_p\colon \mathbb{F}^p(\mathcal{E},a) \to \mathbb{F}^{p-1}(\mathcal{E},a)$ defined by (107), (108) is indeed a surjective homomorphism. The meaning of this homomorphism is explained in Remark 24 below.

According to (107), one has $\tau_p(\mathbb{A}_\alpha^{l_1,l_2}) = 0$ if there is a nonzero entry in the last column of the matrix $\alpha \in \mathcal{M}_p$. According to (108), one has $\tau_p(\mathbb{B}_\beta^{l_1,l_2}) = 0$ if there is a nonzero entry in the last column of the matrix $\beta \in \mathcal{M}_{p+d-1}$.

Recall that the power series $\mathbb{A}$ and $\mathbb{B}$ are given by (94), (95). Taking into account formula (84), we see that formulas (107), (108) say the following. Applying τ_p to the coefficients of the power series $\mathbb{A}\Big|_{u^i_p=a^i_p\ \forall\, i}$, we get the power series (105). Applying τ_p to the coefficients of the power series $\mathbb{B}\Big|_{u^i_{p+d-1}=a^i_{p+d-1}\ \forall\, i}$, we get the power series (106).

Remark 24. Any ZCR of order $\leq p-1$ is at the same time of order $\leq p$. Therefore, (105), (106) can be regarded as an a-normal formal ZCR of order $\leq p$ with coefficients in $\mathbb{F}^{p-1}(\mathcal{E},a)$. The homomorphism $\tau_p\colon\ \mathbb{F}^p(\mathcal{E},a)\to\mathbb{F}^{p-1}(\mathcal{E},a)$ is the homomorphism which corresponds to this ZCR by Theorem 5.

Thus we obtain the following sequence of surjective homomorphisms of Lie algebras

$$\dots \xrightarrow{\tau_{p+1}} \mathbb{F}^p(\mathcal{E},a) \xrightarrow{\tau_p} \mathbb{F}^{p-1}(\mathcal{E},a) \xrightarrow{\tau_{p-1}} \dots \xrightarrow{\tau_2} \mathbb{F}^1(\mathcal{E},a) \xrightarrow{\tau_1} \mathbb{F}^0(\mathcal{E},a). \tag{109}$$

2.5. **Some results on generators of $\mathbb{F}^p(\mathcal{E},a)$.** According to Remark 20, the algebra $\mathbb{F}^p(\mathcal{E},a)$ is given by the generators $\mathbb{A}^{l_1,l_2}_\alpha$, $\mathbb{B}^{l_1,l_2}_\beta$ and the relations arising from (96), (97). Using (5), we can rewrite equation (96) as

$$\frac{\partial}{\partial x}(\mathbb{B}) + \sum_{\substack{i=1,\dots,m,\\ k=0,1,\dots,p+d-1}} u^i_{k+1}\frac{\partial}{\partial u^i_k}(\mathbb{B}) - \frac{\partial}{\partial t}(\mathbb{A}) - \sum_{\substack{i=1,\dots,m,\\ k=0,1,\dots,p}} D^k_x\big(F^i(x,t,u^j_0,u^j_1,\dots,u^j_d)\big)\frac{\partial}{\partial u^i_k}(\mathbb{A}) + [\mathbb{A},\mathbb{B}] = 0. \tag{110}$$

Here we regard $F^i = F^i(x,t,u^j_0,u^j_1,\dots,u^j_d)$ as a power series, using the Taylor series of the function F^i at the point (35).

Theorem 7. *The elements*

$$\mathbb{A}^{l_1,0}_\alpha, \qquad l_1\in\mathbb{Z}_{\geq 0}, \qquad \alpha\in\mathcal{M}_p, \tag{111}$$

generate the algebra $\mathbb{F}^p(\mathcal{E},a)$.

Proof. For each $l\in\mathbb{Z}_{\geq 0}$, denote by $\mathfrak{A}_l\subset\mathbb{F}^p(\mathcal{E},a)$ the subalgebra generated by all the elements $\mathbb{A}^{l_1,l_2}_\alpha$ with $l_2\leq l$. To prove Theorem 7, we need several lemmas.

Lemma 2. *Let $l_1,l_2\in\mathbb{Z}_{\geq 0}$. Let $\beta\in\mathcal{M}_{p+d-1}$ be such that not all entries of the matrix β are zero (i.e., in the matrix β there is a nonzero entry). Then $\mathbb{B}^{l_1,l_2}_\beta\in\mathfrak{A}_{l_2}$.*

Proof. One can prove this lemma, analyzing equation (110) and properties (97), (98). □

Lemma 3. *For all $l_1,l_2\in\mathbb{Z}_{\geq 0}$, one has $\mathbb{B}^{l_1,l_2}_0\in\mathfrak{A}_{l_2}$. Here $0\in\mathcal{M}_{p+d-1}$ is the matrix with zero entries.*

Proof. According to (97), we have $\mathbb{B}^{0,l_2}_0=0$. Therefore, it is sufficient to prove $\mathbb{B}^{l_1,l_2}_0\in\mathfrak{A}_{l_2}$ for $l_1>0$.

Note that property (98) implies

$$\mathbb{A}\Big|_{u^i_k=a^i_k\ \forall\,(i,k)} = 0, \qquad \frac{\partial}{\partial t}(\mathbb{A})\Big|_{u^i_k=a^i_k\ \forall\,(i,k)} = 0. \tag{112}$$

In view of (95), one has

$$\frac{\partial}{\partial x}(\mathbb{B})\Big|_{u^i_k=a^i_k\ \forall\,(i,k)} = \sum_{l_1>0,\ l_2\geq 0} l_1(x-x_a)^{l_1-1}(t-t_a)^{l_2}\cdot\mathbb{B}^{l_1,l_2}_0. \tag{113}$$

Substituting $u^i_k = a^i_k$ for all i, k in (110) and using (112), (113), we get

$$\sum_{l_1>0,\ l_2\geq 0} l_1(x-x_a)^{l_1-1}(t-t_a)^{l_2}\cdot \mathbb{B}_0^{l_1,l_2} = \tag{114}$$
$$= -\left(\sum_{i,k} u^i_{k+1}\frac{\partial}{\partial u^i_k}(\mathbb{B})\right)\Big|_{u^i_k=a^i_k\ \forall\,(i,k)} + \left(\sum_{i,k} D_x^k\big(F^i\big)\frac{\partial}{\partial u^i_k}(\mathbb{A})\right)\Big|_{u^i_k=a^i_k\ \forall\,(i,k)}.$$

For a matrix $\beta \in \mathcal{M}_{p+d-1}$, we denote by $|\beta|$ the sum of all entries of β. Combining (94), (95), (114), we see that for any $l_1 > 0$ and $l_2 \geq 0$ the element $\mathbb{B}_0^{l_1,l_2}$ is equal to a linear combination of elements of the form

$$\mathbb{A}_\alpha^{l_1',l_2},\qquad \mathbb{B}_\beta^{l_1',l_2},\qquad l_1'\in\mathbb{Z}_{\geq 0},\qquad \alpha\in\mathcal{M}_p,\qquad \beta\in\mathcal{M}_{p+d-1},\qquad |\beta|=1. \tag{115}$$

According to Lemma 2 and the definition of $\mathfrak{A}_{l_2}$, the elements (115) belong to $\mathfrak{A}_{l_2}$. Thus $\mathbb{B}_0^{l_1,l_2}\in\mathfrak{A}_{l_2}$. □

Lemma 4. *For all $l_1, l\in\mathbb{Z}_{\geq 0}$ and $\alpha\in\mathcal{M}_p$, we have $\mathbb{A}_\alpha^{l_1,l+1}\in\mathfrak{A}_l$.*

Proof. Using (94), we can rewrite equation (110) as

$$\frac{\partial}{\partial t}(\mathbb{A}) = \sum_{\alpha\in\mathcal{M}_p,\ l_1,l\in\mathbb{Z}_{\geq 0}} (l+1)(x-x_a)^{l_1}(t-t_a)^l\cdot U_a^\alpha\cdot\mathbb{A}_\alpha^{l_1,l+1} =$$
$$= \frac{\partial}{\partial x}(\mathbb{B}) + \sum_{i,k} u^i_{k+1}\frac{\partial}{\partial u^i_k}(\mathbb{B}) - \sum_{i,k} D_x^k\big(F^i\big)\frac{\partial}{\partial u^i_k}(\mathbb{A}) + [\mathbb{A},\mathbb{B}].$$

This implies that $\mathbb{A}_\alpha^{l_1,l+1}$ is equal to a linear combination of elements of the form

$$\mathbb{A}_{\hat\alpha}^{\hat l_1,\hat l_2},\quad \mathbb{B}_{\tilde\beta}^{\tilde l_1,\tilde l_2},\quad \Big[\mathbb{A}_{\hat\alpha}^{\hat l_1,\hat l_2},\mathbb{B}_{\tilde\beta}^{\tilde l_1,\tilde l_2}\Big],\quad \hat l_2\leq l,\quad \tilde l_2\leq l,\quad \hat l_1,\tilde l_1\in\mathbb{Z}_{\geq 0},\quad \hat\alpha\in\mathcal{M}_p,\quad \tilde\beta\in\mathcal{M}_{p+d-1}. \tag{116}$$

Using Lemmas 2, 3 and the condition $\tilde l_2\leq l$, we get $\mathbb{B}_{\tilde\beta}^{\tilde l_1,\tilde l_2}\in\mathfrak{A}_{\tilde l_2}\subset\mathfrak{A}_l$. Therefore, the elements (116) belong to $\mathfrak{A}_l$. Hence $\mathbb{A}_\alpha^{l_1,l+1}\in\mathfrak{A}_l$. □

Now we return to the proof of Theorem 7. According to Lemmas 2, 3 and the definition of $\mathfrak{A}_l$, we have $\mathbb{A}_\alpha^{l_1,l_2},\mathbb{B}_\beta^{l_1,l_2}\in\mathfrak{A}_{l_2}$ for all $l_1,l_2\in\mathbb{Z}_{\geq 0}$, $\alpha\in\mathcal{M}_p$, $\beta\in\mathcal{M}_{p+d-1}$. Lemma 4 implies that

$$\mathfrak{A}_{l_2}\subset\mathfrak{A}_{l_2-1}\subset\mathfrak{A}_{l_2-2}\subset\cdots\subset\mathfrak{A}_0.$$

Therefore, $\mathbb{F}^p(\mathcal{E},a)$ is equal to $\mathfrak{A}_0$, which is generated by the elements (111). □

2.6. **Some constructions with zero-curvature representations.** We continue to work with an evolution PDE (1), which can be written also as (2), according to our notation.

Let $\mathfrak{L}^1$ and $\mathfrak{L}^2$ be Lie algebras. For $i=1,2$, let

$$A^i = A^i(x,t,u^j_0,u^j_1,\dots,u^j_p),\quad B^i = B^i(x,t,u^j_0,u^j_1,\dots,u^j_{p+d-1}),\quad D_x(B)-D_t(A)+[A,B]=0 \tag{117}$$

be an $\mathfrak{L}^i$-valued ZCR for the PDE (1). So the functions A^i, B^i take values in $\mathfrak{L}^i$. The following notions will be needed in the next sections.

The *direct sum* of the $\mathfrak{L}^1$-valued ZCR A^1, B^1 and the $\mathfrak{L}^2$-valued ZCR A^2, B^2 is the $(\mathfrak{L}^1\oplus\mathfrak{L}^2)$-valued ZCR given by the functions $A^1\oplus A^2$, $B^1\oplus B^2$. So the ZCR given by the functions $A^1\oplus A^2$, $B^1\oplus B^2$ takes values in the Lie algebra $\mathfrak{L}^1\oplus\mathfrak{L}^2$.

We say that the ZCR A^2, B^2 is a *reduction* of the ZCR A^1, B^1 if there is a homomorphism $\rho\colon\mathfrak{L}^1\to\mathfrak{L}^2$ such that $A^2=\rho(A^1)$ and $B^2=\rho(B^1)$.

Similarly, one can speak also about direct sums and reductions of formal ZCRs.

Remark 25. For any (possibly infinite-dimensional) Lie algebra $\mathfrak{L}$, there is a (possibly infinite-dimensional) vector space V such that $\mathfrak{L}$ is isomorphic to a Lie subalgebra of $\mathfrak{gl}(V)$. Here $\mathfrak{gl}(V)$ is the algebra of linear maps $V \to V$.

For example, one can use the following construction. Denote by $\mathrm{U}(\mathfrak{L})$ the universal enveloping algebra of $\mathfrak{L}$. We have the injective homomorphism of Lie algebras

$$\xi\colon \mathfrak{L} \hookrightarrow \mathfrak{gl}(\mathrm{U}(\mathfrak{L})), \qquad \xi(v)(w) = vw, \qquad v \in \mathfrak{L}, \qquad w \in \mathrm{U}(\mathfrak{L}).$$

So one can set $V = \mathrm{U}(\mathfrak{L})$.

So we have $\mathfrak{L} \subset \mathfrak{gl}(V)$. Let $k \in \mathbb{Z}_{\geq 0}$. A *formal gauge transformation of order* $\leq k$ is a formal power series of the form

$$G = \sum_{\gamma\in\mathcal{M}_k,\ l_1,l_2\in\mathbb{Z}_{\geq 0}} (x-x_a)^{l_1}(t-t_a)^{l_2}\cdot U_a^\gamma \cdot G_\gamma^{l_1,l_2}, \qquad G_\gamma^{l_1,l_2} \in \mathfrak{gl}(V), \tag{118}$$

such that the map $G_0^{0,0}\colon V \to V$ is invertible. (So the free term $G_0^{0,0}$ of the power series (118) is invertible.) Then G^{-1} is well defined and is a power series with coefficients in $\mathfrak{gl}(V)$ as well.

Recall that a formal ZCR (101), (102) with coefficients in $\mathfrak{L}$ is given by formal power series A, B with coefficients in $\mathfrak{L}$ satisfying $D_x(\mathsf{B}) - D_t(\mathsf{A}) + [\mathsf{A},\mathsf{B}] = 0$. Then

$$\tilde{\mathsf{A}} = G\mathsf{A}G^{-1} - D_x(G)\cdot G^{-1}, \qquad \tilde{\mathsf{B}} = G\mathsf{B}G^{-1} - D_t(G)\cdot G^{-1} \tag{119}$$

are formal power series with coefficients in $\mathfrak{gl}(V)$ and satisfy $D_x(\tilde{\mathsf{B}}) - D_t(\tilde{\mathsf{A}}) + [\tilde{\mathsf{A}},\tilde{\mathsf{B}}] = 0$. Therefore, (119) is a formal ZCR with coefficients in $\mathfrak{gl}(V)$.

The formal ZCR (119) is said to be gauge equivalent to the formal ZCR (101), (102) with respect to the formal gauge transformation (118).

Quite often, it happens that the coefficients of the power series (119) belong to $\mathfrak{L} \subset \mathfrak{gl}(V)$. Then (119) can be regarded as a formal ZCR with coefficients in $\mathfrak{L}$.

This allows us to speak about gauge equivalence for formal ZCRs with coefficients in infinite-dimensional Lie algebras.

3. Relations between $\mathbb{F}^0(\mathcal{E},a)$ and the Wahlquist-Estabrook prolongation algebra

Let m, d be positive integers. Consider an m-component evolution PDE of the form

$$u_t^i = F^i(u_0^j, u_1^j, \dots, u_d^j), \tag{120}$$

$$u^i = u^i(x,t), \qquad u_0^i = u^i, \qquad u_k^i = \frac{\partial^k u^i}{\partial x^k}, \qquad i,j = 1,\dots,m, \qquad k \in \mathbb{Z}_{\geq 0}.$$

Note that the functions F^i in (120) do not depend on x, t.

Let $\mathcal{E}$ be the infinite prolongation of the PDE (120). According Section 2.1, $\mathcal{E}$ is an infinite-dimensional manifold with the coordinates x, t, u_k^i for $i = 1,\dots,m$ and $k \in \mathbb{Z}_{\geq 0}$.

Consider a point $a \in \mathcal{E}$ given by (35). The constants $x_a, t_a, a_k^i \in \mathbb{K}$ from (35) are the coordinates of the point $a \in \mathcal{E}$ in the coordinate system x, t, u_k^i.

As has been said in Section 2.3, for each $q \in \mathbb{Z}_{\geq 0}$, we denote by $\mathcal{M}_q$ the set of matrices of size $m \times (q+1)$ with nonnegative integer entries. For a matrix $\gamma \in \mathcal{M}_q$, its entries are denoted by $\gamma_{i,k} \in \mathbb{Z}_{\geq 0}$, where $i = 1,\dots,m$ and $k = 0,\dots,q$.

According to formula (84), for $\alpha \in \mathcal{M}_0$ and $\beta \in \mathcal{M}_{d-1}$ we have

$$U_a^\alpha = \prod_{i=1,\dots,m} \left(u_0^i - a_0^i\right)^{\alpha_{i,0}}, \qquad U_a^\beta = \prod_{\substack{i=1,\dots,m,\\ k=0,\dots,d-1}} \left(u_k^i - a_k^i\right)^{\beta_{i,k}}.$$

The *Wahlquist-Estabrook prolongation algebra* of the PDE (120) at the point $a \in \mathcal{E}$ can be defined as follows. Consider formal power series

$$\mathrm{A} = \sum_{\alpha \in \mathcal{M}_0} U_a^\alpha \cdot \mathrm{A}_\alpha, \qquad \mathrm{B} = \sum_{\beta \in \mathcal{M}_{d-1}} U_a^\beta \cdot \mathrm{B}_\beta, \tag{121}$$

where A_α, B_β are generators of some Lie algebra, which is described below. The equation

$$D_x(\mathrm{B}) - D_t(\mathrm{A}) + [\mathrm{A}, \mathrm{B}] = 0 \tag{122}$$

is equivalent to some Lie algebraic relations for A_α, B_β. The Wahlquist-Estabrook prolongation algebra (WE algebra for short) at the point $a \in \mathcal{E}$ is the Lie algebra given by the generators A_α, B_β and the relations arising from (122). A more detailed definition of the WE algebra is presented in [9]. We denote this Lie algebra by $\mathfrak{W}_a$.

Then (121), (122) is called the *formal Wahlquist-Estabrook ZCR with coefficients in* $\mathfrak{W}_a$.

For each $p \in \mathbb{Z}_{\geq 0}$, the Lie algebra $\mathbb{F}^p(\mathcal{E}, a)$ has been defined in Section 2.3. In the present section we study $\mathbb{F}^0(\mathcal{E}, a)$. We are going to show that the algebra $\mathbb{F}^0(\mathcal{E}, a)$ for the PDE (120) is isomorphic to some Lie subalgebra of $\mathfrak{W}_a$.

Since the PDE (120) is invariant with respect to the change of variables $x \mapsto x - x_a$, $t \mapsto t - t_a$, we can assume $x_a = t_a = 0$ in (35). Then (94), (95), (96) in the case $p = 0$ can be written as

$$\mathbb{A} = \sum_{\alpha \in \mathcal{M}_0,\ l_1, l_2 \in \mathbb{Z}_{\geq 0}} x^{l_1} t^{l_2} \cdot U_a^\alpha \cdot \mathbb{A}_\alpha^{l_1, l_2}, \qquad \mathbb{B} = \sum_{\beta \in \mathcal{M}_{d-1},\ l_1, l_2 \in \mathbb{Z}_{\geq 0}} x^{l_1} t^{l_2} \cdot U_a^\beta \cdot \mathbb{B}_\beta^{l_1, l_2}, \tag{123}$$

$$D_x(\mathbb{B}) - D_t(\mathbb{A}) + [\mathbb{A}, \mathbb{B}] = 0, \tag{124}$$

where $\mathbb{A}_\alpha^{l_1, l_2}, \mathbb{B}_\beta^{l_1, l_2} \in \mathbb{F}^0(\mathcal{E}, a)$. Note that equation (124) is equivalent to some Lie algebraic relations for $\mathbb{A}_\alpha^{l_1, l_2}$, $\mathbb{B}_\beta^{l_1, l_2}$.

According to Remark 20, the Lie algebra $\mathbb{F}^0(\mathcal{E}, a)$ can be described in terms of generators and relations as follows. The algebra $\mathbb{F}^0(\mathcal{E}, a)$ is given by the generators $\mathbb{A}_\alpha^{l_1, l_2}$, $\mathbb{B}_\beta^{l_1, l_2}$, the relations arising from (124), and the relations $\mathbb{A}_0^{l_1, l_2} = \mathbb{B}_0^{0, l_2} = 0$ for $l_1, l_2 \in \mathbb{Z}_{\geq 0}$.

Lemma 5. *The elements*

$$\mathbb{A}_\alpha^{l_1, 0}, \qquad l_1 \in \mathbb{Z}_{\geq 0}, \qquad \alpha \in \mathcal{M}_0, \qquad \alpha \neq 0, \tag{125}$$

generate the algebra $\mathbb{F}^0(\mathcal{E}, a)$.

Proof. According to Theorem 7, the elements $\mathbb{A}_\alpha^{l_1, 0}$ for $l_1 \in \mathbb{Z}_{\geq 0}$, $\alpha \in \mathcal{M}_0$ generate the algebra $\mathbb{F}^0(\mathcal{E}, a)$. Since $\mathbb{A}_0^{l_1, 0} = 0$, we can assume $\alpha \neq 0$. □

Lemma 6. *Let* $\mathfrak{L}$ *be a Lie algebra. Consider formal power series of the form*

$$P = \sum_{\alpha \in \mathcal{M}_0,\ l_1, l_2 \in \mathbb{Z}_{\geq 0}} x^{l_1} t^{l_2} \cdot U_a^\alpha \cdot P_\alpha^{l_1, l_2}, \qquad Q = \sum_{\beta \in \mathcal{M}_{d-1},\ l_1, l_2 \in \mathbb{Z}_{\geq 0}} x^{l_1} t^{l_2} \cdot U_a^\beta \cdot Q_\beta^{l_1, l_2} \tag{126}$$

satisfying

$$P_\alpha^{l_1, l_2},\ Q_\beta^{l_1, l_2} \in \mathfrak{L}, \qquad P_0^{l_1, l_2} = Q_0^{0, l_2} = 0 \qquad \forall\, l_1, l_2 \in \mathbb{Z}_{\geq 0}, \tag{127}$$

$$D_x(Q) - D_t(P) + [P, Q] = 0. \tag{128}$$

Then the map $\mathbb{A}_\alpha^{l_1, l_2} \mapsto P_\alpha^{l_1, l_2}$, $\mathbb{B}_\beta^{l_1, l_2} \mapsto Q_\beta^{l_1, l_2}$ *determines a homomorphism from* $\mathbb{F}^0(\mathcal{E}, a)$ *to* $\mathfrak{L}$.

Proof. According to Remark 21, formulas (126), (127), (128) say that the power series P, Q constitute an a-normal formal ZCR of order ≤ 0 with coefficients in $\mathfrak{L}$.

According to Theorem 5, this formal ZCR determines a homomorphism $\mathbb{F}^0(\mathcal{E}, a) \to \mathfrak{L}$ given by the formulas $\mathbb{A}_\alpha^{l_1, l_2} \mapsto P_\alpha^{l_1, l_2}$ and $\mathbb{B}_\beta^{l_1, l_2} \mapsto Q_\beta^{l_1, l_2}$. □

Let $\mathfrak{L}$ be a Lie algebra. A *ZCR of Wahlquist-Estabrook type with coefficients in* $\mathfrak{L}$ is given by formal power series

$$P = \sum_{\alpha\in\mathcal{M}_0} U_a^\alpha \cdot P_\alpha, \qquad Q = \sum_{\beta\in\mathcal{M}_{d-1}} U_a^\beta \cdot Q_\beta, \qquad P_\alpha,\, Q_\beta \in \mathfrak{L}, \tag{129}$$

satisfying

$$D_x(Q) - D_t(P) + [P, Q] = 0. \tag{130}$$

The next lemma follows from the definition of the WE algebra $\mathfrak{W}_a$.

Lemma 7. *Recall that the WE algebra* $\mathfrak{W}_a$ *is generated by* A_α, B_β *for* $\alpha \in \mathcal{M}_0$, $\beta \in \mathcal{M}_{d-1}$ *such that one has formulas* (121), (122).

For any Lie algebra $\mathfrak{L}$, *any ZCR of Wahlquist-Estabrook type* (129), (130) *with coefficients in* $\mathfrak{L}$ *determines a homomorphism* $\mathfrak{W}_a \to \mathfrak{L}$ *given by the formulas* $\mathrm{A}_\alpha \mapsto P_\alpha$, $\mathrm{B}_\beta \mapsto Q_\beta$.

Denote by $\mathbf{F}$ the vector space of formal power series in the variables z_1, z_2 with coefficients in $\mathbb{F}^0(\mathcal{E}, a)$. That is, an element of $\mathbf{F}$ is a power series of the form

$$\sum_{l_1,l_2\in\mathbb{Z}_{\geq 0}} z_1^{l_1} z_2^{l_2} C^{l_1 l_2}, \qquad C^{l_1 l_2} \in \mathbb{F}^0(\mathcal{E}, a).$$

The space $\mathbf{F}$ has the Lie algebra structure given by

$$\Big[\sum_{l_1,l_2} z_1^{l_1} z_2^{l_2} C^{l_1 l_2}, \sum_{\tilde{l}_1,\tilde{l}_2} z_1^{\tilde{l}_1} z_2^{\tilde{l}_2} \tilde{C}^{\tilde{l}_1\tilde{l}_2}\Big] = \sum z_1^{l_1+\tilde{l}_1} z_2^{l_2+\tilde{l}_2} \Big[C^{l_1 l_2}, \tilde{C}^{\tilde{l}_1\tilde{l}_2}\Big], \qquad C^{l_1 l_2}, \tilde{C}^{\tilde{l}_1\tilde{l}_2} \in \mathbb{F}^0(\mathcal{E}, a).$$

We have also the following homomorphism of Lie algebras

$$\nu\colon \mathbf{F} \to \mathbb{F}^0(\mathcal{E}, a), \qquad \sum_{l_1,l_2\in\mathbb{Z}_{\geq 0}} z_1^{l_1} z_2^{l_2} C^{l_1 l_2} \;\mapsto\; C^{00}. \tag{131}$$

For $i = 1, 2$, let $\partial_{z_i}\colon \mathbf{F} \to \mathbf{F}$ be the linear map given by

$$\partial_{z_i}\Big(\sum z_1^{l_1} z_2^{l_2} C^{l_1 l_2}\Big) = \sum \frac{\partial}{\partial z_i}\Big(z_1^{l_1} z_2^{l_2}\Big) C^{l_1 l_2}.$$

Let D be the linear span of ∂_{z_1}, ∂_{z_2} in the vector space of linear maps $\mathbf{F} \to \mathbf{F}$. Since the maps ∂_{z_1}, ∂_{z_2} commute, the space D is a 2-dimensional abelian Lie algebra with respect to the commutator of maps.

Denote by $\mathbb{L}$ the vector space $\mathrm{D} \oplus \mathbf{F}$ with the following Lie algebra structure

$$[X_1 + f_1,\, X_2 + f_2] = X_1(f_2) - X_2(f_1) + [f_1, f_2], \qquad X_1, X_2 \in \mathrm{D}, \qquad f_1, f_2 \in \mathbf{F}.$$

An element of $\mathbb{L}$ can be written as a sum of the following form

$$\big(y_1\partial_{z_1} + y_2\partial_{z_2}\big) + \sum z_1^{l_1} z_2^{l_2} C^{l_1 l_2}, \qquad y_1, y_2 \in \mathbb{K}, \qquad C^{l_1 l_2} \in \mathbb{F}^0(\mathcal{E}, a).$$

Theorem 8. *Recall that the WE algebra* $\mathfrak{W}_a$ *is generated by* A_α, B_β *for* $\alpha \in \mathcal{M}_0$, $\beta \in \mathcal{M}_{d-1}$ *such that one has formulas* (121), (122).

Let $\mathfrak{R} \subset \mathfrak{W}_a$ *be the subalgebra generated by the elements*

$$(\operatorname{ad}\mathrm{A}_0)^k(\mathrm{A}_\alpha), \qquad k \in \mathbb{Z}_{\geq 0}, \qquad \alpha \in \mathcal{M}_0, \qquad \alpha \neq 0. \tag{132}$$

(*Note that for* $k = 0$ *we have* $(\operatorname{ad}\mathrm{A}_0)^0(\mathrm{A}_\alpha) = \mathrm{A}_\alpha$, *hence* $\mathrm{A}_\alpha \in \mathfrak{R}$ *for all* $\alpha \neq 0$.)

Then the map $(\operatorname{ad}\mathrm{A}_0)^k(\mathrm{A}_\alpha) \mapsto (k!)\mathbb{A}_\alpha^{k,0}$ *determines an isomorphism between* $\mathfrak{R}$ *and* $\mathbb{F}^0(\mathcal{E}, a)$.

Proof. Since the functions F^i in (120) do not depend on x and t, from (123), (124) it follows that the power series

$$\tilde{\mathrm{A}} = \partial_{z_1} + \sum_{\alpha\in\mathcal{M}_0,\ \alpha\neq 0} U_a^\alpha \cdot \Big(\sum_{l_1,l_2} z_1^{l_1} z_2^{l_2} \mathbb{A}_\alpha^{l_1,l_2}\Big), \tag{133}$$

$$\tilde{\mathrm{B}} = \Big(\partial_{z_2} + \sum_{l_1,l_2} z_1^{l_1} z_2^{l_2} \mathbb{B}_0^{l_1,l_2}\Big) + \sum_{\beta\in\mathcal{M}_{d-1},\ \beta\neq 0} U_a^{\beta} \cdot \Big(\sum_{l_1,l_2} z_1^{l_1} z_2^{l_2} \mathbb{B}_{\beta}^{l_1,l_2}\Big) \tag{134}$$

form a ZCR of Wahlquist-Estabrook type with coefficients in $\mathbb{L}$. Applying Lemma 7 to this ZCR, we obtain the homomorphism

$$\varphi\colon \mathfrak{W}_a \to \mathbb{L}, \qquad \varphi(\mathrm{A}_0) = \partial_{z_1}, \qquad \varphi(\mathrm{A}_\alpha) = \sum_{l_1,l_2} z_1^{l_1} z_2^{l_2} \mathbb{A}_\alpha^{l_1,l_2}, \qquad \alpha\in\mathcal{M}_0, \quad \alpha\neq 0, \tag{135}$$

$$\varphi(\mathrm{B}_0) = \partial_{z_2} + \sum_{l_1,l_2} z_1^{l_1} z_2^{l_2} \mathbb{B}_0^{l_1,l_2}, \qquad \varphi(\mathrm{B}_\beta) = \sum_{l_1,l_2} z_1^{l_1} z_2^{l_2} \mathbb{B}_\beta^{l_1,l_2}, \qquad \beta\in\mathcal{M}_{d-1}, \quad \beta\neq 0.$$

Clearly, $\mathbf{F}$ is a Lie subalgebra of $\mathbb{L} = \mathrm{D}\oplus\mathbf{F}$. In view of (135), for any $\alpha\in\mathcal{M}_0$ and $k\in\mathbb{Z}_{\geq 0}$ such that $\alpha\neq 0$ we have

$$\varphi\Big((\operatorname{ad}\mathrm{A}_0)^k(\mathrm{A}_\alpha)\Big) = \big(\operatorname{ad}\partial_{z_1}\big)^k\Big(\sum_{l_1,l_2} z_1^{l_1} z_2^{l_2} \mathbb{A}_\alpha^{l_1,l_2}\Big) = \big(\partial_{z_1}\big)^k\Big(\sum_{l_1,l_2} z_1^{l_1} z_2^{l_2} \mathbb{A}_\alpha^{l_1,l_2}\Big) \in \mathbf{F}. \tag{136}$$

Since $\mathfrak{R}\subset\mathfrak{W}_a$ is generated by the elements (132), property (136) implies $\varphi(\mathfrak{R})\subset\mathbf{F}\subset\mathbb{L}$. Using the homomorphism (131) and property (136), we obtain

$$\nu\circ\varphi\big|_{\mathfrak{R}}\colon \mathfrak{R}\to\mathbb{F}^0(\mathcal{E},a), \quad (\nu\circ\varphi)\Big((\operatorname{ad}\mathrm{A}_0)^k(\mathrm{A}_\alpha)\Big) = k!\cdot\mathbb{A}_\alpha^{k,0}, \quad k\in\mathbb{Z}_{\geq 0},\ \ \alpha\in\mathcal{M}_0,\ \ \alpha\neq 0. \tag{137}$$

Using Remark 25, we can assume that $\mathfrak{W}_a$ is embedded into the algebra $\mathfrak{gl}(V)$ for some vector space V.

Then the exponentials $\mathrm{e}^{t\mathrm{B}_0}$, $\mathrm{e}^{x\mathrm{A}_0}$ and the expressions (121) can be regarded as power series with coefficients in $\mathfrak{gl}(V)$. If S_1, S_2 are power series with coefficients in $\mathfrak{gl}(V)$, then the product S_1S_2 is a well-defined power series as well. It is easy to check that the following formulas are valid

$$\begin{aligned}\mathrm{e}^{t\mathrm{B}_0}\mathrm{e}^{x\mathrm{A}_0}\Big(D_x + \sum_{\alpha\in\mathcal{M}_0} U_a^{\alpha}\cdot\mathrm{A}_\alpha\Big)\mathrm{e}^{-x\mathrm{A}_0}\mathrm{e}^{-t\mathrm{B}_0} &= \\ = D_x - \mathrm{e}^{t\mathrm{B}_0}\mathrm{A}_0\mathrm{e}^{-t\mathrm{B}_0} + \sum_{\alpha\in\mathcal{M}_0} U_a^{\alpha}\cdot\mathrm{e}^{t\mathrm{B}_0}\mathrm{e}^{x\mathrm{A}_0}\mathrm{A}_\alpha\mathrm{e}^{-x\mathrm{A}_0}\mathrm{e}^{-t\mathrm{B}_0} &= \\ = D_x + \sum_{\alpha\in\mathcal{M}_0,\ \alpha\neq 0} U_a^{\alpha}\cdot\sum_{l_1,l_2}\frac{1}{l_1!l_2!}x^{l_1}t^{l_2}(\operatorname{ad}\mathrm{B}_0)^{l_2}\Big((\operatorname{ad}\mathrm{A}_0)^{l_1}(\mathrm{A}_\alpha)\Big),\end{aligned} \tag{138}$$

$$\begin{aligned}\mathrm{e}^{t\mathrm{B}_0}\mathrm{e}^{x\mathrm{A}_0}\Big(D_t + \sum_{\beta\in\mathcal{M}_{d-1}} U_a^{\beta}\cdot\mathrm{B}_\beta\Big)\mathrm{e}^{-x\mathrm{A}_0}\mathrm{e}^{-t\mathrm{B}_0} &= \\ = D_t - B_0 + \sum_{\beta\in\mathcal{M}_{d-1}} U_a^{\beta}\cdot\sum_{l_1,l_2}\frac{1}{l_1!l_2!}x^{l_1}t^{l_2}(\operatorname{ad}\mathrm{B}_0)^{l_2}\Big((\operatorname{ad}\mathrm{A}_0)^{l_1}(\mathrm{B}_\beta)\Big).\end{aligned} \tag{139}$$

Recall that A, B are given by (121). Set

$$P = \sum_{\alpha\in\mathcal{M}_0,\ \alpha\neq 0} U_a^{\alpha}\cdot\sum_{l_1,l_2}\frac{1}{l_1!l_2!}x^{l_1}t^{l_2}(\operatorname{ad}\mathrm{B}_0)^{l_2}\Big((\operatorname{ad}\mathrm{A}_0)^{l_1}(\mathrm{A}_\alpha)\Big), \tag{140}$$

$$Q = -B_0 + \sum_{\beta\in\mathcal{M}_{d-1}} U_a^{\beta}\cdot\sum_{l_1,l_2}\frac{1}{l_1!l_2!}x^{l_1}t^{l_2}(\operatorname{ad}\mathrm{B}_0)^{l_2}\Big((\operatorname{ad}\mathrm{A}_0)^{l_1}(\mathrm{B}_\beta)\Big). \tag{141}$$

According to (121), (138), (139), (140), (141) we have

$$D_x + P = \mathrm{e}^{t\mathrm{B}_0}\mathrm{e}^{x\mathrm{A}_0}(D_x+\mathrm{A})\mathrm{e}^{-x\mathrm{A}_0}\mathrm{e}^{-t\mathrm{B}_0}, \qquad D_t + Q = \mathrm{e}^{t\mathrm{B}_0}\mathrm{e}^{x\mathrm{A}_0}(D_t+\mathrm{B})\mathrm{e}^{-x\mathrm{A}_0}\mathrm{e}^{-t\mathrm{B}_0}. \tag{142}$$

Note that, since $[D_x, D_t] = 0$, one has

$$[D_x+\mathrm{A}, D_t+\mathrm{B}] = D_x(\mathrm{B}) - D_t(\mathrm{A}) + [\mathrm{A},\mathrm{B}], \qquad [D_x+P, D_t+Q] = D_x(Q) - D_t(P) + [P,Q]. \tag{143}$$

Using (142), (143), (122), we get

$$D_x(Q)-D_t(P)+[P,Q]=[D_x+P,D_t+Q]=$$
$$=\big[\mathrm{e}^{t\mathrm{B}_0}\mathrm{e}^{x\mathrm{A}_0}(D_x+\mathrm{A})\mathrm{e}^{-x\mathrm{A}_0}\mathrm{e}^{-t\mathrm{B}_0},\ \mathrm{e}^{t\mathrm{B}_0}\mathrm{e}^{x\mathrm{A}_0}(D_t+\mathrm{B})\mathrm{e}^{-x\mathrm{A}_0}\mathrm{e}^{-t\mathrm{B}_0}\big]=$$
$$=\mathrm{e}^{t\mathrm{B}_0}\mathrm{e}^{x\mathrm{A}_0}[D_x+\mathrm{A},D_t+\mathrm{B}]\mathrm{e}^{-x\mathrm{A}_0}\mathrm{e}^{-t\mathrm{B}_0}=\mathrm{e}^{t\mathrm{B}_0}\mathrm{e}^{x\mathrm{A}_0}(D_x(\mathrm{B})-D_t(\mathrm{A})+[\mathrm{A},\mathrm{B}])\mathrm{e}^{-x\mathrm{A}_0}\mathrm{e}^{-t\mathrm{B}_0}=0.$$

Therefore, the power series (140), (141) satisfy all conditions of Lemma 6. Applying Lemma 6 to (140), (141), we obtain the homomorphism

(144)
$$\psi\colon\ \mathbb{F}^0(\mathcal{E},a)\to\mathfrak{W}_a,\qquad \psi\big(\mathbb{A}_\alpha^{l_1,l_2}\big)=\frac{1}{l_1!l_2!}(\operatorname{ad}\mathrm{B}_0)^{l_2}\Big((\operatorname{ad}\mathrm{A}_0)^{l_1}(\mathrm{A}_\alpha)\Big),\qquad \alpha\in\mathcal{M}_0,\qquad \alpha\neq 0,$$
$$\psi\big(\mathbb{B}_\beta^{l_1,l_2}\big)=\frac{1}{l_1!l_2!}(\operatorname{ad}\mathrm{B}_0)^{l_2}\Big((\operatorname{ad}\mathrm{A}_0)^{l_1}(\mathrm{B}_\beta)\Big),\qquad \beta\in\mathcal{M}_{d-1},\qquad \beta\neq 0,$$
$$\psi\big(\mathbb{B}_0^{l_1',l_2'}\big)=\frac{1}{l_1'!l_2'!}(\operatorname{ad}\mathrm{B}_0)^{l_2'}\Big((\operatorname{ad}\mathrm{A}_0)^{l_1'}(\mathrm{B}_0)\Big),\qquad l_1'>0.$$

From (144) we get

$$\psi\big(\mathbb{A}_\alpha^{l_1,0}\big)=\frac{1}{l_1!}(\operatorname{ad}\mathrm{A}_0)^{l_1}(\mathrm{A}_\alpha)\in\mathfrak{R},\qquad l_1\in\mathbb{Z}_{\geq 0},\qquad \alpha\in\mathcal{M}_0,\qquad \alpha\neq 0. \tag{145}$$

Since, by Lemma 5, the elements (125) generate the algebra $\mathbb{F}^0(\mathcal{E},a)$, property (145) implies $\psi\big(\mathbb{F}^0(\mathcal{E},a)\big)\subset\mathfrak{R}$.

Then from (137), (145) it follows that the homomorphisms $\psi\colon\mathbb{F}^0(\mathcal{E},a)\to\mathfrak{R}$ and $\nu\circ\varphi\big|_{\mathfrak{R}}\colon\mathfrak{R}\to\mathbb{F}^0(\mathcal{E},a)$ are inverse to each other. □

4. The algebra $\mathbb{F}^0(\mathcal{E},a)$ for the multicomponent Landau-Lifshitz system

For any $m\in\mathbb{Z}_{>0}$ and m-dimensional vectors $v=(v^1,\dots,v^m)$, $w=(w^1,\dots,w^m)$, we set $\langle v,w\rangle=\sum_{i=1}^m v^iw^i$.

In order to describe the algebra $\mathbb{F}^0(\mathcal{E},a)$ for system (24), we need to resolve the constraint $\langle S,S\rangle=1$ for the vector-function $S=\big(s^1(x,t),\dots,s^n(x,t)\big)$. Following [6], we do this as follows

$$s^j=\frac{2u^j}{1+\langle u,u\rangle},\qquad j=1,\dots,n-1,\qquad s^n=\frac{1-\langle u,u\rangle}{1+\langle u,u\rangle}, \tag{146}$$

where $u=\big(u^1(x,t),\dots,u^{n-1}(x,t)\big)$ is an $(n-1)$-dimensional vector-function.

As is shown in [6], using (146), one can rewrite system (24) as follows

$$\begin{aligned}u_t=u_{xxx}-6\langle u,u_x\rangle\Delta^{-1}u_{xx}+\big(-6\langle u,u_{xx}\rangle\Delta^{-1}+24\langle u,u_x\rangle^2\Delta^{-2}-6\langle u,u\rangle\langle u_x,u_x\rangle\Delta^{-2}\big)u_x+\\ +\big(6\langle u_x,u_{xx}\rangle\Delta^{-1}-12\langle u,u_x\rangle\langle u_x,u_x\rangle\Delta^{-2}\big)u+\frac{3}{2}\Big(r_n+4\Delta^{-2}\sum_{i=1}^{n-1}(r_i-r_n)(u^i)^2\Big)u_x,\\ u=\big(u^1(x,t),\dots,u^{n-1}(x,t)\big),\end{aligned} \tag{147}$$

where $\Delta=1+\langle u,u\rangle$ and $r_1,\dots,r_n\in\mathbb{K}$ are the numbers such that $R=\operatorname{diag}(r_1,\dots,r_n)$ in (24). As has been said in Section 1.2, we assume $r_i\neq r_j$ for all $i\neq j$.

In this section we assume $n\geq 3$, because we will use some results of the paper [9], which studied equations (24), (147) in the case $n\geq 3$.

Let $\mathcal{E}$ be the infinite prolongation of the PDE (147). Then $\mathcal{E}$ is a manifold with the coordinates x, t, u^i_k for $i=1,\dots,n-1$ and $k\in\mathbb{Z}_{\geq 0}$. (Recall that $u^i_0=u^i$, according to our notation.) Let $a\in\mathcal{E}$.

According to Theorem 8, the algebra $\mathbb{F}^0(\mathcal{E},a)$ for (147) is isomorphic to a subalgebra of the WE algebra $\mathfrak{W}_a$ of (147). The WE algebra of (147) is described in [9]. To present this description, we need to introduce some auxiliary constructions.

Recall that $\mathfrak{gl}_{n+1}$ is the algebra of matrices of size $(n+1)\times(n+1)$ with entries from $\mathbb{K}$. Let $E_{i,j}\in\mathfrak{gl}_{n+1}$ be the matrix with (i,j)-th entry equal to 1 and all other entries equal to zero.

The Lie subalgebra $\mathfrak{so}_{n,1}\subset\mathfrak{gl}_{n+1}$ has been defined in Remark 6. It has the following basis

$$E_{i,j}-E_{j,i},\qquad i<j\leq n,\qquad\qquad E_{l,n+1}+E_{n+1,l},\qquad l=1,\ldots,n.$$

Consider the algebra $\mathbb{K}[\lambda_1,\ldots,\lambda_n]$ of polynomials in $\lambda_1,\ldots,\lambda_n$. Let $\mathcal{I}\subset\mathbb{K}[\lambda_1,\ldots,\lambda_n]$ be the ideal generated by $\lambda_i^2-\lambda_j^2+r_i-r_j$ for $i,j=1,\ldots,n$.

Consider the quotient algebra $\mathcal{Q}=\mathbb{K}[\lambda_1,\ldots,\lambda_n]/\mathcal{I}$, which is isomorphic to the algebra of polynomial functions on the algebraic curve (25).

The space $\mathfrak{so}_{n,1}\otimes_{\mathbb{K}}\mathcal{Q}$ is an infinite-dimensional Lie algebra over $\mathbb{K}$ with the Lie bracket

$$[M_1\otimes h_1,\,M_2\otimes h_2]=[M_1,M_2]\otimes h_1h_2,\qquad\qquad M_1,M_2\in\mathfrak{so}_{n,1},\qquad\qquad h_1,h_2\in\mathcal{Q}.$$

We have the natural homomorphism $\xi\colon\mathbb{K}[\lambda_1,\ldots,\lambda_n]\to\mathbb{K}[\lambda_1,\ldots,\lambda_n]/\mathcal{I}=\mathcal{Q}$. Set $\hat\lambda_i=\xi(\lambda_i)\in\mathcal{Q}$.

Consider the following elements of $\mathfrak{so}_{n,1}\otimes\mathcal{Q}$

$$Q_i=(E_{i,n+1}+E_{n+1,i})\otimes\hat\lambda_i,\qquad\qquad i=1,\ldots,n. \tag{148}$$

Denote by $L(n)\subset\mathfrak{so}_{n,1}\otimes\mathcal{Q}$ the Lie subalgebra generated by $Q_1,\ldots,Q_n$.

Since $\hat\lambda_i^2-\hat\lambda_j^2+r_i-r_j=0$ in $\mathcal{Q}$, the element $\hat\lambda=\hat\lambda_i^2+r_i\in\mathcal{Q}$ does not depend on i.

Recall that $\mathbb{Z}_{>0}$ is the set of positive integers. For $i,j\in\{1,\ldots,n\}$ and $k\in\mathbb{Z}_{>0}$, consider the following elements of $\mathfrak{so}_{n,1}\otimes_{\mathbb{K}}\mathcal{Q}$

$$Q_i^{2k-1}=(E_{i,n+1}+E_{n+1,i})\otimes\hat\lambda^{k-1}\hat\lambda_i,\qquad\qquad Q_{ij}^{2k}=(E_{i,j}-E_{j,i})\otimes\hat\lambda^{k-1}\hat\lambda_i\hat\lambda_j.$$

For $i,j,l,m\in\{1,\ldots,n\}$ and $k_1,k_2\in\mathbb{Z}_{>0}$ one has

$$\begin{aligned}[Q_{ij}^{2k_1},\,Q_{lm}^{2k_2}]&=\delta_{lj}Q_{im}^{2(k_1+k_2)}-\delta_{im}Q_{lj}^{2(k_1+k_2)}+\delta_{jm}Q_{li}^{2(k_1+k_2)}-\delta_{il}Q_{jm}^{2(k_1+k_2)}+\\&\quad+r_i\delta_{im}Q_{lj}^{2(k_1+k_2-1)}-r_j\delta_{lj}Q_{im}^{2(k_1+k_2-1)}+r_i\delta_{il}Q_{jm}^{2(k_1+k_2-1)}-r_j\delta_{jm}Q_{li}^{2(k_1+k_2-1)},\end{aligned} \tag{149}$$

$$[Q_{ij}^{2k_1},\,Q_l^{2k_2-1}]=\delta_{lj}Q_i^{2k_1+2k_2-1}-\delta_{il}Q_j^{2k_1+2k_2-1}-r_j\delta_{lj}Q_i^{2k_1+2k_2-3}+r_i\delta_{il}Q_j^{2k_1+2k_2-3}, \tag{150}$$

$$[Q_i^{2k_1-1},\,Q_j^{2k_2-1}]=Q_{ij}^{2(k_1+k_2-1)},\qquad\qquad[Q_i^{2k_1-1},\,Q_i^{2k_2-1}]=0. \tag{151}$$

Since $Q_i^1=Q_i$ and $Q_{ij}^{2k}=-Q_{ji}^{2k}$, from (149), (150), (151) we see that the elements

$$Q_l^{2k-1},\qquad Q_{ij}^{2k},\qquad i,j,l\in\{1,\ldots,n\},\qquad i<j,\qquad k\in\mathbb{Z}_{>0}, \tag{152}$$

span the Lie algebra $L(n)$. It is shown in [9] that the elements (152) are linearly independent over $\mathbb{K}$ and, therefore, form a basis of $L(n)$.

Remark 26. In Remark 5 we have said that $L(n)$ consists of certain $\mathfrak{gl}_{n+1}$-valued functions on the curve (25), and in Remark 6 we have shown that these functions take values in the Lie subalgebra $\mathfrak{so}_{n,1}\subset\mathfrak{gl}_{n+1}$.

Here we have defined $L(n)$ as a certain Lie subalgebra of $\mathfrak{so}_{n,1}\otimes\mathcal{Q}$. This is in agreement with Remarks 5, 6, because elements of $\mathfrak{so}_{n,1}\otimes\mathcal{Q}$ can be regarded as $\mathfrak{so}_{n,1}$-valued functions on the curve (25).

Note that the algebra $L(n)$ is very similar to infinite-dimensional Lie algebras that were studied in [22, 23].

Theorem 9. *Let $n\geq 3$. For the PDE* (147)*, the Lie algebra $\mathbb{F}^0(\mathcal{E},a)$ defined in* (109) *is isomorphic to $L(n)$.*

Proof. Let $\mathfrak{W}_a$ be the WE algebra for the PDE (147). According to [9], the algebra $\mathfrak{W}_a$ is isomorphic to the direct sum of $L(n)$ and the 2-dimensional abelian Lie algebra $\mathbb{K}^2$. So $\mathfrak{W}_a \cong L(n) \oplus \mathbb{K}^2$. (Note that the algebra $\mathfrak{W}_a$ is denoted by $\mathfrak{W}$ in [9].)

According to [9], in the formal Wahlquist-Estabrook ZCR with coefficients in $\mathfrak{W}_a$ for the PDE (147), one has

$$\mathrm{A} = C_0 + \sum_{l=1}^{n} C_l \cdot s^l(u^1, \dots, u^{n-1}), \qquad C_0, C_1, \dots, C_n \in \mathfrak{W}_a,$$

where the functions $s^l = s^l(u^1, \dots, u^{n-1})$ are given by (146), the elements $C_1, \dots, C_n \in \mathfrak{W}_a$ generate the Lie subalgebra $L(n) \subset \mathfrak{W}_a \cong L(n) \oplus \mathbb{K}^2$, and one has $[C_0, C_l] = 0$ for all $l = 1, \dots, n$.

This implies that the subalgebra $\mathfrak{R} \subset \mathfrak{W}_a$ defined in Theorem 8 is equal to $L(n) \subset \mathfrak{W}_a$. According to Theorem 8, one has $\mathbb{F}^0(\mathcal{E}, a) \cong \mathfrak{R}$. Since in the considered case we have $\mathfrak{R} = L(n)$, we get $\mathbb{F}^0(\mathcal{E}, a) \cong L(n)$. □

5. The algebra $\mathbb{F}^1(\mathcal{E}, a)$ for the multicomponent Landau-Lifshitz system

5.1. **Preliminary computations.** We continue to use the notation $u^i_k = \dfrac{\partial^k u^i}{\partial x^k}$.

Let $\mathcal{E}$ be the infinite prolongation of the PDE (147). Recall that $u = (u^1, \dots, u^{n-1})$ is an $(n-1)$-dimensional vector in (147). Then $\mathcal{E}$ is an infinite-dimensional manifold with the coordinates

$$x, \quad t, \quad u^i_k, \qquad i = 1, \dots, n-1, \quad k \in \mathbb{Z}_{\geq 0}, \quad u^i_0 = u^i. \tag{153}$$

Consider an arbitrary point $a \in \mathcal{E}$ given by

$$a = (x = x_a,\ t = t_a,\ u^i_k = a^i_k) \in \mathcal{E}, \qquad x_a,\, t_a,\, a^i_k \in \mathbb{K}, \qquad i = 1, \dots, n-1, \qquad k \in \mathbb{Z}_{\geq 0}. \tag{154}$$

Since the PDE (147) is invariant with respect to the change of variables $x \mapsto x - x_a,\ t \mapsto t - t_a$, it is sufficient to consider the case

$$x_a = t_a = 0. \tag{155}$$

For simplicity of exposition, we assume also

$$a^i_k = 0 \qquad \forall\, i, k. \tag{156}$$

(In the case $a^i_k \neq 0$, the computations change very little, and the final result is the same.)

According to Remark 20 and assumptions (155), (156), in order to describe the Lie algebra $\mathbb{F}^1(\mathcal{E}, a)$ for the PDE (147), we need to study the equations

$$D_x(B) - D_t(A) + [A, B] = 0, \tag{157}$$

$$\forall\, i_0 = 1, \dots, n-1, \qquad \left.\frac{\partial A}{\partial u^{i_0}_1}\right|_{u^j_1 = 0\ \forall j,\ u^i_0 = 0\ \forall i > i_0} = 0, \tag{158}$$

$$\left. A \right|_{u^j_0 = 0,\ u^j_1 = 0\ \forall j} = 0, \tag{159}$$

$$\left. B \right|_{x = 0,\ u^j_k = 0\ \forall j,\ \forall k \geq 0} = 0, \tag{160}$$

where

- $A = A(x, t, u^j_0, u^j_1)$ is a power series in the variables x, t, u^j_0, u^j_1 for $j = 1, \dots, n-1$,
- $B = B(x, t, u^j_0, u^j_1, u^j_2, u^j_3)$ is a power series in the variables x, t, u^j_0, u^j_1, u^j_2, u^j_3 for $j = 1, \dots, n-1$.

The coefficients of the power series A, B are generators of the Lie algebra $\mathbb{F}^1(\mathcal{E}, a)$. Relations for these generators are provided by equations (157), (158), (159), (160).

Since, according to our notation, $u_0^j = u^j$, below we sometimes write u^j instead of u_0^j. In particular, we can write $A = A(x, t, u^j, u_1^j)$.

When we consider power series in the variables x, t, u^j, u_k^j, partial derivatives with respect to these variables are often denoted by subscripts. For example,

$$A_{u_1^i} = \frac{\partial A}{\partial u_1^i}, \qquad A_{u_1^i x} = \frac{\partial^2 A}{\partial u_1^i \partial x}, \qquad A_{u_1^i u^j} = \frac{\partial^2 A}{\partial u_1^i \partial u^j}, \qquad i, j = 1, \dots, n-1.$$

Differentiating equation (157) with respect to u_4^i, we obtain that B is of the form

$$B = \sum_i u_3^i A_{u_1^i} + F^1(x, t, u^j, u_1^j, u_2^j), \tag{161}$$

where F^1 is a power series in the variables x, t, u^j, u_1^j, u_2^j.

According to our notation, the symbols u_t, u, u_x, u_{xx}, u_{xxx} in (147) denote the following vectors

$$u_t = (u_t^1, \dots, u_t^{n-1}), \qquad u = (u^1, \dots, u^{n-1}) = (u_0^1, \dots, u_0^{n-1}),$$
$$u_x = (u_1^1, \dots, u_1^{n-1}), \qquad u_{xx} = (u_2^1, \dots, u_2^{n-1}), \qquad u_{xxx} = (u_3^1, \dots, u_3^{n-1}).$$

Therefore, system (147) can be written as

$$u_t^i = u_3^i + G^i(u^j, u_1^j, u_2^j), \qquad i = 1, \dots, n-1, \tag{162}$$

for some functions $G^i(u^j, u_1^j, u_2^j)$ determined by the right-hand side of (147). Then one has

$$D_x(B) = \sum_i \Big(u_4^i A_{u_1^i} + u_3^i \big(A_{u_1^i x} + \sum_j \big(u_1^j A_{u_1^i u^j} + u_2^j A_{u_1^i u_1^j} \big) \big) \Big) + F_x^1 + \sum_i \big(u_1^i F_{u^i}^1 + u_2^i F_{u_1^i}^1 + u_3^i F_{u_2^i}^1 \big),$$

$$D_t(A) = A_t + \sum_i \Big(u_3^i A_{u^i} + G^i A_{u^i} + u_4^i A_{u_1^i} + \sum_j \big(u_1^j G_{u^j}^i A_{u_1^i} + u_2^j G_{u_1^j}^i A_{u_1^i} + u_3^j G_{u_2^j}^i A_{u_1^i} \big) \Big),$$

$$[A, B] = \sum_i u_3^i [A, A_{u_1^i}] + [A, F^1].$$

Differentiating equation (157) with respect to u_3^i, we get

$$F_{u_2^i}^1 = A_{u^i} + \sum_j G_{u_2^i}^j A_{u_1^j} - A_{u_1^i x} - \sum_j u_1^j A_{u_1^i u^j} - \sum_j u_2^j A_{u_1^i u_1^j} - [A, A_{u_1^i}], \qquad i = 1, \dots, n-1. \tag{163}$$

Since $G_{u_2^i u_2^l}^j = 0$ for all i, l, from (163) one obtains that F^1 is of the form

$$F^1 = -\frac{1}{2} \sum_{i,j} u_2^i u_2^j A_{u_1^i u_1^j} + \\ + \sum_i u_2^i \Big(A_{u^i} + \sum_j G_{u_2^i}^j A_{u_1^j} - A_{u_1^i x} - \sum_j u_1^j A_{u_1^i u^j} - [A, A_{u_1^i}] \Big) + F^2(x, t, u^l, u_1^l), \tag{164}$$

where F^2 is a power series in the variables x, t, u^l, u_1^l. Then equation (157) becomes

$$F_x^1 + \sum_i u_1^i F_{u^i}^1 + \sum_i u_2^i F_{u_1^i}^1 - A_t - \sum_i G^i A_{u^i} - \sum_{i,j} u_1^j G_{u^j}^i A_{u_1^i} - \sum_{i,j} u_2^j G_{u_1^j}^i A_{u_1^i} + [A, F^1] = 0. \tag{165}$$

Differentiating (165) with respect to u_2^i, u_2^j, u_2^h and taking into account (164), one gets $A_{u_1^i u_1^j u_1^h} = 0$ for all i, j, h. That is, A is of the form

$$A = \frac{1}{2} \sum_{i,j} u_1^i u_1^j Y_{ij} + \sum_i u_1^i Y_i + Y, \qquad Y_{ij} = Y_{ji}, \tag{166}$$

where Y_{ij}, Y_i, Y are power series in the variables x, t, $u^1,\dots,u^{n-1}$.

From the definition of G^i, for all $h,i,j=1,\dots,n-1$ one has

$$G^h_{u^i_2}=-6\delta^h_i\Delta^{-1}\langle u,u_1\rangle-6\Delta^{-1}u^iu^h_1+6\Delta^{-1}u^i_1u^h, \tag{167}$$

$$\begin{aligned}G^h_{u^j_1}&=-6\Delta^{-1}u^ju^h_2-6\delta^h_j\Delta^{-1}\langle u,u_2\rangle+6\Delta^{-1}u^j_2u^h\\&\quad+48\Delta^{-2}\langle u,u_1\rangle u^ju^h_1+24\delta^h_j\Delta^{-2}\langle u,u_1\rangle^2-12\Delta^{-2}\langle u,u\rangle u^j_1u^h_1-6\delta^h_j\Delta^{-2}\langle u,u\rangle\langle u_1,u_1\rangle\\&\quad-12\Delta^{-2}u^hu^j\langle u_1,u_1\rangle-24\Delta^{-2}\langle u,u_1\rangle u^hu^j_1+\frac{3}{2}\delta^h_j\left(r_n+4\Delta^{-2}\sum_i(r_i-r_n)u^{i2}\right),\end{aligned} \tag{168}$$

where u_k is the vector $(u^1_k,\dots,u^{n-1}_k)$ for $k\in\mathbb{Z}_{\geq 0}$.

Differentiating equation (165) with respect to u^i_2, u^j_2 and taking into account (164), (167), (168), we obtain

$$\begin{aligned}&A_{u^i_1u^j_1x}+\sum_h u^h_1A_{u^i_1u^j_1u^h}+[A,A_{u^i_1u^j_1}]+4\Delta^{-1}A_{u^i_1u^j_1}\langle u,u_1\rangle+2\sum_h u^iu^h_1\Delta^{-1}A_{u^h_1u^j_1}+\\&\qquad+2\sum_h u^ju^h_1\Delta^{-1}A_{u^h_1u^i_1}-2\sum_h u^i_1u^h\Delta^{-1}A_{u^h_1u^j_1}-2\sum_h u^j_1u^h\Delta^{-1}A_{u^h_1u^i_1}=0.\end{aligned} \tag{169}$$

Substituting (166) to (169), one gets

$$\begin{aligned}&Y_{ij,x}+\sum_h u^h_1Y_{ij,h}+[A,Y_{ij}]+\\&+\Delta^{-1}\Big(4\langle u,u_1\rangle Y_{ij}+2\sum_h u^iu^h_1Y_{hj}+2\sum_h u^ju^h_1Y_{hi}-2\sum_h u^i_1u^hY_{hj}-2\sum_h u^j_1u^hY_{hi}\Big)=0.\end{aligned} \tag{170}$$

Here and below we use the following notation

$$Y_{ij,x}=\frac{\partial Y_{ij}}{\partial x},\qquad Y_{ij,h}=\frac{\partial Y_{ij}}{\partial u^h},\qquad Y_{ij,hk}=\frac{\partial^2Y_{ij}}{\partial u^h\partial u^k},\qquad i,j,h,k=1,2,\dots,n-1,$$
$$Y_{h,x}=\frac{\partial Y_h}{\partial x},\qquad Y_{i,j}=\frac{\partial Y_i}{\partial u^j},\qquad Y_{,x}=\frac{\partial Y}{\partial x},\qquad Y_{,h}=\frac{\partial Y}{\partial u^h}.$$

The left-hand side of (170) is a polynomial of degree ≤ 2 with respect to the variables $u^1_1,\dots,u^{n-1}_1$. Equating to zero the coefficients of this polynomial, we obtain

$$[Y_{hk},Y_{ij}]=0\qquad\forall\,h,k,i,j, \tag{171}$$

$$Y_{ij,h}=[Y_{ij},Y_h]-2\Delta^{-1}\big(2u^hY_{ij}+u^iY_{hj}+u^jY_{hi}-\delta^i_h\sum_m u^mY_{mj}-\delta^j_h\sum_m u^mY_{mi}\big)\qquad\forall\,i,j,h, \tag{172}$$

$$Y_{ij,x}=[Y_{ij},Y]\qquad\forall\,i,j. \tag{173}$$

Let us rewrite (172) replacing h by k

$$Y_{ij,k}=[Y_{ij},Y_k]-2\Delta^{-1}\big(2u^kY_{ij}+u^iY_{kj}+u^jY_{ki}-\delta^i_k\sum_m u^mY_{mj}-\delta^j_k\sum_m u^mY_{mi}\big)\qquad\forall\,i,j,k. \tag{174}$$

Now we differentiate (172) with respect to u^k and differentiate (174) with respect to u^h. Then the equality $Y_{ij,hk}=Y_{ij,kh}$ implies

$$[Y_{ij},T_{kh}]+4\Delta^{-2}(-\delta^i_kY_{hj}+\delta^i_hY_{kj}-\delta^j_kY_{hi}+\delta^j_hY_{ki})=0\qquad\forall\,i,j,k,h, \tag{175}$$

where

$$T_{kh}=[Y_k,Y_h]-Y_{k,h}+Y_{h,k}. \tag{176}$$

In other words, we regard (172) as an overdetermined system of PDEs for Y_{ij}, and equations (175) have been obtained from the compatibility condition of system (172).

Remark 27. Another way to obtain (175) from (172) is the following. Set

$$V_k = Y_k + \frac{\partial}{\partial u^k}, \qquad k = 1, \dots, n-1.$$

Then (172) and (176) can be rewritten as follows

$$[V_h, Y_{ij}] = -2\Delta^{-1}\big(2u^h Y_{ij} + u^i Y_{hj} + u^j Y_{hi} - \delta^i_h \sum_m u^m Y_{mj} - \delta^j_h \sum_m u^m Y_{mi}\big), \tag{177}$$

$$T_{kh} = [V_k, V_h]. \tag{178}$$

Using the Jacobi identity and equation (178), we get

$$[V_k, [V_h, Y_{ij}]] = [[V_k, V_h], Y_{ij}] + [V_h, [V_k, Y_{ij}]] = [T_{kh}, Y_{ij}] + [V_h, [V_k, Y_{ij}]]. \tag{179}$$

Let P be a power series in the variables $x, t, u^1, \dots, u^{n-1}$ with coefficients in the Lie algebra $\mathbb{F}^1(\mathcal{E}, a)$. Then the operator $\operatorname{ad} V_k$ can be applied to P in the standard way

$$(\operatorname{ad} V_k)(P) = [V_k, P] = \Big[Y_k + \frac{\partial}{\partial u^k},\, P\Big] = [Y_k, P] + \frac{\partial}{\partial u^k}(P).$$

Applying the operator $\operatorname{ad} V_k$ to equation (177), for all k, h we can express $[V_k, [V_h, Y_{ij}]]$ as a linear combination of Y_{pq}, $p, q = 1, \dots, n-1$. Then, exchanging k and h, we can express $[V_h, [V_k, Y_{ij}]]$ as a linear combination of Y_{pq}, $p, q = 1, \dots, n-1$. Substituting the obtained expressions in (179), one gets (175).

Combining equations (158), (159) with formula (166), we obtain

$$\forall\, i = 1, \dots, n-2, \qquad Y_i \Big|_{u^{i+1} = u^{i+2} = \dots = u^{n-1} = 0} = 0, \qquad Y_{n-1} = 0, \tag{180}$$

$$Y \Big|_{u^1 = u^2 = \dots = u^{n-1} = 0} = 0. \tag{181}$$

In order to study the obtained equations, we need the following lemmas on formal power series, which can be proved straightforwardly by induction on the degrees of the coefficients of these power series.

Lemma 8. *Let $Z_1, \dots, Z_{n-1}$ be formal power series in some variables $v^1, \dots, v^{n-1}$ with coefficients in a Lie algebra $\mathfrak{g}$. Suppose that*

$$\forall\, i = 1, \dots, n-2, \qquad Z_i \Big|_{v^{i+1} = v^{i+2} = \dots = v^{n-1} = 0} = 0, \qquad Z_{n-1} = 0. \tag{182}$$

Set $V_{ij} = [Z_i, Z_j] - \dfrac{\partial Z_i}{\partial v^j} + \dfrac{\partial Z_j}{\partial v^i}$.

Then the Lie subalgebra generated by the coefficients of the power series $Z_1, \dots, Z_{n-1}$ coincides with the Lie subalgebra generated by the coefficients of the power series V_{ij}, $i, j = 1, \dots, n-1$.

Lemma 9. *Let $Z_1, \dots, Z_{n-1}$ and $\Psi_1, \dots, \Psi_m$ be formal power series in some variables $v^1, \dots, v^{n-1}$ with coefficients in a Lie algebra $\mathfrak{g}$.*

Suppose that (182) *holds and*

$$\frac{\partial \Psi_l}{\partial v^h} = [\Psi_l, Z_h] + \sum_{i=1}^{m} f^i_l(v^1, \dots, v^{n-1})\Psi_i \qquad \forall\, l, h, \tag{183}$$

for some power series $f^i_l(v^1, \dots, v^{n-1})$ with coefficients in $\mathbb{K}$. Consider the coefficients of the power series Ψ_l

$$\Psi_l = \sum_{i_1, \dots, i_{n-1} \geq 0} \big(v^1\big)^{i_1} \dots \big(v^{n-1}\big)^{i_{n-1}} \psi^l_{i_1, \dots, i_{n-1}}, \qquad \psi^l_{i_1, \dots, i_{n-1}} \in \mathfrak{g}, \quad l = 1, \dots, n-1.$$

Then any coefficient $\psi^l_{i_1,\dots,i_{n-1}}$ *belongs to the vector subspace spanned by the zero degree coefficients* $\psi^1_{0,\dots,0},\dots,\psi^{n-1}_{0,\dots,0}$.

Recall that Y_{ij} from formula (166) are power series in the variables x, t, $u^1,\dots,u^{n-1}$ with coefficients in the Lie algebra $\mathbb{F}^1(\mathcal{E},a)$. Let $\mathfrak{g}$ be the Lie algebra of formal power series in the variables x, t with coefficients in $\mathbb{F}^1(\mathcal{E},a)$. Then Y_{ij} can be regarded as a power series in the variables $u^1,\dots,u^{n-1}$ with coefficients in $\mathfrak{g}$

$$Y_{ij}=\sum_{i_1,\dots,i_{n-1}\geq 0}\left(u^1\right)^{i_1}\dots\left(u^{n-1}\right)^{i_{n-1}}y^{ij}_{i_1,\dots,i_{n-1}}(x,t),\qquad y^{ij}_{i_1,\dots,i_{n-1}}(x,t)\in\mathfrak{g}. \tag{184}$$

Note that equations (172) are of the type (183) for $v^h=u^h$ and $Z_h=Y_h$. Therefore, by Lemma 9, we obtain the following result.

Lemma 10. *For any* $i,j,i_1,\dots,i_{n-1}$, *the power series* $y^{ij}_{i_1,\dots,i_{n-1}}(x,t)$ *belongs to the vector subspace spanned by the power series* $y^{pq}_{0,\dots,0}(x,t)$, $p,q=1,\dots,n-1$.

From (173), (181), (184) one gets

$$\frac{\partial}{\partial x}\Big(y^{pq}_{0,\dots,0}(x,t)\Big)=Y_{pq,x}\Big|_{u^1=u^2=\dots=u^{n-1}=0}= \tag{185}$$
$$=\Big[Y_{pq}\Big|_{u^1=u^2=\dots=u^{n-1}=0},\,Y\Big|_{u^1=u^2=\dots=u^{n-1}=0}\Big]=0\qquad\forall\,p,q.$$

Combining (185) with Lemma 10, we obtain

$$Y_{pq,x}=0\qquad\forall\,p,q \tag{186}$$

and, by (173),

$$[Y_{ij},Y]=0\qquad\forall\,i,j. \tag{187}$$

Before continuing the analysis of the obtained equations, we need to consider some special cases in the next subsections.

5.2. **Some special cases.** Suppose that A is of the form

$$A=\sum_i u^i_1\tilde{Y}_i+\tilde{Y}, \tag{188}$$

where $\tilde{Y}_i$, $\tilde{Y}$ are power series in the variables x, t, $u^1,\dots,u^{n-1}$ with coefficients in some Lie algebra.

Substituting (188) in (164) we obtain

$$F^1=\sum_i u^i_2H_i+F^2, \tag{189}$$

where

$$H_i=\sum_k u^k_1\tilde{T}_{ik}+\tilde{Y}_{,i}+[\tilde{Y}_i,\tilde{Y}]-\tilde{Y}_{i,x}+\sum_j G^j_{u^i_2}\tilde{Y}_j,$$
$$\tilde{T}_{ik}=[\tilde{Y}_i,\tilde{Y}_k]-\tilde{Y}_{i,k}+\tilde{Y}_{k,i}, \tag{190}$$
$$\tilde{Y}_{i,x}=\frac{\partial\tilde{Y}_i}{\partial x},\qquad\tilde{Y}_{i,k}=\frac{\partial\tilde{Y}_i}{\partial u^k},\qquad\tilde{Y}_{,i}=\frac{\partial\tilde{Y}}{\partial u^i}.$$

Differentiating (165) with respect to u^i_2, one gets

$$-\frac{\partial F^2}{\partial u^i_1}=H_{i,x}+\sum_j u^j_1H_{i,j}-\sum_{j,k}G^j_{u^i_2}u^k_1\tilde{Y}_{k,j}-\sum_j G^j_{u^i_2}\tilde{Y}_{,j}-\sum_{j,k}u^j_1G^k_{u^i_2u^j}\tilde{Y}_k-\sum_k G^k_{u^i_1}\tilde{Y}_k+ \tag{191}$$
$$+\sum_{j,k}G^k_{u^j_2u^i_1}u^j_2\tilde{Y}_k+\sum_k u^k_1[\tilde{Y}_k,H_i]+[\tilde{Y},H_i].$$

Denote the right-hand side of (191) by Ω_i. Then equation (191) reads

$$-\frac{\partial F^2}{\partial u_1^i}=\Omega_i \qquad \forall\, i. \tag{192}$$

Differentiating (192) with respect to u_1^h, u_1^k, we obtain

$$\frac{\partial}{\partial u_1^h}\big(\Omega_i\big)=\frac{\partial}{\partial u_1^i}\big(\Omega_h\big), \tag{193}$$

$$\frac{\partial^2}{\partial u_1^h\partial u_1^k}\big(\Omega_i\big)=\frac{\partial^2}{\partial u_1^i\partial u_1^k}\big(\Omega_h\big). \tag{194}$$

Using formulas (167), (168), it is straightforward to show that equations (194), (193) reduce to

$$\tilde T_{ih,k}=2\Delta^{-1}\Big(2u^k\tilde T_{hi}+u^h\tilde T_{ki}+u^i\tilde T_{hk}+\delta^i_k\sum_m u^m\tilde T_{mh}+\delta^h_k\sum_m u^m\tilde T_{im}\Big)+[\tilde T_{ih},\tilde Y_k], \tag{195}$$

$$\tilde T_{ih,x}=[\tilde T_{ih},\tilde Y], \tag{196}$$

where

$$\tilde T_{ih,k}=\frac{\partial\tilde T_{ih}}{\partial u^k},\qquad \tilde T_{ih,x}=\frac{\partial\tilde T_{ih}}{\partial x}.$$

Set $P_i=\tilde Y_i+\dfrac{\partial}{\partial u^i}$. Then

$$\begin{aligned}\tilde T_{ih}&=[\tilde Y_i,\tilde Y_h]-\tilde Y_{i,h}+\tilde Y_{h,i}=[P_i,P_h],\\ [\tilde T_{ih},\tilde T_{km}]&=[\tilde T_{ih},[P_k,P_m]]=[[\tilde T_{ih},P_k],P_m]+[P_k,[\tilde T_{ih},P_m]]=\\ &=[[\tilde T_{ih},P_k],P_m]-[[\tilde T_{ih},P_m],P_k],\end{aligned} \tag{197}$$

and equation (195) can be written as

$$[\tilde T_{ih},P_k]=2\Delta^{-1}\Big(2u^k\tilde T_{ih}+u^h\tilde T_{ik}+u^i\tilde T_{kh}+\delta^i_k\sum_m u^m\tilde T_{hm}+\delta^h_k\sum_m u^m\tilde T_{mi}\Big). \tag{198}$$

Using (197) and (198), one obtains

$$[\tilde T_{ih},\tilde T_{km}]=4\Delta^{-2}(\delta^h_m\tilde T_{ki}+\delta^h_k\tilde T_{im}+\delta^i_m\tilde T_{hk}+\delta^i_k\tilde T_{mh}). \tag{199}$$

5.3. **A zero-curvature representation.** In order to analyze the structure of $\mathbb{F}^1(\mathcal{E},a)$, we need to construct a ZCR with A of the form

$$A=\sum_i u_1^i\hat Y_i, \tag{200}$$

where $\hat Y_i$ are power series in the variables $u^1,\dots,u^{n-1}$ with coefficients in some Lie algebra. In particular, we assume that $\hat Y_i$ do not depend on x, t. That is,

$$\frac{\partial}{\partial x}\big(\hat Y_i\big)=\frac{\partial}{\partial t}\big(\hat Y_i\big)=0. \tag{201}$$

Also, similarly to (180), we assume

$$\forall\, i=1,\dots,n-2,\qquad \hat Y_i\Big|_{u^{i+1}=u^{i+2}=\dots=u^{n-1}=0}=0,\qquad \hat Y_{n-1}=0. \tag{202}$$

By (161), (164), (200), one has

$$B=\sum_i u_3^i\hat Y_i+\sum_{i,j}u_2^i\left(u_1^jT_{ij}+G^j_{u_2^i}\hat Y_j\right)+F^2. \tag{203}$$

Using (191) and (201), it is straightforward to show that F^2 is of the form

$$F^2 = \sum_{i,j,h} u_1^i u_1^j u_1^h M_{ijh} + \sum_i u_1^i M_i + M, \tag{204}$$

$$M_{ihk} = 2\Delta^{-2}\Big((4(u^k u^i \hat{Y}_h + u^h u^k \hat{Y}_i + u^h u^i \hat{Y}_k) - \langle u,u\rangle(\delta_k^i \hat{Y}_h + \delta_h^i \hat{Y}_k + \delta_k^h \hat{Y}_i)$$
$$- 2\sum_m u^m \hat{Y}_m(\delta_k^h u^i + \delta_k^i u^h + \delta_h^i u^k)\Big) - \frac{2}{3}\sum_m \Delta^{-1} u^m(\delta_k^i \hat{T}_{hm} + \delta_h^i \hat{T}_{km} + \delta_k^h \hat{T}_{im}),$$

$$M_i = \frac{3}{2}\left(r_n + 4\Delta^{-1}\sum_k (r_k - r_n)u^{k^2}\right)\hat{Y}_i,$$

$$\hat{T}_{kh} = [\hat{Y}_k, \hat{Y}_h] - \hat{Y}_{k,h} + \hat{Y}_{h,k}, \tag{205}$$

where M is a power series in the variables $x, t, u^1, \dots, u^{n-1}$.

Equations (195), (199) remain valid if we replace (188) by (200), so one has

$$\hat{T}_{ih,k} = 2\Delta^{-1}\Big(2u^k \hat{T}_{hi} + u^h \hat{T}_{ki} + u^i \hat{T}_{hk} + \delta_k^i \sum_m u^m \hat{T}_{mh} + \delta_k^h \sum_m u^m \hat{T}_{im}\Big) + [\hat{T}_{ih}, \hat{Y}_k], \tag{206}$$

$$[\hat{T}_{ih}, \hat{T}_{km}] = 4\Delta^{-2}(\delta_m^h \hat{T}_{ki} + \delta_k^h \hat{T}_{im} + \delta_m^i \hat{T}_{hk} + \delta_k^i \hat{T}_{mh}). \tag{207}$$

Combining (205) with (201), we obtain also

$$\frac{\partial}{\partial x}(\hat{T}_{kh}) = \frac{\partial}{\partial t}(\hat{T}_{kh}) = 0.$$

Thus $\hat{T}_{kh}$ is a power series in the variables $u^1, \dots, u^{n-1}$.

Let $E_{i,j} \in \mathfrak{gl}_{n-1}(\mathbb{K})$ be the matrix with (i,j)-th entry equal to 1 and all other entries equal to zero. Then the matrices $C_{kh} = 4(E_{k,h} - E_{h,k})$ span the Lie algebra $\mathfrak{so}_{n-1} \subset \mathfrak{gl}_{n-1}(\mathbb{K})$.

Theorem 10. *There is a ZCR with values in $\mathfrak{so}_{n-1}$ such that*

- *A is of the form* (200)*, where the power series $\hat{Y}_i(u^1, \dots, u^{n-1})$ satisfy* (202)*,*
- *B is defined by* (203)*,* (204) *with $M = 0$,*
- *for the power series* (205)*, one has*

$$\hat{T}_{kh}\Big|_{u^1 = \dots = u^{n-1} = 0} = 4(E_{k,h} - E_{h,k}). \tag{208}$$

Proof. For $j = 1, \dots, n-1$, let N^j be the $\mathfrak{so}_{n-1}$-valued matrix-function whose (h,i)-entry is equal to

$$-2\delta_j^h \Delta^{-1} u^i + 2\delta_j^i \Delta^{-1} u^h.$$

Let A be given by (200) with $\hat{Y}_i = N^i$ and B be defined by (203), (204) with $M = 0$. It is straightforward to check that such A, B satisfy the zero-curvature condition (157). In order to achieve condition (202), we can apply a gauge transformation depending on $u^1, \dots, u^{n-1}$, similarly to Theorem 3.

According to Theorem 3, the required gauge transformation is defined on a neighborhood of the point (154) and is equal to the identity transformation at this point. Since we assume (155), (156), in our case the gauge transformation may depend nontrivially on $u^1, \dots, u^{n-1}$ and is equal to the identity transformation at the point $u^1 = \dots = u^{n-1} = 0$.

Computing $\hat{T}_{kh}$ by formula (205) for $\hat{Y}_i = N^i$, one obtains that (208) is valid. It is easy to check that (208) remains valid after applying the gauge transformation, because the gauge transformation is equal to the identity transformation at the point $u^1 = \dots = u^{n-1} = 0$. □

Remark 28. Note that the ZCR described in Theorem 10 is of order ≤ 1 and is a-normal for the point $a \in \mathcal{E}$ satisfying (154), (155), (156).

5.4. **The ideal generated by Y_{ij}.** Continue the analysis started in Section 5.1. Recall that $A = A(x, t, u_0^j, u_1^j)$ and $B = B(x, t, u_0^j, u_1^j, u_2^j, u_3^j)$ are power series with coefficients in the algebra $\mathbb{F}^1(\mathcal{E}, a)$ and satisfy (157), (158), (159), (160). Recall that we have obtained formula (166).

Lemma 11. *The algebra $\mathbb{F}^1(\mathcal{E}, a)$ is generated by the coefficients of the power series*

$$Y_{ij}\Big|_{t=0}, \qquad Y_i\Big|_{t=0}, \qquad Y\Big|_{t=0}, \qquad i, j = 1, \dots, n-1. \tag{209}$$

Proof. Theorem 7 implies that the algebra $\mathbb{F}^1(\mathcal{E}, a)$ is generated by the coefficients of the power series $A\Big|_{t=0}$. According to formula (166), the set of the coefficients of $A\Big|_{t=0}$ consists of the coefficients of the power series (209). □

Lemma 12. *For all i, j, any coefficient of the power series $Y_{ij}\Big|_{t=0}$ belongs to the vector subspace spanned by the zero degree coefficients*

$$z^{pq} = Y_{pq}\Big|_{x=t=u^1=\dots=u^{n-1}=0}, \qquad p, q = 1, \dots, n-1. \tag{210}$$

Proof. The statement follows from Lemma 10 and equation (185). □

Lemma 13. *Let P be a power series in the variables x, u^l. Suppose that*

$$\Big[Y_{ij}\Big|_{t=0},\, P\Big] = 0 \qquad \forall\, i, j. \tag{211}$$

Then any coefficient of the power series $Y_{ij}\Big|_{t=0}$ commutes with any coefficient of P.

Proof. Using Lemma 12, one can prove the statement by induction on the degree of coefficients of P. □

Lemma 14. *For all i, j, any coefficient of the power series $Y_{ij}\Big|_{t=0}$ commutes*

- *with any coefficient of $Y\Big|_{t=0}$,*
- *with any coefficient of $Y_{pq}\Big|_{t=0}$ for all p, q.*

Proof. Substituting $t = 0$ to (187) and (171), we obtain the required statement by Lemma 13. □

Consider the ZCR with values in $\mathfrak{so}_{n-1}$ constructed in Theorem 10. Recall that, according to (200), (205), and Theorem 10, this ZCR determines the power series $\hat{Y}_i$, $\hat{T}_{kh}$ in the variables $u^1, \dots, u^{n-1}$ with coefficients in $\mathfrak{so}_{n-1}$.

According to Remark 28, this $\mathfrak{so}_{n-1}$-valued ZCR is of order ≤ 1 and is a-normal. Therefore, by Remark 23, this ZCR determines a homomorphism $\sigma\colon \mathbb{F}^1(\mathcal{E}, a) \to \mathfrak{so}_{n-1}$.

For a power series P with coefficients in $\mathbb{F}^1(\mathcal{E}, a)$, we can apply σ to each coefficient of P and obtain a power series $\sigma(P)$ with coefficients in $\mathfrak{so}_{n-1}$.

Recall that T_{kh} is defined by (176). By the definition of σ, one has

$$\sigma(Y_i) = \hat{Y}_i, \qquad \sigma(T_{kh}) = \hat{T}_{kh}. \tag{212}$$

Combining (212) with (206), (207), we obtain

(213)

$$\sigma\big(T_{ih,k}\big) = \sigma\Big(2\Delta^{-1}\left(2u^kT_{hi} + u^hT_{ki} + u^iT_{hk} + \delta^i_k\sum_m u^mT_{mh} + \delta^h_k\sum_m u^mT_{im}\right) + [T_{ih}, Y_k]\Big),$$

$$\sigma\big([T_{ih}, T_{km}]\big) = \sigma\big(4\Delta^{-2}(\delta^h_mT_{ki} + \delta^h_kT_{im} + \delta^i_mT_{hk} + \delta^i_kT_{mh})\big). \tag{214}$$

Set

(215)

$$\Gamma_{ihk} = T_{ih,k} - \big(2\Delta^{-1}\left(2u^kT_{hi} + u^hT_{ki} + u^iT_{hk} + \delta^i_k\sum_m u^mT_{mh} + \delta^h_k\sum_m u^mT_{im}\right) + [T_{ih}, Y_k]\big),$$

(216)
$$\Gamma_{ihkm} = [T_{ih}, T_{km}] - 4\Delta^{-2}(\delta^h_mT_{ki} + \delta^h_kT_{im} + \delta^i_mT_{hk} + \delta^i_kT_{mh}).$$

Lemma 15. *One has*

(217)
$$[Y_{pq}, \Gamma_{ihk}] = 0, \qquad [Y_{pq}, \Gamma_{ihkm}] = 0 \qquad \forall\, p, q, i, h, k, m.$$

Proof. Set $P'_i = Y_i + \dfrac{\partial}{\partial u^i}$, then

(218)
$$T_{ih} = [Y_i, Y_h] - Y_{i,h} + Y_{h,i} = [P'_i, P'_h],$$

(219)
$$\Gamma_{ihk} = [P'_k, T_{ih}] - 2\Delta^{-1}\left(2u^kT_{hi} + u^hT_{ki} + u^iT_{hk} + \delta^i_k\sum_m u^mT_{mh} + \delta^h_k\sum_m u^mT_{im}\right).$$

and equation (195) can be written as

(220)
$$[Y_{ij}, P'_h] = 2\Delta^{-1}\big(2u^hY_{ij} + u^iY_{hj} + u^jY_{hi} - \delta^i_h\sum_m u^mY_{mj} - \delta^j_h\sum_m u^mY_{mi}\big).$$

Using (175), (216), (218), (219), (220), one can check relations (217) by a straightforward computation. □

Combining (213), (214), (215), (216) with (201), we obtain

(221)
$$\sigma\big(\Gamma_{ihk}\big) = \sigma\Big(\Gamma_{ihk}\Big|_{t=0}\Big) = 0, \qquad \sigma\big(\Gamma_{ihkm}\big) = \sigma\Big(\Gamma_{ihkm}\Big|_{t=0}\Big) = 0.$$

From (212) and (201) it follows that

(222)
$$\sigma(Y_i) = \sigma\Big(Y_i\Big|_{t=0}\Big) = \sigma\Big(Y_i\Big|_{x=t=0}\Big) = \hat{Y}_i.$$

Set

(223)
$$\Gamma_i = Y_i\Big|_{t=0} - Y_i\Big|_{x=t=0},$$

then equation (222) implies

(224)
$$\sigma(\Gamma_i) = 0.$$

Let $\mathfrak{H} \subset \mathbb{F}^1(\mathcal{E}, a)$ be the subalgebra generated by the coefficients of the power series $Y_i\Big|_{t=0}$, $i = 1, \dots, n-1$. Then $Y_i\Big|_{t=0}$ and $T_{kh}\Big|_{t=0}$ are power series with coefficients in $\mathfrak{H}$. Therefore, the coefficients of the power series

(225)
$$\Gamma_{ihk}\Big|_{t=0}, \qquad \Gamma_{ihkm}\Big|_{t=0}, \qquad \Gamma_i$$

belong to $\mathfrak{H}$ as well.

Lemma 16. *Let $\mathfrak{I} \subset \mathfrak{H}$ be the ideal of $\mathfrak{H}$ generated by the coefficients of the power series* (225)*. Then*

(226)
$$\mathfrak{I} = \mathfrak{H} \cap \ker\sigma.$$

Proof. From (221) and (224) it follows that

(227)
$$\mathfrak{I} \subset \ker\sigma.$$

Consider the quotient Lie algebra $\mathfrak{g} = \mathfrak{H}/\mathfrak{I}$ and the natural projection $\psi\colon \mathfrak{H} \to \mathfrak{g}$. Set

(228)
$$\mathbb{Y}_i = \psi\Big(Y_i\Big|_{t=0}\Big), \qquad \mathbb{T}_{kh} = \psi\Big(T_{kh}\Big|_{t=0}\Big),$$

which are power series with coefficients in $\mathfrak{g}$. From (176), (180) it follows that

$$\mathbb{T}_{kh} = [\mathbb{Y}_k, \mathbb{Y}_h] - \mathbb{Y}_{k,h} + \mathbb{Y}_{h,k}, \tag{229}$$

$$\forall\, i = 1, \dots, n-2, \qquad \mathbb{Y}_i \Big|_{u^{i+1}=u^{i+2}=\dots=u^{n-1}=0} = 0, \qquad \mathbb{Y}_{n-1} = 0. \tag{230}$$

By the definition of $\mathfrak{I}$ and ψ,

$$\psi\Big(\Gamma_{ihk}\Big|_{t=0}\Big) = 0, \qquad \psi\Big(\Gamma_{ihkm}\Big|_{t=0}\Big) = 0, \qquad \psi\big(\Gamma_i\big) = 0. \tag{231}$$

According to (215), (216), (223), (228), equations (231) say that

$$\mathbb{T}_{ih,k} = 2\Delta^{-1}\left(2u^k\mathbb{T}_{hi} + u^h\mathbb{T}_{ki} + u^i\mathbb{T}_{hk} + \delta^i_k\sum_m u^m\mathbb{T}_{mh} + \delta^h_k\sum_m u^m\mathbb{T}_{im}\right) + [\mathbb{T}_{ih}, \mathbb{Y}_k], \tag{232}$$

$$[\mathbb{T}_{ih}, \mathbb{T}_{km}] = 4\Delta^{-2}(\delta^h_m\mathbb{T}_{ki} + \delta^h_k\mathbb{T}_{im} + \delta^i_m\mathbb{T}_{hk} + \delta^i_k\mathbb{T}_{mh}), \tag{233}$$

$$\frac{\partial}{\partial x}\big(\mathbb{Y}_i\big) = 0. \tag{234}$$

From (228), (229), (234) it follows that $\mathbb{Y}_i$, $\mathbb{T}_{kh}$ are power series in the variables $u^1, \dots, u^{n-1}$ with coefficients in $\mathfrak{g}$.

Lemma 17. *The elements*

$$\mathbb{T}_{kh}\Big|_{u^1=u^2=\dots=u^{n-1}=0} \in \mathfrak{g}, \qquad k, h = 1, \dots, n-1, \tag{235}$$

generate the Lie algebra $\mathfrak{g}$.

Proof. From the definition of $\mathfrak{H}$ and $\mathfrak{g}$ it follows that $\mathfrak{g}$ is generated by the coefficients of the power series $\mathbb{Y}_i$.

Relations (229), (230), (232) are of the type considered in Lemmas 8, 9 for $v^i = u^i$. Therefore, by Lemmas 8, 9, any coefficient of $\mathbb{Y}_i$, $\mathbb{T}_{kh}$ belongs to the Lie subalgebra generated by (235). □

From (233) and Lemma 17 it follows that the map

$$\mathfrak{so}_{n-1} \to \mathfrak{g}, \qquad (E_{k,h} - E_{h,k}) \mapsto \frac{1}{4}\Big(\mathbb{T}_{kh}\Big|_{u^1=u^2=\dots=u^{n-1}=0}\Big),$$

is a surjective homomorphism. Therefore,

$$\dim\mathfrak{g} = \dim\big(\mathfrak{H}/\mathfrak{I}\big) \le \dim\mathfrak{so}_{n-1}. \tag{236}$$

According to Theorem 10, the coefficients of the power series $\hat{Y}_i$ generate $\mathfrak{so}_{n-1}$. Combining this with (222) and the definition of $\mathfrak{H}$, we obtain

$$\sigma(\mathfrak{H}) = \mathfrak{so}_{n-1}. \tag{237}$$

Combining (227), (236), (237), one gets (226). □

Substituting $t = 0$ to (172), we obtain

$$Y_{ij,h}\Big|_{t=0} = \tag{238}$$
$$= \Big[Y_{ij}\Big|_{t=0}, Y_h\Big|_{t=0}\Big] - 2\Delta^{-1}\Big(2u^hY_{ij} + u^iY_{hj} + u^jY_{hi} - \delta^i_h\sum_m u^mY_{mj} - \delta^j_h\sum_m u^mY_{mi}\Big)\Big|_{t=0} \quad \forall\, i, j, h.$$

From (186) it follows that $Y_{ij}\Big|_{t=0} = Y_{ij}\Big|_{x=t=0}$ for all i, j. Therefore, substituting $x = t = 0$ to (172), we get

$$Y_{ij,h}\Big|_{t=0} = \tag{239}$$

$$= \Big[Y_{ij}\Big|_{t=0}, Y_h\Big|_{x=t=0}\Big] - 2\Delta^{-1}\Big(2u^hY_{ij} + u^iY_{hj} + u^jY_{hi} - \delta^i_h\sum_m u^mY_{mj} - \delta^j_h\sum_m u^mY_{mi}\Big)\Big|_{t=0} \quad \forall\, i,j,h.$$

Subtracting (239) from (238), one obtains

$$\Big[Y_{ij}\Big|_{t=0}, \Gamma_h\Big] = 0 \qquad \forall\, i,j,h, \tag{240}$$

where $\Gamma_h = Y_h\Big|_{t=0} - Y_h\Big|_{x=t=0}$.

Let $\mathbf{V}$ be the vector subspace spanned by the elements z^{pq}, $p,q = 1,\dots,n-1$, defined in (210). Lemma 14 implies that $\mathbf{V}$ is a commutative subalgebra of the Lie algebra $\mathbb{F}^1(\mathcal{E},a)$.

Recall that $Y_h\Big|_{t=0}$ is a power series in the variables x, $u^1,\dots,u^{n-1}$. So one has

$$Y_h\Big|_{t=0} = \sum_{l,i_1,\dots,i_{n-1}\geq 0} x^l\big(u^1\big)^{i_1}\dots\big(u^{n-1}\big)^{i_{n-1}}\beta^h_{l,i_1,\dots,i_{n-1}} \tag{241}$$

for some elements $\beta^h_{l,i_1,\dots,i_{n-1}} \in \mathbb{F}^1(\mathcal{E},a)$.

Lemma 12 says that for all i,j any coefficient of the power series $Y_{ij}\Big|_{t=0}$ belongs to $\mathbf{V}$. Then equation (238) implies that for all i,j,h the coefficients of the power series $\Big[Y_{ij}\Big|_{t=0}, Y_h\Big|_{t=0}\Big]$ belong to $\mathbf{V}$. Using these facts and the definition of $\mathbf{V}$, by induction on $l+i_1+\dots+i_{n-1}$ one proves that $[\beta^h_{l,i_1,\dots,i_{n-1}}, \mathbf{V}] \subset \mathbf{V}$. Since the Lie algebra $\mathfrak{H}$ is generated by the elements $\beta^h_{l,i_1,\dots,i_{n-1}}$ and $\mathbf{V}$ is spanned by the elements z^{pq}, we see that $[z^{pq},\mathfrak{H}] \subset \mathbf{V}$ for all p,q.

Substituting $t=0$ to (217), one gets

$$\Big[Y_{pq}\Big|_{t=0}, \Gamma_{ihk}\Big|_{t=0}\Big] = 0, \qquad \Big[Y_{pq}\Big|_{t=0}, \Gamma_{ihkm}\Big|_{t=0}\Big] = 0 \qquad \forall\, p,q,i,h,k,m. \tag{242}$$

By Lemma 13, from (240) and (242) it follows that z^{pq} defined in (210) commutes with any coefficient of the power series (225).

Thus $[z^{pq},\mathfrak{H}] \subset \mathbf{V}$ and z^{pq} commutes with any coefficient of the power series (225). Combining this with Lemma 16, one gets

$$\big[z^{pq},\ (\mathfrak{H}\cap\ker\sigma)\big] = 0 \qquad \forall\, p,q. \tag{243}$$

Lemma 18. *The vector space* $\mathbf{V}$ *spanned by the elements* z^{pq}, $p,q = 1,\dots,n-1$, *is a commutative ideal of the Lie algebra* $\mathbb{F}^1(\mathcal{E},a)$.

Proof. We have shown above that $\mathbf{V}$ is a commutative subalgebra $\mathbb{F}^1(\mathcal{E},a)$. Let us show that $\mathbf{V}$ is an ideal of $\mathbb{F}^1(\mathcal{E},a)$.

According to Lemma 11, the Lie algebra $\mathbb{F}^1(\mathcal{E},a)$ is generated by the coefficients of the power series (209). As we have shown above, $[\mathbf{V},\mathfrak{H}] \subset \mathbf{V}$, where $\mathfrak{H} \subset \mathbb{F}^1(\mathcal{E},a)$ is the subalgebra generated by the coefficients of the power series $Y_i\Big|_{t=0}$, $i = 1,\dots,n-1$. Furthermore, Lemma 14 implies that any element of $\mathbf{V}$ commutes with any coefficient of the power series $Y\Big|_{t=0}$ and $Y_{ij}\Big|_{t=0}$. Therefore, $\mathbf{V}$ is an ideal of $\mathbb{F}^1(\mathcal{E},a)$. □

Lemma 19. *For any* $v \in \ker\sigma \subset \mathbb{F}^1(\mathcal{E},a)$, *one has* $[z^{pq},v] = 0$ *for all* p,q.

Proof. By Lemma 11, the algebra $\mathbb{F}^1(\mathcal{E},a)$ is generated by the coefficients of (209). Since $\sigma(Y_{ij}) = \sigma(Y) = 0$ and the coefficients of $Y_i\Big|_{t=0}$ generate $\mathfrak{H}$, the ideal $\ker\sigma$ is generated by the subalgebra $\mathfrak{H}\cap\ker\sigma$ and the coefficients of $Y_{ij}\Big|_{t=0}$, $Y\Big|_{t=0}$. Then the required statement follows from (243), Lemma 14, and Lemma 18. □

From (184) and (186) it follows that $y^{ij}_{i_1,\dots,i_{n-1}}$ is a power series in one variable t.

Lemma 20. *For all i, j, i', j', any coefficient of the power series $y^{ij}_{0,\dots,0}(t)$ commutes with any coefficient of $y^{i'j'}_{0,\dots,0}(t)$.*

Proof. For each $l \in \mathbb{Z}_{\geq 0}$, set

$$z^{ij}_l = \frac{\partial^l y^{ij}_{0,\dots,0}}{\partial t^l}\bigg|_{t=0} \in \mathbb{F}^1(\mathcal{E}, a). \tag{244}$$

Similarly to Lemma 19, by induction on l, one can prove that

$$[z^{ij}_l, \ker\sigma] = 0 \qquad \forall\, i, j, l. \tag{245}$$

Since $\sigma(Y_{i'j'}) = 0$, property (245) implies that z^{ij}_l commutes with any coefficient of the power series $y^{i'j'}_{0,\dots,0}(t)$. Since the elements z^{ij}_l are the coefficients of the power series $y^{ij}_{0,\dots,0}$, we see that any coefficient of $y^{ij}_{0,\dots,0}(t)$ commutes with any coefficient of $y^{i'j'}_{0,\dots,0}(t)$. □

Theorem 11. *The Lie subalgebra $\mathfrak{S}$ generated by the coefficients of Y_{ij} is abelian and satisfies*

$$[\mathfrak{S}, \ker\sigma] = 0. \tag{246}$$

Furthermore, the subalgebra $\mathfrak{S}$ is an ideal of $\mathbb{F}^1(\mathcal{E}, a)$.

Proof. The fact that $\mathfrak{S}$ is abelian follows from Lemma 10, property (186), and Lemma 20. According to Lemma 10 and property (186), the subalgebra $\mathfrak{S}$ is generated by the coefficients (244) of the power series $y^{ij}_{0,\dots,0}(t)$. Then (245) implies (246).

By Lemma 11, the algebra $\mathbb{F}^1(\mathcal{E}, a)$ is generated by the coefficients of (209). Therefore, in order to show that $\mathfrak{S}$ is an ideal, one needs to prove that $[C, \mathfrak{S}] \subset \mathfrak{S}$ for any coefficient C of $Y_i\Big|_{t=0}$ and $Y\Big|_{t=0}$. This can be easily deduced from equations (172) and (187), using Lemma 10 and property (186). □

5.5. **The ideal generated by $\tilde{Y}_i$.** In this subsection we study the quotient Lie algebra $\mathfrak{L} = \mathbb{F}^1(\mathcal{E}, a)/\mathfrak{S}$, where $\mathfrak{S}$ is the ideal generated by the coefficients of Y_{ij}. Therefore, we can assume that A, B are power series with coefficients in $\mathfrak{L}$ and formula (188) holds.

Then $\tilde{T}_{ij}$ defined in (190) are power series in the variables x, t, $u^1, \dots, u^{n-1}$ with coefficients in $\mathfrak{L}$. Let $\tilde{\mathfrak{g}}$ be the Lie algebra of formal power series in the variables x, t with coefficients in $\mathfrak{L}$. Then $\tilde{T}_{ij}$ can be regarded as a power series in the variables $u^1, \dots, u^{n-1}$ with coefficients in $\tilde{\mathfrak{g}}$

$$\tilde{T}_{ij} = \sum_{i_1,\dots,i_{n-1}\geq 0} \big(u^1\big)^{i_1} \dots \big(u^{n-1}\big)^{i_{n-1}} \alpha^{ij}_{i_1,\dots,i_{n-1}}(x,t), \qquad \alpha^{ij}_{i_1,\dots,i_{n-1}}(x,t) \in \tilde{\mathfrak{g}}.$$

From (180), (181) we get

$$\forall\, i = 1, \dots, n-2, \qquad \tilde{Y}_i\Big|_{u^{i+1}=u^{i+2}=\dots=u^{n-1}=0} = 0, \qquad \tilde{Y}_{n-1} = 0, \tag{247}$$

$$\tilde{Y}\Big|_{u^1=u^2=\dots=u^{n-1}=0} = 0. \tag{248}$$

Similarly to Lemma 10, using Lemma 9, from equations (195) we obtain the following.

Lemma 21. *For any $i, j, i_1, \dots, i_{n-1}$, the power series $\alpha^{ij}_{i_1,\dots,i_{n-1}}(x,t)$ belongs to the vector subspace spanned by the power series $\alpha^{pq}_{0,\dots,0}(x,t)$, $p, q = 1, \dots, n-1$.*

Similarly to (185), from (196) and (248) one gets

$$\frac{\partial}{\partial x}\Big(\alpha^{pq}_{0,\dots,0}(x,t)\Big) = \tilde{T}_{pq,x}\Big|_{u^1=u^2=\dots=u^{n-1}=0} = \tag{249}$$
$$= \Big[\tilde{T}_{pq}\Big|_{u^1=u^2=\dots=u^{n-1}=0}, \tilde{Y}\Big|_{u^1=u^2=\dots=u^{n-1}=0}\Big] = 0 \qquad \forall\, p, q.$$

Combining (249) with Lemma 21, we obtain

$$\tilde{T}_{pq,x} = 0 \qquad \forall\, p, q \tag{250}$$

and, by (196),

$$[\tilde{T}_{ij}, \tilde{Y}] = 0 \qquad \forall\, i, j. \tag{251}$$

Lemma 22. *The algebra $\mathfrak{L}$ is generated by the coefficients of the power series*

$$\tilde{T}_{ij}\Big|_{t=0}, \qquad \tilde{Y}\Big|_{t=0}, \qquad i, j = 1, \dots, n-1. \tag{252}$$

Proof. Similarly to Lemma 11, taking into account formula (188), one proves that $\mathfrak{L}$ is generated by the coefficients of $\tilde{Y}_i\Big|_{t=0}$, $\tilde{Y}\Big|_{t=0}$. By Lemma 8, the Lie subalgebra generated by the coefficients of $\tilde{Y}_i\Big|_{t=0}$ coincides with the Lie subalgebra generated by the coefficients of $\tilde{T}_{ij}\Big|_{t=0}$. □

Similarly to Lemma 12, using equation (195), we obtain the following.

Lemma 23. *Any coefficient of the power series $\tilde{T}_{ij}\Big|_{t=0}$ belongs to the vector subspace spanned by the zero degree coefficients*

$$\beta^{pq} = \tilde{T}_{pq}\Big|_{x=t=u^1=\dots=u^{n-1}=0}, \qquad p, q = 1, \dots, n-1. \tag{253}$$

Lemma 24. *Let $\mathfrak{Z} \subset \mathbb{F}^1(\mathcal{E}, a)/\mathfrak{S}$ be the Lie subalgebra generated by the coefficients of the power series $\tilde{T}_{ij}\Big|_{t=0}$. This subalgebra is isomorphic to $\mathfrak{so}_{n-1}$.*

Proof. Recall that $E_{i,j} \in \mathfrak{gl}_{n-1}(\mathbb{K})$ is the matrix with (i, j)-th entry equal to 1 and all other entries equal to zero. The matrices

$$C_{kh} = 4(E_{k,h} - E_{h,k}), \qquad k, h = 1, \dots, n-1, \qquad k < h,$$

form a basis of the Lie algebra $\mathfrak{so}_{n-1} \subset \mathfrak{gl}_{n-1}(\mathbb{K})$.

Substituting $x = t = u^1 = \dots = u^{n-1} = 0$ in equation (199), we see that the elements (253) satisfy

$$[\beta^{ih}, \beta^{km}] = 4(\delta^h_m \beta^{ki} + \delta^h_k \beta^{im} + \delta^i_m \beta^{hk} + \delta^i_k \beta^{mh}). \tag{254}$$

Since $\tilde{T}_{pq} = -\tilde{T}_{qp}$, we have $\beta^{pq} = -\beta^{qp}$. Combining this with equation (254), we see that the map

$$\mathbf{f}\colon \mathfrak{so}_{n-1} \to \mathfrak{Z}, \qquad \mathbf{f}(C_{kh}) = \beta^{kh}, \tag{255}$$

is a homomorphism.

Consider the homomorphism $\sigma\colon \mathbb{F}^1(\mathcal{E}, a) \to \mathfrak{so}_{n-1}$ constructed in Section 5.4. Since $\sigma(Y_{ij}) = 0$ and the ideal $\mathfrak{S} \subset \mathbb{F}^1(\mathcal{E}, a)$ is generated by the coefficients of Y_{ij}, the homomorphism σ determines a homomorphism $\hat{\sigma}\colon \mathbb{F}^1(\mathcal{E}, a)/\mathfrak{S} \to \mathfrak{so}_{n-1}$.

Consider the ZCR with coefficients in $\mathfrak{so}_{n-1}$ constructed in Theorem 10. Recall that, according to (200), (205), and Theorem 10, this ZCR determines the power series $\hat{Y}_i$, $\hat{T}_{kh}$ in the variables $u^1, \dots, u^{n-1}$ with coefficients in $\mathfrak{so}_{n-1}$ such that (208), (212) hold.

For a power series P with coefficients in $\mathbb{F}^1(\mathcal{E}, a)/\mathfrak{S}$, we can apply $\hat{\sigma}$ to each coefficient of P and obtain a power series $\hat{\sigma}(P)$ with coefficients in $\mathfrak{so}_{n-1}$.

The projection $\mathbb{F}^1(\mathcal{E}, a) \to \mathbb{F}^1(\mathcal{E}, a)/\mathfrak{S}$ maps Y_i to $\tilde{Y}_i$ and T_{kh} to $\tilde{T}_{kh}$. Combining this with (212), we obtain

$$\hat{\sigma}(\tilde{Y}_i) = \hat{Y}_i, \qquad \hat{\sigma}(\tilde{T}_{kh}) = \hat{T}_{kh}. \tag{256}$$

From (208), (253), (256) we get

$$\hat{\sigma}(\beta^{kh}) = 4(E_{k,h} - E_{h,k}) = C_{kh}. \tag{257}$$

Relations (255), (257) imply that the Lie subalgebra generated by the elements β^{kh} is isomorphic to $\mathfrak{so}_{n-1}$. According to Lemma 23, the Lie subalgebra generated by β^{kh} coincides with $\mathfrak{Z}$, so $\mathfrak{Z}$ is isomorphic to $\mathfrak{so}_{n-1}$. □

Lemma 25. *Any coefficient of* $\tilde{T}_{ij}\Big|_{t=0}$ *commutes with any coefficient of* $\tilde{Y}\Big|_{t=0}$.

Proof. The statement is proved similarly to Lemma 14, using (251) and Lemma 23. □

Recall that $L(n)$ is the infinite-dimensional Lie algebra generated by (148).

Theorem 12. *Suppose that* $n \geq 4$. *Then the Lie algebra* $\mathfrak{L} = \mathbb{F}^1(\mathcal{E}, a)/\mathfrak{S}$ *is isomorphic to the direct sum* $L(n) \oplus \mathfrak{so}_{n-1}$, *where* $\mathfrak{so}_{n-1}$ *is generated by the coefficients of* $\tilde{T}_{ij}$. *The ideal* $\mathfrak{so}_{n-1}$ *coincides also with the subalgebra generated by the coefficients of* $\tilde{Y}_i$.

The homomorphism $\mathbb{F}^1(\mathcal{E}, a) \to \mathbb{F}^0(\mathcal{E}, a)$ *from* (109) *coincides with the composition of the homomorphisms*

$$\mathbb{F}^1(\mathcal{E}, a) \to \mathbb{F}^1(\mathcal{E}, a)/\mathfrak{S} \cong L(n) \oplus \mathfrak{so}_{n-1} \to L(n) \cong \mathbb{F}^0(\mathcal{E}, a), \tag{258}$$

where $L(n) \cong \mathbb{F}^0(\mathcal{E}, a)$ *is the isomorphism described in Theorem 9.*

Proof. Let S_1 be the subalgebra generated by the coefficients of $\tilde{Y}\Big|_{t=0}$ and S_2 be the subalgebra generated by the coefficients of $\tilde{T}_{ij}\Big|_{t=0}$. By Lemmas 22, 25, one has $\mathfrak{L} = S_1 + S_2$ and $[S_1, S_2] = 0$. By Lemma 24, $S_2 \cong \mathfrak{so}_{n-1}$. Since for $n \geq 4$ the center of the Lie algebra $\mathfrak{so}_{n-1}$ is trivial, we obtain $S_1 \cap S_2 = 0$ and, therefore, $\mathfrak{L} = S_1 \oplus S_2$. In particular, S_2 is an ideal of $\mathfrak{L}$.

For $n \geq 4$, from equations (199), (250) and Lemma 21 it follows that for all $p, q = 1, \ldots, n-1$ any coefficient of $\tilde{T}_{pq}$ belongs to the ideal generated by the coefficients of $\tilde{T}_{ij}\Big|_{t=0}$, $i, j = 1, \ldots, n-1$. (To see this, one needs to differentiate (199) with respect to t several times and substitute $t = 0$.)

Since the subalgebra S_2 generated by the coefficients of $\tilde{T}_{ij}\Big|_{t=0}$ is an ideal, we obtain that all coefficients of $\tilde{T}_{pq}$ belong to S_2.

According to (190) and (247), we can apply Lemma 8 to the power series $\tilde{Y}_i$ and $\tilde{T}_{ij}$. This implies that the subalgebra generated by the coefficients of $\tilde{Y}_i$, $i = 1, \ldots, n-1$, coincides with the subalgebra S_2 generated by the coefficients of $\tilde{T}_{ij}$, $i, j = 1, \ldots, n-1$. Since S_2 is an ideal, we see that the ideal generated by the coefficients of $\tilde{Y}_i$ coincides with S_2. Therefore, taking into account formulas (161) and (188), one obtains that the quotient $\mathfrak{L}/S_2$ is isomorphic to $\mathbb{F}^0(\mathcal{E}, a)$. Combining this with Theorem 9, we get $S_1 \cong \mathfrak{L}/S_2 \cong \mathbb{F}^0(\mathcal{E}, a) \cong L(n)$.

According to formulas (166) and (161), the kernel of the homomorphism $\mathbb{F}^1(\mathcal{E}, a) \to \mathbb{F}^0(\mathcal{E}, a)$ from (109) is generated by the coefficients of the power series Y_{ij}, Y_i. Recall that $\mathfrak{S} \subset \mathbb{F}^1(\mathcal{E}, a)$ is the ideal generated by the coefficients of Y_{ij}. Applying the projection $\mathbb{F}^1(\mathcal{E}, a) \to \mathbb{F}^1(\mathcal{E}, a)/\mathfrak{S}$ to the coefficients of the power series Y_i, we get $\tilde{Y}_i$. As has been shown above,

$$\mathbb{F}^1(\mathcal{E}, a)/\mathfrak{S} \cong L(n) \oplus \mathfrak{so}_{n-1},$$

where $\mathfrak{so}_{n-1}$ coincides with the ideal generated by the coefficients of $\tilde{Y}_i$. According to Theorem 9, the algebra $\mathbb{F}^0(\mathcal{E}, a)$ is isomorphic to $L(n)$. These results show that the homomorphism $\mathbb{F}^1(\mathcal{E}, a) \to \mathbb{F}^0(\mathcal{E}, a)$ from (109) coincides with the composition (258). □

Remark 29. We have two different ZCRs for the same PDE (24), which can be transformed to the PDE (147) by the transformation (29). Namely, we have the $\mathfrak{gl}_{n+1}$-valued ZCR (26), (27) and the $\mathfrak{so}_{n-1}$-valued ZCR described in Theorem 10.

One can embed the Lie algebras $\mathfrak{gl}_{n+1}$ and $\mathfrak{so}_{n-1}$ into the Lie algebra $\mathfrak{gl}_N$ for some $N \geq n+1$, and then one can regard these ZCRs as $\mathfrak{gl}_N$-valued ZCRs. One can ask whether these ZCRs can become gauge equivalent after suitable embeddings $\mathfrak{gl}_{n+1} \hookrightarrow \mathfrak{gl}_N$ and $\mathfrak{so}_{n-1} \hookrightarrow \mathfrak{gl}_N$.

Let us show that these ZCRs cannot become gauge equivalent.

The $\mathfrak{gl}_{n+1}$-valued ZCR (26), (27) is of order ≤ 0. By Theorem 3 and Remark 15, there is an a-normal $\mathfrak{gl}_{n+1}$-valued ZCR $\mathfrak{Z}$ of order ≤ 0 such that the ZCR (26), (27) is gauge equivalent to the ZCR $\mathfrak{Z}$.

Theorem 4 and Remark 16 imply the following. If two a-normal ZCRs $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are gauge equivalent and the ZCR $\mathfrak{C}_1$ is of order ≤ 0, then the ZCR $\mathfrak{C}_2$ is also of order ≤ 0.

The $\mathfrak{so}_{n-1}$-valued ZCR described in Theorem 10 is a-normal and is not of order ≤ 0, because the function A in this ZCR depends nontrivially on u_1^j. Therefore, this $\mathfrak{so}_{n-1}$-valued ZCR cannot become gauge equivalent to the $\mathfrak{gl}_{n+1}$-valued ZCR $\mathfrak{Z}$, after any embeddings $\mathfrak{gl}_{n+1} \hookrightarrow \mathfrak{gl}_N$ and $\mathfrak{so}_{n-1} \hookrightarrow \mathfrak{gl}_N$.

Since the ZCR (26), (27) is gauge equivalent to the ZCR $\mathfrak{Z}$, we see that the $\mathfrak{so}_{n-1}$-valued ZCR described in Theorem 10 cannot become gauge equivalent to the $\mathfrak{gl}_{n+1}$-valued ZCR (26), (27), after any embeddings $\mathfrak{gl}_{n+1} \hookrightarrow \mathfrak{gl}_N$ and $\mathfrak{so}_{n-1} \hookrightarrow \mathfrak{gl}_N$.

6. The algebras $\mathbb{F}^p(\mathcal{E}, a)$ for the multicomponent Landau-Lifshitz system

6.1. **Preliminary computations.** Recall that the infinite prolongation $\mathcal{E}$ of system (147) is an infinite-dimensional manifold with the coordinates (153), where u^i_m corresponds to $\dfrac{\partial^m u^i}{\partial x^m}$ for $m \in \mathbb{Z}_{\ge 0}$ and $i = 1, \dots, n-1$. In particular, $u^i_0 = u^i$.

Consider an arbitrary point $a \in \mathcal{E}$ given by (154). As has been said in Section 5.1, since the PDE (147) is invariant with respect to the change of variables $x \mapsto x - x_a$, $t \mapsto t - t_a$, it is sufficient to consider the case $x_a = t_a = 0$.

For simplicity of exposition, we continue to assume (156), so we assume that $a^i_k = 0$ in (154). (In the case $a^i_k \ne 0$, the computations change very little, and the final result is the same.)

Fix an integer $k \ge 2$. In this section we compute the algebra $\mathbb{F}^k(\mathcal{E}, a)$ of the PDE (147).

According to Remark 20 and assumptions (155), (156), in order to describe the Lie algebra $\mathbb{F}^k(\mathcal{E}, a)$ for the PDE (147), we need to study the equations

$$D_x(B) - D_t(A) + [A, B] = 0, \tag{259}$$

$$\forall\, i_0 = 1, \dots, n-1, \qquad \forall\, k_0 \ge 1, \qquad \frac{\partial A}{\partial u^{i_0}_{k_0}}\,\Big|_{u^i_k = 0\ \forall\,(i,k) \succ (i_0, k_0 - 1)} = 0, \tag{260}$$

$$A\,\Big|_{u^i_k = 0\ \forall\,(i,k)} = 0, \tag{261}$$

$$B\,\Big|_{x=0,\ u^i_k = 0\ \forall\,(i,k)} = 0. \tag{262}$$

where

- $A = A(x, t, u^j_0, u^j_1, \dots, u^j_k)$ is a power series in the variables x, t, u^j_0, $u^j_1, \dots, u^j_k$ for $j = 1, \dots, n-1$,
- $B = B(x, t, u^j_0, u^j_1, \dots, u^j_{k+2})$ is a power series in the variables x, t, u^j_0, $u^j_1, \dots, u^j_{k+2}$ for $j = 1, \dots, n-1$.

The coefficients of the power series A, B are generators of the Lie algebra $\mathbb{F}^k(\mathcal{E}, a)$. Relations for these generators are provided by equations (259), (260), (261), (262).

In this section, summations over repeated indices run from 1 to $n-1$ when referred to the number of dependent variables and from 0 to k when referred to the order of derivatives, unless otherwise specified. For instance, $u^i_{m+1} F^1_{u^i_m}$ means $\sum_{i=1}^{n-1} \sum_{m=0}^{k} u^i_{m+1} F^1_{u^i_m}$. The integer $k \ge 2$ is fixed throughout this section, and there is no summation over k.

Recall that system (147) is of the form

$$u^i_t = u^i_3 + G^i(u^j, u^j_1, u^j_2) = u^i_3 + G^i(u^j_0, u^j_1, u^j_2), \qquad i = 1, \dots, n-1. \tag{263}$$

Then equation (259) reads

$$B_x + \sum_{j=0}^{k+2} u^i_{j+1} B_{u^i_j} - A_t - \sum_{j=0}^{k} u^i_{j+3} A_{u^i_j} - \sum_{j=0}^{k} D^j_x(G^i) A_{u^i_j} + [A, B] = 0. \tag{264}$$

If we put equal to zero the coefficient of u^i_{k+3} in (264), we obtain that $B_{u^i_{k+2}} - A_{u^i_k} = 0$, and, by integrating this, we see that B is of the form

$$B = u^i_{k+2} A_{u^i_k} + F^1(x, t, u^i, u^i_1, \dots, u^i_{k+1}) \tag{265}$$

for some power series $F^1(x, t, u^i, u^i_1, \dots, u^i_{k+1})$. Taking into account the form of A and B above, we rewrite equation (264) as

$$\begin{aligned} u^i_{k+2} A_{u^i_k x} + F^1_x + u^h_{j+1} u^i_{k+2} A_{u^i_k u^h_j} + u^i_{k+3} A_{u^i_k} + \sum_{m=0}^{k+1} u^i_{m+1} F^1_{u^i_m} \\ - A_t - u^i_{j+3} A_{u^i_j} - D^j_x(G^i) A_{u^i_j} + u^i_{k+2} [A, A_{u^i_k}] + [A, F^1] = 0. \end{aligned} \tag{266}$$

Now we compute the coefficient of u^i_{k+2} of the left-hand side of (266). It is

$$A_{u^i_k x} + u^h_{j+1} A_{u^i_k u^h_j} - A_{u^i_{k-1}} + F^1_{u^i_{k+1}} - G^j_{u^i_2} A_{u^j_k} + [A, A_{u^i_k}],$$

and, by putting it equal to zero, one obtains the following system of PDEs

$$F^1_{u^i_{k+1}} = A_{u^i_{k-1}} + A_{u^j_k} G^j_{u^i_2} - A_{u^i_k x} - A_{u^i_k u^h_j} u^h_{j+1} - [A, A_{u^i_k}], \qquad i = 1, \dots, n-1.$$

Integrating this system, we see that F^1 is of the form

$$F^1 = -\frac{1}{2} u^i_{k+1} u^h_{k+1} A_{u^i_k u^h_k} + u^i_{k+1} \Big(A_{u^i_{k-1}} + G^j_{u^i_2} A_{u^j_k} - \sum_{j=0}^{k-1} A_{u^i_k u^h_j} u^h_{j+1} - A_{u^i_k x} - [A, A_{u^i_k}] \Big) + F^2 \tag{267}$$

for some power series $F^2 = F^2(x, t, u^i, u^i_1, \dots, u^i_k)$.

In view of (267), the coefficient of $u^i_{k+1} u^j_{k+1} u^p_{k+1}$ in the left-hand side of (266) is $A_{u^i_k u^j_k u^p_k}$ multiplied by a nonzero scalar. Therefore, $A_{u^i_k u^j_k u^p_k} = 0$ for all i, j, p. Hence A is of the form

$$A = \frac{1}{2} u^i_k u^j_k Y^k_{ij} + u^i_k Y^k_i + Y^k, \qquad Y^k_{ij} = Y^k_{ji}, \tag{268}$$

where Y^k_{ij}, Y^k_i, Y^k are power series in the variables x, t, u^i_m, $m = 0, \dots, k-1$.

Similarly to (180), combining condition (260) with formula (268), we obtain

$$\forall\, i = 1, \dots, n-2, \qquad Y^k_i \Big|_{u^{i+1}_{k-1} = u^{i+2}_{k-1} = \dots = u^{n-1}_{k-1} = 0} = 0, \qquad Y^k_{n-1} = 0. \tag{269}$$

6.2. **The ideal generated by Y^k_{ij}.** The coefficient of $u^i_{k+1} u^h_{k+1}$ of the left-hand side of (266), taking into account that $D^j_x(G^i)$ does not contribute to this coefficient because $k \geq 2$, is

$$-3Y^k_{ih,x} - 3 \sum_{j=0}^{k-1} u^p_{j+1} Y^k_{ih, u^j_p} + G^j_{u^i_2} Y^k_{jh} + G^j_{u^h_2} Y^k_{ji} - 3[A, Y^k_{ih}]. \tag{270}$$

Note that (270) is a polynomial of degree 2 in u^i_k. Equating the coefficient of $u^p_k u^q_k$ of (270) to zero, we get

$$[Y^k_{pq}, Y^k_{ih}] = 0 \qquad \forall\, p, q, i, h. \tag{271}$$

Now, to compute the coefficient of $u^i_{k+1}u^h_{k+1}u^p_k$ of (266), it is enough to look at the coefficient of u^p_k of (270). Note that the quantity $G^j_{u^i_2}Y^k_{jh} + G^j_{u^h_2}Y^k_{ji}$ does not depend on u^p_k, since we suppose $k \geq 2$. Then we have only the system

$$Y^k_{ih,u^p_{k-1}} = [Y^k_{ih}, Y^k_p] \qquad \forall\, p, i, h. \tag{272}$$

We regard (272) as an overdetermined system of PDEs for Y^k_{ih}. The compatibility conditions of this system give the relations

$$[T^k_{pq}, Y^k_{ih}] = 0 \qquad \forall\, p, q, i, h,$$

where

$$T^k_{pq} := [Y^k_p, Y^k_q] - Y^k_{p,u^q_{k-1}} + Y^k_{q,u^p_{k-1}}. \tag{273}$$

Lemma 26. *The Lie subalgebra generated by the coefficients of the power series Y^k_i coincides with the Lie subalgebra generated by the coefficients of T^k_{pq}.*

Proof. The statement follows from Lemma 8 applied to (269), (273). □

The coefficient of $u^i_{k+1}u^h_{k+1}$ of (266) is the coefficient of degree zero in u^p_k of (270). Equating this coefficient to zero, we obtain

$$-3Y^k_{ih,x} - 3\sum_{j=0}^{k-2} u^p_{j+1}Y^k_{ih,u^p_j} + G^j_{u^i_2}Y^k_{jh} + G^j_{u^h_2}Y^k_{ji} - 3[Y^k, Y^k_{ih}] = 0. \tag{274}$$

Lemma 27. *The power series Y^k_{ih} does not depend on u^p_q for $q > 0$. So Y^k_{ih} depends only on $u^p_0 = u^p$ and x, t.*

Proof. This is proved by induction on $l = k - q$, using conditions (260), (261) and equations (272), (274). □

In view of Lemma 27, equation (274) becomes

$$-3Y^k_{ih,x} - 3u^p_1Y^k_{ih,p} + G^j_{u^i_2}Y^k_{jh} + G^j_{u^h_2}Y^k_{ji} - 3[Y^k, Y^k_{ih}] = 0, \tag{275}$$

where $Y^k_{ih,p} = Y^k_{ih,u^p}$.

Furthermore, by differentiating (275) with respect to u^p_l with $l \geq 2$ and with respect to u^p_1, u^q_1 we get, respectively, the following relations

$$[Y^k_{,u^p_l}, Y^k_{ij}] = 0 \qquad \forall\, l \geq 2, \qquad [Y^k_{,u^p_1u^q_1}, Y^k_{ih}] = 0.$$

Therefore, the quantity $[Y^k, Y^k_{ih}]$ is a polynomial of degree ≤ 1 in u^p_1. Its coefficient in u^p_1 is $[Y^k_{,u^p_1}, Y^k_{ih}]$ (recall that Y^k_{ih} depends only on x, t, u^p) and its zero degree term is $[Y^k - Y^k_{,u^p_1}u^p_1, Y^k_{ih}]$. By putting equal to zero the coefficient of u^p_1 and that of degree zero (in u^p_1) of the left-hand side of (275), we get the system

$$Y^k_{ij,h} = [Y^k_{ij}, Y^k_{,u^h_1}] - 2\Delta^{-1}\big(2u^hY^k_{ij} + u^iY^k_{hj} + u^jY^k_{hi} - \delta^i_h u^mY^k_{mj} - \delta^j_h u^mY^k_{mi}\big) \qquad \forall\, i, j, h, \tag{276}$$

$$Y^k_{ij,x} = [Y^k_{ij}, Z], \qquad Z = Y^k - u^p_1Y^k_{,u^p_1}.$$

Similarly to (175), the compatibility conditions of system (276) imply

$$[Y^k_{ij}, W_{mh}] + 4\Delta^{-2}(-\delta^i_m Y^k_{hj} + \delta^i_h Y^k_{mj} - \delta^j_m Y^k_{hi} + \delta^j_h Y^k_{mi}) = 0 \qquad \forall\, i, j, m, h, \tag{277}$$

where

$$W_{mh} := [Y^k_{,u^m_1}, Y^k_{,u^h_1}] - Y^k_{,u^m_1u^h} + Y^k_{,u^h_1u^m}. \tag{278}$$

Recall that in Section 5.4 we have defined the homomorphism $\sigma\colon \mathbb{F}^1(\mathcal{E}, a) \to \mathfrak{so}_{n-1}$. Let $\sigma_k\colon \mathbb{F}^k(\mathcal{E}, a) \to \mathfrak{so}_{n-1}$ be the composition of the homomorphisms $\mathbb{F}^k(\mathcal{E}, a) \to \mathbb{F}^1(\mathcal{E}, a) \to \mathfrak{so}_{n-1}$, where the homomorphism $\mathbb{F}^k(\mathcal{E}, a) \to \mathbb{F}^1(\mathcal{E}, a)$ arises from (109).

Theorem 13. *The Lie subalgebra $\mathfrak{S}_k$ generated by the coefficients of Y^k_{ij} is abelian and satisfies*

$$[\mathfrak{S}_k, \ker \sigma_k] = 0.$$

Furthermore, this subalgebra is an ideal of $\mathbb{F}^k(\mathcal{E}, a)$.

Proof. This is proved similarly to Theorem 11, using the results of this section. □

6.3. **The ideal generated by Y^k_i.** In this subsection we study the quotient Lie algebra $\mathfrak{L}_k = \mathbb{F}^k(\mathcal{E}, a)/\mathfrak{S}_k$, where $\mathfrak{S}_k$ is the ideal generated by the coefficients of Y^k_{ij}. Therefore, we can assume that A, B are power series with coefficients in $\mathfrak{L}_k$ and

$$A = u^i_k Y^k_i + Y^k, \tag{279}$$

where Y^k_i and Y^k depend on $x, t, u^i, \dots, u^i_{k-1}$.

Then from (265), (266), (267) one gets

$$B = u^i_{k+2} Y^k + F^1,$$

$$F^1 = u^i_{k+1}\Big(A_{u^i_{k-1}} + G^j_{u^i_2} A_{u^j_k} - \sum_{j=0}^{k-1} A_{u^i_k u^h_j} u^h_{j+1} - A_{u^i_k x} - [A, A_{u^i_k}]\Big) + F^2, \tag{280}$$

$$F^1_x + \sum_{m=0}^{k} u^i_{m+1} F^1_{u^i_m} - A_t - \sum_{j=0}^{k-2} u^i_{j+3} A_{u^i_j} - D^j_x(G^i) A_{u^i_j} + [A, F^1] = 0. \tag{281}$$

Substituting (279) and (280) in (281), we see that F^2 is a polynomial of degree ≤ 3 in $u^1_k, \dots, u^{n-1}_k$. So F^2 is of the form

$$F^2 = u^i_k u^j_k u^h_k M^k_{ijh} + u^i_k u^j_k M^k_{ij} + u^i_k M^k_i + M^k,$$

where all the $M^k_{\dots}$ are fully symmetric with respect to the lower indices and depend on x, t, u^i_q, $q = 0, \dots, k-1$.

Differentiating (281) with respect to u^i_{k+1}, u^p_k, u^q_k, we get

$$T^k_{ip,u^q_{k-1}} + T^k_{iq,u^p_{k-1}} + [Y^k_p, T^k_{iq}] + [Y^k_q, T^k_{ip}] + 6M^k_{ipq} = 0 \qquad \forall\, i, p, q. \tag{282}$$

Interchanging the indices i and p in (282), one obtains

$$T^k_{pi,u^q_{k-1}} + T^k_{pq,u^i_{k-1}} + [Y^k_i, T^k_{pq}] + [Y^k_q, T^k_{pi}] + 6M^k_{piq} = 0 \qquad \forall\, p, i, q. \tag{283}$$

Subtracting (283) from (282) and using the relations $M^k_{ipq} = M^k_{piq}$, $T^k_{pi} = -T^k_{ip}$, $T^k_{pq} = -T^k_{qp}$, we get

$$T^k_{iq,u^p_{k-1}} + T^k_{qp,u^i_{k-1}} + 2T^k_{ip,u^q_{k-1}} + [Y^k_p, T^k_{iq}] + [Y^k_i, T^k_{qp}] + 2[Y^k_q, T^k_{ip}] = 0. \tag{284}$$

Set

$$V^k_p = Y^k_p + \frac{\partial}{\partial u^p_{k-1}}.$$

Then from (273) we have $T^k_{pq} = [V^k_p, V^k_q]$, and equation (284) can be written in terms of commutators of V^k_p

$$[V^k_p, [V^k_i, V^k_q]] + [V^k_i, [V^k_q, V^k_p]] + 2[V^k_q, [V^k_i, V^k_p]] = 0. \tag{285}$$

By the Jacobi identity, equation (285) can be rewritten as

$$3[V^k_q, [V^k_i, V^k_p]] = 0. \tag{286}$$

Since $[V^k_i, V^k_p] = T^k_{ip}$ and $V^k_q = Y^k_q + \dfrac{\partial}{\partial u^q_{k-1}}$, equation (286) says that

$$T^k_{ip,u^q_{k-1}} + [Y^k_q, T^k_{ip}] = 0 \qquad \forall\, i, p, q. \tag{287}$$

Differentiating (281) with respect to u^i_{k+1}, u^p_k, we obtain

$$
\begin{aligned}
(288)\quad & Y^k_{p,u^i_{k-1}x} - 2Y^k_{i,u^p_{k-1}x} + [Y^k_i, Y^k_p]_{,x} + [Y^k_{i,x}, Y^k_p] - Y^k_{p,u^i_{k-2}} + \sum_{m=0}^{k-2} T^k_{ip,u^q_m} u^q_{m+1} + G^j_{u^i_2} T^k_{pj} + \\
& + [Y^k, T^k_{ip}] + Y^k_{,u^i_{k-1}u^p_{k-1}} - \sum_{m=0}^{k-2} Y^k_{i,u^h_m u^p_{k-1}} u^h_{m+1} - Y^k_{i,u^p_{k-2}} - [Y^k_{,u^p_{k-1}}, Y^k_i] - [Y^k, Y^k_{i,u^p_{k-1}}] + \\
& + [Y^k_p, Y^k_{,u^i_{k-1}}] - \sum_{m=0}^{k-2} [Y^k_p, Y^k_{i,u^h_m}] u^h_{m+1} - [Y^k_p, [Y^k, Y^k_i]] + 2M^k_{ip} = 0 \qquad \forall\, i, p.
\end{aligned}
$$

Denote the left-hand side of (288) by Z^k_{ip}. So equation (288) reads $Z^k_{ip} = 0$ for all i, p. Interchanging the indices i and p, one gets also $Z^k_{pi} = 0$.

Subtracting the equation $Z^k_{pi} = 0$ from the equation $Z^k_{ip} = 0$ and using the relation $M^k_{ip} = M^k_{pi}$, we obtain

$$
(289)\qquad 3T^k_{ip,x} + 3\sum_{m=0}^{k-2} T^k_{ip,u^q_m} u^q_{m+1} + G^h_{u^i_2} T^k_{ph} - G^h_{u^p_2} T^k_{ih} + 3[Y^k, T^k_{ip}] = 0.
$$

Lemma 28. *The power series T^k_{ip} does not depend on u^j_m for $m > 0$. So T^k_{ip} depends only on $u^j_0 = u^j$ and x, t.*

Proof. This is proved by induction on $l = k - m$, using conditions (260), (261) and equations (287), (289). □

Differentiating (289) with respect to u^p_l with $l \geq 2$ and with respect to u^p_1, u^q_1 we get, respectively, the following relations

$$
[Y^k_{,u^p_l}, T^k_{ij}] = 0 \qquad l \geq 2, \qquad\qquad [Y^k_{,u^p_1 u^q_1}, T^k_{ih}] = 0.
$$

Therefore, the left-hand side of (289) is a polynomial of degree 1 in u^p_1. Equating to zero the coefficients of this polynomial, taking into account (167), we obtain the system

$$
\begin{aligned}
(290)\quad & T^k_{ip,u^j} = 2\Delta^{-1}\left(2u^j T^k_{pi} + u^p T^k_{ji} + u^i T^k_{pj} + \delta^i_j u^m T^k_{mp} + \delta^p_j u^m T^k_{im}\right) + [T^k_{ip}, Y^k_{,u^j_1}], \qquad \forall\, i, p, j, \\
& T^k_{ip,x} = [T^k_{ip}, Z], \qquad Z = Y^k - u^q_1 Y^k_{,u^q_1}.
\end{aligned}
$$

The compatibility conditions of system (290) imply

$$
(291)\qquad [T^k_{ip}, W_{jm}] = 4\Delta^{-2}(\delta^p_m T^k_{ji} + \delta^p_j T^k_{im} + \delta^i_m T^k_{pj} + \delta^i_j T^k_{mp})
$$

with W_{jm} defined by (278).

Let $\psi_k\colon \mathbb{F}^k(\mathcal{E}, a) \to \mathbb{F}^1(\mathcal{E}, a)$ be the homomorphism that arises from (109). Formulas (268) and (265) imply that ψ_k maps each coefficient of Y^k_{ij} to zero. Since $\mathfrak{S}_k$ is generated by the coefficients of Y^k_{ij}, we get $\psi_k(\mathfrak{S}_k) = 0$. Therefore, $\psi_k\colon \mathbb{F}^k(\mathcal{E}, a) \to \mathbb{F}^1(\mathcal{E}, a)$ induces a homomorphism from $\mathbb{F}^k(\mathcal{E}, a)/\mathfrak{S}_k$ to $\mathbb{F}^1(\mathcal{E}, a)$, which we denote by $\hat{\psi}_k\colon \mathbb{F}^k(\mathcal{E}, a)/\mathfrak{S}_k \to \mathbb{F}^1(\mathcal{E}, a)$

Recall that in Section 5.4 we have defined the homomorphism $\sigma\colon \mathbb{F}^1(\mathcal{E}, a) \to \mathfrak{so}_{n-1}$. Let

$$
\hat{\sigma}_k\colon \mathbb{F}^k(\mathcal{E}, a)/\mathfrak{S}_k \to \mathfrak{so}_{n-1}
$$

be the composition of the homomorphisms $\sigma\colon \mathbb{F}^1(\mathcal{E}, a) \to \mathfrak{so}_{n-1}$ and $\hat{\psi}_k\colon \mathbb{F}^k(\mathcal{E}, a)/\mathfrak{S}_k \to \mathbb{F}^1(\mathcal{E}, a)$.

Theorem 14. *Recall that $k \geq 2$. Let $\mathfrak{H}_k \subset \mathbb{F}^k(\mathcal{E}, a)/\mathfrak{S}_k$ be the Lie subalgebra generated by the coefficients of Y^k_i. Then $\mathfrak{H}_k$ is abelian and satisfies*

$$
(292)\qquad [\mathfrak{H}_k, \ker \hat{\sigma}_k] = 0.
$$

Furthermore, this subalgebra is an ideal of the Lie algebra $\mathbb{F}^k(\mathcal{E}, a)/\mathfrak{S}_k$.

Proof. According to Lemma 26, the Lie subalgebra generated by the coefficients of the power series Y_i^k coincides with the Lie subalgebra generated by the coefficients of T_{pq}^k.

Therefore, it remains to prove that the subalgebra generated by the coefficients of T_{pq}^k is an abelian ideal and satisfies (292). This is proved similarly to Theorem 11, using the results of the present section. □

6.4. **The structure of the algebras $\mathbb{F}^p(\mathcal{E},a)$.**

Theorem 15. *Let $n\geq 4$. Recall that the n-component Landau-Lifshitz system* (24) *is transformed to the PDE* (147) *by means of the change of variables* (146).

The Lie algebras $\mathbb{F}^p(\mathcal{E},a)$, $p\in\mathbb{Z}_{\geq 0}$, for the PDE (147) *have the following structure.*

The algebra $\mathbb{F}^0(\mathcal{E},a)$ is isomorphic to $L(n)$.

There is an abelian ideal $\mathfrak{S}$ of $\mathbb{F}^1(\mathcal{E},a)$ such that $\mathbb{F}^1(\mathcal{E},a)/\mathfrak{S}\cong L(n)\oplus\mathfrak{so}_{n-1}$, where $\mathfrak{so}_{n-1}$ is the Lie algebra of skew-symmetric $(n-1)\times(n-1)$ matrices. The homomorphism $\mathbb{F}^1(\mathcal{E},a)\to\mathbb{F}^0(\mathcal{E},a)$ from (109) *coincides with the composition of the homomorphisms*

$$\mathbb{F}^1(\mathcal{E},a)\to\mathbb{F}^1(\mathcal{E},a)/\mathfrak{S}\cong L(n)\oplus\mathfrak{so}_{n-1}\to L(n)\cong\mathbb{F}^0(\mathcal{E},a).$$

Let $\tau_k\colon\ \mathbb{F}^k(\mathcal{E},a)\to\mathbb{F}^{k-1}(\mathcal{E},a)$ be the surjective homomorphism from (109). *Then for any $k\geq 2$ we have*

$$[v_1,[v_2,v_3]]=0\qquad\forall\, v_1,v_2,v_3\in\ker\tau_k. \tag{293}$$

In particular, the kernel of τ_k is nilpotent.

For each $k\geq 1$, let $\varphi_k\colon\ \mathbb{F}^k(\mathcal{E},a)\to L(n)\oplus\mathfrak{so}_{n-1}$ be the composition of the homomorphisms

$$\mathbb{F}^k(\mathcal{E},a)\to\mathbb{F}^1(\mathcal{E},a)\to\mathbb{F}^1(\mathcal{E},a)/\mathfrak{S}\cong L(n)\oplus\mathfrak{so}_{n-1}, \tag{294}$$

where $\mathbb{F}^k(\mathcal{E},a)\to\mathbb{F}^1(\mathcal{E},a)$ arises from (109). *Then*

$$[h_1,[h_2,\dots,[h_{2k-2},[h_{2k-1},h_{2k}]]\dots]]=0\qquad\forall\, h_1,h_2,\dots,h_{2k}\in\ker\varphi_k. \tag{295}$$

In particular, the kernel of φ_k is nilpotent.

Proof. The isomorphism $\mathbb{F}^0(\mathcal{E},a)\cong L(n)$ is proved in Theorem 9.

In particular, the abelian ideal $\mathfrak{S}\subset\mathbb{F}^1(\mathcal{E},a)$ is defined in Theorem 11, and the isomorphism $\mathbb{F}^1(\mathcal{E},a)/\mathfrak{S}\cong L(n)\oplus\mathfrak{so}_{n-1}$ is described in Theorem 12. Also, in Theorem 12 it is shown that the homomorphism $\mathbb{F}^1(\mathcal{E},a)\to\mathbb{F}^0(\mathcal{E},a)$ from (109) coincides with the composition (293).

Recall that in Section 5.4 we have defined the homomorphism $\sigma\colon\ \mathbb{F}^1(\mathcal{E},a)\to\mathfrak{so}_{n-1}$. For any $r\geq 1$, let $\sigma_r\colon\ \mathbb{F}^r(\mathcal{E},a)\to\mathfrak{so}_{n-1}$ be the composition of the homomorphisms $\mathbb{F}^r(\mathcal{E},a)\to\mathbb{F}^1(\mathcal{E},a)\to\mathfrak{so}_{n-1}$, where $\mathbb{F}^r(\mathcal{E},a)\to\mathbb{F}^1(\mathcal{E},a)$ arises from (109).

Since for any $k\geq 2$ the composition

$$\mathbb{F}^k(\mathcal{E},a)\xrightarrow{\tau_k}\mathbb{F}^{k-1}(\mathcal{E},a)\xrightarrow{\sigma_{k-1}}\mathfrak{so}_{n-1}$$

coincides with $\sigma_k\colon\ \mathbb{F}^k(\mathcal{E},a)\to\mathfrak{so}_{n-1}$, we have

$$\ker\tau_k\subset\ker\sigma_k\qquad\forall\, k\geq 2. \tag{296}$$

Formulas (268), (265) imply that $\ker\tau_k$ is generated by the coefficients of the power series Y_{ij}^k, Y_i^k.

Recall that $\mathfrak{S}_k\subset\mathbb{F}^k(\mathcal{E},a)$ is the ideal generated by the coefficients of Y_{ij}^k for any $k\geq 2$. According to Theorem 13,

$$[\mathfrak{S}_k,\ker\sigma_k]=0\qquad\forall\, k\geq 2. \tag{297}$$

Since $\ker\tau_k\subset\ker\sigma_k$, we get

$$[\mathfrak{S}_k,\ker\tau_k]=0\qquad\forall\, k\geq 2. \tag{298}$$

Theorem 14 implies

$$[\ker\tau_k, \ker\sigma_k] \subset \mathfrak{S}_k \qquad \forall\, k \geq 2. \tag{299}$$

Combining (296), (298), (299), one obtains $[\ker\tau_k, \ker\tau_k] \subset \mathfrak{S}_k$ and $[\mathfrak{S}_k, \ker\tau_k] = 0$. This yields (293) and implies that $\ker\tau_k$ is nilpotent.

Since $\varphi_k\colon \mathbb{F}^k(\mathcal{E}, a) \to L(n) \oplus \mathfrak{so}_{n-1}$ is the composition (294), we have

$$\ker\varphi_k \subset \ker\sigma_k \qquad \forall\, k \geq 2. \tag{300}$$

For $k = 1$, the homomorphism $\varphi_1\colon \mathbb{F}^1(\mathcal{E}, a) \to L(n) \oplus \mathfrak{so}_{n-1}$ is the composition of the homomorphisms

$$\mathbb{F}^1(\mathcal{E}, a) \to \mathbb{F}^1(\mathcal{E}, a)/\mathfrak{S} \cong L(n) \oplus \mathfrak{so}_{n-1}, \tag{301}$$

hence $\ker\varphi_1 = \mathfrak{S} \subset \mathbb{F}^1(\mathcal{E}, a)$. According to Theorem 11, the ideal $\mathfrak{S}$ is abelian, so $[h_1, h_2] = 0$ for all $h_1, h_2 \in \ker\varphi_1 = \mathfrak{S}$. This proves (295) for $k = 1$.

Now let us prove (295) by induction on $k \geq 2$.

In the case $k = 2$, consider any elements

$$h_1, h_2, h_3, h_4 \in \ker\varphi_2 \subset \mathbb{F}^2(\mathcal{E}, a). \tag{302}$$

According to the definition of φ_k, the homomorphism φ_2 is the composition of the homomorphisms

$$\mathbb{F}^2(\mathcal{E}, a) \xrightarrow{\tau_2} \mathbb{F}^1(\mathcal{E}, a) \to \mathbb{F}^1(\mathcal{E}, a)/\mathfrak{S} \cong L(n) \oplus \mathfrak{so}_{n-1}.$$

Therefore, the condition $h_3, h_4 \in \ker\varphi_2$ means that $\tau_2(h_3), \tau_2(h_4) \in \mathfrak{S}$. Since $[\mathfrak{S}, \mathfrak{S}] = 0$, we have $\tau_2([h_3, h_4]) = 0$, so $[h_3, h_4] \in \ker\tau_2$.

From (300), (302) it follows that $h_1, h_2 \in \ker\sigma_2$. According to (299), since $h_2 \in \ker\sigma_2$ and $[h_3, h_4] \in \ker\tau_2$, we have $[h_2, [h_3, h_4]] \in \mathfrak{S}_2$. According to (297), since $h_1 \in \ker\sigma_2$ and $[h_2, [h_3, h_4]] \in \mathfrak{S}_2$, we get $[h_1, [h_2, [h_3, h_4]]] = 0$, which means that we have proved (295) for $k = 2$.

Let $r \geq 2$ be such that (295) is valid for $k = r$. Then for any elements

$$h_1, h_2, h_3, \ldots, h_{2r+2} \in \ker\varphi_{r+1} \subset \mathbb{F}^{r+1}(\mathcal{E}, a)$$

we have

$$\big[\tau_{r+1}(h_3), \big[\tau_{r+1}(h_4), \ldots, \big[\tau_{r+1}(h_{2r}), \big[\tau_{r+1}(h_{2r+1}), \tau_{r+1}(h_{2r+2})\big]\big]\ldots\big]\big] = 0, \tag{303}$$

because $\tau_{r+1}(h_i) \in \ker\varphi_r$ for $i = 3, 4, \ldots, 2r+2$. Equation (303) says that

$$[h_3, [h_4, \ldots, [h_{2r}, [h_{2r+1}, h_{2r+2}]]\ldots]] \in \ker\tau_{r+1}. \tag{304}$$

According to (300), since $h_2 \in \ker\varphi_{r+1}$, we have $h_2 \in \ker\sigma_{r+1}$. Combining this with (299) and (304), one obtains

$$[h_2, [h_3, [h_4, \ldots, [h_{2r}, [h_{2r+1}, h_{2r+2}]]\ldots]]] \in \mathfrak{S}_{r+1}. \tag{305}$$

According to (300), since $h_1 \in \ker\varphi_{r+1}$, we have $h_1 \in \ker\sigma_{r+1}$. Combining this with (297) and (305), one gets

$$[h_1, [h_2, [h_3, \ldots, [h_{2r}, [h_{2r+1}, h_{2r+2}]]\ldots]]] = 0.$$

Thus we have proved (295) for $k = r + 1$. Clearly, property (295) implies that $\ker\varphi_k$ is nilpotent. □

7. On zero-curvature representations for the multicomponent Landau-Lifshitz system

In this section we use the notions of reductions and direct sums of ZCRs. These notions have been introduced in Section 2.6.

Let $n \geq 4$. Recall that the n-component Landau-Lifshitz system (24) is transformed to the PDE (147) by means of the change of variables (146). In what follows we study ZCRs for this PDE.

Let $\mathfrak{Z}_1$ be the $L(n)$-valued ZCR defined in Remark 5. So the ZCR $\mathfrak{Z}_1$ is given by the functions (26), (27), which take values in the infinite-dimensional Lie algebra $L(n)$ introduced in Remark 5.

Let $\mathfrak{Z}_2$ be the $\mathfrak{so}_{n-1}$-valued ZCR defined in Theorem 10. We can consider also the direct sum $\mathfrak{Z}_1 \oplus \mathfrak{Z}_2$ of the ZCRs $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$. The ZCR $\mathfrak{Z}_1 \oplus \mathfrak{Z}_2$ takes values in the Lie algebra $L(n) \oplus \mathfrak{so}_{n-1}$.

For any given ZCR $\mathfrak{R}$ of the PDE (147), we are going to show that, after suitable gauge transformations and after killing some nilpotent ideal in the corresponding Lie algebra, the ZCR $\mathfrak{R}$ becomes isomorphic to a reduction of the ZCR $\mathfrak{Z}_1 \oplus \mathfrak{Z}_2$.

Let $\mathcal{E}$ be the infinite prolongation of the PDE (147). Fix a point $a \in \mathcal{E}$ given by (154). We are going to study ZCRs defined on a neighborhood of $a \in \mathcal{E}$.

Fix $p \in \mathbb{Z}_{\geq 0}$. In what follows, we study ZCRs of order $\leq p$. Without loss of generality, we can assume $p \geq 1$, because any ZCR of order ≤ 0 is at the same time of order ≤ 1.

Recall that the theory of Section 2 has been developed for an arbitrary m-component PDE (2) such that the right-hand side of (2) may depend on x, t, u^j, u^j_k for $k \leq d$. Since we study now the $(n-1)$-component PDE (147) of order 3, we will use the theory of Section 2 for $m = n-1$ and $d = 3$.

According to (94), (95), one has the power series $\mathbb{A} = \mathbb{A}(x, t, u^j_0, u^j_1, \ldots, u^j_p)$, $\mathbb{B} = \mathbb{B}(x, t, u^j_0, u^j_1, \ldots, u^j_{p+2})$ with coefficients in $\mathbb{F}^p(\mathcal{E}, a)$ satisfying (96), so $\mathbb{A}$, $\mathbb{B}$ constitute a formal ZCR with coefficients in $\mathbb{F}^p(\mathcal{E}, a)$.

According to Theorem 15, we have the surjective homomorphism

$$\varphi_p \colon \mathbb{F}^p(\mathcal{E}, a) \to L(n) \oplus \mathfrak{so}_{n-1} \tag{306}$$

such that $\ker \varphi_p$ is a nilpotent ideal of $\mathbb{F}^p(\mathcal{E}, a)$. Then

$$\check{\mathbb{A}} = \varphi_p(\mathbb{A}), \qquad \check{\mathbb{B}} = \varphi_p(\mathbb{B}) \tag{307}$$

are power series with coefficients in $L(n) \oplus \mathfrak{so}_{n-1}$ and constitute a formal ZCR with coefficients in $L(n) \oplus \mathfrak{so}_{n-1}$.

According to Remark 22, the $\big(L(n) \oplus \mathfrak{so}_{n-1}\big)$-valued ZCR $\mathfrak{Z}_1 \oplus \mathfrak{Z}_2$ can be regarded also as a formal ZCR with coefficients in $L(n) \oplus \mathfrak{so}_{n-1}$.

Recall that in Remark 25 we have defined formal gauge transformations and the notion of gauge equivalence for formal ZCRs with coefficients in arbitrary (possibly infinite-dimensional) Lie algebras. The results of Sections 5, 6 imply that the formal ZCR $\check{\mathbb{A}}$, $\check{\mathbb{B}}$ with coefficients in $L(n) \oplus \mathfrak{so}_{n-1}$ is gauge equivalent to the formal ZCR $\mathfrak{Z}_1 \oplus \mathfrak{Z}_2$.

Let $\mathfrak{g} \subset \mathfrak{gl}_N$ be a matrix Lie algebra. Let $\mathfrak{R}$ be a $\mathfrak{g}$-valued ZCR of order $\leq p$. So the ZCR $\mathfrak{R}$ is given by $\mathfrak{g}$-valued functions $A = A(x, t, u^j_0, u^j_1, \ldots, u^j_p)$, $B = B(x, t, u^j_0, u^j_1, \ldots, u^j_{p+2})$ satisfying $D_x(B) - D_t(A) + [A, B] = 0$.

According to Theorem 6, there is a homomorphism $\mu \colon \mathbb{F}^p(\mathcal{E}, a) \to \mathfrak{g}$ such that the ZCR A, B is gauge equivalent to the $\mu\big(\mathbb{F}^p(\mathcal{E}, a)\big)$-valued ZCR

$$\tilde{A} = \mu(\mathbb{A}), \qquad \tilde{B} = \mu(\mathbb{B}). \tag{308}$$

Here $\tilde{A}$, $\tilde{B}$ are $\mu\big(\mathbb{F}^p(\mathcal{E}, a)\big)$-valued functions, and formulas (308) mean that μ maps the coefficients of the power series $\mathbb{A}$, $\mathbb{B}$ to the corresponding coefficients of the Taylor series of the functions $\tilde{A}$, $\tilde{B}$. The ZCR $\tilde{A}$, $\tilde{B}$ is a-normal.

Recall that $\ker\varphi_p$ is a nilpotent ideal of $\mathbb{F}^p(\mathcal{E},a)$. This implies that $\mu\big(\ker\varphi_p\big)$ is a nilpotent ideal of the Lie subalgebra $\mu\big(\mathbb{F}^p(\mathcal{E},a)\big)\subset\mathfrak{g}$.

Since the homomorphism (306) is surjective, the algebra $L(n)\oplus\mathfrak{so}_{n-1}$ is isomorphic to $\mathbb{F}^p(\mathcal{E},a)/\ker\varphi_p$, and the surjective homomorphism $\mu\colon\mathbb{F}^p(\mathcal{E},a)\to\mu\big(\mathbb{F}^p(\mathcal{E},a)\big)$ induces a surjective homomorphism

$$\hat\mu\colon L(n)\oplus\mathfrak{so}_{n-1}\to\mu\big(\mathbb{F}^p(\mathcal{E},a)\big)/\mu\big(\ker\varphi_p\big). \tag{309}$$

Consider the natural surjective homomorphism

$$\psi\colon\mu\big(\mathbb{F}^p(\mathcal{E},a)\big)\to\mu\big(\mathbb{F}^p(\mathcal{E},a)\big)/\mu\big(\ker\varphi_p\big). \tag{310}$$

Using this homomorphism and the $\mu\big(\mathbb{F}^p(\mathcal{E},a)\big)$-valued ZCR (308), we see that the functions $\psi(\tilde A)$, $\psi(\tilde B)$ form a ZCR with values in the Lie algebra $\mu\big(\mathbb{F}^p(\mathcal{E},a)\big)/\mu\big(\ker\varphi_p\big)$.

Formulas (307), (308) and the definition of the homomorphism (309) imply the following

$$\psi(\tilde A)=\hat\mu\big(\check{\mathbb{A}}\big),\qquad\qquad\psi(\tilde B)=\hat\mu\big(\check{\mathbb{B}}\big). \tag{311}$$

Equations (311) say that the ZCR $\psi(\tilde A)$, $\psi(\tilde B)$ is a reduction of the ZCR $\check{\mathbb{A}}$, $\check{\mathbb{B}}$, which is gauge equivalent to the ZCR $\mathfrak{Z}_1\oplus\mathfrak{Z}_2$, as has been discussed above.

Recall that two ZCRs are called gauge equivalent if one is obtained from the other by means of a gauge transformation. Thus, starting from an arbitrary ZCR $\mathfrak{R}$ given by functions A, B, we have done the following steps:

(1) Using Theorem 6, we have obtained a homomorphism $\mu\colon\mathbb{F}^p(\mathcal{E},a)\to\mathfrak{g}$ and a gauge equivalent ZCR $\tilde A$, $\tilde B$ satisfying (308). The ZCR $\tilde A$, $\tilde B$ takes values in the Lie subalgebra $\mu\big(\mathbb{F}^p(\mathcal{E},a)\big)\subset\mathfrak{g}$ and is obtained from the ZCR A, B by means of a gauge transformation.
(2) Killing the nilpotent ideal $\mu\big(\ker\varphi_p\big)\subset\mu\big(\mathbb{F}^p(\mathcal{E},a)\big)$, we have obtained the ZCR $\psi(\tilde A)$, $\psi(\tilde B)$, where ψ is defined by (310).
(3) We have shown that the ZCR $\psi(\tilde A)$, $\psi(\tilde B)$ is a reduction of the ZCR $\check{\mathbb{A}}$, $\check{\mathbb{B}}$, which is gauge equivalent to the ZCR $\mathfrak{Z}_1\oplus\mathfrak{Z}_2$. The ZCR $\check{\mathbb{A}}$, $\check{\mathbb{B}}$ is obtained from the ZCR $\mathfrak{Z}_1\oplus\mathfrak{Z}_2$ by means of a gauge transformation.

Thus, for any given ZCR $\mathfrak{R}$, we have shown that, after suitable gauge transformations and after killing some nilpotent ideal in the corresponding Lie algebra, the ZCR $\mathfrak{R}$ becomes isomorphic to a reduction of the ZCR $\mathfrak{Z}_1\oplus\mathfrak{Z}_2$.

Remark 30. One has also a similar result in the case when a ZCR $\mathfrak{R}$ takes values in a matrix Lie algebra $\mathfrak{g}$ and depends on a parameter λ.

Suppose that the ZCR $\mathfrak{R}$ is given by functions

$$A=A(\lambda,x,t,u_0^j,u_1^j,\dots,u_p^j),\quad B=B(\lambda,x,t,u_0^j,u_1^j,\dots,u_{p+2}^j),\quad D_x(B)-D_t(A)+[A,B]=0, \tag{312}$$

where $\mathfrak{g}$-valued functions A, B depend on x, t, u_k^i and a parameter λ.

Let $\tilde{\mathfrak{g}}$ be the infinite-dimensional Lie algebra of functions $h(\lambda)$ with values in $\mathfrak{g}$. (Depending on the problem under study, one can consider analytic or meromorphic functions $h(\lambda)$. Or one can assume that λ runs through an open subset of some algebraic curve and consider $\mathfrak{g}$-valued functions $h(\lambda)$ on this algebraic curve.)

Then (312) can be regarded as a ZCR with values in $\tilde{\mathfrak{g}}$. Using parameter-dependent versions of Theorems 3, 6, one can show that the ZCR (312) is gauge equivalent to an a-normal ZCR

$$\tilde A=\tilde A(\lambda,x,t,u_0^j,u_1^j,\dots,u_p^j),\quad \tilde B=\tilde B(\lambda,x,t,u_0^j,u_1^j,\dots,u_{p+2}^j),\quad D_x(\tilde B)-D_t(\tilde A)+[\tilde A,\tilde B]=0,$$

satisfying $\tilde A=\mu(\mathbb{A})$, $\tilde B=\mu(\mathbb{B})$ for some homomorphism $\mu\colon\mathbb{F}^p(\mathcal{E},a)\to\tilde{\mathfrak{g}}$. The functions $\tilde A$, $\tilde B$ take values in the Lie subalgebra $\mu\big(\mathbb{F}^p(\mathcal{E},a)\big)\subset\tilde{\mathfrak{g}}$.

Then one can kill the nilpotent ideal $\mu\big(\ker\varphi_p\big)\subset\mu\big(\mathbb{F}^p(\mathcal{E},a)\big)$ and proceed similarly to the steps described above.

8. The algebras $\mathbb{F}^p(\mathcal{E},a)$ for the classical Landau-Lifshitz and nonlinear Schrödinger equations

In this section we assume $\mathbb{K}=\mathbb{C}$ and study the algebras $\mathbb{F}^p(\mathcal{E},a)$ for the classical Landau-Lifshitz and nonlinear Schrödinger equations.

The classical Landau-Lifshitz equation reads

$$S_t = S\times S_{xx}+S\times JS, \qquad S=\big(S^1(x,t),S^2(x,t),S^3(x,t)\big), \qquad (S^1)^2+(S^2)^2+(S^3)^2=1, \tag{313}$$

where $J=\operatorname{diag}(j_1,j_2,j_3)$ is a constant diagonal (3×3)-matrix with $j_1,j_2,j_3\in\mathbb{C}$, and the symbol $\times$ denotes the vector product. We consider the *fully anisotropic* case $j_1\neq j_2\neq j_3\neq j_1$.

Let $\mathcal{E}$ be the infinite prolongation of the PDE (313). Let $a\in\mathcal{E}$. We are going to describe the algebras $\mathbb{F}^p(\mathcal{E},a)$ for this PDE.

Remark 31. Strictly speaking, in order to define $\mathcal{E}$ and $\mathbb{F}^p(\mathcal{E},a)$ for the PDE (313), we need to resolve the constraint $(S^1)^2+(S^2)^2+(S^3)^2=1$ and to rewrite (313) as a 2-component evolution PDE. For example, Roelofs and Martini [16] use the spherical coordinates

$$S^1=\cos v\sin u, \qquad S^2=\sin v\sin u, \qquad S^3=\cos u. \tag{314}$$

According to [16], using the transformation (314), one can rewrite the PDE (313) as the 2-component evolution PDE

$$\begin{aligned} u_t &= -(\sin u)v_{xx}-2(\cos u)u_xv_x+(j_1-j_2)\sin u\cos v\sin v,\\ v_t &= \frac{1}{\sin u}u_{xx}-(\cos u)v_x^2+(\cos u)(j_1\cos^2 v+j_2\sin^2 v-j_3). \end{aligned} \tag{315}$$

One can use also some other way to resolve the constraint $(S^1)^2+(S^2)^2+(S^3)^2=1$ and to rewrite (313) as a 2-component evolution PDE, then $\mathcal{E}$ and $\mathbb{F}^p(\mathcal{E},a)$ will be the same (up to isomorphism).

We need some auxiliary constructions. Let $\mathbb{C}[v_1,v_2,v_3]$ be the algebra of polynomials in the variables v_1, v_2, v_3. Recall that, by our assumption, the constants $j_1,j_2,j_3\in\mathbb{C}$ satisfy $j_1\neq j_2\neq j_3\neq j_1$. Consider the ideal $\mathcal{I}_{j_1,j_2,j_3}\subset\mathbb{C}[v_1,v_2,v_3]$ generated by the polynomials

$$v_\alpha^2-v_\beta^2+j_\alpha-j_\beta, \qquad \alpha,\,\beta=1,2,3. \tag{316}$$

Set $E_{j_1,j_2,j_3}=\mathbb{C}[v_1,v_2,v_3]/\mathcal{I}_{j_1,j_2,j_3}$. In other words, E_{j_1,j_2,j_3} is the commutative associative algebra of polynomial functions on the algebraic curve in $\mathbb{C}^3$ defined by the polynomials (316).

Since we assume $j_1\neq j_2\neq j_3\neq j_1$, this curve is nonsingular and is of genus 1, so this is an elliptic curve. It is well known that the Landau-Lifshitz equation (313) possesses an $\mathfrak{so}_3(\mathbb{C})$-valued ZCR parametrized by points of this curve [21, 5].

We have the natural surjective homomorphism $\mu\colon\mathbb{C}[v_1,v_2,v_3]\to\mathbb{C}[v_1,v_2,v_3]/\mathcal{I}_{j_1,j_2,j_3}=E_{j_1,j_2,j_3}$. Set $\bar v_i=\mu(v_i)\in E_{j_1,j_2,j_3}$ for $i=1,2,3$.

Consider also a basis x_1, x_2, x_3 of the Lie algebra $\mathfrak{so}_3(\mathbb{C})$ such that $[x_1,x_2]=x_3$, $[x_2,x_3]=x_1$, $[x_3,x_1]=x_2$. We endow the space $\mathfrak{so}_3(\mathbb{C})\otimes_{\mathbb{C}}E_{j_1,j_2,j_3}$ with the following Lie algebra structure

$$[z_1\otimes h_1,\ z_2\otimes h_2]=[z_1,z_2]\otimes h_1h_2, \qquad z_1,z_2\in\mathfrak{so}_3(\mathbb{C}), \qquad h_1,h_2\in E_{j_1,j_2,j_3}.$$

Denote by $\mathfrak{R}_{j_1,j_2,j_3}$ the Lie subalgebra of $\mathfrak{so}_3(\mathbb{C})\otimes_{\mathbb{C}}E_{j_1,j_2,j_3}$ generated by the elements

$$x_i\otimes\bar v_i\ \in\ \mathfrak{so}_3(\mathbb{C})\otimes_{\mathbb{C}}E_{j_1,j_2,j_3}, \qquad i=1,2,3.$$

Since $\mathfrak{R}_{j_1,j_2,j_3}\subset\mathfrak{so}_3(\mathbb{C})\otimes_{\mathbb{C}}E_{j_1,j_2,j_3}$, we can regard elements of $\mathfrak{R}_{j_1,j_2,j_3}$ as $\mathfrak{so}_3(\mathbb{C})$-valued functions on the elliptic curve in $\mathbb{C}^3$ determined by the polynomials (316). The paper [16] describes a basis for $\mathfrak{R}_{j_1,j_2,j_3}$, which implies that the algebra $\mathfrak{R}_{j_1,j_2,j_3}$ is infinite-dimensional.

Theorem 16. *Recall that $J=\operatorname{diag}(j_1,j_2,j_3)$ in (313) is a constant diagonal (3×3)-matrix with $j_1,j_2,j_3\in\mathbb{C}$. We consider the case $j_1\neq j_2\neq j_3\neq j_1$.*

Let $\mathcal{E}$ be the infinite prolongation of the Landau-Lifshitz equation (313). Let $a\in\mathcal{E}$.

The Lie algebras $\mathbb{F}^p(\mathcal{E},a)$, $p\in\mathbb{Z}_{\geq 0}$, for this PDE have the following structure.

The algebra $\mathbb{F}^0(\mathcal{E},a)$ is isomorphic to $\mathfrak{R}_{j_1,j_2,j_3}$.

For each $p\in\mathbb{Z}_{\geq 0}$, there are an abelian ideal $\mathcal{I}_{p+1}$ of the Lie algebra $\mathbb{F}^{p+1}(\mathcal{E},a)$ and an abelian ideal $\mathcal{J}_{p+1}$ of the quotient Lie algebra $\mathbb{F}^{p+1}(\mathcal{E},a)/\mathcal{I}_{p+1}$ such that

- $\big[\mathcal{I}_{p+1},\,\mathbb{F}^{p+1}(\mathcal{E},a)\big]=0$, *so $\mathcal{I}_{p+1}$ lies in the center of the Lie algebra $\mathbb{F}^{p+1}(\mathcal{E},a)$,*
- $\big[\mathcal{J}_{p+1},\,\mathbb{F}^{p+1}(\mathcal{E},a)/\mathcal{I}_{p+1}\big]=0$, *so $\mathcal{J}_{p+1}$ lies in the center of the Lie algebra $\mathbb{F}^{p+1}(\mathcal{E},a)/\mathcal{I}_{p+1}$,*
- *the algebra $\mathbb{F}^p(\mathcal{E},a)$ is isomorphic to the quotient algebra $\big(\mathbb{F}^{p+1}(\mathcal{E},a)/\mathcal{I}_{p+1}\big)/\mathcal{J}_{p+1}$, and the surjective homomorphism $\mathbb{F}^{p+1}(\mathcal{E},a)\to\mathbb{F}^p(\mathcal{E},a)$ from (109) coincides with the composition of the natural homomorphisms*

$$\mathbb{F}^{p+1}(\mathcal{E},a)\to\mathbb{F}^{p+1}(\mathcal{E},a)/\mathcal{I}_{p+1}\to\big(\mathbb{F}^{p+1}(\mathcal{E},a)/\mathcal{I}_{p+1}\big)/\mathcal{J}_{p+1}\cong\mathbb{F}^p(\mathcal{E},a),$$

which means that the Lie algebra $\mathbb{F}^{p+1}(\mathcal{E},a)$ is obtained from the Lie algebra $\mathbb{F}^p(\mathcal{E},a)$ by applying two times the operation of central extension.

For each $q\in\mathbb{Z}_{>0}$, the Lie algebra $\mathbb{F}^q(\mathcal{E},a)$ is obtained from the Lie algebra $\mathbb{F}^0(\mathcal{E},a)$ by applying several times the operation of central extension. The kernel of the surjective homomorphism $\mathbb{F}^q(\mathcal{E},a)\to\mathbb{F}^0(\mathcal{E},a)$ from (109) is nilpotent.

Proof. Recall that, using the transformation (314), one can rewrite the PDE (313) in the form (315). In what follows, when we speak about the WE algebra and the algebras $\mathbb{F}^p(\mathcal{E},a)$ for the PDE (313), we assume that the PDE (313) is written in the form (315).

Let $\mathfrak{W}_a$ be the Wahlquist-Estabrook prolongation algebra (WE algebra) for the PDE (313). It is shown in [16] that this algebra is isomorphic to the direct sum of $\mathfrak{R}_{j_1,j_2,j_3}$ and the 2-dimensional abelian Lie algebra $\mathbb{C}^2$. So $\mathfrak{W}_a\cong\mathfrak{R}_{j_1,j_2,j_3}\oplus\mathbb{C}^2$.

In Section 3 for any evolution PDE of the form (120) we have defined the notion of formal Wahlquist-Estabrook ZCR, which is given by formulas (121), (122). According to [16], in the formal Wahlquist-Estabrook ZCR with coefficients in $\mathfrak{W}_a$ for the PDE (313) one has

$$\mathrm{A}=p_1S^1+p_2S^2+p_3S^3+p_4,\qquad p_1,p_2,p_3,p_4\in\mathfrak{W}_a\cong\mathfrak{R}_{j_1,j_2,j_3}\oplus\mathbb{C}^2,$$

where S^1, S^2, S^3 are given by (314), the elements p_1, p_2, p_3 generate the Lie subalgebra $\mathfrak{R}_{j_1,j_2,j_3}\subset\mathfrak{W}_a\cong\mathfrak{R}_{j_1,j_2,j_3}\oplus\mathbb{C}^2$, and one has $[p_4,p_l]=0$ for all $l=1,2,3$.

This implies that the subalgebra $\mathfrak{R}\subset\mathfrak{W}_a$ defined in Theorem 8 in Section 3 is equal to $\mathfrak{R}_{j_1,j_2,j_3}\subset\mathfrak{W}_a$. According to Theorem 8, one has $\mathbb{F}^0(\mathcal{E},a)\cong\mathfrak{R}$. Since in the considered case we have $\mathfrak{R}=\mathfrak{R}_{j_1,j_2,j_3}$, we see that $\mathbb{F}^0(\mathcal{E},a)$ is isomorphic to $\mathfrak{R}_{j_1,j_2,j_3}$.

It remains to prove the statements about $\mathbb{F}^{p+1}(\mathcal{E},a)$, $\mathcal{I}_{p+1}$, $\mathcal{J}_{p+1}$, $\mathbb{F}^p(\mathcal{E},a)$, $\mathbb{F}^q(\mathcal{E},a)$, $\mathbb{F}^q(\mathcal{E},a)\to\mathbb{F}^0(\mathcal{E},a)$ in Theorem 16.

The statements about $\mathbb{F}^{p+1}(\mathcal{E},a)$, $\mathcal{I}_{p+1}$, $\mathcal{J}_{p+1}$, $\mathbb{F}^p(\mathcal{E},a)$ can be proved similarly to the results of Sections 5, 6.

The homomorphism $\mathbb{F}^q(\mathcal{E},a)\to\mathbb{F}^0(\mathcal{E},a)$ is equal to the composition of the surjective homomorphisms

$$\mathbb{F}^q(\mathcal{E},a)\to\mathbb{F}^{q-1}(\mathcal{E},a)\to\cdots\to\mathbb{F}^1(\mathcal{E},a)\to\mathbb{F}^0(\mathcal{E},a)$$

from (109). It is easily seen that the statements about $\mathbb{F}^{p+1}(\mathcal{E},a)$, $\mathcal{I}_{p+1}$, $\mathcal{J}_{p+1}$, $\mathbb{F}^p(\mathcal{E},a)$ imply the statements about $\mathbb{F}^q(\mathcal{E},a)$ and $\mathbb{F}^q(\mathcal{E},a)\to\mathbb{F}^0(\mathcal{E},a)$. □

The nonlinear Schrödinger (NLS) equation is another well-known PDE from mathematical physics (see, e.g, [5]). It can be written as follows

$$i\psi_t+\psi_{xx}-\kappa\bar{\psi}\psi^2=0,\qquad i=\sqrt{-1}\in\mathbb{C},\tag{317}$$

where $\kappa\in\mathbb{R}$ is a nonzero constant. The function $\psi=\psi(x,t)$ takes values in $\mathbb{C}$, and one has

$$\psi=u^1+iu^2,\qquad\bar{\psi}=u^1-iu^2,\qquad i=\sqrt{-1}\in\mathbb{C},\tag{318}$$

for some $\mathbb{R}$-valued functions $u^1=u^1(x,t)$ and $u^2=u^2(x,t)$.

Taking into account (318), we see that the NLS equation (317) is equivalent to the 2-component evolution PDE

$$\begin{aligned} u^1_t &= -u^2_{xx} + \kappa u^2\big((u^1)^2 + (u^2)^2\big),\\ u^2_t &= u^1_{xx} - \kappa u^1\big((u^1)^2 + (u^2)^2\big). \end{aligned} \tag{319}$$

In what follows, when we speak about the NLS equation, we mean the PDE (319).

So we will study the evolution PDE (319), where κ is a nonzero constant. Since in this section we work over $\mathbb{C}$, we assume that u^1, u^2 in (319) take values in $\mathbb{C}$.

Let $\mathbb{C}[\lambda]$ be the algebra of polynomials in the variable λ. Consider the infinite-dimensional Lie algebra

$$\mathfrak{sl}_2(\mathbb{C}[\lambda]) = \mathfrak{sl}_2(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\lambda].$$

Theorem 17. *Let $\mathcal{E}$ be the infinite prolongation of the NLS equation* (319)*. Let $a \in \mathcal{E}$.*

The Lie algebras $\mathbb{F}^p(\mathcal{E},a)$, $p \in \mathbb{Z}_{\geq 0}$, for this PDE have the following structure.

The algebra $\mathbb{F}^0(\mathcal{E},a)$ is isomorphic to the direct sum of $\mathfrak{sl}_2(\mathbb{C}[\lambda])$ and a one-dimensional abelian Lie algebra.

For each $p \in \mathbb{Z}_{\geq 0}$, there are an abelian ideal $\mathcal{I}_{p+1}$ of the Lie algebra $\mathbb{F}^{p+1}(\mathcal{E},a)$ and an abelian ideal $\mathcal{J}_{p+1}$ of the quotient Lie algebra $\mathbb{F}^{p+1}(\mathcal{E},a)/\mathcal{I}_{p+1}$ such that

- $\big[\mathcal{I}_{p+1},\, \mathbb{F}^{p+1}(\mathcal{E},a)\big] = 0$*, so $\mathcal{I}_{p+1}$ lies in the center of the Lie algebra $\mathbb{F}^{p+1}(\mathcal{E},a)$,*
- $\big[\mathcal{J}_{p+1},\, \mathbb{F}^{p+1}(\mathcal{E},a)/\mathcal{I}_{p+1}\big] = 0$*, so $\mathcal{J}_{p+1}$ lies in the center of the Lie algebra $\mathbb{F}^{p+1}(\mathcal{E},a)/\mathcal{I}_{p+1}$,*
- *the algebra $\mathbb{F}^p(\mathcal{E},a)$ is isomorphic to the quotient algebra $\big(\mathbb{F}^{p+1}(\mathcal{E},a)/\mathcal{I}_{p+1}\big)/\mathcal{J}_{p+1}$, and the surjective homomorphism $\mathbb{F}^{p+1}(\mathcal{E},a) \to \mathbb{F}^p(\mathcal{E},a)$ from* (109) *coincides with the composition of the natural homomorphisms*

$$\mathbb{F}^{p+1}(\mathcal{E},a) \to \mathbb{F}^{p+1}(\mathcal{E},a)/\mathcal{I}_{p+1} \to \big(\mathbb{F}^{p+1}(\mathcal{E},a)/\mathcal{I}_{p+1}\big)/\mathcal{J}_{p+1} \cong \mathbb{F}^p(\mathcal{E},a),$$

which means that the Lie algebra $\mathbb{F}^{p+1}(\mathcal{E},a)$ is obtained from the Lie algebra $\mathbb{F}^p(\mathcal{E},a)$ by applying two times the operation of central extension.

For each $q \in \mathbb{Z}_{>0}$, the Lie algebra $\mathbb{F}^q(\mathcal{E},a)$ is obtained from the Lie algebra $\mathbb{F}^0(\mathcal{E},a)$ by applying several times the operation of central extension. The kernel of the surjective homomorphism $\mathbb{F}^q(\mathcal{E},a) \to \mathbb{F}^0(\mathcal{E},a)$ from (109) *is nilpotent.*

Proof. Let $\mathfrak{W}_a$ be the Wahlquist-Estabrook prolongation algebra (WE algebra) for the NLS equation (319). It is shown in [3, 4] that this algebra is isomorphic to the direct sum of $\mathfrak{sl}_2(\mathbb{C}[\lambda])$ and a 3-dimensional abelian Lie algebra.

To describe $\mathbb{F}^0(\mathcal{E},a)$, we can use again Theorem 8, which allows us to describe $\mathbb{F}^0(\mathcal{E},a)$ as a certain subalgebra of $\mathfrak{W}_a$. Applying Theorem 8 to the description of the Wahlquist-Estabrook prolongation algebra $\mathfrak{W}_a$ in [3, 4], one obtains that $\mathbb{F}^0(\mathcal{E},a)$ is isomorphic to the direct sum of $\mathfrak{sl}_2(\mathbb{C}[\lambda])$ and a one-dimensional abelian Lie algebra.

It remains to prove the statements about $\mathbb{F}^{p+1}(\mathcal{E},a)$, $\mathcal{I}_{p+1}$, $\mathcal{J}_{p+1}$, $\mathbb{F}^p(\mathcal{E},a)$, $\mathbb{F}^q(\mathcal{E},a)$, $\mathbb{F}^q(\mathcal{E},a) \to \mathbb{F}^0(\mathcal{E},a)$ in Theorem 17.

The statements about $\mathbb{F}^{p+1}(\mathcal{E},a)$, $\mathcal{I}_{p+1}$, $\mathcal{J}_{p+1}$, $\mathbb{F}^p(\mathcal{E},a)$ can be proved similarly to the results of Sections 5, 6.

The homomorphism $\mathbb{F}^q(\mathcal{E},a) \to \mathbb{F}^0(\mathcal{E},a)$ is equal to the composition of the surjective homomorphisms

$$\mathbb{F}^q(\mathcal{E},a) \to \mathbb{F}^{q-1}(\mathcal{E},a) \to \cdots \to \mathbb{F}^1(\mathcal{E},a) \to \mathbb{F}^0(\mathcal{E},a)$$

from (109). It is easily seen that the statements about $\mathbb{F}^{p+1}(\mathcal{E},a)$, $\mathcal{I}_{p+1}$, $\mathcal{J}_{p+1}$, $\mathbb{F}^p(\mathcal{E},a)$ imply the statements about $\mathbb{F}^q(\mathcal{E},a)$ and $\mathbb{F}^q(\mathcal{E},a) \to \mathbb{F}^0(\mathcal{E},a)$. □

Acknowledgments

Sergei Igonin is a research fellow of Istituto Nazionale di Alta Matematica (INdAM), Italy. Sergei Igonin and Gianni Manno are members of GNSAGA of INdAM.

The authors thank T. Skrypnyk and V. V. Sokolov for useful discussions.

Sergei Igonin is grateful to the Max Planck Institute for Mathematics (Bonn, Germany) for its hospitality and excellent working conditions during 02.2006–01.2007 and 06.2010–09.2010, when part of this research was done.

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