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Original
Threshold models of cascades in large-scale networks / Rossi, WILBERT SAMUEL; Como, Giacomo; Fagnani, Fabio. - In: IEEE TRANSACTIONS ON NETWORK SCIENCE AND ENGINEERING. - ISSN 2327-4697. - STAMPA. - (2017).

Availability:
This version is available at: 11583/2693627 since: 2017-11-24T15:43:43Z

Publisher:
IEEE

Published
DOI:10.1109/TNSE.2017.2777941

(Article begins on next page)

Threshold Models of Cascades in Large-Scale Networks

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Abstract—The spread of new beliefs, behaviors, conventions, norms, and technologies in social and economic networks are often driven by cascading mechanisms, and so are contagion dynamics in financial networks. Global behaviors generally emerge from the interplay between the structure of the interconnection topology and the local agents' interactions. We focus on the Threshold Model (TM) of cascades first introduced by Granovetter (1978). This can be interpreted as the best response dynamics in a network game whereby agents choose strategically between two actions. Each agent is equipped with an individual threshold representing the number of her neighbors who must have adopted a certain action for that to become the agent's best response. We analyze the TM dynamics on large-scale networks with heterogeneous agents. Through a local mean-field approach, we obtain a nonlinear, one-dimensional, recursive equation that approximates the evolution of the TM dynamics on most of the networks of a given size and distribution of degrees and thresholds. We prove that, on all but a fraction of networks with given degree and threshold statistics that is vanishing as the network size grows large, the actual fraction of adopters of a given action is arbitrarily close to the output of the aforementioned recursion. Numerical simulations on some real network testbeds show good adherence to the theoretical predictions.

Index Terms—cascades; social networks; threshold model; coordination game; best response; random graphs; local mean-field.

1 Introduction

Cascading phenomena permeate the dynamics of social and economic networks. Notable examples are the adoption of new technologies and social norms, the spread of fads and behaviors, participation to riots [1], [2]. Such phenomena have been largely recognized to spread through networks of individual interactions [1], [3], [4]. However, in contrast to standard network epidemic models based on pairwise contact mechanisms [5], [6] —whereby diffusion of a new state occurs independently on the links among the agents— complex neighborhood effects —whereby the propensity of an agent to adopt a new state grows nonlinearly with the fraction of adopters among her neighbors— play a central role in the mechanisms underlying such cascading phenomena [7], [8], [9].

One of the most studied models of cascading mechanisms capturing such complex neighborhood effects is the Threshold Model (TM) of [1]. The original work of Granovetter [1] is concerned with a fully mixed population of n interacting agents, each holding a binary state $Z_i(t) = 0, 1$, for $i = 1, \dots, n$, and updating it at every discrete time instant $t = 0, 1, \dots$ according to the following threshold rule: $Z_i(t + 1) = 1$ if the current fraction of state-1 adopters in the population is not less than a certain value Θ_i, i.e., if $\frac{1}{n}\sum_{j=1}^{n} Z_j(t) \geq \Theta_i$ and $Z_i(t + 1) = 0$ otherwise, i.e., if $\frac{1}{n}\sum_{j=1}^{n} Z_j(t) < \Theta_i$. Here $\Theta_i \in [0, 1]$ is a normalized threshold value that measures the reluctance of agent i in choosing state 1, equivalently, her propensity to choose state 0. In more realistic scenarios, the population is not fully mixed and agents interact on an interconnection network that can be represented as a, generally directed, graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ whose node set $\mathcal{V} = \{1, 2, \dots, n\}$ is identified with the set of agents themselves and where the presence of a link $(i, j) \in \mathcal{E}$ represents the fact that agent i observes agent j and gets directly influenced by her state. In this setting, the TM dynamics reads as follows:

$$Z_i(t+1) = \begin{cases} 1 & \text{if} \quad \sum_{j:(i,j)\in\mathcal{E}} Z_j(t) \geq \Theta_i k_i \\ 0 & \text{if} \quad \sum_{j:(i,j)\in\mathcal{E}} Z_j(t) < \Theta_i k_i \,, \end{cases} \tag{1}$$

where k_i stands for node i's out-degree, see, e.g., [10], [11]. This can be interpreted as the best response dynamics in a network game whereby agents choose strategically between two actions, 0 and 1, and their payoff is an increasing function of the number of their neighbors choosing the same action. A variant of the TM, that is referred to as Progressive Threshold Model (PTM) or as Bootstrap Percolation in statistical physics, allows for state transitions from 0 to 1 only, but not from 1 to 0, so that when an agent adopts state 1, she keeps it ever after [12], [13], [14], [15].

As illustrated by [1], there is a simple way to analyze the TM in fully mixed populations. If one denotes by $z(t) := \frac{1}{n}\sum_i Z_i(t)$ the fraction of state-1 adopters at time t, and if $F(\theta) := \frac{1}{n}|\{i : \Theta_i \leq \theta\}|$, for $0 \leq \theta \leq 1$, stands for the cumulative distribution function of the normalized thresholds, then

$$z(t+1) = F(z(t))\,, \qquad t \geq 0\,. \tag{2}$$

Hence, the evolution of the fraction of state-1 adopters in the population can be determined by the above one-dimesional non-linear recursion. This is a dramatic reduction of complexity with respect to the original TM dynamics whose discrete state space has cardinality 2^n growing exponentially fast in the population size. In fact, an analogous result can be

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verified to hold true for the PTM, provided that agents with initial state $Z_i(0) = 1$ are considered as if having threshold 0, which is consistent with the fact they will always keep their state equal to 1. More precisely, if one introduces the distribution function $\tilde{F}(\theta) = \frac{1}{n}|\{i : \Theta_i(1 - Z_i(0)) \leq \theta\}|$ then the fraction $z(t)$ of state-1 adopters in the PTM satisfies the recursion[1] $z(t+1) = \tilde{F}(z(t))$.

In the more complex case where the population is not fully mixed but rather interacts along a given graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the simple recursion (2) does not hold true any longer for the fraction of state-1 adopters $z(t)$ in the TM (1). In fact, for undirected (possibly infinite) graphs $\mathcal{G}$ and homogeneous normalized thresholds $\Theta_i = \theta$, [10] characterizes the fixed points of the TM dynamics as those configurations in $\{0,1\}^n$ whose support $\mathcal{U} \subseteq \mathcal{V}$ is a θ-cohesive subset of $\mathcal{V}$ with $(1-\theta)$-cohesive complement $\mathcal{V} \setminus \mathcal{U}$, meaning that all nodes in $\mathcal{U}$ have at least a fraction θ of neighbors in $\mathcal{U}$ and all nodes in $\mathcal{V} \setminus \mathcal{U}$ have less than a fraction θ of neighbors in $\mathcal{U}$. While such a characterization provides fundamental insight into the structure of the equilibria of the TM, finding θ-cohesive subsets of nodes with $(1-\theta)$-cohesive complement in an arbitrary graph $\mathcal{G}$ is a computationally hard problem. Computational complexity issues also arise in the PTM dynamics, for which, e.g., [12] prove NP-hardness of the selection problem of the k 'most influential' nodes, i.e., the choice of the cardinality-k subset of nodes that, if initiated as state-1 adopters, lead to the largest set of final state-1 adopters. Building on submodularity properties of the number of final state-1 adopters as a function of the set of initial state-1 adopters, provable approximation guarantees are then provided by [12] for the k 'most influential' nodes selection problem. Such influence maximization problem has attracted a large amount of attention recently, see, e.g., [16], [17] and has also been tackled in the statistical physics literature [18], [19], [20]. Asymptotic analysis of the TM dynamics and associated complexity issues have also been addressed by [11].

As the aforementioned results point out, analysis and optimization of the TM and of the PTM on general networks is typically a hard problem. On the other hand, in practical large-scale applications, complete information on the network structure and on the specific threshold configuration might not be available, while only aggregate statistics such as degree and threshold distributions might be known. With this motivation in mind, the present paper deals with the analysis of the TM and of the PTM dynamics on the *ensemble* of all graphs with a given joint degree/threshold distribution (formally we will consider the so-called *configuration model* of interconnections, cf. [6], [21]), rather than on a specific graph $\mathcal{G}$. Our main result shows that for all but a vanishingly small (as the network size n grows large) fraction of networks from the configuration model ensemble of given joint degree-threshold distribution, the fraction $z(t)$ of state-1 adopters in the TM dynamics can be approximated, to an arbitrary small tolerance level, by the solution of the recursion

$$x(t+1) = \phi(x(t)), \qquad y(t+1) = \psi(x(t)), \tag{3}$$

where $\phi(x)$ and $\psi(x)$ are suitably defined polynomial functions that map the interval $[0,1]$ in itself, whose form depends only on the joint degree-threshold distribution (see (13) and (14)). An analogous result for the PTM is proved as well, provided that agents with initial state $Z_i(0) = 1$ are treated as if having threshold 0, equivalently, that the functions $\phi(x)$ and $\psi(x)$ are defined based on the joint distribution of node degrees and the product $(1 - \Theta_i)Z_i(0)$.

Our results should be compared to the literature on the analysis of the TM or the PTM on large-scale random networks with given degree distribution. [13] and [22], [23] study the asymptotic behavior of the PTM in random undirected networks. In particular, the paper [22] focuses on the asymptotic effect of two vaccination strategies equivalent to the *a priori* removal of nodes, whereas the papers [13] and [23] both rigorously provide conditions, in the large-scale limit, for the PTM contagion to eventually reach a sizeable fraction of nodes when started from a single node or a fraction of nodes that is sublinear in n. [24] present analogous results for a version of the PTM on random weighted directed networks, proposed as a model for cascading failures in financial networks. Building on the approach of [13], [25] rigorously investigates how the the presence of tighter communities in the random network affects the extension of the final PTM contagion. In contrast with those results, ours are concerned with approximation of the dynamics rather than with the asymptotics of the fraction of state-1 adopters. For the PTM, our recursive equations are similar to the generating function approach of [15], that is strictly accurate on tree structure and gives a reasonably well approximation on networks without dense loops. Our major contribution is a mathematical proof of the approximation accuracy along the dynamic. The other major difference is that they are not limited to the PTM but cover also the original TM on the directed configuration model ensemble of networks. On the other hand, it should be stressed that our results do not extend to the analysis of the general TM on the undirected configuration model ensemble. In fact, as pointed out by [26], the analysis of the TM on undirected trees presents itself additional challenges beyond the scope of the approach proposed here.

In summary, the main contributions of this paper consist in providing a rigorous approximation result in terms of the output $y(t)$ of the recursion (3) for the fraction $z(t)$ of state-1 adopters in the TM and the PTM dynamics on the ensemble of directed networks (Theorem 1) and of the PTM on the ensemble of undirected networks (Theorem 2). Such theoretical results are then supported by numerical simulations on an actual large-scale network topology (see Section 5). In the course of building up the tools for such analysis, we also prove that the PTM can be regarded as a special case of the TM (Lemma 2), a result of potential independent interest.

The rest of this paper is organized as follows. The final part of this section gathers some notational conventions to be used throughout; Section 2 formally introduces the TM and the PTM dynamics, proves some fundamental monotonicity properties (Lemma 1), and builds on them to show that the PTM can be regarded as a special case of the TM when all agents with initial state 1 have threshold 0 (Lemma 2); in Section 3 we present our main result,

1. Formally, the result follows from Lemma 2 in Section 2.

Theorem 1, which guarantees that the output $y(t)$ of the recursion (3) provides a good approximation of the fraction of state-1 adopters in both the TM and PTM dynamics on the ensemble of directed networks; in Section 4 we formally prove Theorem 1, and extend it to the PTM dynamics on the ensemble of undirected networks (Theorem 2) and to networks with time-varying thresholds; in Section 5 we present numerical simulations on an actual large-scale network.

Notational conventions We denote the transpose of a matrix M by M' and the all-one vector by $\mathbb{1}$. We model interconnection topologies as directed multi-graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{1, \dots, n\}$ is a finite set of nodes representing the interacting agents and $\mathcal{E}$ is a multi-set of directed links $e = (i,j) \in \mathcal{V} \times \mathcal{V}$. Here, the use of the prefix *multi* reflects the fact that links (i,j) directed from the same tail node i to the same head node j may occur with multiplicity larger than 1, i.e., we allow for the possible presence of parallel links. The adjacency matrix $A \in \mathbb{R}^{n\times n}$ of $\mathcal{G}$ has then nonnegative-integer entries A_{ij} whose value represents the multiplicity with which link (i,j) appears in $\mathcal{E}$.[2] Observe that we also allow for the possibility of selfloops, i.e., links of the form (i,i) that correspond to nonzero diagonal entries $A_{ii} > 0$ of the adjacency matrix. Of course, directed graphs with no self-loops can be recovered as a special case when A has binary entries $A_{ij} \in \{0,1\}$ and zero diagonal, whereas undirected graphs can be recovered as a special case when the adjacency matrix $A' = A$ is symmetric. In particular, simple graphs (undirected and with no self-loops) correspond to the case when the adjacency matrix is symmetric and has zero diagonal and binary entries. The in-degree and out-degree vectors of a graph are then denoted by $\delta = A'\mathbb{1}$ and $\kappa = A\mathbb{1}$, respectively, so that $\delta_i = \sum_j A_{ji}$ and $\kappa_i = \sum_j A_{ij}$ are the in- and out-degree, respectively, of node i. Whenever the interconnection topology contains a link $(i,j) \in \mathcal{E}$ we refer to node j as an out-neighbor of i and to node i as an in-neighbor of j. An l-tuple of nodes $i_0, i_1, \dots i_l$ is referred to as a length-l *walk* from i_0 to i_l if $(i_{h-1}, i_h) \in \mathcal{E}$ for $1 \le h \le l$. Finally, the *depth-t neighborhood* $\mathcal{N}^i_t$ of a node i is the subgraph of $\mathcal{G}$ containing all nodes j such that there exists a walk from i to j of length $l \le t$.

2 The Threshold Model of cascades

In this section, we introduce the TM dynamics on arbitrary interconnection networks. We then prove some basic monotonicity properties of the TM and use them to show how the PTM can be recovered as a special case of the TM with the proper choice of thresholds.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an interconnection topology. We follow the convention that the link direction is the opposite of the one of the influence, so that the presence of a link $(i,j) \in \mathcal{E}$ indicates that agent i observes, and is influenced by, agent j. The behavior of each agent $i = 1, \dots, n$ in the TM dynamics is characterized by a threshold value $\rho_i \in \{0, 1, \dots, \kappa_i\}$ that represents the minimum number of state-1 adopters that she needs to observe among her neighbors in order to adopt state 1 at the next time instant. Such threshold is related to the normalized threshold $\Theta_i \in [0,1]$ mentioned in Section 1 by the identity $\rho_i = \lceil \Theta_i \kappa_i \rceil$. The vector of all agents' thresholds is then denoted by $\rho \in \mathbb{R}^n$. In order to introduce the TM dynamics, we are left to specify an initial state $\sigma_i \in \{0,1\}$ for every agent i. Let the vector of all agents' initial states be denoted by $\sigma \in \{0,1\}^n$. We will refer to a network as the 4-tuple $\mathcal{N} = (\mathcal{V}, \mathcal{E}, \rho, \sigma)$ of a set of agents $\mathcal{V}$, a multiset of links $\mathcal{E}$, a threshold vector ρ, and a vector of initial states σ. The TM on a network $\mathcal{N} = (\mathcal{V}, \mathcal{E}, \rho, \sigma)$ is then defined as the discrete-time dynamical system with state space $\{0,1\}^n$ and update rule

$$Z_i(0) = \sigma_i\,, \quad Z_i(t+1) = \begin{cases} 1 & \text{if } \sum_j A_{ij} Z_j(t) \ge \rho_i \\ 0 & \text{if } \sum_j A_{ij} Z_j(t) < \rho_i \end{cases}, \quad (4)$$

for $t \ge 0$ and $i = 1, \dots, n$.

Remark 1. *The TM can be interpreted as the best response dynamics in so-called semi-anonymous network games with strategic complements [27], whereby the agents $i \in \mathcal{V}$ have utilities that are increasing functions of the number of their out-neighbors taking the same action. E.g., [2], [10], [23] consider best response dynamics for network coordination games where agents i choose their binary action $Z_i \in \{0,1\}$ so as to maximize their utilities*

$$u_i(Z_i, Z_{-i}) = a_i Z_i \sum\nolimits_j A_{ij} Z_j + b_i (1 - Z_i) \sum\nolimits_j A_{ij}(1 - Z_j)\,,$$

where Z_{-i} is the vector of all but agent i's actions and $a_i > 0$ and $b_i > 0$ are constants. In such games, it is easy to verify that the best response dynamics is given by (4) *with $\rho_i = \kappa_i b_i/(a_i + b_i)$.*

The following lemma captures some basic monotonicity properties of the TM dynamics that prove particularly useful in its analysis. In stating and proving it we will adopt the notational convention that an inequality between vectors is meant to hold true entry-wise.

Lemma 1. *Let $\mathcal{N} = (\mathcal{V}, \mathcal{E}, \rho, \sigma)$ and $\mathcal{N}^+ = (\mathcal{V}, \mathcal{E}, \rho, \sigma^+)$ be two networks differing only (possibly) for the initial state vector. Let $Z(t)$ and $Z^+(t)$ be the state vectors of the TM dynamics* (4) *on $\mathcal{N}$ and $\mathcal{N}^+$, respectively. Then,*

(i) *if $\sigma^+ \ge \sigma$, then $Z^+(t) \ge Z(t)$ for all $t \ge 0$;*
(ii) *if $\rho_i \le (1 - \sigma_i)\kappa_i$ for all i, then $Z(t)$ is non-decreasing in t, hence, in particular, it is eventually constant.*

Proof. (i) Let A be the adjacency matrix of $\mathcal{N}$ and $\mathcal{N}^+$. Observe that, since A is nonnegative, if $Z^+(t) \ge Z(t)$ for some $t \ge 0$, then $AZ^+(t) \ge AZ(t)$ and $Z^+(t+1) \ge Z(t+1)$ (as $Z^+_i(t+1) = 0$ implies $\sum_j A_{ij} Z_j(t) \le \sum_j A_{ij} Z^+_j(t) < \rho_i$ so that $Z_i(t+1) = 0$). The claim follows by induction on t.

(ii) Let $Z(0) = \sigma$ and $Z^+(0) = \sigma^+ = Z(1)$. Observe that, if $\rho_i \le (1 - \sigma_i)\kappa_i$ for every i, then for all those i such that $Z_i(0) = \sigma_i = 1$ one has $\rho_i = 0 \le \sum_j A_{ij} Z_j(0)$ so that $\sigma^+_i = Z_i(1) = 1$. Hence, necessarily $\sigma^+ = Z(1) \ge \sigma$. It then follows from (i) that $Z^+(t) = Z(t+1) \ge Z(t)$ for all $t \ge 0$, i.e., $Z(t)$ is non-decreasing, hence eventually constant. □

We now introduce a variation of the TM known as *Progressive* TM (PTM), whereby only state transitions from 0 to 1 are allowed, but not from 1 to 0. Formally, the PTM

2. In fact, one could easily relax the integer constraint on the entries of the adjacency matrix A and consider weighted graphs, whereby each positive entry A_{ij} stands for the weight of the link from node i to node j. For the sake of simplicity in the exposition we will not consider this generalization explicitly in this paper.

on a network $\mathcal{N} = (\mathcal{V}, \mathcal{E}, \rho, \sigma)$ is defined by the following recursive relations

$$Z_i(0) = \sigma_i \,,$$
$$Z_i(t+1) = \begin{cases} 1 & \text{if } \sum_j A_{ij} Z_j(t) \geq (1 - Z_i(t))\rho_i \\ 0 & \text{if } \sum_j A_{ij} Z_j(t) < (1 - Z_i(t))\rho_i \end{cases} , \quad (5)$$

valid for $t \geq 0$ and $i = 1, \dots, n$. Observe that in the PTM dynamics the state update rule of every agent i depends on her own current state, regardless of the presence of self-loops in the network. This is in contrast with the TM update rule, whereby the new state of every agent i such that $A_{ii} = 0$ depends on the current state of its out-neighbors only and not on itself. In spite of these differences, the following result shows that the PTM dynamics coincides with the TM provided that agents with initial state 1 are treated as if having effective threshold 0.

Lemma 2. *The PTM dynamics* (5) *on a network* $\mathcal{N} = (\mathcal{V}, \mathcal{E}, \rho, \sigma)$ *coincide with the dynamics defined by*

$$Z_i(0) = \sigma_i \,,$$
$$Z_i(t+1) = \begin{cases} 1 & \text{if } \sum_j A_{ij} Z_j(t) \geq (1 - \sigma_i)\rho_i \\ 0 & \text{if } \sum_j A_{ij} Z_j(t) < (1 - \sigma_i)\rho_i \end{cases} \quad t \geq 0 \,. \quad (6)$$

for $i = 1, \dots, n$. *In particular, if* $\rho_i \leq (1 - \sigma_i)\kappa_i$ *for every* $i \in \mathcal{V}$, *then the TM dynamics* (4) *and the PTM dynamics* (5) *coincide.*

Proof. Let us denote by $Z(t)$ and $\tilde{Z}(t)$ the state vectors generated by the recursions (5) and (6), respectively. It follows from applying part (ii) of Lemma 1 to the network $\tilde{\mathcal{N}} = (\mathcal{V}, \mathcal{E}, \tilde{\rho}, \sigma)$ where $\tilde{\rho}_i = \rho_i(1 - \sigma_i)$ that $\tilde{Z}(t)$ is non-decreasing in t. On the other hand, $Z(t)$ is non-decreasing by construction, since only transitions from 0 to 1 are allowed by (5) but not the other way around. Now, we shall proceed by an induction argument, assuming that $Z(s) = \tilde{Z}(s)$ for $s = 0, 1, \dots, t$ and showing that then $Z(t+1) = \tilde{Z}(t+1)$. For all those i such that $Z_i(t) = \tilde{Z}_i(t) = 0$ monotonicity of $\tilde{Z}(t)$ implies that $\sigma_i = \tilde{Z}_i(0) \leq \tilde{Z}_i(t) = 0$ and therefore the updates in (5) and in (6) coincide, yielding $Z_i(t+1) = \tilde{Z}_i(t+1)$. On the other hand, for all those i such that $Z_i(t) = \tilde{Z}_i(t) = 1$, monotonicity implies that $Z_i(t+1) \geq Z_i(t) = 1$ and $\tilde{Z}_i(t+1) \geq \tilde{Z}_i(t) = 1$ so that $\tilde{Z}_i(t+1) = Z_i(t+1)$. This proves the first claim. The second part of the Lemma simply follows from the fact that $\rho_i \leq (1 - \sigma_i)\kappa_i$ and $\sigma_i \in \{0, 1\}$ imply $(1 - \sigma_i)\rho_i = \rho_i$. □

Lemma 2 is particularly significant in that it implies that the study of the PTM dynamics (5) can be reduced to that of a special case of the TM dynamics (4), where all agents with initial state $\sigma_i = 1$ have threshold $\rho_i = 0$. Observe that, if an agent i has threshold $\rho_i = 0$, then her state in the TM dynamics (1) satisfies $Z_i(t) = 1$ for $t \geq 1$. Hence, it is intuitive that, for the TM dynamics to coincide with the PTM ones, agents with initial state $\sigma_i = 1$ should have threshold $\rho_i = 0$, so that they will keep their state equal to 1 throughout the process. The less intuitive and deeper part of Lemma 2 consists in showing that the condition that all agents with initial state $\sigma_i = 1$ have threshold $\rho_i = 0$ is also sufficient for the state $Z_i(t)$ of all other agents —i.e., of those i with initial state $\sigma_i = 0$— to have the same dynamics under both the TM (1) and the PTM (5) update rules, hence, in particular, to switch at most once — from state $Z_i(t) = 0$ to $Z_i(t+1) = 1$ but never from $Z_i(t) = 1$ to $Z_i(t+1) = 0$. In fact, this is nothing but the statement of part (ii) of Lemma 1, which is turn a consequence of the monotonicity properties of the TM dynamics stated in part (i) of Lemma 1.

3 Main result

As mentioned in Section 1, the TM on a complete network lends itself to a simple analysis enabled by the fact that the fraction of state-1 adopters $z(t) = \frac{1}{n}\sum_i Z_i(t)$ evolves according to the one-dimensional recursion (2), where F is the cumulative distribution of the normalized thresholds across the population [1]. While such a one-dimensional recursion does not hold true for the TM dynamics on general networks, the main contribution of this paper consists in showing that the fraction of state-1 adopters $z(t)$ in the TM and the PTM dynamics on most directed networks can be approximated in a quantitatively precise sense by the output $y(t)$ of another one-dimensional recursion of the form

$$x(t+1) = \phi(x(t)) \,, \qquad y(t+1) = \psi(x(t)) \,, \qquad (7)$$

where (cf. (13) and (14)) $\phi(x)$ and $\psi(x)$ are polynomials with nonnegative coefficients that depend on the network's statistics $\boldsymbol{p}$ defined below. In this section, after introducing the notation for the network's statistics, we provide the expression of the recursion (7), introduce the directed version of the configuration model ensemble of interconnection and enunciate the main result. We postpone to Section 4 the formal derivation of the recursion and the intermediate results that form the proof of the main theorem.

Throughout, we will use the following notation. For a network $\mathcal{N} = (\mathcal{V}, \mathcal{E}, \rho, \sigma)$ of size n,

$$p_{d,k,r,s} = \frac{1}{n} \left|\{i \in \mathcal{V} : \delta_i = d, \, \kappa_i = k, \, \rho_i = r, \, \sigma_i = s\}\right| , \quad (8)$$

for $d \geq 0\,,\ 0 \leq r \leq k\,,\ s = 0, 1$ stands for the fraction of agents having in-degree d, out-degree k, threshold r, and initial state s and

$$l := \sum_{i \in \mathcal{V}} \delta_i = \sum_{i \in \mathcal{V}} \kappa_i \,, \qquad \overline{d} = \frac{l}{n} \,,$$

denote the network's total and average degree, respectively. We refer to $\boldsymbol{p} = \{p_{d,k,r,s}\}$ as the *network's statistics* and let

$$p_{k,r} := \sum_{d \geq 0} \sum_{s=0,1} p_{d,k,r,s} \,, \qquad q_{k,r} := \frac{1}{\overline{d}} \sum_{d \geq 0} \sum_{s=0,1} d p_{d,k,r,s} \,, \quad (9)$$

for $k, r \geq 0$, be the fractions of agents and, respectively, of links pointing to agents, of out-degree k and threshold r. Moreover, let

$$\upsilon := \sum_{d \geq 0} \sum_{k \geq 0} \sum_{r \geq 0} p_{d,k,r,1} \,, \qquad \xi := \frac{1}{\overline{d}} \sum_{d \geq 0} \sum_{k \geq 0} \sum_{r \geq 0} d p_{d,k,r,1} \quad (10)$$

be the fractions of agents and, respectively, of links pointing towards agents, with initial state $\sigma_i = 1$.

3.1 A heuristic derivation of the recursion

In order to get a quick, not thorough yet intuitive derivation of the recursion (7), consider the following random network dynamics with state vector $Y(t) \in \{0, 1\}^n$ whose initial state is $Y(0) = \sigma$ and whereby, at each time $t \geq 0$, agents

$i \in \mathcal{V}$ select κ_i agents $J^i_1, \ldots, J^i_{\kappa_i}$ independently at random from the population with probability $\mathbb{P}(J^i_h = j) = \delta_j / l$ and update its state as $Y_i(t+1) = 1$ if $\sum_{0 \le h \le \kappa_i} Y_{J^i_h}(t) \ge \rho_i$ and as $Y_i(t+1) = 0$ if $\sum_{0 \le h \le \kappa_i} Y_{J^i_h}(t) < \rho_i$. Let $y(t) = \frac{1}{n}\sum_i Y_i(t)$ and $x(t) = \frac{1}{l}\sum_i \delta_i Y_i(t)$ be the fractions of state-1 adopters and links pointing towards state-1 adopters, respectively. It is immediate to verify that

$$x(0) = \xi\,, \qquad y(0) = \upsilon\,. \tag{11}$$

On the other hand, if I is a random agent selected from $\mathcal{V}$ with uniform probability $\mathbb{P}(I = i) = 1/n$, then

$$\begin{aligned}
\mathbb{E}[y(t+1)|Y(t)] &= \mathbb{P}(Y_I(t+1) = 1|Y(t)) \\
&= \sum_{k\ge 0}\sum_{r \ge 0} p_{k,r}\mathbb{P}\left(\textstyle\sum_{h=1}^k Y_{J^I_h}(t) \ge r \big| Y(t)\right) \\
&= \sum_{k\ge 0}\sum_{r \ge 0} p_{k,r} \sum_{r \le u \le k} \mathbb{P}\left(\textstyle\sum_{h=1}^k Y_{J^I_h}(t) = u \big| Y(t)\right) \\
&= \sum_{k\ge 0}\sum_{r \ge 0} p_{k,r}\varphi_{k,r}(x(t)) = \psi(x(t))\,,
\end{aligned}$$

where

$$\varphi_{k,r}(x) := \sum_{u=r}^{k}\binom{k}{u} x^u (1-x)^{k-u}\,, \qquad 0 \le r \le k\,, \tag{12}$$

$$\psi(x) := \sum_{k\ge 0}\sum_{r\ge 0} p_{k,r}\varphi_{k,r}(x)\,, \tag{13}$$

and the fourth identity above follows from the fact that, conditioned on $Y(t)$, the $Y_{J^i_h}(t)$ are independent Bernoulli random variables with $\mathbb{P}(Y_{J^i_h}(t) = 1|Y(t)) = x(t)$. An analogous computation shows that, if M is a random agent selected with probability $\mathbb{P}(M = m) = \delta_m / l$, then

$$\begin{aligned}
\mathbb{E}[x(t+1)|Y(t)] &= \mathbb{P}(Y_M(t+1) = 1|Y(t)) \\
&= \sum_{k\ge 0}\sum_{r \ge 0} q_{k,r}\mathbb{P}\left(\textstyle\sum_{h=1}^k Y_{J^M_h}(t) \ge r \big| Y(t)\right) \\
&= \sum_{k\ge 0}\sum_{r \ge 0} q_{k,r} \sum_{r \le u \le k} \mathbb{P}\left(\textstyle\sum_{h=1}^k Y_{J^M_h}(t) = u \big| Y(t)\right) \\
&= \sum_{k\ge 0}\sum_{r \ge 0} q_{k,r}\varphi_{k,r}(x(t)) = \phi(x(t))\,,
\end{aligned}$$

where

$$\phi(x) := \sum_{k\ge 0}\sum_{r\ge 0} q_{k,r}\varphi_{k,r}(x)\,, \tag{14}$$

and in the second identity we used the fraction $q_{k,r}$ of links pointing to agents of out-degree k and threshold r.

While the above computations are merely concerned with the conditional expected fractions of state-1 adopters, and links pointing towards state-1 adopters, in the random network dynamics $Y(t)$, the output $y(t)$ of the recursion (7) with initial condition (11) does in fact provide a good approximation of the evolution of the fraction of state-1 adopters for the actual TM dynamics (1) on most of the networks with given statistics $\boldsymbol{p}$.

3.2 Formal statement of the main result

We start by introducing the configuration model ensemble $\mathcal{C}_{n,\boldsymbol{p}}$ of all networks with given size n and compatible statistics $\boldsymbol{p}$. We refer to $\boldsymbol{p}$ and n as compatible if $np_{d,k,r,s}$ is an integer for all non-negative values of d, k, $0 \le r \le k$, and $s \in \{0,1\}$. We construct a random network $\mathcal{N} = (\mathcal{V}, \mathcal{E}, \rho, \sigma)$ of compatible size n and statistics $\boldsymbol{p}$ as follows. Let $\mathcal{V} = \{1, \ldots, n\}$ be a node set and let δ, κ, ρ, and σ be a designed vectors of in-degrees, out-degrees, thresholds, and initial states, such that (8) holds true, i.e., there is exactly a fraction $p_{d,k,r,s}$ of agents $i \in \mathcal{V}$ with $(\delta_i, \kappa_i, \rho_i, \sigma_i) = (d, k, r, s)$. Let $l = \overline{d}n$ be the number of directed links, put $\mathcal{L} = \{1, 2, \ldots, l\}$, and let $\nu, \lambda : \mathcal{L} \to \mathcal{V}$ be two maps such that $|\nu^{-1}(i)| = \delta_i$ and $|\lambda^{-1}(i)| = \kappa_i$. Then, let π be a uniform random permutation of $\mathcal{L}$ and let the network $\mathcal{N} = (\mathcal{V}, \mathcal{E}, \rho, \sigma)$ have node set $\mathcal{V}$, link multiset $\mathcal{E} = \{(\lambda(h), \nu(\pi(h)))\}_{1 \le h \le l}$, threshold vector ρ, and initial state vector σ. Figure 1 illustrates the above construction. We refer to such network $\mathcal{N}$ as being sampled from the configuration model ensemble $\mathcal{C}_{n,\boldsymbol{p}}$.

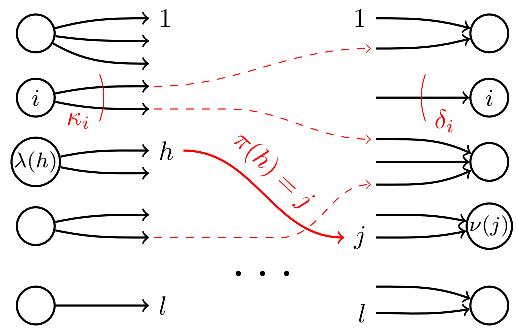

Fig. 1. The Configuration Model, with each node represented twice, on the left and on the right side of the picture. The picture contains the link $(\lambda(h), \nu(\pi(h)))$ and a few other dashed links.

The next theorem is our main contribution. It guarantees that the fraction of state-1 adopters $z(t)$ after a finite number of iterations of the TM dynamics (4) is arbitrarily close to the output $y(t)$ of the recursion (7) on all but a fraction of networks in $\mathcal{C}_{n,\boldsymbol{p}}$ that vanishes as n grows large.

Theorem 1. *Let $\mathcal{N}$ be a network sampled from configuration model ensemble $\mathcal{C}_{n,\boldsymbol{p}}$ of size n and statistics $\boldsymbol{p}$. Let $Z(t)$, for $t \ge 0$ be the state vector of the TM dynamics (4) on $\mathcal{N}$, let $z(t) = \frac{1}{n}\sum_i Z_i(t)$, and let $y(t)$ be the output of the recursion (7). Then, for $\varepsilon > 0$ and $n \ge \gamma_t / \varepsilon$ where $\gamma_t = d_{\max} k_{\max}^{2t+3} / \overline{d}$, it holds true*

$$|z(t) - y(t)| \le \varepsilon$$

for all but at most a fraction $2e^{-\varepsilon^2 \beta n}$ of networks $\mathcal{N}$ from the configuration model ensemble $\mathcal{C}_{n,\boldsymbol{p}}$, where $\beta = (32\overline{d}d_{\max}^{2t})^{-1}$.

While the proof of Theorem 1 is postponed to Section 4, we conclude this section with a few remarks.

Remark 2. *While Theorem 1 is stated for the TM dynamics, it follows from Lemma 2 that the same result remains valid for the fraction of state-1 adopters in the PTM dynamics as long as*

$$p_{d,k,r,1} = 0\,, \qquad d \ge 0\,, \quad 1 \le r \le k\,, \tag{15}$$

i.e., when $\rho_i \le \kappa_i(1 - \sigma_i)$ for all agents $i \in \mathcal{V}$, so for those nodes with $\sigma = 1$ it is required that $\rho_i = 0$ to exclude any switch to state-0. Two further important extensions of Theorem 1 to time-varying thresholds and, respectively, the undirected configuration model will be discussed at the end of Section 4.

Remark 3. *While proving a bound for finite-size networks, Theorem 1 readily implies that, for sequences of networks whose* *network statistics converge to a given limit as the network size grows large, the fraction of state-*1 *adopters in the TM on the configuration models ensemble concentrates around the output of the recursion* (3) *associated to such limit network statistics for finite values of* t*, provided that the maximum in- and out-degrees remain bounded or grow slower than* $n^{1/(2t)}$ *as* n *grows large. As numerical simulations suggest that the result might remain true also for faster growth rates of the minimum and maximum degree, extensions of our result in this direction remain an interesting research question.*

Remark 4. *Theorem 1 provides an approximation result for possibly large but finite values of* t *(if one considers sequences of networks of increasing size, the result applies up to values of* t *growing at most as* $\log n/(2\log d_{\max})$*). In fact, by combining Theorem 1 with techniques for the exchange of limits in* t *and* n *such as those in [28] would allow to show that with high probability as the network size grows large, the asymptotic fraction of state-*1 *adopters in the configuration model ensemble with bounded maximum degree concentrates on the set of all stationary points of the recursion* (3)*. When* (3) *has a unique (globally attractive) stationary point, concentration is guaranteed in that point for every initial fraction of state-*1 *adopters. When* (3) *has multiple stationary points, this approach does not allow one to relate the initial fraction of state-*1 *adopters to the highly probable limit and more ad-hoc techniques should be used to prove it (see, e.g., [23], [24], [29]). It should noted, however, that from a practical viewpoint our numerical simulations reported in Section 5 suggest that the asymptotic behavior of the recursion* (3) *still provides a very good indication of the asymptotic behavior of the TM on finite-size networks.*

Remark 5. *Theorem 1 states that for all but an exponentially small fraction of networks from the configuration model ensemble the fraction of state-*1 *adopters in the TM can be approximated by the output of the recursion* (3)*, see for example the simulations in Figure 2. In fact, our numerical results reported in Section 5, strongly suggest that such approximation remains valid not only for artificial networks sampled from the configuration model ensemble, but for actual large-scale social networks.*

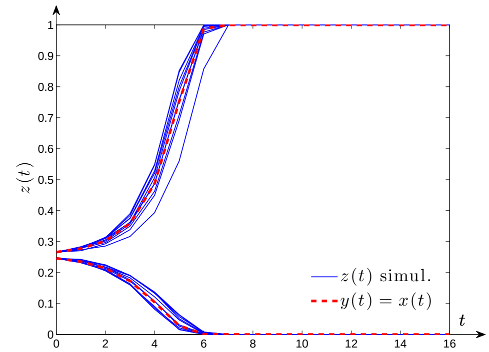

Fig. 2. Simulations comparing the dynamics of the fraction of state-1 adopters $z(t)$ in the TM (4) (blue solid lines) with the output $y(t)$ of the recursion (7) (dashed red lines). The synthetic random networks have $n = 2000$ nodes, each with in-degree $d = 7$, out-degree $k = 7$, and threshold $r = 3$. The recursion reduces to $y(t+1) = x(t+1) = \varphi_{7,3}(x(t))$ with $\upsilon = \xi$. The initial conditions are such that $\upsilon = 0.246$ or $\upsilon = 0.266$. The corresponding simulations converge to zero or one, respectively: in both case the recursion captures the behavior and timing of the simulated dynamic. The theoretical predictions are less accurate if υ is chosen very close to 0.256: the simulations $z(t)$ return close to the recursion $y(t)$ if the network size n is increased. The values of $z(T)$ for large T and various initial conditions can be compared with the limit of the recursion output, for the same synthetic networks. If υ is not very close to 0.256, the predicted limit always matches the simulations.

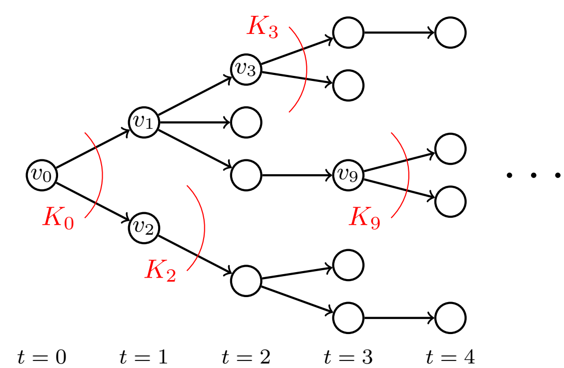

Fig. 3. A directed two-stage branching process $\mathcal{T}$ with root node v_0. The triples (K_h, R_h, S_h), for $h \geq 0$, of the agents' outdegrees, thresholds, and initial states are mutually independent and have distribution (16) and (17). The state $X_{v_0}(t)$ of the root node at time $t \geq 0$ is a deterministic function of the initial states S_j of the agents j in generation t.

4 Proofs and Extensions

This section is devoted to the proof of Theorem 1 and to two extensions. In Section 4.1 we introduce a different random graph model with rooted tree structure, the *two-stage branching process* $\mathcal{T}_{\boldsymbol{p}}$, and show that the output $y(t)$ of the recursion (7) gives the *exact* expression of the *expected value* of the root node's state in the TM dynamics (4) on $\mathcal{T}_{\boldsymbol{p}}$. In Section 4.2, we consider the configuration model $\mathcal{C}_{n,\boldsymbol{p}}$ and prove that, after t iterations of the TM dynamics (4) on the configuration model ensemble, the *average* fraction $\overline{z}(t)$ of state-1 adopters is arbitrarily close to $y(t)$, i.e., the expected value of the root node's state on $\mathcal{T}_{\boldsymbol{p}}$. Then, a concentration result is obtained, showing that on most of the networks in $\mathcal{C}_{n,\boldsymbol{p}}$, the fraction $z(t)$ of state-1 adopters after t iterations of the TM dynamics is arbitrarily close to its average $\overline{z}(t)$, hence to the output $y(t)$ of the recursion (7). Finally, Section 4.3 extends Theorem 1 to the PTM on the undirected configuration model ensemble and to networks with time-varying thresholds.

4.1 The TM on the two-stage branching process

In this subsection we first introduce a random graph model with rooted directed tree structure, to be referred to as the two-stage branching process $\mathcal{T}_{\boldsymbol{p}}$. Then, we provide a complete theoretical analysis of the TM dynamics on $\mathcal{T}_{\boldsymbol{p}}$ that will be the basis for then analysing, in the next subsection, the configuration model ensemble $\mathcal{C}_{n,\boldsymbol{p}}$ which exhibits a local tree-like structure.

Let $\boldsymbol{p}$ be the network statistics with average degree $\overline{d}$ and

$$p_{k,r,s} = \sum_{d\geq 0} p_{d,k,r,s}\,, \qquad q_{k,r,s} = \frac{1}{\overline{d}}\sum_{d\geq 0} d p_{d,k,r,s}\,,$$

for $0 \leq r \leq k$, $s = 0, 1$, be the fractions of agents and, respectively, of links pointing to agents, of out-degree k, threshold r, and initial state s. In order to define the associated two-stage branching process $\mathcal{T}_{\boldsymbol{p}}$, we start from a root node v_0 and randomly generate a directed tree graph according to the following rule (compare Figure 3). First, we assign to the root node v_0 a random out-degree $\kappa_{v_0} = K_0$, threshold $\rho_{v_0} = R_0$ and initial state $\sigma_{v_0} = S_0$ such that the triple (K_0, R_0, S_0) has joint probability distribution

$$\mathbb{P}(K_0 = k, R_0 = r, S_0 = s) = p_{k,r,s}\,, \tag{16}$$

for $0 \le r \le k$ and $s = 0, 1$. Then, we connect the root node v_0 with K_0 directed links pointing to new nodes $v_1, \ldots, v_{K_0}$, and assign to each such generation-1 node v_h, $1 \le h \le K_0$, out-degree $\kappa_{v_h} = K_h$, threshold $\rho_{v_h} = R_h$, and initial state $\sigma_{v_h} = S_h$ such that the triples (K_h, R_h, S_h) are mutually independent, independent from (K_0, R_0, S_0), and identically distributed with

$$\mathbb{P}(K_h = k, R_h = r, S_h = s) = q_{k,r,s}\,, \tag{17}$$

for $0 \le r \le k$ and $s = 0, 1$. We then connect each of the generation-1 nodes v_h with K_h directed links pointing to distinct new nodes, and assign to such generation-2 nodes $v_{J_1+1}, \ldots, v_{J_2}$, where $J_1 = K_0$ and $J_2 = \sum_{0 \le j \le J_1} K_j$, out-degree $\kappa_{v_h} = K_h$, threshold $\rho_{v_h} = R_h$, and initial state $\sigma_{v_h} = S_h$ such that the triples (K_h, R_h, S_h), for $J_1 + 1 \le h \le J_2$, are mutually independent, independent from $(K_0, R_0, S_0), \ldots, (K_{J_1}, R_{J_1}, S_{J_1})$, and identically distributed with $\mathbb{P}(K_h = k, R_h = r, S_h = s) = q_{k,r,s}$ for $k \ge 0$, $0 \le r \le k$, and $s = 0, 1$. We then keep on repeating the same procedure over and over, thus generating, in a breadth-first manner, a possibly infinite random tree network $\mathcal{T}_{\boldsymbol{p}}$ with node set $\mathcal{V} = \{v_0, v_1, \ldots\}$, thresholds $\rho_{v_0}, \rho_{v_1}, \ldots$, and initial states $\sigma_{v_0}, \sigma_{v_1}, \ldots$. For $t \ge 0$, we let $\mathcal{T}_{\boldsymbol{p},t}$ be the finite random tree network obtained by truncating $\mathcal{T}_{\boldsymbol{p}}$ at the t-th generation. Observe that the specific realization of the two-stage branching process is uniquely determined by the sequence of mutually independent triples $(K_0, R_0, S_0), (K_1, R_1, S_1), (K_2, R_2, S_2) \ldots$, which are distributed according to $\mathbb{P}(K_0 = k, R_0 = r, S_0 = s) = p_{k,r,s}$ and $\mathbb{P}(K_h = k, R_h = r, S_h = s) = q_{k,r,s}$ for $h \ge 1$.

The following result shows that the state $x(t)$ and output $y(t)$ of the recursion (7) coincide with the exact expected states of the TM dynamics on $\mathcal{T}_{\boldsymbol{p}}$. Observe that the TM dynamics (4) is a deterministic process, hence the only randomness concernes the generation of $\mathcal{T}_{\boldsymbol{p}}$.

Proposition 1. *Let $\boldsymbol{p}$ be the network statistics and $\mathcal{T}_{\boldsymbol{p}} = (\mathcal{V}, \mathcal{E}, \rho, \sigma)$ be the associated two-stage branching process with node set $\mathcal{V} = \{v_0, v_1, \ldots\}$, where v_0 is the root node. Let $Z(t)$, for $t \ge 0$, be the state vector of the TM dynamics on $\mathcal{T}_{\boldsymbol{p}}$, and let $x(t)$ and $y(t)$ be respectively the state and output of the recursion* (7). *Then, for every fixed time $t \ge 0$, the following holds:*

- (i) *For every $i \in \mathcal{V}$, the states $\{Z_j(t)\}_{j:\,(i,j)\in\mathcal{E}}$ of the offsprings v_j of v_i in $\mathcal{T}_{\boldsymbol{p}}$ are independent and identically distributed Bernoulli random variables with expected value $x(t)$;*
- (ii) *The state $Z_{v_0}(t)$ of the root node v_0 is a Bernoulli random variable with expected value $y(t)$.*

Proof. (i) First notice that the state $Z_i(t)$ of any node $i \in \mathcal{V}$ is a deterministic function of the threshold and of the initial states of the descendants of node i in $\mathcal{T}_{\boldsymbol{p}}$ up to generation t. It follows that, given any two non-root nodes $j, l \in \mathcal{V} \setminus \{v_0\}$, $Z_j(t)$ and $Z_l(t)$ are Bernoulli random variables with identical distribution, since the two subnetworks of their descendants are branching processes with the same statistics. Moreover, for every node $i \in \mathcal{V}$, let $\mathcal{N}_i$ be the set of its out-neighbors in $\mathcal{T}_{\boldsymbol{p}}$ and observe that the variables $Z_j(t)$, for $j \in \mathcal{N}_i$, are mutually independent since each pair of the subnetworks of their descendants have empty intersection. Let $\zeta(t) = \mathbb{E}[Z_j(t)]$, $j \in \mathcal{V} \setminus \{v_0\}$, be the expected value of all these r.v.'s. For any $i \in \mathcal{V}$ and $j \in \mathcal{N}_i$, (4)) implies that

$$\begin{aligned}\zeta(t+1) &= \mathbb{P}\left(\textstyle\sum_{h\in\mathcal{V}} A_{jh} Z_h(t) \ge \rho_j\right)\\ &= \sum_{k\ge 0}\sum_{0\le r\le k} q_{k,r}\,\mathbb{P}\left(\textstyle\sum_{h\in\mathcal{N}_j} Z_h(t) \ge r \,\Big|\, k_j = k,\ \rho_j = r\right).\end{aligned}$$

Now, observe that the conditional probability in the rightmost summation above is simply the probability that a sum of k independent and identically distributed Bernoulli random variables having mean $\zeta(t)$ is not below the threshold r. Therefore, such conditional probability is equal to $\varphi_{k,r}(\zeta(t))$. Substituting we get

$$\zeta(t+1) = \sum_{k\ge 0}\sum_{0\le r\le k} q_{k,r}\varphi_{k,r}(\zeta(t)) = \phi(\zeta(t))\,.$$

Since $\zeta(0) = \mathbb{P}(Z_j(0) = 1) = \mathbb{P}(\sigma_j = 1) = x(0)$, it follows that $\zeta(t) = x(t)$ for every $t \ge 0$.

(ii) Put $\nu(t) = \mathbb{E}[Z_{v_0}(t)]$. Then, (4) and point (i) yield

$$\begin{aligned}\nu(t+1) &= \mathbb{P}\left(\textstyle\sum_{h\in\mathcal{V}} A_{v_0 h} Z_h(t) \ge \rho_{v_0}\right)\\ &= \sum_{k\ge 0}\sum_{0\le r\le k} p_{k,r}\,\mathbb{P}\left(\textstyle\sum_{h\in\mathcal{N}_{v_0}} Z_h(t) \ge \rho_{v_0} \,\Big|\, k_{v_0} = k,\ \rho_{v_0} = r\right)\\ &= \sum_{k\ge 0}\sum_{0\le r\le k} p_{k,r}\varphi_{k,r}(\zeta(t)) = \psi(\zeta(t))\,,\end{aligned}$$

thus completing the proof. □

4.2 The TM on the configuration model

We analyse, in this subsection, the configuration model ensemble $\mathcal{C}_{n,\boldsymbol{p}}$ introduced in Section 3 and prove Theorem 1.

Lemma 3. *Let $\mathcal{N}$ be a network sampled from the configuration model ensemble $\mathcal{C}_{n,\boldsymbol{p}}$ of compatible size n and statistics $\boldsymbol{p}$. For $t \ge 0$, let $\mathcal{N}_t$ be the depth-t neighborhood of a node in $\mathcal{N}$ chosen uniformly at random from the node set $\mathcal{V}$, and let $\mu_{\mathcal{N}_t}$ its probability distribution. Let $\mathcal{T}_{\boldsymbol{p},t}$ be a two-stage branching process truncated at depth t, and let $\mu_{\mathcal{T}_{\boldsymbol{p},t}}$ be its distribution. Then, the total variation distance $||\mu_{\mathcal{N}_t} - \mu_{\mathcal{T}_{\boldsymbol{p},t}}||_{TV}$ between $\mu_{\mathcal{N}_t}$ and $\mu_{\mathcal{T}_{\boldsymbol{p},t}}$ satisfies*

$$||\mu_{\mathcal{N}_t} - \mu_{\mathcal{T}_{\boldsymbol{p},t}}||_{TV} \le \frac{\gamma_t}{2n}\,, \qquad \gamma_t = \frac{d_{\max} k_{\max}^{2t+3}}{\overline{d}}\,,$$

where $d_{\max} = \max\{d \ge 0 : \sum_{k,r,s} p_{d,k,r,s} > 0\}$ is the maximum in-degree and $k_{\max} = \max\{k \ge 0 : \sum_{d,r,s} p_{d,k,r,s} > 0\}$ is the maximum out-degree.

Proof. We will construct a coupling of the configuration model $\mathcal{C}_{n,\boldsymbol{p}}$ and the two-stage branching process $\mathcal{T}$ such that the depth-t neighborhood $\mathcal{N}_t$ of a uniform random node in $\mathcal{N}$ and the depth-t truncated branching process $\mathcal{T}_{\boldsymbol{p},t}$ satisfy $\mathbb{P}(\mathcal{N}_t \neq \mathcal{T}_{\boldsymbol{p},t}) \le \gamma_t/n$. The claim will then follow from the well-known bound $||\mu_{\mathcal{N}_t} - \mu_{\mathcal{T}_{\boldsymbol{p},t}}||_{TV} \le \mathbb{P}(\mathcal{N}_t \neq \mathcal{T}_{\boldsymbol{p},t})$ valid for every coupling of $\mathcal{N}_t$ and $\mathcal{T}_{\boldsymbol{p},t}$ (cf. [30, Proposition 4.7]).

In order to sample a network $\mathcal{N}$ from $\mathcal{C}_{n,\boldsymbol{p}}$ and define the coupling altogether, let us assign in-degree δ_i, out-degree κ_i, threshold ρ_i, and initial state σ_i to each of the n nodes $i \in \mathcal{V}$ in such a way that there are exactly $np_{d,k,r,s}$

nodes of in-degree d, out-degree k, threshold r, and initial state s. Let $l = n\overline{d} = \mathbb{1}'\delta$, $\mathcal{L} = \{1, 2, \dots, l\}$, and let $\nu : \mathcal{L} \to \mathcal{V}$ be a map such that $|\nu^{-1}(i)| = \delta_i$. Let w_0 be a random node chosen uniformly from $\mathcal{V}$, and let $K_0 = \kappa_{w_0}$, $R_0 = \rho_{w_0}$, and $S_0 = \sigma_{w_0}$ be its out-degree, threshold, and initial state, respectively. Let $(L_h)_{h=1,2,\dots}$ be a sequence of mutually independent random variables with identical uniform distribution on the set $\mathcal{L}$ and independent from w_0. Let $(M_h)_{h=1,2,\dots,l}$ be a finite sequence of $\mathcal{L}$-valued random variables such that, conditioned on w_0, $L_1, \dots, L_h$ and $M_1, \dots, M_{h-1}$, one has $M_h = L_h$ if $L_h \notin \{M_1, \dots, M_{h-1}\}$, while, if $L_h \in \{M_1, \dots, M_{h-1}\}$, M_h is conditionally uniformly distributed on the set $\mathcal{L} \setminus \{M_1, \dots, M_{h-1}\}$. Notice that the marginal probability distributions of the two sequences $(L_h)_{h=1,2,\dots}$ and $(M_h)_{h=1,2,\dots,l}$ correspond to sampling with replacement and, respectively, sampling without replacement, from the same set $\mathcal{L}$ (note that $(M_h)_{h=1,2,\dots,l}$ represents a permutation on $\mathcal{L}$). Moreover, for $1 \le h < l$,

$$\mathbb{P}\big(L_{h+1} \neq M_{h+1} | (L_1, \dots, L_h) = (M_1, \dots, M_h)\big) \le \frac{h}{l}. \quad (18)$$

Let $\mathcal{T}_{\boldsymbol{p},t}$ be the random directed tree whose root v_0 has out-degree K_0, threshold R_0 and initial state S_0, and that is then generated starting from v_0 in a breadth-first fashion, by assigning to each node v_h, $h \ge 1$ at generation $1 \le u \le t$ out-degree $K_h = \kappa_{\nu(L_h)}$, threshold $R_h = \rho_{\nu(L_h)}$ and initial state $S_h = \sigma_{\nu(L_h)}$. Observe that the triples (K_h, R_h, S_h) for $h \ge 0$ are mutually independent and have distribution $\mathbb{P}(K_0 = k, R_0 = r, S_0 = s) = p_{k,r,s}$ and $\mathbb{P}(K_h = k, R_h = r, S_h = s) = \frac{1}{\overline{d}} \sum_d d p_{d,k,r,s} = q_{k,r,s}$ for $h \ge 1$. Hence, $\mathcal{T}_{\boldsymbol{p},t}$ generated in this way has indeed the desired distribution $\mu_{\mathcal{T}_{\boldsymbol{p},t}}$.

On the other hand, let the network $\mathcal{N}$, and hence $\mathcal{N}_t$, be generated starting from w_0 and exploring its neighborhood in a breadth-first fashion. First let the $J_0 = K_0$ outgoing links of v_0 point to the nodes $v_1 = \nu(M_1), \dots, v_{J_0} = \nu(M_{J_0})$; then let the J_1 links outgoing from the set $\{v_1, \dots, v_{J_0}\} \setminus \{v_0\}$ of new out-neighbors of v_0 point to the nodes $\nu(M_{J_0+1}), \dots, \nu(M_{J_0+J_1})$; then let the J_2 links outgoing from the set $\{v_{J_0+1}, \dots, v_{J_0+J_1}\} \setminus \{v_0, v_1, \dots v_{J_0}\}$ point to the nodes $\nu(M_{J_0+J_1+1}), \dots, \nu(M_{J_0+J_1+J_2})$, and so on, possibly restarting from one of the unreached nodes in $\mathcal{V}$ if the process has arrived to a point where $J_u = 0$ and $\sum_{h \le u} J_h < l$ (so that not all nodes have been reached from v_0). Now, let $H_t = \sum_{0 \le u \le t-1} J_u$ and $N_t = |\{v_0, v_1, \dots, v_{H_t}\}|$ be the total number of links and, respectively, nodes in $\mathcal{N}_t$. Observe that $\mathcal{N}_t$ is a directed tree if and only if $N_t = H_t + 1$, which is in turn equivalent to $\nu(M_h) \neq \nu(M_{h'}) \neq w_0$ for all $1 \le h < h' \le N_t$.

If we define the events

$$\begin{aligned} E_h \quad &:= \{(L_1, \dots, L_h) = (M_1, \dots, M_h)\}, \\ F_{h+1} &:= \nu(M_{h+1}) \in \{w_0, \nu(M_1), \dots, \nu(M_h)\}, \end{aligned}$$

we notice that, for $0 \le h < l$,

$$P\big(F_{h+1} | E_h \text{ and } L_{h+1} = M_{h+1}\big) \le \frac{(h+1)(d_{\max} - 1) + 1}{l}.$$

The above together with (18) gives

$$\begin{aligned} \varsigma_h &:= \mathbb{P}\big(L_{h+1} \neq M_{h+1} \text{ or } F_{h+1} | E_h\big) \\ &= \mathbb{P}\big(L_{h+1} \neq M_{h+1} | E_h\big) \\ &\quad + \mathbb{P}\big(L_{h+1} = M_{h+1} \text{ and } F_{h+1} | E_h\big) \\ &\le \mathbb{P}\big(L_{h+1} \neq M_{h+1} | E_h\big) + \mathbb{P}\big(F_{h+1} | E_h\big) \\ &\le \frac{h}{l} + \frac{(h+1)(d_{\max} - 1) + 1}{l} \le \frac{(h+1) d_{\max}}{l}. \end{aligned}$$

The key observation is that, upon identifying node $v_h \in \mathcal{N}$ with node $w_h \in \mathcal{T}_{\boldsymbol{p},t}$ for all $0 \le h < N_t$, in order for $\mathcal{N}_t \neq \mathcal{T}_{\boldsymbol{p},t}$ it is necessary that either $N_t \neq H_t + 1$ (in which case $\mathcal{N}_t$ is not a tree) or $(L_1, \dots, L_{H_t}) \neq (M_1, \dots, M_{H_t})$ (in which case the nodes v_h and w_h might have different outdegree, threshold, or initial state). In order to estimate the probability that any of this occurs, first observe that a standard induction argument shows that $J_u \le k_{\max}^{u+1}$ for all $u \ge 0$, so that $H_t \le \sum_{1 \le u \le t} k_{\max}^u \le k_{\max}^{t+1}$. Then,

$$\begin{aligned} \mathbb{P}(\mathcal{N}_t \neq \mathcal{T}_{\boldsymbol{p},t}) \quad &\le \quad \mathbb{P}\left(\bigcup_{1 \le h \le H_t} E_{H_t}^c \cup F_h \right) \le \sum_{h=0}^{k_{\max}^{t+1} - 1} \varsigma_h \\ &\le \quad \sum_{h=1}^{k_{\max}^{t+1}} \frac{d_{\max} h}{l} = \frac{d_{\max} k_{\max}^{t+1} (k_{\max}^{t+1} + 1)}{2n\overline{d}} \\ &\le \quad \frac{d_{\max}}{2n\overline{d}} k_{\max}^{2t+3}. \end{aligned}$$

Hence, the claim follows from the above and the aforementioned bound on $||\mu_{\mathcal{N}_t} - \mu_{\mathcal{T}_{\boldsymbol{p},t}}||_{TV}$ $\square$

As a consequence of Lemma 3, we get the following.

Proposition 2. *Let $\mathcal{N}$ be a network sampled from the configuration model ensemble $\mathcal{C}_{n,\boldsymbol{p}}$ of compatible size n and statistics $\boldsymbol{p}$. Let $Z(t)$, for $t \ge 0$, be the state vector of the TM dynamics* (4) *on $\mathcal{N}$, $z(t) = \frac{1}{n} \sum_i Z_i(t)$ be the fraction of state-1 adopters at time t, and $\overline{z}(t) = \mathbb{E}[z(t)]$ be its expectation. Then,*

$$|\overline{z}(t) - y(t)| \le \frac{\gamma_t}{2n},$$

where $y(t)$ is the output of the recursion (7) *and $\gamma_t = d_{\max} k_{\max}^{2t+3} / \overline{d}$ as in Lemma 3.*

Proof. Observe that, in the TM dynamics, the state $Z_i(t)$ of an agent i in a network $\mathcal{N} = (\mathcal{V}, \mathcal{E}, \rho, \sigma)$ is a deterministic function of the initial states $Z_j(0) = \sigma_j$ of the agents j reachable from i with t hops in $\mathcal{N}$ and of the thresholds ρ_k of the agents k reachable from i with less than t hops in $\mathcal{N}$. In particular, if $\mathcal{N}_t^i$ is the depth-t neighborhood of node i in $\mathcal{N}$, then $Z_i(t) = \chi(\mathcal{N}_t^i)$, where χ is a certain deterministic $\{0, 1\}$-valued function. It follows that, if $\mathcal{N}$ is a network sampled from the configuration model ensemble $\mathcal{C}_{n,\boldsymbol{p}}$, $\mathcal{N}_t$ is the depth-t neighborhood of uniform random node in $\mathcal{N}$, and $\mu_{\mathcal{N}_t}$ is its distribution, then

$$\overline{z}(t) = \mathbb{E}\left[\frac{1}{n} \sum_{i \in \mathcal{V}} Z_i(t) \right] = \int \chi(\omega) \mathrm{d}\mu_{\mathcal{N}_t}(\omega).$$

On the other hand, it follows from Proposition 1 that, if $\mathcal{T}_{\boldsymbol{p},t}$ is a two-stage directed branching process with offspring distribution $p_{k,r,s} = \sum_d p_{d,k,r,s}$ for the first generation and $q_{k,r,s} = \frac{1}{\overline{d}} \sum_{d \ge 0} d p_{d,k,r,s}$ for the following generations, truncated at depth t, and $\mu_{\mathcal{T}_{\boldsymbol{p},t}}$ is its distribution, then the output $y(t)$ of the recursion (7) satisfies

$$y(t) = \int \chi(\omega) \mathrm{d}\mu_{\mathcal{T}_{\boldsymbol{p},t}}(\omega) \,.$$

It then follows from the fact that χ is a $\{0,1\}$-valued random variable and Lemma 3 that

$$\begin{aligned} &|\overline{z}(t) - y(t)| \\ &= \left| \int \left(\chi(\omega) - \tfrac{1}{2}\right) \mathrm{d}\mu_{\mathcal{N}_t}(\omega) - \int \left(\chi(\omega) - \tfrac{1}{2}\right) \mathrm{d}\mu_{\mathcal{T}_{\boldsymbol{p},t}}(\omega) \right| \\ &\leq ||\mu_{\mathcal{N}_t} - \mu_{\mathcal{T}_{\boldsymbol{p},t}}||_{TV} \leq \frac{\gamma_t}{2n} \,, \end{aligned}$$

thus completing the proof. □

The following result establishes concentration of the fraction of state-1 adopters in the TM dynamics on a random network drawn from the configuration model ensemble and its expectation.

Proposition 3. *Let n and $\boldsymbol{p}$ be compatible network size and statistics. Then, for all $\varepsilon > 0$, for at least a fraction*

$$1 - 2e^{-\varepsilon^2 \beta n} \qquad \textit{with} \qquad \beta = (32\overline{d} d_{\max}^{2t})^{-1}$$

of networks $\mathcal{N}$ from the configuration model ensemble $\mathcal{C}_{n,\boldsymbol{p}}$, the fraction of $z(t) = \frac{1}{n}\sum_{i\in\mathcal{V}} Z_i(t)$ of state-1 adopters in the TM dynamics (4) on $\mathcal{N}$ satisfies

$$|z(t) - \overline{z}(t)| \leq \varepsilon/2 \,,$$

where $\overline{z}(t)$ is the average of $z(t)$ over the choice of $\mathcal{N}$ from $\mathcal{C}_{n,\boldsymbol{p}}$.

Proof. Let $a(t) = nz(t) = \sum_{i\in\mathcal{V}} Z_i(t)$ be the total number of agents in state 1 at time t in the network $\mathcal{N}$ drawn uniformly from the configuration model ensemble, and let $\overline{a}(t) = n\overline{z}(t)$ be its average over the ensemble. In order to prove the result we will construct a martingale $A_0, A_1, \ldots, A_l$, where $l = n\overline{d}$ is the total number of links, such that $A_0 = \overline{a}(t)$, $A_l = a(t)$, and

$$|A_h - A_{h-1}| \leq \alpha \,, \qquad \alpha := \frac{2d_{\max}^t}{d_{\max} - 1} \,, \qquad h = 1, 2, \ldots, l \,. \tag{19}$$

The result will then follow from the Hoeffding-Azuma inequality [31, Theorem 7.2.1] which implies that the fraction of networks from the configuration model ensemble for which $|A_0 - A_l| \geq \eta = n\varepsilon/2$ is upper bounded by

$$\begin{aligned} 2\exp\left(-\frac{\eta^2}{2l\alpha^2}\right) &= 2\exp\left(-\frac{n\varepsilon^2}{8\overline{d}\alpha^2}\right) \\ &= 2\exp\left(-\frac{n\varepsilon^2 (d_{\max}-1)^2}{32\overline{d} d_{\max}^{2t}}\right) \\ &\leq 2\exp(-\varepsilon^2 \beta n) \,, \end{aligned}$$

where $\beta = (32\overline{d} d_{\max}^{2t})^{-1}$.

In order to define the aforementioned martingale, let $\mathcal{L} = \{1, 2, \ldots, l\}$ and recall that the configuration model ensemble is defined starting from in-degree, out-degree, threshold, and initial state vectors $\delta, \kappa, \rho, \sigma \in \mathbb{R}^n$ with empirical frequency coinciding with the prescribed distribution $\{p_{d,k,r,s}\}$ and two maps $\nu, \lambda : \mathcal{L} \to \mathcal{V}$ such that $|\nu^{-1}(i)| = \delta_i$ and $|\lambda^{-1}(i)| = \kappa_i$ for all $i \in \mathcal{V}$. The ensemble is then defined by taking a uniform permutation π of the set $\mathcal{L}$ and wiring the h-th link from node $\lambda(h)$ to node $\nu(\pi(h))$ for $h = 1, \ldots, l$. Let $\pi_{[h]} = (\pi(1), \pi(2), \ldots, \pi(h))$ be the vector obtained by unveiling the first h values of π. Then, define $A_h = \mathbb{E}[a(t)|\pi_{[h]}]$, for $h = 0, 1, \ldots, l$ and observe that $A_0, A_1, \ldots, A_l$ is indeed a (Doob) martingale, generally referred to as the link-exposure martingale. It is easily verified that $A_0 = \mathbb{E}[a(t)] = \overline{a}(t)$ and $A_l = \mathbb{E}[a(t)|\pi] = a(t)$.

What remains to be proven is the bound (19). For a given $h = 1, \ldots, l$, let $\tilde{\pi}$ be a random permutation of $\mathcal{L}$ which is obtained from π by choosing some j uniformly at random from the set $\mathcal{L} \setminus \{\pi(1), \ldots, \pi(h-1)\}$ and putting $\tilde{\pi}(h) = j$ and $\tilde{\pi}(\pi^{-1}(j)) = \pi(h)$, and $\tilde{\pi}(k) = \pi(k)$ for all $k \in \mathcal{L}\setminus\{h, \pi^{-1}(j)\}$. Notice that $\tilde{\pi}$ and π differ in at most two positions, h and $\pi^{-1}(j) \geq h$, the latter inequality following from the fact that $j \in \mathcal{L} \setminus \{\pi(1), \ldots, \pi(h-1)\}$. Hence, in particular, $\tilde{\pi}_{[h-1]} = \pi_{[h-1]}$. Moreover, $\tilde{\pi}$ and π have the same conditional distribution given $\pi_{[h-1]}$ (since they both correspond to choosing a bijection of $\{h, h+1, \ldots, l\}$ to $\mathcal{L} \setminus \{\pi(1), \ldots, \pi(h-1)\}$ uniformly) and $\tilde{\pi}$ is conditionally independent from $\pi_{[h]}$ given $\pi_{[h-1]}$. Therefore,

$$\begin{aligned} A_h - A_{h-1} &= \mathbb{E}[A(t)|\pi_{[h]}] - \mathbb{E}\left[A(t)|\pi_{[h-1]}\right] \\ &= \mathbb{E}[A(t)|\pi_{[h]}] - \mathbb{E}[\tilde{A}(t)|\pi_{[h-1]}] \\ &= \mathbb{E}[A(t) - \tilde{A}(t)|\pi_{[h]}] \,, \end{aligned} \tag{20}$$

for all $h = 1, \ldots, l$.

Now, observe that the value of $\pi(h)$ affects the depth-t neighborhoods of the node $\lambda(h)$, of its in-neighbors, the in-neighbors of its in-neighbors and so on, until those nodes from which $\lambda(h)$ can be reached in less than t hops, for a total of at most

$$\sum_{s=0}^{t-1} d_{\max}^s = \frac{d_{\max}^t - 1}{d_{\max} - 1} < \frac{d_{\max}^t}{d_{\max} - 1} = c$$

nodes in $\mathcal{N}$. Analogously, the value of j affects the depth-t neighborhoods of the node $\lambda(\pi^{-1}(j))$ as well as its in-neighbors, the in-neighbors of its in-neighbors and so on, for a total of less than c nodes in $\mathcal{N}$. It follows that, if $\tilde{A}(t) = \sum_i \tilde{Z}_i(t)$ where $\tilde{Z}(t)$ is the state vector of the TM dynamics on the network $\tilde{N}$ associated to the permutation $\tilde{\pi}$ in the configuration model, then $|A(t) - \tilde{A}(t)| \leq 2c$. It then follows from (20) and the above that

$$\begin{aligned} |A_h - A_{h-1}| &\leq \left| \mathbb{E}\left[A(t) - \tilde{A}(t)|\pi_{[h]}\right] \right| \\ &\leq \mathbb{E}\left[|A(t) - \tilde{A}(t)||\pi_{[h]}\right] \leq 2c \,. \end{aligned}$$

which proves (19). The claim follows from the Hoeffding-Azuma inequality as outlined earlier. □

By combining Propositions 2 and 3 we get the proof of Theorem 1, which was stated at the end of Section 3.

Proof of Theorem 1. Proposition 3 implies that $|z(t) - \overline{z}(t)| \leq \varepsilon/2$ for all but at most a fraction $2e^{-\varepsilon^2\beta n}$ of networks from the configuration model ensemble $\mathcal{C}_{n,\boldsymbol{p}}$. On the other hand, Proposition 2 implies that $|\overline{z}(t) - y(t)| \leq \varepsilon/2$ for $\gamma_t \leq n\varepsilon$. □

4.3 Extentions

We conclude this section by discussing how Theorem 1 can be extended to including two variants of the model: undirected configuration model and time-varying thresholds.

4.3.1 The PTM on the undirected configuration model

While Theorem 1 concerns the approximation of the average fraction of state-1 adopters in the TM dynamics for most networks in the directed configuration model ensemble $\mathcal{C}_{n,\boldsymbol{p}}$, for the PTM only the result can be extended to the undirected configuration model ensemble as defined below.

Let $u_{k,r,s} = p_{k,k,r,s}$ for $k \geq 0$, $0 \leq r \leq k$, and $s \in \{0,1\}$, denote the fraction of agents of degree k, threshold r and initial state s in an undirected network. We shall refer to $\boldsymbol{u} = \{u_{k,r,s}\}$ as undirected network statistics. A network size n and undirected network statistics $\boldsymbol{u}$ are said to be compatible if $nu_{k,r,s}$ is an integer for all $0 \leq r \leq k$ and $s = 0, 1$, and $l = \sum_{k\geq 0}\sum_{0\leq r\leq k}\sum_{s=0,1} nku_{k,r,s}$ is even. For compatible undirected network statistics $\boldsymbol{u}$ and size n, let $\mathcal{V} = \{1, \ldots, n\}$ be a node set and let κ, ρ, and σ be designed vectors of degrees, thresholds, and initial states, such that there is exactly a fraction $u_{k,r,s}$ of agents $i \in \mathcal{V}$ with $(\kappa_i, \rho_i, \sigma_i) = (k, r, s)$. Put $\mathcal{L} = \{1, 2, \ldots, l\}$, and let $\lambda : \mathcal{L} \to \mathcal{V}$ be a map such that $|\lambda^{-1}(i)| = \kappa_i$ for all agents $i \in \mathcal{V}$. Let π be a uniform random permutation of $\mathcal{L}$ and let the network $\mathcal{N} = (\mathcal{V}, \mathcal{E}, \rho, \sigma)$ have node set $\mathcal{V}$, link multiset $\mathcal{E} = \{(\lambda(\pi(2h-1)), \lambda(\pi(2h))), (\lambda(\pi(2h)), \lambda(\pi(2h-1))) : 1 \leq h \leq l/2\}$, threshold vector ρ, and initial state vector σ. Observe that, for every realization of the permutation π, the resulting network $\mathcal{N}$ is undirected, has size n and statistics $\boldsymbol{u}$. We refer to such network $\mathcal{N}$ as being sampled from the undirected configuration model ensemble $\mathcal{M}_{n,\boldsymbol{u}}$.

The key step for extending Theorem 1 to the PTM dynamics on undirected configuration model ensemble $\mathcal{M}_{n,\boldsymbol{u}}$ is the following result showing that the PTM dynamics on a rooted undirected tree coincides with PTM dynamics on the directed version of the tree.

Lemma 4. *For every network $\mathcal{T} = (\mathcal{V}, \mathcal{E}, \rho, \sigma)$ with undirected tree topology and every node $i \in \mathcal{V}$, the state vector $Z(t)$ of the PTM dynamics* (5) *on $\mathcal{T}$ satisfies*

$$Z_i(t) = Z_i^{(i)}(t)\,, \qquad t \geq 0\,,$$

where $Z^{(i)}(t)$ is the state vector of the PTM dynamics on the network $\overrightarrow{\mathcal{T}_{(i)}} = (\mathcal{V}, \overrightarrow{\mathcal{E}_{(i)}}, \rho, \sigma)$ with directed tree topology rooted in i, obtained from $\mathcal{T}$ by making all its links directed from nodes at lower distance from i to nodes at higher distance from it.

Proof. We proceed by induction on t. The case $t = 0$ is trivial as the initial condition is the same $Z_i(0) = \sigma_i = Z_i^{(i)}(0)$ for all $i \in \mathcal{V}$. Now, assuming that, for some given $t \geq 0$, the PTM dynamics on every network with undirected tree topology satisfies

$$Z_i(t) = Z_i^{(i)}(t)\,, \qquad \forall i \in \mathcal{V}$$

we will prove that

$$Z_i(t+1) = Z_i^{(i)}(t+1)\,, \qquad \forall i \in \mathcal{V}$$

for all networks with undirected tree topology $\mathcal{T} = (\mathcal{V}, \mathcal{E}, \rho, \sigma)$. We separately deal with the two cases: (a) $Z_i(t) = Z_i^{(i)}(t) = 1$; and (b) $Z_i(t) = Z_i^{(i)}(t) = 0$. Since we are considering the PTM dynamics, case (a) is easily dealt with, as $Z_i(t) = 1 = Z_i^{(i)}(t)$ implies $Z_i(t+1) = 1 = Z_i^{(i)}(t+1)$. On the other hand, in order to address case (b), let $\mathcal{J}$ be the set of neighbors of i in $\mathcal{T}$, which coincides with the set of offsprings of node i in $\overrightarrow{\mathcal{T}_{(i)}}$. For every $j \in \mathcal{J}$, let $\overrightarrow{\mathcal{T}_{(i,j)}} = (\mathcal{V}_{(i,j)}, \overrightarrow{\mathcal{E}_{(i,j)}}, \sigma, \rho)$ be the network obtained by restricting $\overrightarrow{\mathcal{T}_{(i)}}$ to node j and all its offsprings, let $\mathcal{T}_{(i,j)} = (\mathcal{V}_{(i,j)}, \mathcal{E}_{(i,j)}, \sigma, \rho)$ be the undirected version of $\overrightarrow{\mathcal{T}_{(i,j)}}$, and let $W(t)$ and $W^{(j)}(t)$ be the vector states of the PTM dynamics on $\mathcal{T}_{(i,j)}$ and $\overrightarrow{\mathcal{T}_{(i,j)}}$, respectively. Now, note that $Z_j^{(i)}(t) = W_j^{(j)}(t)$, since j has the same t-depth neighborhood in the two networks. On the other hand, note that, if the state of the PTM dynamics on $\mathcal{T}$ is such that $Z_i(t) = 0$, then $Z_i(s) = 0$ for all $0 \leq s \leq t$, so that the state of node j in the PTM dynamics on $\mathcal{T}$ depends only on the thresholds ρ_h and the initial states σ_h of agents $h \in \mathcal{V}_{(i,j)}$, and is the same as the state of node j in PTM dynamics on the original network $\mathcal{T}_{(i,j)}$, i.e., $Z_j(t) = W_j(t)$. Finally, observe that the inductive assumption applied to the restricted network $\mathcal{T}_{(i,j)}$ implies that $W_j(t) = W_j^{(j)}(t)$. It then follows that, if $Z_i(t) = Z_i^{(i)}(t) = 0$, then

$$Z_j(t) = W_j(t) = W_j^{(j)}(t) = Z_j^{(i)}(t)\,, \qquad \forall j \in \mathcal{J}\,.$$

This implies, by the structure of the recursive equation (5) that $Z_i(t+1) = Z_i^{(i)}(t+1)$. This completes the proof. $\square$

Using Lemma 4 it is straightforward to extend Proposition 1 to the undirected two-stage branching process. Then, the results in Section 4.2 carry over to the undirected configuration model ensemble without signficant changes, leading the following result.

Theorem 2. *Let $\mathcal{N}$ be a network sampled from the undirected configuration model ensemble $\mathcal{M}_{n,\boldsymbol{u}}$ of size n and statistics $\boldsymbol{u}$. Let $Z(t)$, for $t \geq 0$ be the state vector of the PTM dynamics* (5) *on $\mathcal{N}$, let $z(t) = \frac{1}{n}\sum_i Z_i(t)$, and let $y(t)$ be the output of the recursion* (7). *Then, for $\varepsilon > 0$ and $n \geq \gamma_t/\varepsilon$ where $\gamma_t = k_{\max}^{2t+4}/\overline{k}$,*

$$|z(t) - y(t)| \leq \varepsilon$$

for all but at most a fraction $2e^{-\varepsilon^2\beta n}$ of networks $\mathcal{N}$ from the $\mathcal{M}_{n,\boldsymbol{u}}$, where $\beta = (32\overline{k}k_{\max}^{2t})^{-1}$.

We stress the fact that the proposed extension of the approximation results for the undirected configuration model ensemble is strictly limited to the PTM and does not apply to the general TM. The key step where the structure of the PTM model is used is in the proof of Lemma 4 which allows one to reduce the study of the PTM on undirected trees to the one of PTM on directed trees. An analogous result does not hold true for the TM without permanent activation and indeed the analysis on undirected trees is known to face relevant additional challenges, see [26] for the majority dynamics (that can be considered a special case of the TM).

4.3.2 Time-varying thresholds

We first observe that, while we have not made it explicit yet, all the results discussed in this section carry over, along with their proofs, also for networks with time-varying thresholds $\rho_i(t)$. In this case, the network statistics

$$p_{d,k,r,s}(t) = \frac{1}{n}\left|\{i \in \mathcal{V} : \delta_i = d,\, \kappa_i = k,\, \rho_i(t) = r,\, \sigma_i = s\}\right|,$$

for $d \geq 0\,,\ 0 \leq r \leq k\,,\ s = 0, 1$, become time-varying, and so do their marginals

$$\begin{aligned} p_{k,r}(t) &:= \sum_{d\geq 0} \sum_{s=0,1} p_{d,k,r,s}(t)\,, \\ q_{k,r}(t) &:= \frac{1}{\overline{d}} \sum_{d\geq 0} \sum_{s=0,1} d q_{d,k,r,s}(t)\,, \end{aligned} \tag{21}$$

for $k, r \geq 0$. In contrast, $p_{d,k,s} = \sum_{0\leq r\leq k} p_{d,k,r,s}(t)$ remains constant in time since so do the degrees δ_i and κ_i and the initial states σ_i of all agents i. For networks with such time-varying thresholds, Theorem 1 continues to hold true with $y(t)$ equal to the output of the modified recursion

$$\begin{cases} x(t+1) = \phi(x(t), t)\,, \\ y(t+1) = \psi(x(t), t)\,, \end{cases} \qquad t \geq 0\,, \tag{22}$$

where $\phi(x,t) := \sum_{k\geq 0} \sum_{r\geq 0} q_{k,r}(t)\varphi_{k,r}(x)$ and $\psi(x,t) := \sum_{k\geq 0} \sum_{r\geq 0} p_{k,r}(t)\varphi_{k,r}(x)$.

A note of caution concerns extensions of Lemma 2 to networks with time-varying thresholds. This result, allowing one to identify the TM dynamics with the *progressive* TM (PTM) dynamics whenever the condition $\rho_i \leq \delta_i(1-\sigma_i)$ is met for all agents i, continues to hold true for time-varying networks only with the additional assumption that the thresholds are monotonically non-increasing in time, i.e., $\rho_i(t+1) \leq \rho_i(t)$ for every node i and time instant $t \geq 0$.

5 Numerical simulations on a real network

In this section, we discuss some numerical simulations testing the prediction capability of our theoretical results for the TM on the topology of the online social network Epinions.com. This was a general consumer review website with a community of users, operating until 2014. Members of the community could submit product reviews for any of over 100 000 products, rate other reviews and list the reviewers they trusted. The directed graph of trust relationships between users, called the "Web of Trust", was used in combination with the review's ratings to determine which reviews were shown to the users. The entire "Web of Trust" directed graph was obtained by crawling the website and is available from the online collection of [32]. The dataset[3] is a list of directed links representing the *who-trusts-whom* relations between users: the list contains 508 837 directed links corresponding to $n = 75\,879$ different users.

From the dataset topology, we computed the empirical joint degree statistic $p_{d,k} = n^{-1}\left|\{i : \delta_i = d, \kappa_i = k\}\right|$, i.e., the fractions of nodes with in-degree d and out-degree k. The marginals $\sum_k p_{d,k}$ and $\sum_d p_{d,k}$ follow an approximate power law distribution with exponent 1.6. About 32% of nodes has no in-neighbors while about 20% of nodes has no out-neighbors; 99% of the nodes have in and out-degree within $0 \leq d, k \leq 150$. The maximum in-degree and out-degree are 3 035 and 1 801 respectively; the average in/out-degree is 6.705. We also computed the fraction of links pointing to nodes with given in-degree d and out-degree k, i.e. the in-degree weighted, joint degree statistic $q_{d,k} = dp_{d,k}/\overline{d}$.

To simulate the TM we chose thresholds and initial states as follows. We introduce a vector $\Theta \in [0,1]^n$, of normalized thresholds with cumulative distribution function $F(\theta) := \frac{1}{n}|\{i :\, \Theta_i \leq \theta\}|$. Given the fraction $\upsilon \in [0,1]$, we consider the binary vector $\Sigma \in \{0,1\}^n$ such that $\upsilon = \frac{1}{n}\sum_i \Sigma_i$, i.e. a fraction υ of entries is equal to one. We define the network $\mathcal{N} = (\mathcal{V}, \mathcal{E}, \rho, \sigma)$ as follows. The agents' set $\mathcal{V}$ and the links' set $\mathcal{E}$ are those of the Epinions.com dataset. Let π' and π'' be two independent and uniformly chosen permutations on the set $\mathcal{V} = \{1, 2, \ldots, n\}$ The threshold vector ρ has entries $\rho_i = \lceil \Theta_{\pi'(i)}\kappa_i \rceil$, and the initial state vector σ has entries $\sigma_i = \Sigma_{\pi''(i)}$. Given $\mathcal{N}$, we compute the evolution of the configuration $Z(t) \in \{0,1\}^n$ according to (4) until a fixed time horizon T. From $Z(t)$ we compute the fraction $z(t)$ of state-1 adopters at time t, as well as the fraction $a(t) := \frac{1}{|\mathcal{E}|}\sum_i \delta_i Z_i(t)$ of links pointing to state-1 adopters.

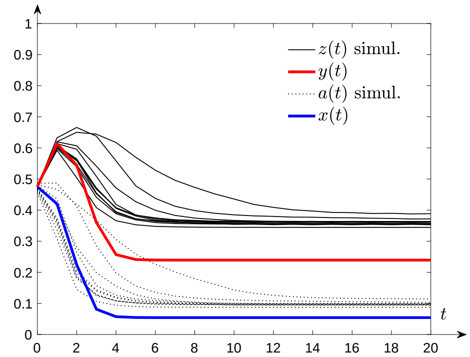

Fig. 4. Simulations of the TM dynamics on the Epinions.com topology, with agents endowed with the thresholds $\rho_i = \lceil \frac{1}{2}\kappa_i \rceil$. The initial states are randomly selected, conditioned on a fraction $\upsilon = 0.475$ of nodes having $\sigma_i = 1$. The plot contains the simulations of the fraction of state-1 adopters $z(t)$ (thin black lines) and the corresponding fraction of links pointing to state-1 adopters $a(t)$ (thin dotted lines). These simulations shall be compared with the recursion's output dynamic $y(t)$ (thick red line) and with the recursion's state dynamic $x(t)$ (thick blue line) respectively; the latter is initialized using $\xi = \upsilon$. The recursion captures the qualitative behavior of these simulations fairly well, with a mismatch of about 15% between the limit of $y(t)$ and the values to which the simulated $z(t)$ are about to settle. A close look reveals that somer simulations show a little ripple with period two.

The following examples describe three group of simulations. We will use $h(x)$ to denote the right-continuous unit step function, $h(x) = 1$ for $x \geq 0$, $h(x) = 0$ for $x < 0$.

Example 1. *In the first group of simulations we assume that every agent has normalized threshold* $\Theta_i = 0.500$*, corresponding to a distribution function* $F(\theta) = h(\theta - \frac{1}{2})$*. Hence, the threshold of agent* i *is* $\rho_i = \lceil \frac{1}{2}\kappa_i \rceil$*. Given* $\upsilon \in [0,1]$*, each simulation consists in choosing a random initial state assignment such that exactly a fraction* υ *of nodes has* $\sigma_i = 1$ *and in computing the TM dynamic until a prearranged time horizon* T*. For each* υ *we typically produce some simulations and compare them with the dynamic predicted with the recursion, initialized with* $\xi = \upsilon$*. Figure 4 represents some simulations with* $\upsilon = 0.475$*: the top plot contains the simulated dynamics* $a(t)$ *to be compared with the recursion's state dynamic* $x(t)$*; the bottom plot contains the corresponding simulated fraction of active nodes,* $z(t)$*, to be compared with the recursion's output dynamic* $y(t)$*. The recursion captures the qualitative behavior of the simulations. The top plot of Figure 5 represents the recursion's functions* $\phi(x)$ *and* $\psi(x)$ *corresponding to this group of simulations. The bottom plot of the same figure compares the asymptotic activation predicted by the recursion with several simulations, obtained for various* υ *and*

3. Retrieved from snap.stanford.edu/data/soc-Epinions1.html.

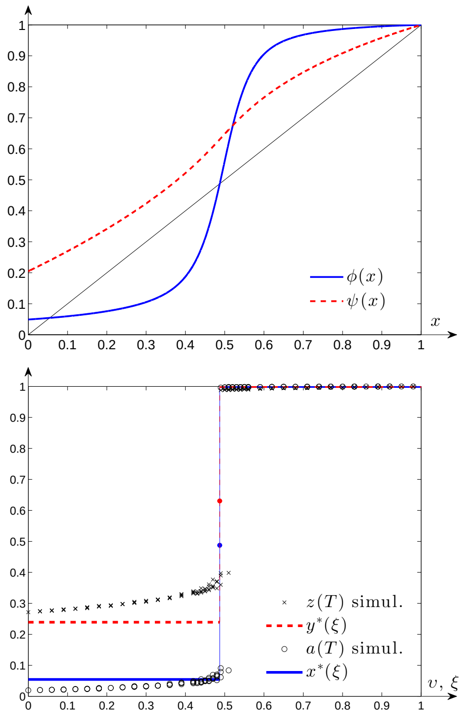

Fig. 5. The top plot reports the functions $\phi(x)$ (solid blue) and $\psi(x)$ (dashed red), corresponding to the Epinions.com network where each agent i is endowed with the thresholds $\rho_i = \lceil \frac{1}{2}\kappa_i \rceil$. The bottom plot compares the values reached by the simulations at the time horizon $T = 100$, for various value of the fraction υ of initially active nodes, with the asymptotic activation predicted by the recursion initialized with $\xi = \upsilon$. The black crosses represent $z(T)$, i.e., the fraction of state-1 adopters, to be compared with the recursion limits $y^*(\xi)$ in dashed red. The black circles represent $a(T)$, i.e., the fraction of links pointing to state-1 adopters, to be compared with the recursion limits $x^*(\xi)$ in dashed red. Near the discontinuity, predicted in $\xi^* \approx 0.487$ and well matched by the simulations, the starting values of υ are more dense.

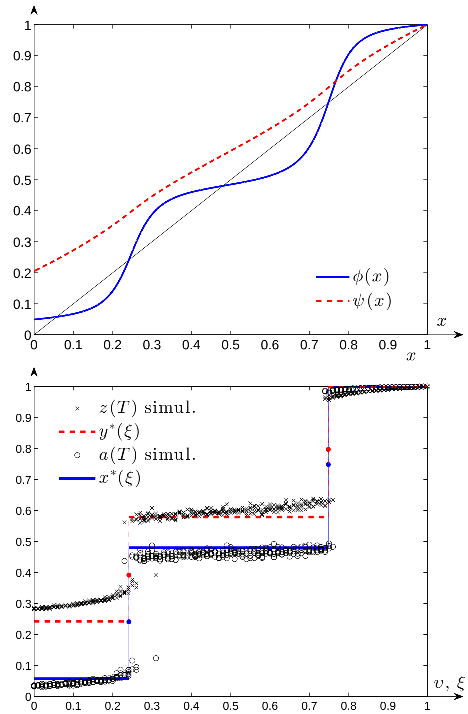

Fig. 6. The top plot contains the functions $\phi(x)$ (solid blue) and $\psi(x)$ (dashed red), corresponding to the Epinions.com network where 40% of the nodes is endowed with the normalized threshold $\frac{1}{4}$ and the remaining 60% by $\frac{3}{4}$. The bottom plot compares the values reached by the simulations at the time horizon $T = 100$, for various value of the fraction υ of initially active nodes, with the asymptotic activation predicted by the recursion initialized with $\xi = \upsilon$. The black crosses represent the fraction of state-1 adopters $z(T)$, to be compared with the recursion limits $y^*(\xi)$ in dashed red. The black circles represent the fraction of links pointing to state-1 adopters $a(T)$, to be compared with the recursion limits $x^*(\xi)$ in dashed red. The predicted limits $y^*(\xi)$ and $x^*(\xi)$ are discontinuous for $\xi_1^* \approx 0.241$ and $\xi_2^* \approx 0.7482$, which are the two unstable equilibria of $\phi(x)$ (cf. top plot). The discontinuities are well matched by the simulations, except for one point obtained with $\upsilon = 0.310$. Apart from matching the discontinuities, the simulated values show a slowly increasing trend, unexpected from the recursion limits.

using a time horizon $T = 100$. The fractions of state-1 adopters $z(T)$ shall be compared with the recursion's output asymptotic value $y^*(\xi)$, while the corresponding fraction of links pointing at state-1 adopters, $a(T)$, shall be compared with the recursion's state asymptotic value $x^*(\xi)$. The simulations match well the discontinuity predicted in $\xi^* \approx 0.487$. Before the discontinuity, the simulated values of $z(T)$ are higher that the limit $y^*(\xi)$, showing an increasing trend. The same trend is present in the corresponding values of $a(T)$, that are however closer to the limit $x^*(\xi)$. After the discontinuity, simulations and limits agree.

Example 2. *In the second group of simulations we allow the normalized thresholds to take two different values: to 40% of the nodes we assign $\frac{1}{4}$ as normalized threshold; the remaining 60% of nodes gets $\frac{3}{4}$. The choice corresponds to the cumulative distribution of the normalized threshold $F(\theta) = \frac{4}{10}h(\theta - \frac{1}{4}) + \frac{6}{10}h(\theta - \frac{3}{4})$. The top plot of Figure 6 represents the functions $\phi(x)$ and $\psi(x)$ corresponding to the thresholds chosen: the recursion predicts the presence of two discontinuities in the asymptotic activation for the TM, for the seed values $\xi_1^* \approx 0.241$ and $\xi_2^* \approx 0.7482$, that correspond to the unstable equilibria of $\phi(x)$. The bottom plot of Figure 6 compares the predicted asymptotic activation with the simulations, computed for various υ up to time $T = 100$. The fractions of state-1 adopters $z(T)$ shall be compared with the recursion's output asymptotic value $y^*(\xi)$, while the corresponding fraction of links pointing at state-1 adopters, $a(T)$, is nearly superimposed to recursion's state asymptotic value $x^*(\xi)$. The plot shows a good agreement between $a(T)$ and $x^*(\xi)$, while $z(T)$ seems a bit underestimated by $y^*(\xi)$. The values $z(T)$ and $a(T)$ of a simulation with $\upsilon = 0.310$ settled to a smaller limit, compatible with those obtained for $\upsilon < 0.270$. Apart from this simulation, the discontinuities are matched well. Also here the values of $z(T)$ (and less markedly those of $a(T)$) show an increasing trend with respect to the fraction of initially active nodes υ, a behavior not predicted by the recursion limits.*

Example 3. *Finally, we present a group of simulations where we allow the normalized thresholds to take three different values: 30% of the nodes are endowed with the normalized threshold $\frac{1}{5}$, 30% by $\frac{1}{2}$ and the remaining 40% by $\frac{4}{5}$. The corresponding cumulative distribution is $F(\theta) = \frac{3}{10}h(\theta - \frac{1}{5}) + \frac{3}{10}h(\theta - \frac{1}{2}) + \frac{4}{10}h(\theta - \frac{4}{5})$. The top plot of Figure 7 represents the functions $\phi(x)$ and $\psi(x)$, with $\phi(x)$ showing seven fixed points. The bottom plot of the same figure contains the dynamic of the fraction of state-1 node $z(t)$, starting from a fraction $\upsilon = 0.700$ of initial adopters. The simulations are compared with the output $y(t)$ of the recursion: the majority of the simulations tend to a limit*

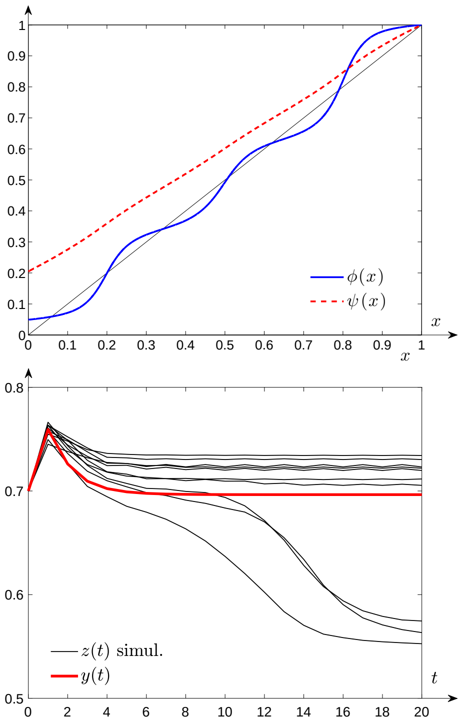

Fig. 7. Top: the functions $\phi(x)$ (solid blue) and $\psi(x)$ (dashed red), corresponding to the Epinions.com network where 30% of the nodes is endowed with the normalized threshold $\frac{1}{5}$, 30% by $\frac{1}{2}$ and the remaining 40% by $\frac{4}{5}$. Bottom: some simulations (thin black lines) of the dynamic of the fraction of state-1 adopters, $z(t)$, starting from a fraction $\upsilon = 0.700$ of nodes with state one. The majority of these simulations tend to a limit just above the recursion and some show a ripple with period two; three simulations tend to a smaller value. The simulations are compared with the output $y(t)$ of (7) (thick red line).

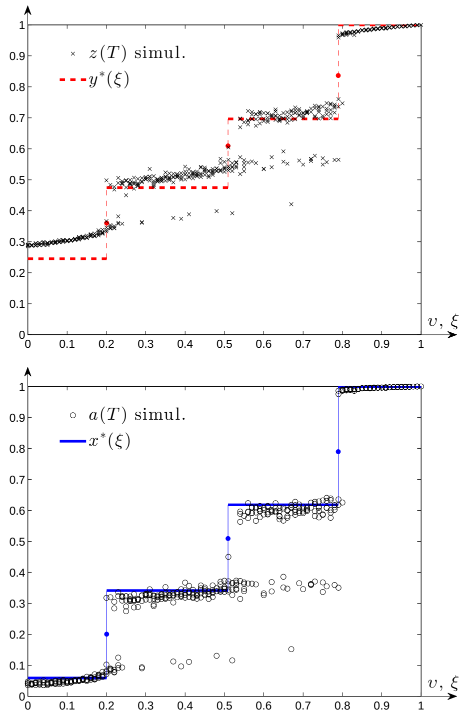

Fig. 8. Comparison between the predicted asymptotic activation and the actual simulations, on the Epinions.com graph where 30% of the nodes is endowed with the normalized threshold $\frac{1}{5}$, 30% by $\frac{1}{2}$ and the remaining 40% by $\frac{4}{5}$. The top plot contains the simulated values of the fraction of state-1 adopters $z(T)$ at time $T = 100$ (black crosses), for various υ, compared with the limit $y^*(\xi)$ (red dashed line) of the recursion output, obtained assuming $\xi = \upsilon$. The bottom plot represents the values of the fraction of links pointing to state-1 adopters, $a(t)$, for the corresponding simulations, to be compared with the asymptotic value of the recursion's state $x^*(\xi)$. We observe that some simulationst settle to values that are smaller than those of the points having similar υ. With this choice of normalized thresholds, the limits $y^*(\xi)$ and $x^*(\xi)$ have three discontinuities, in $\xi_1^* \approx 0.201$, $\xi_2^* \approx 0.509$ and $\xi_3^* \approx 0.789$.

just above the recursion, while showing a ripple with period two; three simulations tend to a smaller value. With this choice of normalized thresholds, the recursion predicts the presence of three discontinuities in the asymptotic activation for the TM, in $\xi_1^* \approx 0.201$, $\xi_2^* \approx 0.509$ *and* $\xi_3^* \approx 0.789$. *Figure 8 compares the simulations with the predicted asymptotic activation. The top plot represents the simulated values of* $z(T)$ *at time* $T = 100$, *for various* υ, *and the limit* $y^*(\xi)$, *obtained assuming the initial condition* $\xi = \upsilon$. *The bottom plot represents the corresponding simulated values of* $a(T)$, *at* $T = 100$, *and the recursion's limit* $x^*(\xi)$. *Some of the simulations in Figure 8 settle to values smaller than the those of the points having similar* υ, *values that might be expected from a smaller initial condition.*

Overall, the simulations of the TM on Epinions.com give some interesting insights and are in good agreement with the prediction obtained with the recursion (7). A few differences between the simulations and the predictions remain. In several simulations we observed that the dynamics of $z(t)$ and $a(t)$ presents an oscillation of period two, superimposed to the settling value. For few simulations, in particular in the last example, the supposed settling value, evaluated with $z(T)$ and $a(T)$ at time $T = 100$, seemed to be smaller that what expected.Finally, besides the expected jumps, the values $z(T)$ and $a(T)$ seem to have an increasing trend with respect to the initial value υ while the values $z(T)$ seem to be a little but consistently underestimated by the recursion.

There are some possible reasons for these behaviors. The graph used in these simulations comes from an online network which is not a completely random network. The recursion does not take into account the community structure of the network, which may play a role in the oscillations observed as well as in the increasing trend of the settling values. Furthermore, the network contains few nodes with extremely high in/out-degree. These nodes can bias a simulation depending on their initial state and threshold. This may contribute to explain the presence of trajectories with smaller-than-expected settling value. The recursion has however a good predicting capability: the discontinuities in the settling values of the simulations match well with the jumps in the recursion's limits.

6 Conclusion

We have shown that, for all but an asymptotically vanishing fraction of networks with given degree and threshold statistics, the fraction of state-1 adopters in the TM can be approximated by the output of a one-dimensional nonlinear recursion. Our results apply both to the original TM and to the Progressive TM on the configuration model ensemble of directed networks and for the Progressive TM (but not to the original TM) on the configuration model ensemble of undirected networks. Simulations run on the social network Epinions.com confirm the validity of our theoretical results. Ongoing work is concerned with the use of the obtained one-dimensional recursion for the design of feedback control policies for the TM – see [33, ch. 4] for preliminary results. Another direction consists in applying our approach to large-scale networks containing communities, [25], [34].

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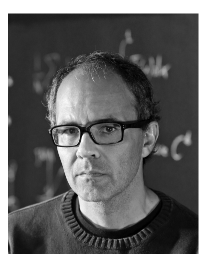

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