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Abstract

The following fact is known for large classes of distributed control systems: when the target is regular, there exists a regular steering control. This fact is important to prove convergence estimates of numerical algorithms for the approximate computation of the steering control.

In this paper we extend this property to a class of systems with persistent memory (of Maxwell/Boltzmann type) and we show that it is possible to construct such smooth control via the solution of an optimization problem.

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1 Introduction

We study the following system where $x \in (0,\pi)$ and $t > 0$:

$$\begin{cases} w''(x,t) = w_{xx}(x,t) + \int_0^t M(t-s)w_{xx}(x,s)\,\mathrm{d}s\,, \quad w(0,t) = f(t)\,, \quad w(\pi,t) = 0 \\ w(x,0) = 0\,, \quad w'(x,0) = 0\,. \end{cases} \tag{1}$$

We assume $M(t) \in H^2(0,T)$ and $f(t) \in L^2(0,T)$ for every $T > 0$. As proved for example in [5], $w(x,t) \in C([0,T];L^2(0,\pi)) \cap C^1([0,T];H^{-1}(0,\pi))$ and, for every $(\xi,\eta) \in L^2(0,\pi) \times H^{-1}(0,\pi)$ and $T > 2\pi$, there exists $f \in L^2(0,T)$ such that $w(T) = \xi\,, \quad w'(T) = \eta$. We prove:

Theorem 1. *The following properties hold:*

1. *Let $(\xi,\eta) \in H_0^1(0,\pi) \times L^2(0,\pi)$ and let $T > 2\pi$. There exists a steering control $f \in H_0^1(0,T)$.*

2. *One of the smooth steering controls is the integral of the function g which realizes the minimum of a suitable quadratic functional introduced in Sect. 3.*

The statement 1 is proved in Section 2 while statement 2 is in Section 3.

We conclude this introduction with few comments. First we note that system (1) is often encountered in the study of viscoelasticity and diffusion equations with memory. When $M(t) = 0$ of course it reduces to the string equation. In the case of the wave equation (even when x in regions of $\mathbb{R}^d$, $d > 1$) theorem 1 is known. The proof that we give, based on moment methods, shows in particular controllability (in $H_0^1(0,\pi) \times L^2(0,\pi)$) of the cascade connection of system (1) with an integrator. We refer to [7, Ch. 11] and references therein for this idea and to [8] for a precise analysis of the reachable set using smooth controls in the case of the wave equation.

For memoryless systems, a result analogous to Theorem 1 is the key for a numerical analysis of the construction of steering controls via otimization methods, see [1].

Finally, it is easy to guess that Theorem 1 can be extended to the case $\dim x > 1$ and to higher regularity degree of the target. This will be the subject of a future analysis.

2 The moment problem and the proof of Theorem 1 item 1

The following computations are a bit simplified if we integrate the first equation of (1) on $[0,t]$ and we write it in the equivalent form (here $N(t) = 1 + \int_0^t M(s)\,\mathrm{d}s$)

$$w'(x,t) = \int_0^t N(t-s) w_{xx}(x,s)\,\mathrm{d}s\,, \qquad w(x,0) = 0\,, \quad w(0,t) = f(t)\,,\ w(\pi,t) = 0\,. \tag{2}$$

We use the orthonormal basis of $L^2(0,\pi)$ whose elements are $\Phi_n = \sqrt{(2/\pi)}\sin nx$, $n \in \mathbb{N}$, and we expand

$$w(x,t) = \sum_{n\in\mathbb{N}} \Phi_n(x) w_n(t)\,, \quad w_n(t) = \sqrt{\frac{2}{\pi}} \int_0^\pi \Phi_n(x) w(x)\,\mathrm{d}x\,.$$

Then $w_n(x,t)$ must satisfy

$$w_n'(t) = -n^2 \int_0^t N(t-s) w_n(s)\,\mathrm{d}s + n \int_0^t N(t-s)\left(\sqrt{2/\pi} f(s)\right)\,\mathrm{d}s\,.$$

The function $\sqrt{2/\pi}f$ will be renamed f.

Let $z_n(t)$ solve

$$z_n'(t) = -n^2 \int_0^t N(t-s) z_n(s)\, \mathrm{d}s\,, \quad z_n(0) = 1\,. \tag{3}$$

We have (see [3])

$$\begin{aligned} w_n(t) &= n \int_0^t \left(\int_0^{t-s} N(t-s-\tau) z_n(\tau) d\tau \right) f(s)\, \mathrm{d}s = \\ &= \frac{1}{n} \int_0^t \left(\frac{\mathrm{d}}{\mathrm{d}s} z_n(t-s) \right) f(s)\, \mathrm{d}s\,, \quad (4) \\ w_n'(t) &= n \int_0^t \left(-\frac{\mathrm{d}}{\mathrm{d}s} \int_0^{t-s} N(t-s-\tau) z_n(\tau) d\tau \right) f(s)\, \mathrm{d}s\,. \quad (5) \end{aligned}$$

We require that a target $(\xi, \eta) \in H_0^1(0,\pi) \times L^2(0,\pi)$ is reached at time T, i.e. we require $(w(T), w'(T)) = (\xi, \eta)$.

The Fourier expansion of the targets is

$$\xi = \sum_{n=1}^{+\infty} \frac{\xi_n}{n} \Phi_n\,, \quad \text{and} \quad \eta = \sum_{n=1}^{+\infty} \eta_n \Phi_n\,, \qquad (\{\xi_n\}, \{\eta_n\}) \in l^2(\mathbb{N}) \times l^2(\mathbb{N})\,.$$

So, controllability to (ξ, η) at time T is equivalent to the existence of a control $f \in L^2(0,T)$ such that $w_n(T) = \xi_n/n$, $w_n'(T) = \eta_n$ for every n. The expression we found for $w_n(t)$ and $w_n'(t)$ suggest that we investigate whether is it possible to solve this problem with

$$f(t) = \int_0^t g(s)\, \mathrm{d}s\,, \quad g \in L^2(0,T)\,. \tag{6}$$

If this is possible then we have the existence of an H^1-steering control, and we get a steering control in $H_0^1(0,T)$ if we can find g which satisfies the additional condition

$$\int_0^T g(s)\, \mathrm{d}s = 0\,. \tag{7}$$

We replace the expression (6) in $w_n(T)$ and $w_n'(T)$ and we integrate by parts. We see that f is an H^1 steering control to (ξ, η) if the following *moment problem* is solvable:

$$\xi_n = \int_0^T g(r)\,\mathrm{d}r - \int_0^T z_n(T-s)g(s)\,\mathrm{d}s\,, \tag{8}$$

$$\eta_n = \int_0^T \left[n\int_0^{T-s} N(T-s-r)z_n(r)\,\mathrm{d}r \right] g(s)\,\mathrm{d}s = \int_0^T g(T-s)\left(\frac{-z_n'(s)}{n}\right)\,\mathrm{d}s\,. \tag{9}$$

We multiply equation (9) by i and we sum to (8). Furthermore we impose the additional condition (7). We find the moment problem:

$$\int_0^T Z_n(s)g(T-s)\,\mathrm{d}s = c_0\,, \quad c_n = \begin{cases} -\xi_n - i\eta_n & \text{if} \quad n>0 \\ 0 & \text{if} \quad n=0 \end{cases} \tag{10}$$

and $Z_n(t) = \left(z_n(s) + \frac{i}{n}z_n'(s)\right)$ if $n>0$, $Z_0(t) = 1$. In order to prove statement 1 of Theorem 1, we prove solvability of the moment problem (10).

We note that $\{c_n\}_{n>0}$ is an arbitray *complex valued* $l^2(\mathbb{N})$ sequence while g is real (when ξ and η are real). We reformulate the moment problem (10) with $n\in\mathbb{Z}$. We proceed as follows: for $n<0$ we define:

$$z_n(t) = z_{-n}(t), \quad \Phi_n(x) = \Phi_{-n}(x)\,, \quad Z_{-n}(t) = \bar{Z}_n(t)\,.$$

As in [5, Lemma 5.1], we see that the moment problem (10) can be equivalently studied with $n\in\mathbb{Z}$ and g complex valued.

Our goal is the proof that the moment problem (10), $n\in\mathbb{Z}$, is solvable. Even more, we prove that $\{Z_n(t)\}_{n\in\mathbb{Z}}$ is a Riesz sequence in $L^2(0,T)$, provided that $T>2\pi$.

Remark 2. *The fact that $\{Z_n(t)\}_{n\in\mathbb{Z}}$ is a Riesz sequence in $L^2(0,T)$ implies the following additional information:* **1)** *the transformation from $g\in L^2(0,T)$ (and so also from $f\in H_0^1(0,T)$) to $(w(T),w'(T))\in H_0^1(0,\pi)\times L^2(0,\pi)$ is linear and continuous;* **2)** *the solution $g\in L^2(0,T)$ of minimal norm of the moment problem depends continuously on the target $(\xi,\eta)\in H_0^1(0,\pi)\times L^2(0,\pi)$. Integrating this function g as in (6) we get the steering control f of minimal norm in $H_0^1(0,T)$ and so the solution $f\in H_0^1(0,T)$ of minimal norm depends continuously on the target $(\xi,\eta)\in H_0^1(0,\pi)\times L^2(0,\pi)$;* **3)** *any solution g of the moment problem belongs to* $L^2_{0,T} = \left\{h\in L^2(0,T)\,:\,\int_0^T h(s)\,\mathrm{d}s = 0\right\}$.

2.1 The proof that $\{Z_n\}_{n\in\mathbb{Z}}$ is a Riesz sequence in $L^2(0,T)$, $T>2\pi$

The proof that $\{Z_n\}_{n\in\mathbb{Z}}$ is a Riesz sequence in $L^2(0,T)$, $T>2\pi$, is divided in two steps: in the first one we show that the sequence $\{Z_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ is a Riesz sequence in $L^2(0,T)$. Then we will prove that $\{Z_n\}_{n\in\mathbb{Z}}$ is a Riesz sequence in $L^2(0,T)$ too. In the proof we use the following definitions and results (see [5, Chp. 3]): a sequence $\{x_n\}$ in a Hilbert space H is:

- a *Riesz sequence* when it is the image of an orthonormal sequence under a linear bounded and boundedly invertible transformation;
- ω-*independent* when the following holds: if $\{\alpha_n\}\in l^2$ and if $\sum_{n=1}^{+\infty}\alpha_n x_n = 0$ (convergence in the norm of H) then $\{\alpha_n\}=0$.

Let $\{x_n\}$ be a Riesz sequence in the Hilbert space H and let $\{y_n\}$ be quadratically close to $\{x_n\}$, i.e. $\sum\|x_n-y_n\|_H^2<+\infty$. Then there exists N such that $\{y_n\}_{|n|>N}$ is a Riesz sequence. If furthermore $\{y_n\}$ is ω-independent then it is a Riesz sequence too.

We introduce the notation and $\mathbb{Z}'=\mathbb{Z}\setminus\{0\}$.

Step 1: $\{Z_n\}_{n\in\mathbb{Z}'}$ is a Riesz sequence in $L^2(0,T)$, $T>2\pi$ This part of the proof is contained in [5]. The proof in [5] is quite complex since there $x\in\Omega\subseteq\mathbb{R}^d$, $d\geq 1$. When $d=1$ the proof is much simplified and goes as we sketch here for completeness.

We put $N'(0)=\gamma$. Using [4, Lemmas 5.2 and 5.5] we get that for every $T>0$ there exists C such that

$$\sum_{n\in\mathbb{Z}'}\left\|Z_n(t)-e^{\gamma t}e^{int}\right\|^2_{L^2(0,T)}\leq C\,. \tag{11}$$

Then there exists $N>0$ such that $\{Z_n\}_{|n|\geq N}$ is a Riesz sequence in $L^2(0,T)$.

We prove that $\{Z_n\}_{n\in\mathbb{Z}'}$ is ω-independent i.e. we prove that $\{\alpha_n\}_{n\in\mathbb{Z}'}=0$ when $\{\alpha_n\}\in l^2(\mathbb{Z}')$ and

$$\sum_{n\in\mathbb{Z}'}\alpha_n Z_n=0 \quad \text{i.e.} \quad \sum_{n\in\mathbb{Z}'}\alpha_n\left(z_n+\frac{i}{n}z_n'\right)=0\,. \tag{12}$$

Using $T>2\pi$ and [5, Lemma 3.4] applied twice it is possible to prove that $\alpha_n=\frac{\gamma_n}{n^2}$ with $\{\gamma_n\}\in l^2(\mathbb{Z}')$ (see also [3]). This fact justifies the termwise differentiation of the series (12). Using

$$z_n''(t)=-n^2N(t)-n^2\int_0^t N(t-s)z_n'(s)\,\mathrm{d}s \tag{13}$$

we get

$$\int_0^t N(t-s)\left[\sum_{n\in\mathbb{Z}'}\gamma_n\left(z_n(s)+\frac{i}{n}z_n'(s)\right)\right]\,\mathrm{d}s-iN(t)\left[\sum_{n\in\mathbb{Z}'}\frac{\gamma_n}{n}\right]=0\,. \quad (14)$$

Computing with $t=0$ we see that $\sum_{n\in\mathbb{Z}'}n\alpha_n=\sum_{n\in\mathbb{Z}'}\frac{\gamma_n}{n}=0$ and so, using $N(0)\neq 0$, we get

$$\sum_{n\in\mathbb{Z}'}\left[n^2\alpha_n z_n(s)+in\alpha_n z_n'(s)\right]=0 \quad \text{hence} \quad \sum_{n\neq\pm1,\ n\in\mathbb{Z}'}\alpha_n(n^2-1)\left[z_n+\frac{iz_n'}{n}\right]=0\,.$$

Note that $\{\alpha_n(n^2-1)\}=\{\alpha_n^{(1)}\}\in l^2(\mathbb{Z}')$. Hence we can start a boostrap argument and repeat this procedure. After at most $2N$ iteration of the process we get

$$\sum_{|n|>N}\alpha_n^{(N)}Z_n=0$$

and so $\alpha_n^{(N)}=0$ when $|n|>N$ since we noted that $\{Z_n\}_{|n|>N}$ is a Riesz sequence in $L^2(0,T)$. We have $\alpha_n^{(N)}=0$ if and only if $\alpha_n=0$ and this shows that the series (12) is a finite sum, $\sum_{n\in\mathbb{Z}',\ |n|\leq N}\alpha_n Z_n=0$. The proof is now finished since it is easy to prove, as in [5, 6], that *the sequence* $\{Z_n(t)\}_{n\in\mathbb{Z}'}$ *is linearly independent.*

Step 2: $\{Z_n\}_{n\in\mathbb{Z}}$ is a Riesz sequence Of course, $\{Z_n\}_{n\in\mathbb{Z}}$ is quadratically close to $\{e^{\gamma t}e^{int}\}_{n\in\mathbb{Z}}$. It remains to prove ω-independence, when $T>2\pi$. We prove $\{\alpha_n\}_{n\in\mathbb{Z}}=0$ when $\{\alpha_n\}\in l^2(\mathbb{Z})$ and

$$\alpha_0+\sum_{n\in\mathbb{Z}'}\alpha_n Z_n=0\,. \quad (15)$$

Using that constant functions belong to H^1 and [5, Lemma 3.4] applied twice we see that $\alpha_n=\gamma_n/n^2$, $\{\gamma_n\}\in l^2$. So, we can compute termwise the derivatives of both the sides of (15) and we get

$$\sum_{n\in\mathbb{Z}'}\alpha_n\left(z_n'(t)+\frac{i}{n}\left[-n^2N(t)-n^2\int_0^t N(t-s)z_n'(s)\,\mathrm{d}s\right]\right)=0\,. \quad (16)$$

Computing with $t=0$ we get $\sum_{n\in\mathbb{Z}'}\alpha_n n=0$. Then (using (3)) the equation (16) becomes

$$\int_0^t N(t-s)\left[\sum_{n\in\mathbb{Z}'}\left(\alpha_n n^2 z_n(s)+i\alpha_n n z_n'(s)\right)\right]\,\mathrm{d}s=0$$

so that (using again $N(0) \neq 0$ and $\{\alpha_n n^2\} \in l^2$)

$$\sum_{n\in\mathbb{Z}'} \alpha_n n^2 \left[z_n(t) + i\frac{1}{n} z_n'(t) \right] = \sum_{n\in\mathbb{Z}'} \alpha_n n^2 Z_n(t) = 0\,. \tag{17}$$

The fact that $\{Z_n(t)\}_{n\in\mathbb{Z}'}$ is a Riesz sequence implies that $\{\alpha_n\} = 0$ and so also $\alpha_0 = 0$, as we wanted to prove.

This ends the proof of Statement 1 in Theorem 1.

3 Variational characterization of the steering control and the proof of item 2 of Theorem 1

The fact that $\{Z_n\}_{n\in\mathbb{Z}}$ is a Riesz sequence implies that the moment problem (10) admits solutions $g \in L^2(0,T)$ when $T > 2\pi$. Each one of these functions, once integrated, provides a steering control $f \in H_0^1(0,T)$. In this section we give a variational characterization of a solution g of the moment problem (10) as the minimizer of a quadratic functional, as in [2].

We recall the following definition from Remark 2:

$$L^2_{0,T} = \left\{ h \in L^2(0,T)\,, \quad \int_0^T h(s)\,\mathrm{d}s = 0 \right\} \subseteq L^2(0,T)$$

and we consider the problem

$$\begin{cases} w''(x,t) = w_{xx}(x,t) + \int_0^t M(t-s) w_{xx}(x,s)\,\mathrm{d}s\,, \\ y'(t) = g(t) \in L^2_{0,T}\,, \\ w(0,t) = y(t)\,, \quad w(\pi,t) = 0\,, \\ w(x,0) = 0\,,\; w'(x,0) = 0\,,\; y(0) = 0\,. \end{cases} \tag{18}$$

We proved that $(w(T), w'(T)) = (\xi,\eta) \in H_0^1(0,\pi) \times L^2(0,\pi)$ (and $y(T) = 0$) if and only if g solves the moment problem (10) with $n \in \mathbb{Z}$ (note that the condition $y(T) = 0$ comes for free, implied by $g \in L^2_{0,T}$). The first statement in Remark 2 implies that

$$\Lambda_T \in \mathcal{L}\left(L^2_{0,T}, H_0^1(0,\pi) \times L^2(0,\pi) \right) \text{ where } \Lambda_T g = \big(w(T), w'(T) \big)\,. \tag{19}$$

Let (W_0, W_1) be any element of $L^2(0,\pi) \times H^{-1}(0,\pi)$ and consider

$$\begin{cases} W''(x,t) = W_{xx}(x,t) + \int_0^t M(t-s) W_{xx}(x,s)\,\mathrm{d}s\,, \\ Y'(t) = \int_0^t M(t-s) W_x(0,s)\,\mathrm{d}s + W_x(0,t)\,, \\ W(0,t) = W(\pi,t) = 0\,, \\ W(x,0) = W_0 = \sum_{n=1}^{+\infty} W_n^0 \Phi_n\,,\ W'(x,0) = W_1 = \sum_{n=1}^{+\infty} \left(n W_n^1\right) \Phi_n\,,\ Y(0) = 0 \end{cases} \tag{20}$$

(note that $\{W_n^0\}$, $\{W_n^1\}$ belong to l^2).

We introduce the notations $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)$ to denote respectively the duality pairing of $H_0^1(0,\pi)$ and $H^{-1}(0,\pi)$ and the inner product in $L^2(0,\pi)$. Assuming first $g \in \mathcal{D}(0,T)$, W_0, W_1 in $\mathcal{D}(0,\pi)$ we multiply the first equation of (18) with $W(x,T-t)$ and the second one with $Y(T-t)$. Then we integrate respectively on $(0,\pi)\times(0,T)$ and on $(0,T)$ and we sum. Standard integrations by parts show that

$$(w'(T), W_0) + \langle w(T), W_1\rangle = \int_0^T g(s) Y(T-s)\,\mathrm{d}s \tag{21}$$

Using statement 1) in Remark 2, i.e. the continuous dependence of $(w(T), w'(T)) \in H_0^1(0,\pi) \times L^2(0,\pi)$ on $g \in L^2_{0,T}$, we see that

$$\begin{aligned} \left|\int_0^T g(s) Y(T-s)\,\mathrm{d}s\right| &= \left|(w'(T), W_0) + \langle w(T), W_1\rangle\right| \le \\ \le & |w'(T)|_{L^2(0,\pi)} |W_0|_{L^2(0,\pi)} + |w(T)|_{H_0^1(0,\pi)} |W_1|_{H^{-1}(0,\pi)} \le \\ \le & M\left[|W_0|_{L^2(0,\pi)} + |W_1|_{H^{-1}(0,T)}\right] |g|_{L^2_{0,T}}\,. \end{aligned}$$

So, the transformation $(W_0, W_1) \to Y(\cdot) \in L^2(0,T)$ admits a continuous extension to $L^2(0,\pi) \times H^{-1}(0,\pi)$ and we see also that $g \in L^2_{0,T}$ steers the solution of (18) to the target $(w(T), w'(T)) = (\xi,\eta) \in H_0^1(0,\pi) \times L^2(0,\pi)$ if and only if the following equality holds for every $W_0 \in L^2(0,\pi)$, $W_1 \in H^{-1}(0,\pi)$:

$$(\eta, W_0) + \langle \xi, W_1\rangle = \int_0^T g(s) Y(T-s)\,\mathrm{d}s = \int_0^T g(s)\left(P_0 Y(T-\cdot)\right)\,\mathrm{d}s \tag{22}$$

where P_0 is the orthogonal projection of $L^2(0,T)$ onto $L^2_{0,T}$ (easily computed from cosine Fourier expansion).

We introduce the duality pairing of $H_0^1(0,\pi) \times L^2(0,\pi)$ and its dual $H^{-1}(0,\pi) \times L^2(0,\pi)$:

$$\langle\!\langle (\xi,\eta), (W_1, W_0)\rangle\!\rangle = (\eta, W_0) + \langle \xi, W_1\rangle$$

so that Equality (21) takes the form

$$\langle\!\langle \Lambda_T g, (W_1, W_0) \rangle\!\rangle = \int_0^T g(s)\,(P_0 Y(T-\cdot))\ \mathrm{d}s\,,\ \text{hence}\ \Lambda_T^*(W_1,W_0) = P_0 Y(T-\cdot)\,.$$

Similar to [2], we consider the quadratic functional $(W_1, W_0) \mapsto \mathcal{J}(W_1, W_0)$ on $H^{-1}(0,\pi) \times L^2(0,\pi)$ defined by

$$\begin{aligned}\mathcal{J}(W_1,W_0) &= \frac{1}{2}\int_0^T |P_0 Y(T-\cdot)|^2\ \mathrm{d}t - (\eta, W_0) - \langle \xi, W_1\rangle = \\ &= \frac{1}{2}\int_0^T |\Lambda_T^*(W_0,W_1)|^2\ \mathrm{d}t - \langle\!\langle (\xi,\eta),(W_1,W_0)\rangle\!\rangle\,.\end{aligned}$$

Computing the Frèchet derivative of $\mathcal{J}$ we see that $(\hat W_1, \hat W_0) \in H^{-1}(0,\pi)\times L^2(0,\pi)$ is a stationary point if and only if

$$\int_0^T (P_0 Y(T-\cdot))\left(P_0 \hat Y(T-\cdot)\right)\ \mathrm{d}t - (\eta, W_0) - \langle \xi, W_1\rangle = 0 \quad \forall (W_1,W_0) \in H^{-1}(0,\pi)\times L^2(0,\pi)$$

(here Y and $\hat Y$ are the functions computed from (20) and initial conditions respectively $(W_0, W_1, 0)$ and $(\hat W_0, \hat W_1, 0)$). We see from here that if $(\hat W_1, \hat W_0)$ is a stationary point of $\mathcal{J}$ then $\hat g(t) = P_0 \hat Y(T-\cdot)$ realizes the equality (22), and so it is a steering control.

In order to complete the proof of item 2 of Theorem 1 we note the following result, which implies that *$\mathcal{J}$ has a unique stationary point in $H^{-1}(0,\pi)\times L^2(0,\pi)$, which is a minimum point.*

Theorem 3. *The functional $\mathcal{J}$ is continuous, coercive and strictly convex on $H^{-1}(0,\pi)\times L^2(0,\pi)$.*

Proof: Convexity is obvious and continuity follows since (19) implies $\Lambda_T^* \in \mathcal{L}\left(H^{-1}(0,\pi)\times L^2(0,\pi), L^2_{0,T}\right)$. The proof of strict convexity is the same as in [2].

The operator $\Lambda_T g = (w(T), w'(T))$ from $g \in L^2_{0,T}$ to $H^1_0(0,\pi)\times L^2(0,\pi)$ is surjective so that its adjoint Λ_T^* is coercive. So, we have coercivity of the quadratic part of $\mathcal{J}$, hence of the functional $\mathcal{J}$ itself. ∎

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