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A FICTITIOUS DOMAIN APPROACH FOR WAVE PROPAGATION PROBLEMS IN UNBOUNDED DOMAINS

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Abstract. *We consider the scattering of waves by an obstacle $\mathcal{O} \subset \mathbb{R}^2$ having a sufficiently smooth boundary Γ. By using a fictitious domain approach, we artificially extend the solution inside the obstacle $\mathcal{O}$. Then we solve the original problem in the simpler domain that includes $\mathcal{O}$ and that is artificially bounded by a perfectly transparent boundary $\mathcal{B}$, delimiting the region of interest. We prescribe on $\mathcal{B}$ a Non Reflecting Boundary Condition (NRBC) which is based on a space-time integral equation and that defines a relationship that the solution of the differential problem and its normal derivative must satisfy on $\mathcal{B}$ (see [2]). Further, we enforce the Dirichlet boundary condition on Γ in a weak sense, by means of Lagrange multipliers. The NRBC is discretized on $\mathcal{B}$ by combining a special second order (in time) convolution quadrature and a standard collocation method in space. In the enlarged domain, this discretization is coupled with an unconditionally stable ODE time integrator and a FEM in space. The constraint on Γ is imposed by a matrix B_h that represents a discrete trace operator.*

A particularly useful application of this approach is the scattering of a wave by rotating rigid bodies. In this case the method avoids the complexity of constructing at each time step a new finite element computational mesh and requires only the construction of the discrete trace operator B_h. We present some results we have obtained for rotating (even multiple) scatterers and non trivial data.

1 INTRODUCTION

In recent years, numerical methods for the solution of time dependent problems describing waves scattered by an obstacle have received considerable attention. Boundary Element Methods are particularly useful for treating infinite domain problems and have been successfully used in the solution of linear wave propagation problems in two and three dimensions. In such problems the method offers the great advantage of describing the solution only by its boundary values, thus reducing the problem dimensions by one; hence, only a discretization of the boundary is necessary, which significantly reduces the number of unknowns, if compared to domain-discretization-based methods such as finite element methods. Once the density function is retrieved (by solving the corresponding boundary integral equation), the solution at any point is obtained by computing boundary integrals. This procedure may result costly, especially when the solution is required at many points of the infinite domain.

Alternatively, the use of domain-discretization-based methods requires a finite computational domain with prescribed boundary conditions. A key issue is therefore the choice of a bounded computational domain, where one is interested in studying the behavior of the solution, and the introduction of boundary conditions which guarantee that the solution of the initial boundary value problem inside the finite computational domain coincides with the restriction to the computational domain of the solution of the original problem, which is defined in the infinite region. The method of Artificial (or Absorbing) Boundary Condition (ABC) consists of introducing an artificial boundary $\mathcal{B}$ that truncates the infinite domain and determines two distinct regions: a bounded domain of interest Ω and a residual infinite domain $\mathcal{D}$. By analyzing the problem in $\mathcal{D}$, a Non Reflecting Boundary Condition (NRBC) on $\mathcal{B}$ is derived in order to avoid spurious reflections. Once the NRBC is given, it is used to solve the problem in Ω by using a numerical method such as, for example, finite differences or finite elements.

Many NRBCs have been proposed in the last two-three decades, and most of them are local, both in time and space. For a review, see for example [5], [6], [7]. All these papers, except for [12], Sections 5.5, 5.6, [8], [9], [2], deal with the construction of NRBC with the property of absorbing only outgoing waves, not waves that are either outgoing or incoming. Therefore, known sources must necessarily be included in the computational domain. However, this can be a severe drawback when, for example, sources are far away from the physical domain. Moreover, the NRBC holds only for a single convex artificial boundary having a special shape, like a circle (sphere) or ellipse (ellipsoid). Only in the last years multiple scattering problems have been examined (see [8], [9]). Very recently, in ([2]) we have proposed a (fully) non local NRBC, based on a boundary integral relationship. For its discretization, we have constructed a numerical scheme which is based on a second order Lubich discrete convolution quadrature formula for the discretization of the time integral, coupled with a classical collocation method in space. Among the advantages of this transparent condition we recall the following: it allows the use of (smooth) curve (surfaces) of arbitrary shape; it can be used also in situations of multiple scattering; it allows the treatment of sources and initial data that must not be necessarily included in the finite computational domain; its computational cost is much lower that what it might first appear, due to some special properties of the coefficients of the Lubich convolution quadratures. Indeed, if in the 2D case we choose as $\mathcal{B}$ a circle, the CPU required for the solution of some test problems is similar to that of local NBRCs.

In this paper we aim at using the NRBC proposed in [2] as a transparent condition coupled with a fictitious domain method. In particular, we consider the scattering of a wave by an obstacle $\mathcal{O} \subset \mathbb{R}^2$ having a smooth boundary Γ. By using the fictitious domain approach, we artificially extend the solution inside the obstacle $\mathcal{O}$ and we impose the boundary conditions on Γ in the weak form, by means of Lagrange multipliers. Then, we truncate the infinite external domain by an artificial boundary $\mathcal{B}$ where we impose the transparent boundary conditions proposed in [2]. We solve the original problem in the finite computational domain $\widetilde{\Omega}$ that includes $\mathcal{O}$ and is bounded by the artificial boundary $\mathcal{B}$. To this end, we discretize the space-time integral equation on $\mathcal{B}$ by combining a second order (in time) BDF convolution quadrature and a collocation method in space. Such a discretization is then coupled with an unconditionally stable ODE time integrator and a FEM in space. The main point is that the FEM mesh is defined on the enlarged domain $\widetilde{\Omega}$ and that the constraint on Γ is imposed by a matrix B_h that represents a discrete trace operator.

A significant application of this approach is the scattering of a wave by rotating rigid bodies. In this case the method avoids the complexity of constructing at each time step a new finite element computational mesh and requires only the construction of the discrete trace operator B_h. We will present some new results that we have obtained for rotating (even multiple) scatterers and non trivial data.

2 The model problem

Let $\mathcal{O} \subset \mathbb{R}^2$ be an open bounded domain with a sufficiently smooth boundary Γ; define $\Omega^e = \mathbb{R}^2 \setminus \bar{\mathcal{O}}$. We consider the following wave propagation problem in Ω^e:

$$\begin{cases} u^e_{tt}(\mathbf{x},t) - \Delta u^e(\mathbf{x},t) & = f(\mathbf{x},t) \quad \text{in } \Omega^e \times (0,T) \\ u(\mathbf{x},t) & = 0 \quad \text{in } \Gamma \times (0,T) \\ u^e(\mathbf{x},0) & = u_0(\mathbf{x}) \quad \text{in } \Omega^e \\ u^e_t(\mathbf{x},0) & = v_0(\mathbf{x}) \quad \text{in } \Omega^e. \end{cases} \tag{1}$$

Since in general one has to determine the solution u^e of the above problem in a bounded subregion of Ω^e, surrounding the physical domain $\mathcal{O}$, we truncate the infinite domain Ω^e by introducing an artificial smooth boundary $\mathcal{B}$. This boundary divides Ω^e into two (open) sub-domains: a finite computational domain Ω, which is bounded internally by Γ and externally by $\mathcal{B}$, and an infinite residual domain $\mathcal{D}$. For simplicity, we assume that the initial condition u_0, the initial velocity v_0 and the source term f have a local support which is included in Ω. We impose on $\mathcal{B}$ the exact non reflecting boundary condition

$$\frac{1}{2}u(\mathbf{x},t) = \boldsymbol{\mathcal{V}}\partial_{\mathbf{n}_\mathcal{D}} u(\mathbf{x},t) - \boldsymbol{\mathcal{K}} u(\mathbf{x},t) \quad \mathbf{x} \in \mathcal{B}, \tag{2}$$

where

$$\boldsymbol{\mathcal{V}}\psi(\mathbf{x},t) := \int_0^t \int_{\mathcal{B}} G(\mathbf{x}-\mathbf{y},t-\tau)\psi(\mathbf{y},\tau)d\mathcal{B}_{\mathbf{y}}d\tau,$$

and

$$\boldsymbol{\mathcal{K}}\varphi(\mathbf{x},t) := \int_0^t \int_{\mathcal{B}} \partial_{\mathbf{n}_\mathcal{D}} G(\mathbf{x}-\mathbf{y},t-\tau)\varphi(\mathbf{y},\tau)d\mathcal{B}_{\mathbf{y}}d\tau,$$

are the single and double layer integral operators;

$$G(\mathbf{x},t) = \frac{1}{2\pi}\frac{H(t-\|\mathbf{x}\|)}{\sqrt{t^2-\|\mathbf{x}\|^2}},$$

is the fundamental solution of the wave equation (1) (being $H(\cdot)$ the Heaviside functions); $\partial_{\mathbf{n}_\mathcal{D}} u$ is the outward normal derivative to $\mathcal{B} = \partial\mathcal{D}$. We refer the reader to [2] for details on the derivation of the NRBC and on the regularity of the operators $\boldsymbol{\mathcal{V}}$ and $\boldsymbol{\mathcal{K}}$.

Denoting by $u_\mathcal{B}(\mathbf{x}, t) = u(\mathbf{x}, t)_{|\mathcal{B}}$ and $\lambda_\mathcal{B}(\mathbf{x}, t) = -\partial_{\mathbf{n}_\mathcal{D}} u(\mathbf{x}, t)$, the model problem (defined in the domain of interest Ω) takes the form:

$$\begin{cases} u_{tt}(\mathbf{x},t) - \Delta u(\mathbf{x},t) & = f(\mathbf{x},t) & \text{in}\, \Omega \times (0,T) \\ u(\mathbf{x},t) & = 0 & \text{in}\, \Gamma \times (0,T) \\ \frac{1}{2} u_\mathcal{B}(\mathbf{x},t) + \boldsymbol{\mathcal{V}}\lambda_\mathcal{B}(\mathbf{x},t) + \boldsymbol{\mathcal{K}} u_\mathcal{B}(\mathbf{x},t) & = 0 & \text{in}\, \mathcal{B} \times (0,T) \\ u(\mathbf{x},0) & = u_0(\mathbf{x}) & \text{in}\, \Omega \\ u_t(\mathbf{x},0) & = v_0(\mathbf{x}) & \text{in}\, \Omega, \end{cases}$$

3 The fictitious domain approach

In order to describe the fictitious domain approach, we introduce the extended domain $\widetilde{\Omega} := \Omega \cup \mathcal{O}$ which is bounded by the artificial boundary $\mathcal{B}$ (see Figure 1). The main idea of the fictitious domain method (or domain embedding method) consists in extending artificially the solution of the exterior problem inside the obstacle, and to solve the new problem in the whole extended domain $\widetilde{\Omega}$ (see [4], [11] and their references). The main advantage of this approach is the possibility of solving the problem in a simpler domain by treating the Dirichlet boundary conditions on Γ by Lagrange multipliers, working with a given fixed mesh in the enlarged domain.

Figure 1: Geometry of the problem (left plot) and the fictitious domain approach (right plot)

For a generic function w, we set $w(t)(\mathbf{x}) := w(\mathbf{x}, t)$. Following [2] and [4], the problem defined in the domain of interest $\widetilde{\Omega}$ consists in finding the triad of unknown functions $(u(t), \lambda_\Gamma(t), \lambda_\mathcal{B}(t)) \in H^1(\widetilde{\Omega}) \times H^{-1/2}(\Gamma) \times H^{-1/2}(\mathcal{B})$ such that, for all $w \in H^1(\widetilde{\Omega}), \varphi \in H^{-1/2}(\Gamma), \mu \in H^{-1/2}(\mathcal{B})$, the following generalized saddle-point evolution problem

$$\begin{cases} \dfrac{d^2}{dt^2}(u(t), w)_{\widetilde{\Omega}} + a(u(t), w) - < \lambda_\Gamma(t), w >_\Gamma - < \lambda_\mathcal{B}(t), w >_\mathcal{B} & = (f(t), w) \\ < \varphi, u(t) >_\Gamma & = 0 \\ \frac{1}{2} < \mu, u_\mathcal{B}(t) >_\mathcal{B} + < \mu, \boldsymbol{\mathcal{V}}\lambda_\mathcal{B}(t) >_\mathcal{B} + < \mu, \boldsymbol{\mathcal{K}} u_\mathcal{B}(t) >_\mathcal{B} & = 0 \\ u(0) & = u_0 \\ \frac{du}{dt}(0) & = v_0. \end{cases}, \qquad (3)$$

holds in the distributional sense in $(0,T)$, where $a : H^1(\widetilde{\Omega}) \times H^1(\widetilde{\Omega}) \to \mathbb{R}$ is the bilinear form

$$a(v,w) = \int_{\widetilde{\Omega}} \nabla v \cdot \nabla w,$$

and $(v,w)_{\widetilde{\Omega}} = \int_{\widetilde{\Omega}} vw$. The bilinear forms $< \cdot, \cdot >_\Gamma$ and $< \cdot, \cdot >_{\mathcal{B}}$ denote the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, and $H^{-1/2}(\mathcal{B})$ and $H^{1/2}(\mathcal{B})$, respectively.

The weak formulation (3) is the natural setting for the analysis of the method and to deal with convergence issues and error estimates of the associated finite element numerical approximation. However, its numerical solution using methods such as finite and boundary elements, makes it hardly competitive with existing NRBC approaches of local type, in particular when the problem does not have far field sources. From this point of view, replacing the weak formulation of the NRBC with a strong one makes the scheme certainly more appealing, although stronger smoothness conditions on the solution are required. Therefore, in the next section we will consider the coupling of the weak form of the differential equation with the strong form (2) of the NRBC.

At the moment, the theory to justify the validity of the presented approach is still at an early stage. For this reason we have performed an intensive numerical testing to validate the proposed method.

4 Approximation

4.1 Discretization of the NRBC

We start by briefly recalling the main steps of the Lubich-collocation method for the discretization of the NRBC (2) (for more details we refer to [10] and [3]).

The Lubich convolution quadrature formulas have the fundamental property of not using explicitly the expression of the kernel of the integral equation they are applied to, which is instead replaced by that of its Laplace transform. In particular, the discretization in time is based on the splitting of the interval $[0,T]$ into N steps of equal length $\Delta_t = T/N$ and in the collocation of the equation at the discrete time levels $t_n = n\Delta_t$, $n = 0,\dots,N$.

The time integrals appearing in the definition of the single and double layer operators are discretized by means of the convolution quadrature formula associated with the second order Backward Differentiation Method (BDF) for ordinary differential equations (see [3]):

$$\begin{aligned}
(\boldsymbol{\mathcal{V}}\lambda_{\mathcal{B}})(\mathbf{x},t_n) &\approx \sum_{j=0}^{n} \int_{\mathcal{B}} \omega_{n-j}^{\boldsymbol{\mathcal{V}}}(\Delta_t; ||\mathbf{x}-\mathbf{y}||)\lambda_{\mathcal{B}}(\mathbf{y},t_j)\, d\mathcal{B}_{\mathbf{y}}, \quad n = 0,\dots,N \\
(\boldsymbol{\mathcal{K}}u_{\mathcal{B}})(\mathbf{x},t_n) &\approx \sum_{j=0}^{n} \int_{\mathcal{B}} \omega_{n-j}^{\boldsymbol{\mathcal{K}}}(\Delta_t; ||\mathbf{x}-\mathbf{y}||)u_{\mathcal{B}}(\mathbf{y},t_j)\, d\mathcal{B}_{\mathbf{y}}, \quad n = 0,\dots,N
\end{aligned}$$

The coefficients $\omega_n^{\mathcal{J}}$, $\mathcal{J} = \boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{K}}$, are given by

$$\omega_n^{\mathcal{J}}(\Delta_t; ||\mathbf{x}-\mathbf{y}||) = \frac{1}{2\pi\imath} \int_{|z|=\rho} K^{\mathcal{J}}\left(||\mathbf{x}-\mathbf{y}||, \frac{\gamma(z)}{\Delta_t}\right) z^{-(n+1)}\, dz \tag{4}$$

where in this case $K^{\boldsymbol{\mathcal{V}}} = \widehat{G}$ is the Laplace transform of the kernel G appearing in the definition of the single layer operator $\boldsymbol{\mathcal{V}}$, and $K^{\boldsymbol{\mathcal{K}}} = \widehat{\partial G/\partial \mathbf{n}}$ is the Laplace transform of the kernel $\partial G/\partial \mathbf{n}$ appearing in the definition of the double layer operator $\boldsymbol{\mathcal{K}}$. In particular,

$$K^{\boldsymbol{\mathcal{V}}}(r,s) = \frac{1}{2\pi} K_0(rs), \qquad K^{\boldsymbol{\mathcal{K}}}(r,s) = -\frac{1}{2\pi} s K_1(rs) \frac{\partial r}{\partial \mathbf{n}},$$

where $K_0(z)$ and $K_1(z)$ are the second kind modified Bessel function of order 0 and 1, respectively. The function $\gamma(z) = 3/2 - 2z + 1/2z^2$ is the so called characteristic quotient of the BDF method of order 2. The parameter ρ is such that for $|z| \leq \rho$ the corresponding $\gamma(z)$ lies in the domain of analyticity of $K^{\mathcal{J}}$.

For the space discretization, we introduce a parametrization of the curve $\mathcal{B}$, $\mathbf{x} = \boldsymbol{\psi}(x) = (\psi_1(x), \psi_2(x))$ and $\mathbf{y} = \boldsymbol{\psi}(y) = (\psi_1(y), \psi_2(y))$ with $x, y \in [a, b]$. Then, at every time instant t_j we approximate the unknown function $u_{\mathcal{B}}(\boldsymbol{\psi}(x), t_j)$ and its normal derivative $\lambda_{\mathcal{B}}(\boldsymbol{\psi}(x), t_j)$ by continuous piecewise linear interpolants, associated with a partition $\{x_k\}_{k=1}^{M+1}$ of the parametrization interval $[a, b]$. Denoting by $\{N_i(x)\}$ the classical Lagrangian basis functions of local degree 1 associated with the spatial partition, and collocating the fully discretized equation at the points $\xi_h = x_h$, we obtain (see [2]) the following absorbing condition at time t_n (written in matrix notation):

$$\left(\frac{1}{2}\mathbf{I} + \mathbf{K}_0\right)\mathbf{u}_{\mathcal{B}}^n + \mathbf{V}_0\boldsymbol{\lambda}_{\mathcal{B}}^n = -\sum_{j=0}^{n-1}\mathbf{K}_{n-j}\mathbf{u}_{\mathcal{B}}^j - \sum_{j=0}^{n-1}\mathbf{V}_{n-j}\boldsymbol{\lambda}_{\mathcal{B}}^j, \quad n = 1, \ldots, N. \tag{5}$$

where $\mathbf{I}$ denotes the identity matrix and the matrices $\mathbf{V}$ and $\mathbf{K}$ are given by

$$(\mathbf{V}_{n-j})_{hi} = \int_a^b \omega_{n-j}^{\boldsymbol{\mathcal{V}}}(\Delta_t; \|\boldsymbol{\psi}(x_h) - \boldsymbol{\psi}(y)\|)N_i(y)\|\boldsymbol{\psi}'(y)\|dy,$$

and

$$(\mathbf{K}_{n-j})_{hi} = \int_a^b \omega_{n-j}^{\boldsymbol{\mathcal{K}}}(\Delta_t; \|\boldsymbol{\psi}(x_h) - \boldsymbol{\psi}(y)\|)N_i(y)\|\boldsymbol{\psi}'(y)\|dy.$$

We remark that, since the role of the NRBC is to define on $\mathcal{B}$ a relationship between the wave and its normal derivative, to prevent the raising of spurious waves, the more accurate is the discretized relationship the more transparent this will be. To this end, having chosen a continuous piecewise linear approximant for $u_{\mathcal{B}}$, we use an approximant of the same type also for $\lambda_{\mathcal{B}}$. The integrals appearing in the definition (4) of the coefficients $\omega_n^{\mathcal{J}}$ of the quadrature formula are efficiently computed by using the trapezoidal rule with $L \geq N$ equal steps of length $2\pi/L$. Moreover, they are computed simultaneously by using the FFT algorithm, with $O(N \log N)$ flops.

4.2 Finite element approximation and time discretization of the complete scheme

For the space discretization we consider a regular decomposition of the enlarged domain $\widetilde{\Omega} = \cup_{K \in \mathcal{T}_h} K$ into triangles K of edge length h. We introduce the finite dimensional space

$$X_h = \{w_h \in C^0(\widetilde{\Omega}) : w_{h|_K} \in \mathbb{P}^1(K), K \in \mathcal{T}_h\} \subset H^1(\widetilde{\Omega}),$$

for the finite element approximation of u. We also introduce the space $W_h \in H^{-1/2}(\mathcal{B})$ of piecewise linear (continuous) functions defined on the boundary $\mathcal{B}$ (the traces of the piecewise linear functions defined in X_h) and the space $M_\delta \in H^{-1/2}(\Gamma)$ of piecewise constant functions defined on a partition of Γ into segments of stepsize δ. We remark that the two stepsize parameters h and δ are a priori independent, but their choice is subject to a compatibility relation between the two spaces X_h and M_δ which guarantees the well posedness of the problem (see Lemma 4.1).

We approximate the variational problem (3) by: find $u_h \in X_h$, $\lambda_{\Gamma,\delta} \in M_\delta$, $\lambda_{\mathcal{B},h} \in W_h$ such that

$$\begin{cases} \dfrac{d^2}{dt^2}(u_h, w_h)_{\widetilde{\Omega}} + a(u_h, w_h) - < \lambda_{\Gamma,\delta}, w_h >_\Gamma - < \lambda_{\mathcal{B},h}, w_h >_{\mathcal{B}} & = (f(t), w_h) \quad \forall\, w_h \in X_h, \\ < \varphi_\delta, u_h >_\Gamma & = 0 \qquad\quad \forall \varphi_\delta \in M_\delta \end{cases}$$

For the time discretization we consider the Crank-Nicolson integration method; by introducing the new variable $v := \partial u / \partial t$, with obvious notation, we approximate the variational problem (3) by: find $u_h^{n+1} \in X_h$, $\lambda_{\Gamma,\delta}^{n+1} \in M_\delta$, $\lambda_{\mathcal{B},h}^{n+1} \in W_h$ such that for all $w_h \in X_h$ and for all $\varphi_\delta \in M_\delta$

$$\begin{cases} (u_h^{n+1}, w_h)_{\widetilde{\Omega}} + \frac{\Delta_t^2}{4} a(u_h^{n+1}, w_h) - & \frac{\Delta_t^2}{4} < \lambda_{\Gamma,\delta}^{n+1}, w_h >_\Gamma - \frac{\Delta_t^2}{4} < \lambda_{\mathcal{B},h}^{n+1}, w_h >_{\mathcal{B}} = \\ & (u_h^n, w_h)_{\widetilde{\Omega}} - \frac{\Delta_t^2}{4} a(u_h^n, w_h) + \frac{\Delta_t^2}{4} < \lambda_{\Gamma,\delta}^n, w_h >_\Gamma \\ & + \frac{\Delta_t^2}{4} < \lambda_{\mathcal{B},h}^n, w_h >_{\mathcal{B}} + \Delta_t (v_h^n, w_h)_{\widetilde{\Omega}} + \frac{\Delta_t^2}{4} (f^{n+1} + f^n, w_h)_{\widetilde{\Omega}} \\ < \varphi_\delta, u_h^{n+1} >_\Gamma = 0 & \\ v_h^{n+1} = \frac{2}{\Delta_t}(u_h^{n+1} - u_h^n) - v_h^n & \end{cases}$$

which, in matrix form, reads

$$\begin{cases} \left(\mathbf{M_h} + \frac{\Delta_t^2}{4}\mathbf{A_h}\right)\mathbf{u}^{n+1} - \frac{\Delta_t^2}{4}\mathbf{B}_\delta \boldsymbol{\lambda}_\Gamma^{n+1} - \frac{\Delta_t^2}{4}\mathbf{Q_h}\boldsymbol{\lambda}_{\mathcal{B}}^{n+1} & = \left(\mathbf{M_h} - \frac{\Delta_t^2}{4}\mathbf{A_h}\right)\mathbf{u}^n + \frac{\Delta_t^2}{4}\mathbf{B}_\delta \boldsymbol{\lambda}_\Gamma^n \\ & \quad + \frac{\Delta_t^2}{4}\mathbf{Q_h}\boldsymbol{\lambda}_{\mathcal{B}}^n + \Delta_t \mathbf{M_h} \mathbf{v}^n \\ & \quad + \frac{\Delta_t^2}{4}\left(\mathbf{f}^{n+1} + \mathbf{f}^n\right) \\ {\mathbf{B}_\delta}^T \mathbf{u}^{n+1} = 0 & \\ \mathbf{v}_k^{n+1} = \frac{2}{\Delta_t}(\mathbf{u}_k^{n+1} - \mathbf{u}_k^n) - \mathbf{v}_k^n & \end{cases} \tag{6}$$

where:

- $\mathbf{M_h}$ denotes the mass matrix;
- $\mathbf{A_h}$ denotes the stiffness matrix;
- the rectangular matrix $\mathbf{B}_\delta$ represents a discrete trace operator on Γ;
- the rectangular matrix $\mathbf{Q_h}$ represents a discrete trace operator on $\mathcal{B}$.

Equation (6) is finally coupled with the discretized NRBC equation (5).

We recall that the following result holds (see [4]):

Lemma 4.1. *Suppose that the spaces X_h and M_δ satisfy that the space mesh size used for the discretization of the obstacle Γ is three times larger than the space mesh size used for the discretization of the computational domain $\widetilde{\Omega}$, then there exists a constant $C > 0$ (independent of h and δ) such that the following discrete inf-sup condition holds*

$$\inf_{\lambda_\delta \in M_\delta} \sup_{v_h \in X_h} \frac{< \lambda_\delta, v_h >_\Gamma}{\|\lambda_\delta\|_{-1/2,\Gamma} \|v_h\|_{1,\widetilde{\Omega}}} \geq C \tag{7}$$

Condition (7) guarantees that the rank of the matrix $\mathbf{B}_\delta$ is maximum.

5 Numerical results

In this section, we present some examples of the numerical testing we have performed by using the approach discussed in the previous section. We consider both fix and rotating obstacles. In the first case, to test the accuracy of the approximation obtained by the fictitious domain approach, we construct a reference "exact" solution by applying the Lubich-collocation boundary element method described in [3] with a very fine space and time discretization. Once the density function is retrieved, the solution at any point in the infinite domain is obtained by computing the associated potential (see [3] for details). This solution will be denoted by the acronym BEM.

Example 1 As a first test we consider the scattering of a wave by a fix obstacle, represented by a disk of radius 2. The wave is propagating radially, starting from an initial configuration $u_0(x, y) = e^{-5((x-5)^2+y^2)}$, without the presence of any external source ($f = 0$). Although u_0 does not have a local support (and thus contradicts one of our assumptions), it decays exponentially fast away from its center $x = (5, 0)$, in such a way that, from the computational point of view, it can be regarded as compact and supported in a disk with radius smaller than 3 (at distance 2.7 from its center it assumes approximately values of the order $1e - 16$). We choose the artificial boundary $\mathcal{B}$ as a circle of radius $R = 10$, so that the support of u_0 is included in $\widetilde{\Omega}$. The disk bounded by Γ represents a soft obstacle that acts as a reflecting body. The enlarged domain $\widetilde{\Omega}$ is the whole disk of radius 10. In Figure 2 we show snapshots of the solution at some time instants. In Figure 3, left plot, we compare the solution obtained by the fictitious approach with the reference one (BEM) at the boundary mesh point $x \approx (10, 0)$ and for $t \in [0, 20]$. The approximate solution has been obtained by a decomposition of the domain $\widetilde{\Omega}$ into 68724 triangles and by choosing a uniform partition of Γ into 128 segments. With such a choice the spatial step size is $h \approx 7.6e - 02$, while $\delta \approx 9.8e - 02$. The time interval has been decomposed into $N = 256$ time steps. We note that the solution is zero until the initial data reaches the artificial boundary (around $t = 4$). Approximatively at time $t = 2.5$, the wave reaches the boundary Γ and is perfectly reflected back, so that around $t = 9$ we see another outgoing wave at the artificial boundary $\mathcal{B}$. After that time, the wave is completely out of the annulus, as the reference solution and the approximate solution with the exact NRBC show. In the right plot we show the energy of the system. Since the system is a conservative one, the energy remains constant for the time instants $t \in [0, 4]$ after which it starts dissipating because the wave reaches the artificial boundary and leaves the finite computational domain. It is worthwhile noting that the wave bumps the obstacle approximatively at $t = 2.5$ but, since the obstacle is fix, the energy is perfectly preserved up to $t \approx 4$.

In the next two examples we apply the proposed scheme to the diffraction of a wave by rotating bodies. In this case the location of the obstacles depends on t. Boundary element methods seem difficult to apply to such types of problem and standard finite elements would require the reconstruction of the computational mesh at each time step. To avoid such complexity, the fictitious domain approach seems a very attractive solution. In case of rotating rigid obstacles, the modification of the previously described scheme simply consists in replacing the boundary matrix $\mathbf{B_h}$ by a time dependent one.

For the treatment of rotating obstacles, an alternative approach is the one of embedding the rotating body in a domain that rotates together with the scatterer. Such a domain is in turn placed inside a stationary residual domain (see for example [1] where a similar strategy is applied to

Figure 2: Example 1: Snapshots of the solution at different times.

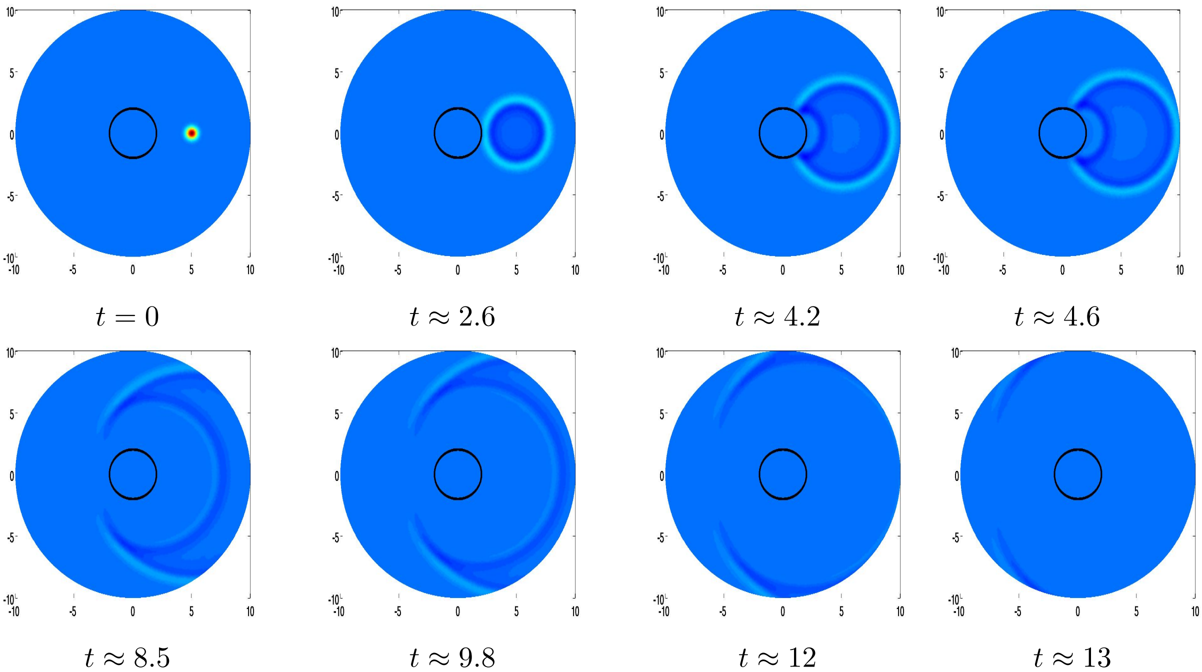
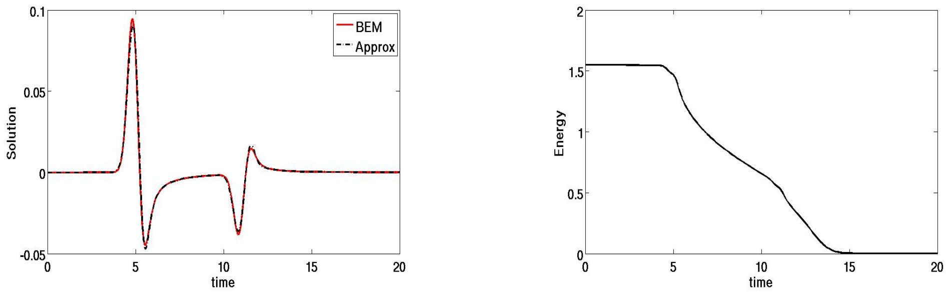


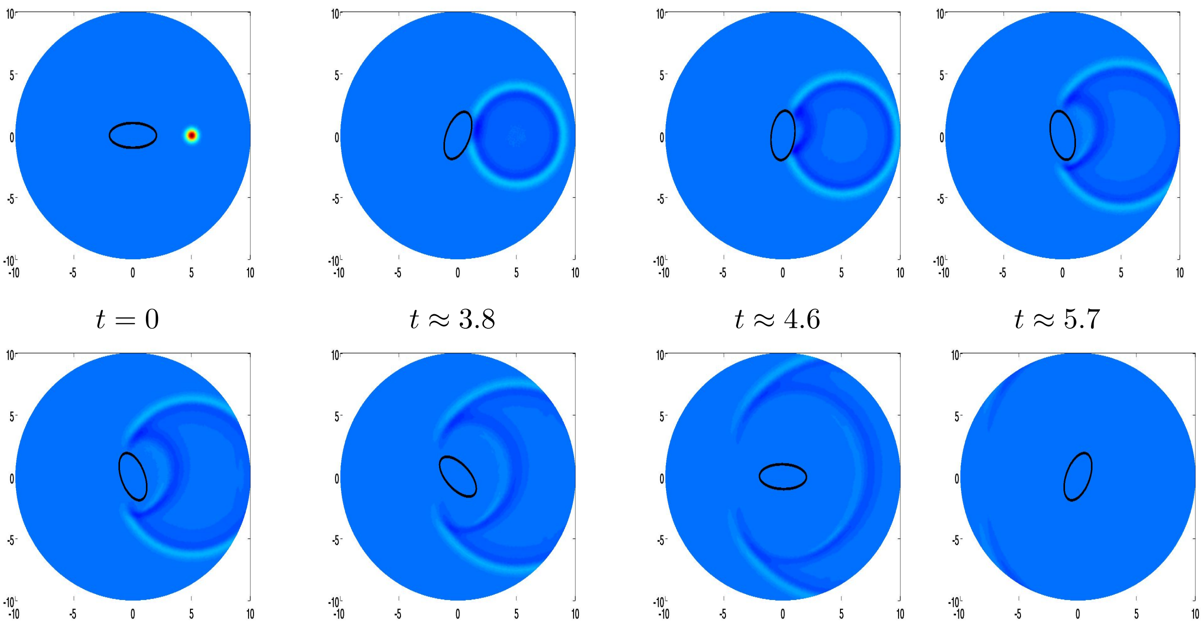
Figure 3: Example 1. Behavior of the solution at $P \approx (10, 0)$ (left plot) and energy dissipation (right plot).



the computation of flows induced by rotating components). By imposing the continuity of the discrete solution weakly on the interface between the rotating and the stationary subdomains, no compatible discretizations are required at the interface. However, in the time marching scheme, an interpolation technique is required to upload the solution at each time step. This increases the computational overhead especially when the discretization of the rotating domain must be chosen sufficiently fine to approximate accurately the boundary of the rotating scatterer.

Example 2 We consider a soft ellipsoidal obstacle whose boundary Γ is the ellipse of equation $x^2/a^2 + y^2/b^2 = 1$ with $a = 2$ and $b = 1$. The scatterer rotates around its center with a constant angular velocity equals to $\omega = \pi/128$. We consider a wave with the initial configuration u_0 as in Example 1 and that impinges upon the rotating obstacle. The transparent artificial boundary $\mathcal{B}$ is the circle of radius 10. In Figure 4 we show the snapshots of the solution at different times instants. In Figure 6 we show the behavior of the solution at a point $P \approx (10, 0)$ that belongs to the artificial boundary (left plot) and the energy behavior of the system with respect to time (right plot). The space/time discretization parameters are the same of Example 1. The wave bumps the rotating obstacle around $t = 3.5$, and the energy is preserved up to the time instant $t \approx 5$, when the wave reaches the transparent boundary and leaves the computational domain. We remark that, even if in this case the system is not a conservative one, the velocity of rotation of the obstacle is small if compared to the speed of propagation of the wave, so that the energy remains constant until the wave starts vanishing from the computational domain.

Figure 4: Example 2: Snapshots of the solution at different times.

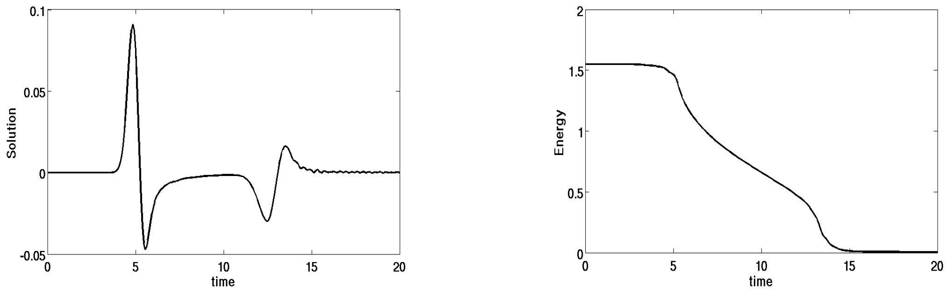


$t = 0$ $t \approx 3.8$ $t \approx 4.6$ $t \approx 5.7$

$t \approx 6.2$ $t \approx 7.4$ $t \approx 10$ $t \approx 14$

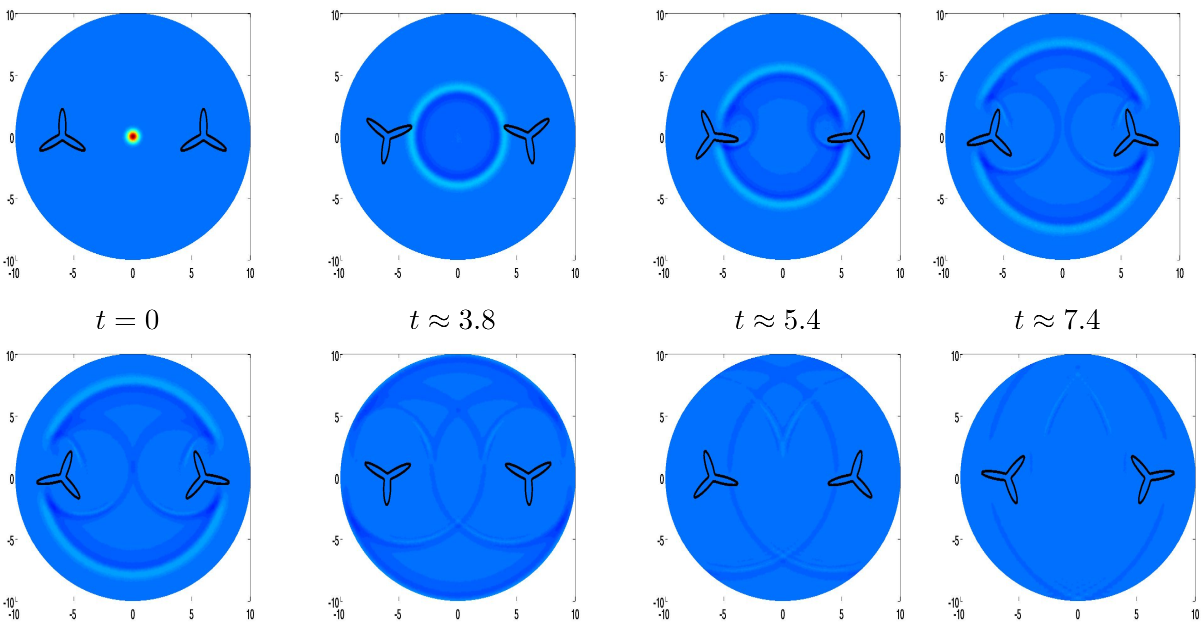
Example 3 In this last example we consider two scatterers, both having helicoidal shape, that rotate around their own center with constant angular velocity $\omega = \pi/128$ and in opposite directions (clockwise direction for the left obstacle and anticlockwise direction for the right one). The two obstacles are surrounded by an artificial circular boundary of radius 10. The initial configuration of the wave is given by the function u_0 of Examples 1 and 2, now centered

Figure 5: Example 2. Behavior of the solution at $P \approx (10, 0)$ (left plot) and energy dissipation (right plot).



at the origin of the axis. In Figure 6 we show the snapshots of the wave propagation at some instants. In Figure 7, left plot, we show the behavior of the solution at a point $P \approx (0, 4)$ for $t \in [0, 20]$ and in the right plot the energy of the system. The solution has been obtained by a decomposition of $\widetilde{\Omega}$ into 69176 triangles, by choosing a uniform partition of the two boundaries of the obstacles into 128 segments and with $N = 256$ time steps. In this case, since the system is not a conservative one, we observe that the energy increases after it bumps the two obstacles (around $t = 4$) and is reflected back.

Figure 6: Example 3. Snapshots of the solution at different times.



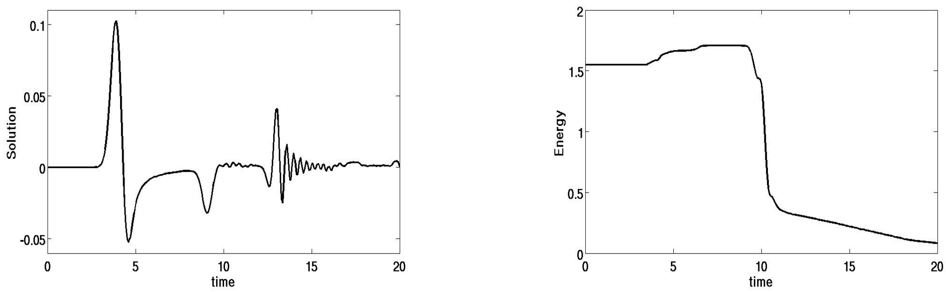
$t = 0$ $\quad$ $t \approx 3.8$ $\quad$ $t \approx 5.4$ $\quad$ $t \approx 7.4$

$t \approx 7.7$ $\quad$ $t \approx 10$ $\quad$ $t \approx 12.4$ $\quad$ $t \approx 15.5$

6 Conclusions

We have considered a fictitious domain method for the solution of wave equation problems in unbounded domains, coupled with a NRBC on a suitably chosen artificial boundary. For its solution, we have used standard finite element methods and an unconditionally stable time marching scheme for the approximation of the domain method, and a convolution quadrature technique in time coupled with a collocation method in space for the approximation of the NRBC. The coupling of the two schemes is a new topic and, although fictitious domain methods

Figure 7: Example 3. Behavior of the solution at $P \approx (0, 4)$ (left plot) and energy dissipation (right plot).



have been successfully applied to time dependent problems with stationary obstacles (see for example [13]), there is still much work to do on numerical methods for the treatment of rotating scatterers. At the moment the theory to justify the validity of the presented approach is still at an early stage, but the first numerical results we have obtained are very promising. Moreover, we emphasize that for the other NRBCs (of local type) that have been proposed in literature, it is not clear whether their application can be extended to the case of rotating obstacles. For these reasons we believe that the proposed method represents a new efficient approach to the diffraction of waves by rotating bodies.

REFERENCES

[1] Y. Bazilevs, T.J.R. Hughes, NURBS-based isogeometric analysis for the computation of flows about rotating components, Comput. Mec. 43 (2008), 143–150.

[2] S. Falletta, G. Monegato, An exact non reflecting boundary condition for 2D time-dependent wave equation problems. Wave Motion 51 (2014), no. 1, 168–192.

[3] S. Falletta, G. Monegato, L. Scuderi, A space-time BIE method for nonhomogeneous exterior wave equation problems. The Dirichlet case. IMA J. Numer. Anal. 32 (2012), no. 1, 202–226.

[4] V. Girault, R. Glowinski, Error analysis of a fictitious domain method applied to a Dirichlet problem. Japan J. Indust. Appl. Math 12 (1995), 487–524.

[5] D. Givoli, Numerical Methods for Problems in Infinite Domains, Elsevier, Amsterdam, 1992.

[6] D. Givoli, Recent advances in the DtN FE method, Archives of Computational Methods in Engineering 6 (1999) 71-116.

[7] D. Givoli, High-order local non-reflecting boundary conditions: A review, Wave Motion 39 (2004) 319-326.

[8] M.J. Grote, Ch. Kirsch, Nonreflecting boundary condition for time-dependent multiple scattering, J. Comput. Phys. 221 (2007) 41-62.

[9] M.J. Grote, I. Sim, Local nonreflecting boundary condition for time-dependent multiple scattering, J. Comput. Phys. 230 (2011) 3135-3154.

[10] Ch. Lubich, Convolution quadrature and discretized operational calculus. I, Num. Math. 52 (1988) 129-145.

[11] P.Joly, Variational methods for time-dependent wave propagation problems. Topics in computational wave propagation, 201264, Lect. Notes Comput. Sci. Eng., 31, Springer, Berlin (2003).

[12] A. Taflove, S.C. Hagness, Computational Electrodynamics. The finite-difference time-domain method, Hartech House, Boston-London, 2005.

[13] Y. Zhang, A fictitious domain method for acoustic wave propagation problems, Mathematical and Computer Modelling 50 (2009) 351–359.