From Finite to Linear Elastic Fracture Mechanics by Scaling

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Abstract. In the setting of finite elasticity we study the asymptotic behaviour of a crack that propagates quasi-statically in a brittle material. With a natural scaling of size and boundary conditions we prove that for large domains the evolution with finite elasticity converges to the evolution with linearized elasticity. In the proof the crucial step is the (locally uniform) convergence of the non-linear to the linear energy release rate, which follows from the combination of several ingredients: the Γ-convergence of re-scaled energies, the strong convergence of minimizers, the Euler-Lagrange equation for non-linear elasticity and the volume integral representation of the energy release.

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1 Introduction

Since its origin, the theory of crack propagation in elastic solids has been developed within linearized elasticity. The story begins in 1913 when Inglis [19] proved that the stress around elliptical holes and cracks in an (ideal) infinite linear elastic solid is proportional to the inverse of the square of the radius of curvature. At first glance this property leads to think that linearized elasticity is not applicable in the presence of a large curvature, since strain and stress are very large (infinite in the case of a crack). Surely, behind the adoption of linearized elasticity there is a sort of "theoretical convenience", however, Linear Elastic Fracture Mechanics, shortly LEFM, has been employed for almost one century in plenty of realistic applications (e.g. in aerospace and nuclear engineering). The intuition is therefore that the effect of the non-linearities on crack propagation is often negligible. In this perspective, the goal of this paper is to prove, on a rigorous basis, that for large brittle solids there holds a clear relationship between the quasi-static propagations obtained with finite and linearized elasticity; this relationship holds (at least) for homogeneous materials or loading conditions in which microstructures and/or cavitations do not occur.

Before describing the content of our paper, let us recall the pieces of literature which are closer to this subject and the ones which provide the technical background.

We start with elasto-statics. Usually in the textbooks on elasticity theory, e.g. [17], linear elasticity is introduced by linearizing the non-linear constitutive law around the identity: denoting by v and u respectively the deformation and the displacement field, if $W(Dv)$ is the non-linear energy density then the stress is expanded as

$$S(Dv) = DW(Dv) = DW(I) + D^2W(I)(I - Dv) + o(|I - Dv|)\,.$$

If the residual stress $DW(I)$ vanishes and if $|I - Dv| = |Du|$ is small then

$$S(Dv) \approx D^2W(I)(I - Dv)$$

and (up to an additive constant)

$$W(Dv) \approx \tfrac{1}{2} Du : D^2W(I)Du\,.$$

This is a "pointwise approach" since it is based on the pointwise expansion of the stress (or equivalently of the energy). An alternative way has been followed in [10] where linearized elasticity is obtained by the following scaling argument. In a two dimensional setting consider, for $\varepsilon > 0$, a family of re-scaled non-linear densities

$$W_\varepsilon(Du) = \varepsilon^{-2} W(I + \varepsilon Du)$$

and the corresponding energies

$$\mathcal{F}_\varepsilon(u) = \int_\Omega W_\varepsilon(Du)\,dx\,.$$

This scaling is natural for small boundary data, of the form $u = \varepsilon g$, or for large bodies, of the form Ω/ε. In [10] it is shown that the energies $\mathcal{F}_\varepsilon$ Γ-converge to the linear elastic energy F, given by

$$F(u) = \tfrac{1}{2} \int_\Omega Du : D^2W(I)Du\,dx\,,$$

and, most importantly, that the minimizers u_ε of $\mathcal{F}_\varepsilon$ converge weakly in H^1 to the minimizer u of F. Later, it was proven in [28] that the minimizers actually converge strongly in H^1. The mechanical interpretation is clear: when the scaling applies (e.g. for small boundary conditions or large bodies) linearized elasticity is a good approximation of non-linear elasticity.

We refer to the original references and to §5 for the details. We could say that this is a "global approach" since it does not depend on pointwise a-priori bounds on the displacement gradient, in particular it holds also in the presence of re-entrant corners and cracks.

In a different spirit, but focused on the role of finite elasticity in Fracture Mechanics it is worth to mention [23], where the reader can find a detailed study of the singularities of the stress field around the crack tip for a class of neo-Hookean materials. This work actually opened an entire line of research, leading to several results on stress singularities under a variety of hypotheses. Conversely, in another context, that is micromagnetics, but still in the spirit of linking different models through scaling and Γ-convergence we could mention [12] and [11] among the many, where a clear separation of scales occurs in the limit.

Let us now switch to the quasi-static propagation of a brittle crack. In the last years this classical problem in mechanics has attracted much interest in the context of energetic solutions first and (non-convex) rate-independent problems later. We start with the latter. Existence results for the quasi-static propagation of a brittle crack in a linear elastic solid were provided in [27] using a gradient flow approach, close to the original statement of Griffith [16], and in [21] by means of a viscosity approach, together with a Lipschitz parametrization of the graph (originally introduced in [13]). For a detailed analysis and for an exhaustive list of references we refer the reader to [26]. At this point, it is important to remark that the solutions of non-convex rate-independent problems can easily present discontinuities. Mechanically, discontinuities represent the unstable regimes in which the balance between elastic energy and dissipation breaks. Explicit examples can be provided, for instance in the case of an ASTM-CT specimen with a short initial crack [26]. Mathematically, the system of equations that describes the evolution requires some special care: if $\ell(t)$ denotes the length of the crack at time t and $G(t, \ell(t))$ is the energy release rate then the solution satisfies the following conditions

$$G(t, \ell(t)) \le G_c \quad \text{for } t \in [0, T] \setminus J(\ell)\,,$$

$$\ell'(t) = 0 \qquad \text{if } G(t, \ell(t)) < G_c\,,$$

$$G(t, l) \ge G_c \quad \text{for } t \in J(\ell) \text{ and } l \in [\ell(t^-), \ell(t^+)],$$

where G_c is the fracture toughness and $J(\ell)$ denotes the set of discontinuity points of ℓ. Loosely speaking, the latter inequality says that the solution ℓ jumps, from $\ell(t^-)$ to $\ell(t^+)$ if and only if it is unstable. An analogous result holds true for the quasi-static evolution of a brittle crack with polyconvex energies [22] upon replacing G with the energy release rate $\mathcal{G}$ for polyconvex energies, studied in [20]. At the current stage of development the above mentioned results hold true if the crack path is known a priori and sufficiently regular.

Now, let us turn to energetic solutions: this class is based on a minimality criterion [14], which is actually not equivalent to Griffith's criterion. The approach however allows to employ the very general setting of SBV fields together with the technical apparatus of the calculus of variations. Typically, the evolution is found by a time incremental procedure: if K_n denotes the crack at time t_n, the crack at time t_{n+1} is given by $K_n \cup J(u)$where $u \in SBV$ is a global minimizer of the energy

$$\int_\Omega W(Du)\, dx + \mathcal{H}^1(J(u) \setminus K_n)\,.$$

For details and a complete list of references we refer the reader to [5], as far as fracture mechanics is concerned, and to [24, 25], for an abstract approach. Here we mention in particular [8] and [9], where the above scheme is applied to non-linear elastic energies, and [18] where the attention is focused on the relationship between cavitation and crack nucleation.

Now, let us present the idea behind our convergence analysis. Consider a family of scaled domains $\Omega_\lambda = \lambda\widehat{\Omega}$ for $\lambda \in \mathbb{N}$. Given a field $\hat{g}$ on a subset $\partial_D\widehat{\Omega}$ of $\partial\widehat{\Omega}$ together with a scalar control $\alpha(t)$ on a reference time interval $[0,T]$, consider the scaled boundary conditions $u_\lambda(x) = \alpha(t)\lambda^{1/2}\hat{g}(x/\lambda)$ on $\partial_D\Omega_\lambda = \lambda\partial_D\widehat{\Omega}$ (the reason for choosing $\lambda^{1/2}$ will be clarified in the sequel). Let $\mathcal{G}_\lambda$ be the non-linear energy release rate. Denote by ℓ_λ the corresponding quasi-static evolution, which solves

$$\begin{gathered} \mathcal{G}_\lambda(t,\ell_\lambda(t)) \le G_c \quad \text{for } t \in [0,T] \setminus J(\ell_\lambda)\,, \\ \ell'_\lambda(t) = 0 \qquad \text{if } \mathcal{G}_\lambda(t,\ell_\lambda(t)) < G_c\,, \\ \mathcal{G}_\lambda(t,l) \ge G_c \quad \text{for } t \in J(\ell_\lambda) \text{ and } l \in [\ell_\lambda(t^-),\ell_\lambda(t^+)]. \end{gathered}$$

In order to analyze the behavior of the solutions ℓ_λ for large values of λ we first write our problem in a reference setting: Ω_λ is scaled back to $\widehat{\Omega} = \Omega_\lambda/\lambda$, ℓ_λ to $\hat{\ell}_\lambda = \ell_\lambda/\lambda$ etc. The rescaled energy density takes the form $\widehat{W}_\lambda(Du) = \lambda^2 W(I + \lambda^{-1/2}Du)$; indeed, this is the scaling for which the energy release rates satisfy $\widehat{\mathcal{G}}_\lambda(t,\hat{\ell}_\lambda(t)) = \mathcal{G}_\lambda(t,\ell_\lambda(t))$. Then, re-writing the equations above, the rescaled evolutions $\hat{\ell}_\lambda$ solve

$$\begin{gathered} \widehat{\mathcal{G}}_\lambda(t,\hat{\ell}_\lambda(t)) \le G_c \quad \text{for } t \in [0,T] \setminus J(\hat{\ell}_\lambda)\,, \\ \hat{\ell}'_\lambda(t) = 0 \qquad \text{if } \widehat{\mathcal{G}}_\lambda(t,\hat{\ell}_\lambda(t)) < G_c\,, \\ \widehat{\mathcal{G}}_\lambda(t,l) \ge G_c \quad \text{for } t \in J(\hat{\ell}_\lambda) \text{ and } l \in [\hat{\ell}_\lambda(t^-),\hat{\ell}_\lambda(t^+)]. \end{gathered}$$

Our goal is now to characterize the limit of the evolutions $\hat{\ell}_\lambda$, to this end the crucial point is the convergence of the energy release rates. Specifically, we show that $\widehat{\mathcal{G}}_\lambda$ converge to $\widehat{G}$, the energy release rate for linearized elasticity, uniformly in the variables $(t,\hat{\ell})$. Technically, this is proved with the aid of several ingredients: the Γ-convergence result of [10], the strong convergence of minimizers of [28], the Euler-Lagrange equation for non-linear elasticity and the volume integral representation of the energy release [20] in terms of the (non-linear) Eshelby tensor. As a consequence, $\hat{\ell}_\lambda \to \hat{\ell}$ pointwise in $[0,T]$, where $\hat{\ell}$ is the quasi-static evolution in LEFM, which solves

$$\begin{gathered} \widehat{G}(t,\hat{\ell}(t)) \le G_c \quad \text{for } t \in [0,T] \setminus J(\hat{\ell})\,, \\ \hat{\ell}'(t) = 0 \qquad \text{if } \widehat{G}(t,\hat{\ell}(t)) < G_c\,, \\ \widehat{G}(t,l) \ge G_c \quad \text{for } t \in J(\hat{\ell}) \text{ and } l \in [\hat{\ell}(t^-),\hat{\ell}(t^+)]. \end{gathered}$$

Finally, let us mention that the scaling law employed above, known as Bazant's law [4], and the asymptotic behaviour are consistent with several experimental observations; in a theoretical perspective our result provides a rigorous link beetween finite and linearized elasticity in fracture mechanics, along the lines of Problem 11 in [3].

2 Setting

2.1 Hypotheses

In this section we collect the assumptions on the material and on the elastic energy density.

The reference configuration $\Omega \subset \mathbb{R}^2$ is a bounded, open, connected set with Lipschitz boundary. The boundary $\partial\Omega$ is decomposed into the union of two disjoint subsets, $\partial_D\Omega$ and $\partial_N\Omega$, with $\mathcal{H}^1(\partial_D\Omega) > 0$, where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. Without loss of generality, we fix a system of Cartesian coordinates with the origin $(0,0)$ on $\partial\Omega$. We assume that the line segment $K_L = (0,L] \times \{0\} \subset \Omega$ and that $\partial_D\Omega \subset (\partial\Omega \setminus [0,L] \times \{0\})$. With this notation, for $0 < l_0 < L$, the initial crack is the set $K_0 = (0,l_0] \times \{0\} \subset \Omega$ while the admissible cracks will be of the form

$$K_\ell = (0,\ell] \times \{0\} \quad \text{for } \ell \in [l_0, L].$$

Note that, since the endpoint $(L,0)$ of K_L belongs to Ω, the domains $\Omega \setminus K_\ell$ are connected, for all $\ell \in [l_0, L]$. We also assume that the sets $\Omega^\pm = \{x = (x_1,x_2) \in \Omega : \text{sign}(x_2) = \pm 1\}$ are connected, Lipschitz and that $\partial_D\Omega^\pm$ are not empty.

Let $u : \Omega \to \mathbb{R}^2$ be the displacement from the reference configuration. The deformation $v : \Omega \to \mathbb{R}^2$ and the deformation gradient $Dv : \Omega \to \mathbb{R}^{2\times 2}$ then take respectively the form

$$v(x) = x + u(x)\,, \qquad Dv = I + Du\,, \tag{1}$$

where I is the 2×2 identity matrix. We consider the case of hyperelastic materials and we do the following assumptions on the stored energy density W (cf. [10, 20, 22]):

$$W : \mathbb{R}^{2\times 2} \to [0,+\infty] \text{ is polyconvex, and of class } C^2 \text{ in } \mathbb{R}^{2\times 2}_+, \tag{2}$$

where $\mathbb{R}^{2\times 2}_+$ denotes the set of 2×2 matrices having positive determinant, while W polyconvex means that $W(F) = \psi(F, \det F)$ for all $F \in \mathbb{R}^{2\times 2}$ and some convex function ψ. For $F \in \mathbb{R}^{2\times 2}$, we assume that W is orientation preserving

$$W(F) = +\infty \quad \text{if } \det(F) \leq 0, \tag{3}$$

and frame invariant

$$W(F) = W(QF) \quad \text{for every } Q \in SO(2), \tag{4}$$

where $SO(2)$ is the set of orthogonal 2×2 matrices with positive determinant (rotations). Further, we assume that W has a single energy well at $SO(2)$, precisely

$$W(F) = 0 \Leftrightarrow F \in SO(2), \tag{5}$$

and that it is "quadratically coercive" near the well, i.e. there exists $C_1 > 0$ such that

$$W(F) \geq C_1|\sqrt{F^T F} - I|^2 \tag{6}$$

for every F in a (small) neighborhood of $SO(2)$. We do also the following global coercivity assumption: there exist $p \geq 2$, $q > 1$ and constants $C_i > 0$, $i = 2,3,4$ such that

$$C_2|F|^p + C_3|\det F|^q - C_4 \leq W(F), \tag{7}$$

for every $F \in \mathbb{R}^{2\times 2}$. At last, we assume that the derivatives of W satisfy the following growth condition for a positive constant C_5,

$$|F^T DW(F)| + |F^T D^2W(F)[FH]||H|^{-1} \leq C_5(W(F)+1), \tag{8}$$

for every $F \in \mathbb{R}^{2\times 2}_+$ and every $H \in \mathbb{R}^{2\times 2} \setminus \{0\}$.

Remark 2.1 A recent result in [1] uses the coerciveness assumption (7) with $p > 1$ and $C_3 = 0$ for Γ-convergence. This is however not enough in the present paper since our proof (see §8) relies on the existence of minimizers and the representation of the energy release by means of the Eshebly tensor, both these facts depend on the stronger assumption (7) adopted here. Actually, for the purpose of this paper it is not restrictive to assume $p = 2$.

2.2 Notation and preliminaries

Throughout the paper we use the following notations. The first Piola-Kirchhoff stress is denoted by

$$S(v) = DW(Dv),$$

while the elasticity tensor is

$$\mathbf{C} = D^2W(I).$$

By frame indifference, invoking for instance [6, Theorem 4.2-1], there exists a density $\widetilde{W} : \mathbb{R}^{2\times 2}_{sym^*} \to \mathbb{R}$ such that $W(F) = \widetilde{W}(F^TF)$ for every $F \in \mathbb{R}^{2\times 2}_+$, where $\mathbb{R}^{2\times 2}_{sym^*}$ (the domain of $\widetilde{W}$) denotes the set of 2 by 2, symmetric, positive definte matrices. For our pourpose it is more convenient to employ $E = (F^TF - I)/2$, instead of F^TF. Accordingly we introduce also the density $V(E) = \widetilde{W}(2E + I)$. The domain of V, denoted by $\mathbb{R}^{2\times 2}_V$, is thus

$$\mathbb{R}^{2\times 2}_V = \{E \in \mathbb{R}^{2\times 2}_{sym} : E = (F^TF - I)/2 \text{ for } F \in \mathbb{R}^{2\times 2}_+\}.$$

Thus, for every deformation $v = \mathrm{id} + u$ with $\det(Dv) > 0$ we can write

$$W(Dv) = W(I + Du) = V(E(u)) \tag{9}$$

where

$$E(u) = (Dv^TDv - I)/2 = \boldsymbol{\varepsilon}(u) + (Du^TDu)/2$$

is the Green-St.Venant strain tensor while $\boldsymbol{\varepsilon}(u) = (Du^T + Du)/2$ is the linearized strain. The linearized stress is denoted by

$$\boldsymbol{\sigma}(u) = \mathbf{C}[\boldsymbol{\varepsilon}(u)] = D^2W(I)[\boldsymbol{\varepsilon}(u)].$$

For a given displacement $u$the non-linear elastic energy is given by

$$\int_\Omega W(I + Du)\, dx = \int_\Omega V(E(u))\, dx, \tag{10}$$

where the equivalence holds if $\det(I + Du) > 0$. The linearized elastic energy density $W_{\rm lin}$ defined on the set $\mathbb{R}^{2\times 2}_{\rm sym}$ is

$$W_{\rm lin}(E) = \frac{1}{2} E : \mathbf{C}[E] \quad \text{for every } E \in \mathbb{R}^{2\times 2}_{\rm sym},$$

and the associated linearized elastic energy is given by

$$\int_\Omega W_{\rm lin}(Du)\,dx := \int_\Omega W_{\rm lin}(\boldsymbol{\varepsilon}(u))\,dx = \tfrac{1}{2}\int_\Omega \boldsymbol{\varepsilon}(u) : \mathbf{C}[\boldsymbol{\varepsilon}(u)]\,dx = \tfrac{1}{2}\int_\Omega \boldsymbol{\varepsilon}(u) : \boldsymbol{\sigma}(u)\,dx. \tag{11}$$

Before proceeding, it is useful to recall the following result on the distance from $F \in \mathbb{R}^{2\times 2}_+$ to $SO(2)$, denoted by $d(F, SO(2))$.

Lemma 2.2 *Let $F \in \mathbb{R}^{2\times 2}_+$ and let $F = RU$ be its polar decomposition ($R \in SO(2)$ and $U = \sqrt{F^T F}$). Then $d(F, SO(2)) = |\sqrt{F^T F} - I|^2 = |U - I|^2 = |F - R|^2$.*

At this point, we provide the technical properties of the non-linear energy density which will be employed in the sequel.

Lemma 2.3 (expansion) *Suppose W satisfies the hypotheses (2)–(8). Then*

$$W(I) = 0, \qquad DW(I) = 0, \tag{12}$$

moreover there exists $C > 0$ such that

$$H : \mathbf{C}[H] = H_{sym} : \mathbf{C}[H_{sym}] \geq C|H_{sym}|^2 \tag{13}$$

for every $H \in \mathbb{R}^{2\times 2}$, where $H_{sym} := (H + H^T)/2$.

Proof. Conditions (12) follow by (2) and (5). To prove (13) let $F \in \mathbb{R}^{2\times 2}_{\rm sym}$ be such that $|F - I| \ll 1$. By Taylor expansion of $W(F)$ around the identity I together with (6) and (12), it follows that (being F symmetric)

$$W(F) = \tfrac{1}{2}(F - I) : \mathbf{C}[F - I] + o(|F - I|^2) \geq C_1|\sqrt{F^T F} - I|^2 = C_1|F - I|^2.$$

Hence, for $|F - I| \ll 1$ we get $(F - I) : \mathbf{C}[F - I] \geq C_1'|F - I|^2$, for $C_1' < C_1$. Given $H \in \mathbb{R}^{2\times 2}$ let $F = \delta H_{\rm sym} + I$ for $\delta \ll 1$; the previous estimate gives (13). ■

Lemma 2.4 (coercivity) *The function V associated to W via (9) has the following coercivity property: for suitable positive values a and b*

$$V(E) \geq \begin{cases} a|E|^2 & \text{if } |E| < b, \\ 2ab|E| - ab^2 & \text{if } |E| \geq b, \end{cases} \tag{14}$$

for every $E \in \mathbb{R}^{2\times 2}_V$.

Proof. Invoking [10, Section 3] the proof of (14) is a consequence of the following conditions:

(*a*) $\inf_{|E|\geq c} V(E) > 0$ for every $c > 0$;

(*b*) there exists $\alpha > 0$ such that $V(E) \geq \alpha|E|^2$ for every $|E| \ll 1$;

(*c*) $\liminf_{|E|\to+\infty} V(E)|E|^{-1} > 0$.

We first note that for given $F \in \mathbb{R}^{2\times2}_+$ by definition $F^TF = 2E + I$, which implies that $E = 0$ if and only if $F \in SO(2)$. Then, by (9), (2) and (5) $W(F) = V(E) \geq 0$, and $V(E) = 0$ if and only if $E = 0$. By (9) and (6), if $|E| \ll 1$ there exists $\alpha > 0$ such that

$$V(E) = W(F) \geq C_1|\sqrt{F^TF} - I|^2 = C_1|\sqrt{I + 2E} - I|^2 \geq \alpha|E|^2\,,$$

and this proves condition (*b*). When $|E|$ is large, using (9) and (7) we get

$$\liminf_{|E|\to+\infty} V(E)|E|^{-1} = 2\liminf_{|F|\to+\infty} W(F)|F^TF - I|^{-1} \geq 2\liminf_{|F|\to+\infty} W(F)|F|^{-2} > 0,$$

yielding condition (*c*). Note that in first limit $E \in \mathbb{R}^{2\times2}_V$ while in the others $F \in \mathbb{R}^{2\times2}_+$. Recalling that $V(E) = 0$ if and only if $E = 0$, condition (a) follows. ■

The next lemma collects two growth estimates which are employed, as a key tool, in §7, to prove the convergence of the energy release rates.

Lemma 2.5 *There exist positive constants* c_1, c_2 *and* $c_3 > 0$ *such that*

$$|F^TDW(F)| \leq c_1W(F) + c_2|F - I| \leq c_3(W(F) + |F - I|)\,. \tag{15}$$

Moreover, there exist $c_4, c_5 > 0$ *such that*

$$|F^TDW(F) - DW(F)F^T| \leq 2c_1W(F) + c_4|F - I|^2 \leq c_5(W(F) + |F - I|^2)\,. \tag{16}$$

Proof. To prove (15), let $F \in \mathbb{R}^{2\times2}$. Since W is of class C^2 (at least in a neighborhood of $SO(2)$) by (2), there exists $0 < C' \ll 1$ such that, if $|\sqrt{F^TF} - I|^2 \leq C'$ then, for $F = RU$ there exists $\bar{c} > 0$, depending only on C', such that

$$|DW(F)| = |DW(F) - DW(R)| \leq \bar{c}|F - R| \leq \bar{c}|F - I|. \tag{17}$$

For $|\sqrt{F^TF} - I|^2 \leq C'$ there exists $c_2 > 0$ such that $\bar{c}|F| \leq c_2$ and therefore

$$|F^TDW(F)| \leq \bar{c}|F||F - I| \leq c_2|F - I| \qquad \text{if } |\sqrt{F^TF} - I|^2 \leq C'.$$

On the other hand, if $|\sqrt{F^TF} - I|^2 \geq C'$ then, by Lemma 2.4 there exists $C'' > 0$ such that $W(F) \geq C''$. For the constant $C_5 > 0$ in the growth condition (8) let $C^* > 0$ be such that $C_5 = C''C^*$. Then setting $c_1 = C_5 + C^*$, from (8) we get

$$|F^TDW(F)| \leq C_5(W(F) + 1) \leq (C_5 + C^*)W(F) = c_1W(F) \qquad \text{if } |\sqrt{F^TF} - I|^2 \geq C',$$

and (15) is proved. Before proceeding, note that $DW(F)F^T$ still enjoys the growth estimate (8); this is standard, see for instance [3, Proposition 2.3]. Thus for the constant $C' > 0$ as above, if $|\sqrt{F^TF} - I|^2 \geq C'$ then

$$|F^T DW(F) - DW(F)F^T| \leq 2c_1 W(F)\,.$$

If $|\sqrt{F^TF} - I|^2 \leq C' \ll 1$ then, using (17), there exists $c_4 > 0$ such that

$$\begin{aligned}|F^T DW(F) - DW(F)F^T| &= |(F-I)^T DW(F) - DW(F)(F-I)^T| \\ &\leq 2|F-I||DW(F)| \leq c_4|F-I|^2\,.\end{aligned}$$

Putting together the last two estimates (16) is proved. ■

2.3 Examples

We show in this section that the set of stored energy densities W that satisfy assumptions (2)–(8) contains some relevant examples used in the applications.

Example 2.6 *(Ogden materials)* Let

$$W(F) := \begin{cases} +\infty & \text{if } \det F \leq 0, \\ \alpha(|F|^2) + \beta(\det^2 F) + \gamma & \text{else,} \end{cases} \tag{18}$$

where $\gamma = -\alpha(2) - \beta(1)$ and $\alpha, \beta : (0, +\infty) \to \mathbb{R}$ are functions of class C^2 such that

(h_1) α is convex and increasing,

(h_2) the function $s \mapsto \begin{cases} +\infty & \text{if } s \leq 0 \\ \beta(s^2) & \text{if } s > 0 \end{cases}$ is convex,

(h_3) $\alpha'(2s) + \beta'(s^2)s = 0 \Leftrightarrow s = 1$ and $\alpha''(2) + \beta''(1) - \alpha'(2) > 0$,

(h_4) $\beta(s) \geq C(s^r - 1)$ for $r > 1/2$ and $s|\beta'(s)| + s^2|\beta''(s)| \leq C(\beta(s) + 1)$,

(h_5) $\alpha(s) \leq C(s+1)$, $s^{1/2}\alpha'(s) \leq C(s^{1/2} + 1)$ and $|\alpha''(s)s| \leq C$,

where the value of the positive constant C may change from place to place.

Clearly W is polyconvex and of class C^2 on $\mathbb{R}^{2\times2}_+$, it is also orientation preserving and frame-invariant, namely it satisfies conditions (2)–(4), except the non-negativity which will be discussed hereafter together with condition (5).

Let $F \in \mathbb{R}^{2\times2}_+$ and let $\lambda_i > 0$, $i = 1, 2$, be the eigenvalues of F^TF. Then $|F|^2 = \mathrm{tr}\,(F^TF) = \lambda_1 + \lambda_2$ and $\det^2 F = \det F^TF = \lambda_1\lambda_2$. Thus $W(F) = \alpha(\lambda_1 + \lambda_2) + \beta(\lambda_1\lambda_2) + \gamma = w(\lambda_1, \lambda_2)$. Moreover $F \in SO(2) \Leftrightarrow F^TF = I \Leftrightarrow \lambda_1 = \lambda_2 = 1$. Hence, by the definition of γ we get that $W(F) = 0$ when $F \in SO(2)$. In order to have condition (5) satisfied we impose that $(\lambda_1, \lambda_2) = (1, 1)$ is the only critical point of w. From this condition it follows that $(1, 1)$ is also the only minimum of w and hence w is non-negative. Thus, $\nabla w(\lambda_1, \lambda_2) =$

$\alpha'(\lambda_1+\lambda_2)(1,1)+\beta'(\lambda_1\lambda_2)(\lambda_2,\lambda_1)$ and recalling that $\alpha'(s)>0$ for every s, by (h_1), we can write

$$\nabla w = 0 \Leftrightarrow (1,1) = -\frac{\beta'(\lambda_1\lambda_2)}{\alpha'(\lambda_1+\lambda_2)}(\lambda_2,\lambda_1)\,.$$

Thus $\nabla w = 0$ implies $\lambda_1 = \lambda_2$. Moreover, for $s = \lambda_1 = \lambda_2$ we get by condition (h_3) that $\nabla w(\lambda_1,\lambda_2) = 0 \Leftrightarrow \lambda_1 = \lambda_2 = 1$, as desired.

Passing to condition (6), it is useful to write $W(F) = V(E)$ where $E = (F^TF - I)/2$:

$$V(E) = \alpha\big(2\mathrm{tr}\,E + 2\big) + \beta\big(1 + 2\mathrm{tr}\,E + 4\mathrm{det}E\big) + \gamma.$$

By the previous step it follows that $V(0) = \alpha(2) + \beta(1) + \gamma = 0$, while $DV(0) = 2I(\alpha'(2) + \beta'(1)) = 0$ by (h_3). Hence, the Taylor expansion around 0 gives $V(E) = \frac{1}{2}E : D^2V(0)[E] + o(|E|^2)$ and $V(E) \geq C|E|^2$ (for $|E|$ small enough) if the tensor $D^2V(0)$ is positive definite. Now

$$D^2V(0) = 4(\alpha''(2) + \beta''(1) + \beta'(1))\mathbf{T} - 4\beta'(1)\mathbf{I},$$

where $\mathbf{T}$ is the tensor $\mathbf{T}_{ijkl} = \delta_{ij}\delta_{kl}$, while $\mathbf{I}$ is the identity tensor ($\mathbf{I}_{ijkl} = \delta_{ij\,kl}$). Then

$$\begin{aligned} E : D^2V(0)[E] &= 4(\alpha''(2) + \beta''(1) + \beta'(1))\mathrm{tr}\,^2E - 4\beta'(1)|E|^2 \\ &= 4(\alpha''(2) + \beta''(1) - \alpha'(2))\mathrm{tr}\,^2E + 4\alpha'(2)|E|^2\,, \end{aligned}$$

where the second equality follows from (h_3). Since $\alpha'(2) > 0$ by (h_1) and $\alpha''(2)+\beta''(1)-\alpha'(2) > 0$ by (h_3) it follows that $E : D^2V(0)[E] \geq C|E|^2$, for some positive constant C. Thus, for $|E|$ small enough, $V(E) \geq c|E|^2 \geq c'|F^TF - I|^2 \geq c''|\sqrt{F^TF} - I|^2$, where the last inequality holds since $\sqrt{F^TF} - I$ is small enough. The previous estimate, written in terms of W and F, gives (6).

The coercivity condition (7) holds true with $p = 2$ by (h_4) and since $\alpha(s) \geq C(s-1)$ by convexity (cf. (h_1)).

According to [20, Example 2.7]), in the case $p = 2$ condition (8) is guaranteed provided (h_4) holds and if $W_1(F) := \alpha(|F|^2)$ satisfies

$$W_1(F) \leq C'(|F|^2 + 1), \qquad |DW_1(F)| \leq C(|F|^1 + 1), \qquad |D^2W_1(F)| \leq C.$$

In terms of α, this is equivalent to (h_5).

In conclusion, if $\alpha, \beta : (0,+\infty) \to \mathbb{R}$ are C^2 functions satisfying (h_1)–(h_5) then W defined in (18) satisfies (2)–(8).

Example 2.7 *(Mooney-Rivlin)* Let $\alpha(s) := as$, and $\beta(s) := bs - \ln s$ for $a = 1 - b$ with $0 < b < 1$. It is easy to check that (h_1)–(h_5) from Example 2.6 are satisfied. This is the energy density of Mooney-Rivlin materials.

3 Finite elasticity: quasi-static evolution

3.1 Existence of minimizers (and non-uniqueness)

We introduce the non-linear elastic energy in terms of the displacement u associated to the deformation v via $v(x) = x + u(x)$. Given $t \in [0,T]$ and $\ell \in [l_0, L]$ the set of admissible displacements is

$$\mathcal{U}(t,\ell) := \{u \in H^1(\Omega \setminus K_\ell; \mathbb{R}^2) : u = \alpha(t) g \text{ on } \partial_D \Omega\}\,.$$

The analysis is performed in the case when $\alpha \in C^{1,1}([0,T])$ and $g \in W^{1,\infty}(\Omega; \mathbb{R}^2)$. Throughout the paper we will also assume that there exists $\alpha_0 > 0$ such that $\alpha(t) \geq \alpha_0$ for every $t \in [0,T]$. Note that for every $\ell \in [l_0, L]$, $K(\ell) \subseteq K(L)$ and $H^1(\Omega \setminus K_\ell; \mathbb{R}^2) \subseteq H^1(\Omega \setminus K_L; \mathbb{R}^2)$. The non-linear energy $\mathcal{F} : [0,T] \times [l_0, L] \times H^1(\Omega \setminus K_L; \mathbb{R}^2) \to [0, +\infty)$ is defined by

$$\mathcal{F}(t,\ell,u) := \begin{cases} \displaystyle\int_{\Omega \setminus K_\ell} W(I + Du)\, dx & \text{if } u \in \mathcal{U}(t,\ell), \\ +\infty & \text{else.} \end{cases} \tag{19}$$

The reduced energy $\mathcal{F}_{\min} : [0,T] \times [l_0, L] \to [0, +\infty]$ is defined as the minimal value of the energy functional for given t and ℓ, that is

$$\mathcal{F}_{\min}(t,\ell) := \min\{\mathcal{F}(t,\ell,u) : u \in \mathcal{U}(t,\ell)\}. \tag{20}$$

Note that a minimizer of $\mathcal{F}(t,\ell,\cdot)$ exists [2] thanks to the coercivity of W but in general, in the context of finite-strains, it is not unique [2, 29].

3.2 Outer variations

For the convergence of the energy release we will need an Euler-Lagrange equation in the context of non-linear elasticity. Note that the usual additive variations are generally not allowed since they may not preserve the orientation. It is instead necessary to employ the following type of variations. For $\delta \in \mathbb{R}$ and $\phi \in C_0^\infty(\mathbb{R}^2; \mathbb{R}^2)$, let $\Phi^\delta = \mathrm{id} + \delta\phi$. For $t \in [0,T]$ and $\ell \in [l_0, L]$, let u be a minimizer of $\mathcal{F}(t,\ell,\cdot)$ and let $v := \mathrm{id} + u$ be the corresponding deformation. We will consider the variation of v given by $v^\delta := \Phi^\delta \circ v$. With this choice

$$Dv^\delta = D\Phi^\delta(v) Dv = Dv + \delta D\phi(v) Dv\,.$$

This is an admissible variation since $\det(Dv^\delta) = \det(Dv)\det(I + \delta D\phi) > 0$ and $v^\delta(x) - x = u(x) + \delta\phi(x + u(x)) = \alpha(t)\hat{g}(x)$ on $\partial_D \Omega$. Then (cfr. [3, Theorem 2.4]) the deformation v satisfies the Euler-Lagrange condition

$$\int_\Omega DW(Dv) : (D\phi \circ v) Dv\, dx = 0\,. \tag{21}$$

3.3 Representation of the energy release

We first recall the regularity properties of the reduced energy $\mathcal{F}_{\min}$, cf. [22, Corollary 3.9].

Proposition 3.1 *The reduced energy functional $\mathcal{F}_{\min}$ is Lipschitz continuous on $[0,T]\times[l_0,L]$ and has left and right partial derivatives with respect to t and ℓ, for every $t \in [0,T]$ and $\ell \in [l_0,L]$.*

Next, for fixed t and ℓ the energy release rate $\mathcal{G}(t,\ell)$ is defined by

$$\mathcal{G}(t,\ell) := \lim_{h\to 0^+} \frac{\mathcal{F}_{\min}(t,\ell) - \mathcal{F}_{\min}(t,\ell+h)}{h} = -\partial_\ell^+ \mathcal{F}_{\min}(t,\ell), \tag{22}$$

where $\partial_\ell^+ \mathcal{F}_{\min}$ is the right partial derivative with respect to ℓ. A crucial point to prove the convergence of the evolutions is the convergence of the energy release rates; to this end the first step is a convenient representation of the energy release. Let $\psi \in C_0^\infty(\mathbb{R}^2;\mathbb{R}^2)$ with $\psi = \hat{e}_1 = (1,0)$ in a neighborhood of the origin and let $\psi_\ell(x) := \psi(x_1 - \ell, x_2)$ be its translation to the crack tip $(\ell,0)$. It is shown in [20, Theorem 3.3] that the energy release rate $\mathcal{G}$ is well defined and satisfies

$$\mathcal{G}(t,\ell) = \max\{\mathscr{G}(t,\ell,u,\psi) : u \in \operatorname{argmin} \mathcal{F}(t,\ell,\cdot)\},$$

where

$$\mathscr{G}(t,\ell,u,\psi) := \int_{\Omega\setminus K_\ell} \Big[(I+Du)^T DW(I+Du) - W(I+Du)\boldsymbol{I}\Big] : D\psi_\ell \, dx .$$

We remark that $\mathscr{G}(t,\ell,u,\psi)$ is independent of the choice of ψ and that, at the current stage, it is not clear whether it is really necessary to select a maximizer in the above formula. This difficult technical point is clearly out of the scope of our paper. Here we will just select (for every t ad ℓ) the minimizer $u_{t,\ell}$ that satisfies $\mathcal{G}(t,\ell) = \mathscr{G}(t,\ell,u,\psi)$. In conclusion, the energy release rate $\mathcal{G}(t,\ell)$ can be expressed by

$$\mathcal{G}(t,\ell) = \int_{\Omega\setminus K_\ell} \Big[(I+Du_{t,\ell})^T DW(I+Du_{t,\ell}) - W(I+Du_{t,\ell})\boldsymbol{I}\Big] : D\psi_\ell \, dx . \tag{23}$$

3.4 Quasi-static evolution

For brittle materials the energy dissipated by the crack is proportional to the length of the crack, i.e. it is of the form $G_c\,\ell$ where $G_c > 0$ is a material parameter, called fracture toughness. Taking also into account the irreversibility of the process, the dissipation (rate of dissipated energy) is thus

$$\mathcal{D}(\ell') = \begin{cases} G_c \ell' & \text{if } \ell' \geq 0, \\ +\infty & \text{else.} \end{cases}$$

Now we can introduce the notion of quasi-static evolution, associated to the non-linear (reduced) energy $\mathcal{F}_{\min}$ and the dissipation $\mathcal{D}$.

Definition 3.2 *A quasi-static evolution associated to $\mathcal{F}_{\min}$ and $\mathcal{D}$ is a map $\ell \in BV([0,T];[l_0,L])$ such that for every $t \in [0,T]$ the following conditions hold true.*

(a) ℓ is non-decreasing,

(b) if $t \notin J(\ell)$ then $\mathcal{G}(t,\ell(t)) \leq G_c$,

(c) if $\mathcal{G}(t,\ell(t)) < G_c$ then $\ell'(t) = 0$,

(d) for every $t_* \in J(\ell)$ and $\ell_* \in [\ell(t_*^-), \ell(t_*^+)]$ we have $\mathcal{G}(t_*,\ell_*) \geq G_c$,

where $J(\ell)$ denotes the jump set of ℓ.

Existence of an evolution according to this Definition is proved in [22, Theorem 2.6] (where it is named *special local energetic solution*).

4 Linear elasticity: quasi-static evolution

The linearized elastic energy $F : [0,T] \times [l_0,L] \times H^1(\Omega \setminus K_L; \mathbb{R}^2) \to [0,+\infty]$ is defined by

$$F(t,\ell,u) := \begin{cases} \displaystyle\int_{\Omega\setminus K_\ell} W_{\mathrm{lin}}(Du)\,dx & \text{if } u \in \mathcal{U}(t,\ell), \\ +\infty & \text{else,} \end{cases} \tag{24}$$

where $W_{\mathrm{lin}}(Du)$ is the linear elastic density (11). The coercivity of W_{lin} (see Lemma 2.3) and the Dirichlet datum guarantee that for every $t \in [0,T]$ and $\ell \in [l_0,L]$ there exists a unique minimizer of $F(t,\ell,\cdot)$. As we did before we introduce also the reduced linearized energy functional $F_{\min} : [0,T] \times [l_0,L] \to \mathbb{R}$

$$F_{\min}(t,\ell) := \min\{F(t,\ell,u) : u \in \mathcal{U}(t,\ell)\}. \tag{25}$$

The regularity of $F_{\min}$ is stated in the next result, e.g. [21, Theorem 3.2].

Proposition 4.1 *The reduced functional $F_{\min}$ belongs to $C^1([0,T]\times[l_0,L])$. Moreover, the energy release rate*

$$G(t,\ell) := -\partial_\ell F_{\min}(t,\ell) \tag{26}$$

can be represented in terms of the Eshelby tensor via the following formula

$$G(t,\ell) = \int_{\Omega\setminus K_\ell} \Big[Du^T \boldsymbol{\sigma}(u) - W_{\mathrm{lin}}(Du)\boldsymbol{I} \Big] : D\psi_\ell \, dx, \tag{27}$$

where u denotes the unique minimizer of $F(t,\ell,\cdot)$ and ψ_ℓ is defined as in the previous section.

4.1 Quasi-static evolution

A quasi-static evolution is characterized as we did for the non-linear case.

Definition 4.2 *A quasi-static evolution associated to $F_{\min}$ and $\mathcal{D}$ is a map $\ell \in BV([0,T];[l_0,L])$ such that for every $t \in [0,T]$ the following conditions hold true.*

(a) *ℓ is non-decreasing,*

(b) *if $t \notin J(\ell)$ then $G(t,\ell(t)) \le G_c$,*

(c) *if $G(t,\ell(t)) < G_c$ then $\ell'(t) = 0$,*

(d) *for every $t_* \in J(\ell)$ and $\ell_* \in [\ell(t_*^-), \ell(t_*^+)]$ we have $G(t_*,\ell_*) \ge G_c$,*

where $J(\ell)$ denote the jump set of ℓ.

Existence of this evolution is nowadays well known, cf. [21, 27].

5 Scaling laws

The aim of this section is to present the scaling laws which link the linear and non-linear theories in elasticity and fracture. We begin with the classical small strain argument both in the small displacement §5.1 and large domain formulation §5.2. Finally, we introduce in §5.3 the natural scaling in Fracture Mechanics.

5.1 Scaling in elasticity: small displacements

Fix $t \in [0,T]$ and $\ell \in [l_0,L]$. For $g \in W^{1,\infty}(\Omega;\mathbb{R}^2)$ and $\delta > 0$ (a dimensionless parameter) consider the set of displacements

$$\mathcal{U}_\delta(t,\ell) = \left\{ u \in H^1(\Omega \setminus K_\ell;\mathbb{R}^2) \, : \, u = \delta\alpha(t) g \ \text{ on } \partial_D\Omega \ \right\}$$

endowed with the strong topology of H^1. Let

$$u_\delta \in \operatorname{argmin} \{ \mathcal{F}(t,\ell,u) : u \in \mathcal{U}_\delta(t,\ell) \}$$

(to shorten the notation we omit the dependence on t and ℓ). As discussed in §3 for every $\delta > 0$ a solution u_δ exists, even if in general it is not unique. In order to study the behavior of the family $\{u_\delta\}$ as $\delta \searrow 0$ it is more convenient to write the problem in terms of a family of energies $\mathcal{F}_\delta$ (depending on δ) defined on the (reference) functional space

$$\mathcal{U}(t,\ell) = \left\{ u \in H^1(\Omega \setminus K_\ell;\mathbb{R}^2) \, : \, u = \alpha(t) g \ \text{ on } \partial_D\Omega \ \right\} .$$

Clearly if $u \in \mathcal{U}_\delta(t,\ell)$ then $\hat{u} = u/\delta \in \mathcal{U}(t,\ell)$ and

$$\mathcal{F}(t,\ell,\delta\hat{u}) = \int_{\Omega\setminus K_\ell} W(I + \delta D\hat{u}) \, dx \, .$$

For $\hat{u} \in \mathcal{U}(t,\ell)$ it is natural to introduce the energy

$$\mathcal{F}_\delta(t,\ell,\hat{u}) := \delta^{-2}\mathcal{F}(t,\ell,\delta\hat{u}) = \delta^{-2}\int_{\Omega\setminus K_\ell} W(I+\delta D\hat{u})\,dx\,.$$

Clearly,

$$u_\delta \in \operatorname{argmin}\{\mathcal{F}(t,\ell,u) : u \in \mathcal{U}_\delta(t,\ell)\}$$

if and only if

$$\hat{u}_\delta = u_\delta/\delta \in \operatorname{argmin}\{\mathcal{F}_\delta(t,\ell,\hat{u}) : \hat{u} \in \mathcal{U}(t,\ell)\}\ .$$

Following the classical presentation, let us write a Taylor expansion of the energy density W around the identity. Using the hypotheses listed in §2 (and Lemma 2.3 therein) we get

$$\begin{aligned} W(I+\delta D\hat{u}) &= W(I) + \delta DW(I)D\hat{u} + \tfrac{1}{2}\delta^2 D\hat{u} : D^2W(I)[D\hat{u}] + o(|\delta D\hat{u}|^2) \\ &= \delta^2 W_{\text{lin}}(D\hat{u}) + o(|\delta D\hat{u}|^2)\,. \end{aligned}$$

Thus

$$\mathcal{F}_\delta(t,\ell,\hat{u}) = \delta^{-2}\int_{\Omega\setminus K_\ell} W(I+\delta D\hat{u})\,dx = \int_{\Omega\setminus K_\ell} W_{\text{lin}}(D\hat{u}) + o(|D\hat{u}|^2)\,dx.$$

So, if $D\hat{u}$ is small a.e. in Ω then $\mathcal{F}_\delta(t,\ell,\hat{u}) \approx F(t,\ell,\hat{u})$, where

$$F(t,\ell,\hat{u}) = \int_{\Omega\setminus K_\ell} W_{\text{lin}}(D\hat{u})\,dx,$$

and we may reasonably expect that $\hat{u}_\delta \approx \hat{u}_{\text{lin}}$, where $\hat{u}_{\text{lin}}$ is the unique minimizer of F in $\mathcal{U}(t,\ell)$. In particular $u_\delta \approx \delta\hat{u}_{\text{lin}}$. This is the classical small strain argument employed in elasticity theory. Clearly, in the presence of a singularity, e.g. close to the crack tip, the small strain assumption is not valid. Nonetheless linearized elasticity is widely employed in fracture mechanics and provides very accurate solutions: the fitting between fringe patterns and theoretical solutions is usually very good. This seems to suggest that linearized elasticity is fairly good also above the small strain assumption. In order to prove that it is actually so, some more effort is needed to make the above argument both more rigorous and more general. This task has been pursued in [10] and [28] with the aid of Γ-convergence [7] and rigidity [15].

Proposition 5.1 *Fix $t \in [0,T]$ and $\ell \in [l_0,L]$. Under the hypotheses listed in §2, the functionals $\mathcal{F}_\delta(t,\ell,\cdot)$ Γ-converge to $F(t,\ell,\cdot)$ (as $\delta \searrow 0$) with respect to the weak topology of H^1 (induced on $\mathcal{U}(t,\ell)$). Moreover, if $\hat{u}_\delta \in \operatorname{argmin}\{\mathcal{F}_\delta(t,\ell,\hat{u}) : \hat{u} \in \mathcal{U}(t,\ell)\}$ then (up to subsequences) $\hat{u}_\delta \to \hat{u}$ strongly in H^1, where $\hat{u}$ is the unique minimizer of the energy $F(t,\ell,\cdot)$.*

5.2 Scaling in elasticity: large domains

Besides the small displacement setting presented above, the convergence result of Proposition 5.1 can be obtained also by a scaling argument in which the dimension of the domain change but the order of the boundary conditions remains the same. Fix $t \in [0,T]$ and $\hat{\ell} \in [l_0,L]$. For notational convenience, from now on we rename the reference domain Ω by $\widehat{\Omega}$. Denote by $\hat{u}$ and $\hat{v}$ respectively the displacement and the deformation defined on $\widehat{\Omega}$, and let $\hat{g}$ be the boundary condition imposed on $\partial_D\widehat{\Omega}$. For $\lambda > 1$ consider the family of rescaled domains $\Omega_\lambda = \lambda\widehat{\Omega}$ and let $\ell = \lambda\hat{\ell}$. Consider the set of admissible configurations

$$\mathcal{U}_\lambda(t,\ell) = \left\{ u \in H^1(\Omega_\lambda \setminus K_\ell; \mathbb{R}^2) \,:\, u = \alpha(t) g_\lambda \ \text{ on } \partial_D\Omega_\lambda \ \right\}$$

for $g_\lambda(x) = \hat{g}(x/\lambda)$. For $u \in \mathcal{U}_\lambda(t,\ell)$ the non-linear elastic energy, with density W, is

$$\mathcal{F}_\lambda(t,\ell,u) = \int_{\Omega_\lambda \setminus K_\ell} W(I + Du)\, dx\,.$$

By a linear change of variable, denoting $\hat{u}(x) = u(\lambda x)$ and $\hat{x} = x/\lambda$ we obtain

$$\mathcal{F}_\lambda(t,\ell,u) = \int_{\widehat{\Omega} \setminus K_{\hat{\ell}}} \lambda^2 W(I + \lambda^{-1} D\hat{u})\, d\hat{x} = \widehat{\mathcal{F}}_\lambda(t,\hat{\ell},\hat{u})\,.$$

Note that $\hat{u} = \alpha(t)\hat{g}$ on $\partial_D\widehat{\Omega}$, hence $\hat{u} \in \mathcal{U}(t,\hat{\ell})$. In this case Proposition 5.1 reads as follows.

Proposition 5.2 *Fix $t \in [0,T]$ and $\hat{\ell} \in [l_0, L]$. Under the hypotheses listed in §2, the functionals $\widehat{\mathcal{F}}_\lambda(t,\hat{\ell},\cdot)$ Γ-converge to $F(t,\hat{\ell},\cdot)$ (as $\lambda \to +\infty$) with respect to the weak topology of H^1 (induced on $\mathcal{U}(t,\hat{\ell})$). Moreover, if $\hat{u}_\lambda \in \operatorname{argmin}\{\widehat{\mathcal{F}}_\lambda(t,\hat{\ell},\hat{u}) \,:\, \hat{u} \in \mathcal{U}(t,\hat{\ell})\}$ then (up to subsequences) $\hat{u}_\lambda \to \hat{u}$ strongly in H^1, where $\hat{u}$ is the unique minimizer of $F(t,\hat{\ell},\cdot)$.*

The physical interpretation is clear: for large domains linearized elasticity is a good approximation of non-linear elasticity.

5.3 Scaling in fracture

Consider again a family of domains $\Omega_\lambda = \lambda\widehat{\Omega}$, $\lambda > 1$. In the linear theory of Fracture Mechanics the natural scaling of the Dirichlet datum is given by $g_\lambda(x) = \lambda^{1/2}\hat{g}(x/\lambda)$ for $x \in \partial_D\Omega_\lambda$. In this way dissipated energy and elastic energy both scale linearly with λ. Our setting is not linear, however, as we will see in the sequel, this is the natural scaling which connects the non-linear and the linear theory. (It is interesting to note that this is also the theoretical and experimental scaling for the transition between the quasi-fragile (or cohesive) and the fragile regime, e.g. [4]). Fit $t \in [0,T]$. Let $\hat{\ell} \in [l_0, L]$ and $\ell = \lambda\hat{\ell}$. We consider admissible configurations to belong to the sets

$$\mathcal{U}_\lambda(t,\ell) = \left\{ u \in H^1(\Omega_\lambda \setminus K_\ell; \mathbb{R}^2) \,:\, u = \alpha(t) g_\lambda \ \text{ on } \partial_D\Omega_\lambda \ \right\}\,.$$

As above, for $u \in \mathcal{U}_\lambda(t,\ell)$ the (non-linear) elastic energy is

$$\mathcal{F}_\lambda(t,\ell,u) = \int_{\Omega_\lambda \setminus K_\ell} W(I + Du)\, dx\,.$$

Set $u(x) = \lambda^{1/2}\hat{u}(x/\lambda)$, so that $Du(x) = \lambda^{-1/2} D\hat{u}(x/\lambda) = \lambda^{-1/2} D\hat{u}(\hat{x})$, with $\hat{x} = x/\lambda \in \widehat{\Omega}$. Then

$$\begin{aligned}\mathcal{F}_\lambda(t,\ell,u) &= \int_{\Omega_\lambda \setminus K_\ell} W(I + Du)\, dx \\ &= \lambda^2 \int_{\widehat{\Omega} \setminus K_{\hat{\ell}}} W(I + \lambda^{-1/2} D\hat{u})\, d\hat{x} = \lambda \widehat{\mathcal{F}}_\lambda(t,\hat{\ell},\hat{u}).\end{aligned}$$

In the sequel we are going to work with the energy $\widehat{\mathcal{F}}_\lambda$ which differs from $\mathcal{F}_\lambda$ by a factor λ. This is the natural scaling since the fracture energy $G_c \ell = \lambda G_c \hat\ell$ scales linearly with respect to λ. Therefore the balance between elastic and fracture energy, which provides the "force" for fracture propagation, remains the same. Note also that, if $u_\lambda \in \operatorname{argmin} \mathcal{F}_\lambda(t, \ell, \cdot)$ then its rescaling $\hat u_\lambda \in \operatorname{argmin} \widehat{\mathcal{F}}_\lambda(t, \hat\ell, \cdot)$. In particular, by Proposition 5.2, the sequence of minimizers $\{\hat u_\lambda\}$ is strongly precompact in $H^1(\widehat\Omega \setminus K_{\hat\ell}; \mathbb{R}^2)$.

6 Convergence of energies and minimizers

In this section we provide the convergence results that will be needed in the sequel for the convergence of the energy release rates. For $t \in [0, T]$ and $\hat\ell \in [l_0, L]$ let

$$\widehat{\mathcal{U}}(t, \hat\ell\,) = \{\hat u \in H^1(\widehat\Omega \setminus K_{\hat\ell}\,; \mathbb{R}^2) \,:\, \hat u = \alpha(t)\hat g \;\text{ on } \partial_D \widehat\Omega\}$$

be the set of admissible displacements, and let $\widehat{\mathcal{F}}_\lambda : [0, T] \times [l_0, L] \times H^1(\Omega \setminus K_L; \mathbb{R}^2) \to [0, +\infty]$ be defined as

$$\widehat{\mathcal{F}}_\lambda(t, \hat\ell, \hat u) = \begin{cases} \displaystyle\int_{\widehat\Omega \setminus K_{\hat\ell}} \lambda W(I + \lambda^{-1/2} D\hat u)\, d\hat x & \text{if } \hat u \in \widehat{\mathcal{U}}(t, \hat\ell\,), \\ +\infty & \text{else.} \end{cases}$$

For technical reasons that will be clear later, it is necessary to consider sequences of functionals $\widehat{\mathcal{F}}_\lambda(t_\lambda, \hat\ell_\lambda, \cdot)$ rather than the usual $\widehat{\mathcal{F}}_\lambda(t, \hat\ell, \cdot)$. Let $t_\lambda \in [0, T]$ and $\hat\ell_\lambda \in [l_0, L]$ be such that $t_\lambda \to t_\infty$ and $\hat\ell_\lambda \to \hat\ell_\infty$ (as $\lambda \to +\infty$). First of all we provide the weak compactness of equibounded sequences.

Proposition 6.1 *Let $\hat u_\lambda \in \widehat{\mathcal{U}}(t_\lambda, \hat\ell_\lambda)$ be such that $\widehat{\mathcal{F}}_\lambda(t_\lambda, \hat\ell_\lambda, \hat u_\lambda)$ is bounded uniformly with respect to λ. Then there exists $\hat u \in \widehat{\mathcal{U}}(t_\infty, \hat\ell_\infty)$ such that (up to subsequences) $D\hat u_\lambda$ converges to $D\hat u$ weakly in $L^2(\widehat\Omega; \mathbb{R}^{2\times 2})$.*

Proof. Observe that if $\hat u \in \widehat{\mathcal{U}}(t_\lambda, \hat\ell_\lambda)$ then $\hat u \in \widehat{\mathcal{U}}(t_\lambda, L)$ and $\widehat{\mathcal{F}}_\lambda(t_\lambda, \hat\ell_\lambda, \hat u) = \widehat{\mathcal{F}}_\lambda(t_\lambda, L, \hat u)$. By our assumptions the sets $\widehat\Omega^\pm = \{\hat x = (\hat x_1, \hat x_2) \in \widehat\Omega \,:\, \operatorname{sign}(\hat x_2) = \pm 1\}$ are connected and Lipschitz continuous, therefore Proposition 3.4 in [10] applies, yielding the estimate

$$\int_{\widehat\Omega^\pm} |D\hat u_\lambda|^2\, d\hat x \le C\widehat{\mathcal{F}}_\lambda(t_\lambda, L, \hat u_\lambda) + C\,\alpha^2(t_\lambda) \int_{\partial_D \widehat\Omega^\pm} |\hat g|^2\, d\hat s\,, \tag{28}$$

where the positive constant C depends only on the density W, the sets $\widehat\Omega^\pm$ and $\partial_D\widehat\Omega^\pm$. Note that this estimate relies on the quantitative rigidity lemma of [15]. We remark also that Proposition 3.4 in [10] follows from [10, Lemma 3.1] which in turn follows from a coercivity estimate, like (14), holding in $\mathbb{R}^{2\times 2}_{sym}$. In our formulation (14) holds only in $\mathbb{R}^{2\times 2}_V$; however in the proof of [10, Lemma 3.1] coercivity is applied to the Green-St.Venant tensor which belongs to $\mathbb{R}^{2\times 2}_V$. Therefore the same proof works also in our setting. As a consequence, since by assumption $\widehat{\mathcal{F}}_\lambda(t_\lambda, \hat\ell_\lambda, \hat u_\lambda) = \widehat{\mathcal{F}}_\lambda(t_\lambda, L, \hat u_\lambda)$ is uniformly bounded with respect to λ, the sequence $\hat u_\lambda$ is uniformly bounded in $H^1(\widehat\Omega^\pm \setminus K_L; \mathbb{R}^2)$ and thus weakly precompact. Therefore, since K_L is negligible, there exists a subsequence (not relabeled) such that $\hat{u}_\lambda$ converge to $\hat{u}$ strongly in $L^2(\widehat{\Omega};\mathbb{R}^2)$ and $D\hat{u}_\lambda$ converge weakly to $D\hat{u}$ in $L^2(\widehat{\Omega};\mathbb{R}^{2\times 2})$.

It remains to show that $\hat{u} \in \widehat{\mathcal{U}}(t_\infty, \hat{\ell}_\infty)$. The idea is to apply a change of variable that maps $\widehat{\mathcal{U}}(t_\lambda, \hat{\ell}_\lambda)$ onto $\widehat{\mathcal{U}}(t_\infty, \hat{\ell}_\infty)$. As in §3 let $\psi \in C_0^\infty(\mathbb{R}^2;\mathbb{R}^2)$ with the support contained in a ball B_r of radius r centered at the origin, and such that $\psi(0,0) = \hat{e}_1 = (1,0)$. Let $\Psi_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ denote the map defined by

$$\Psi_\lambda(x) := (x_1, x_2) + (\hat{\ell}_\lambda - \hat{\ell}_\infty)\psi(x_1 - \hat{\ell}_\infty, x_2)\,. \tag{29}$$

Then, for $r \ll 1$ and $\lambda \gg 1$, Ψ_λ is a smooth diffeomorphism such that $\Psi_\lambda(K_{\hat{\ell}_\infty}) = K_{\hat{\ell}_\lambda}$, $\Psi_\lambda(\widehat{\Omega} \setminus K_{\hat{\ell}_\infty}) = \widehat{\Omega} \setminus K_{\hat{\ell}_\lambda}$ and $\Psi_\lambda(x) = x$ on $\partial_D\widehat{\Omega}$. Moreover, $D\Psi_\lambda^{-1} \to I$ and $\det D\Psi_\lambda \to 1$ uniformly in $\widehat{\Omega}$. Now, for $\hat{u}_\lambda \in \widehat{\mathcal{U}}(t_\lambda, \hat{\ell}_\lambda)$ let

$$\tilde{u}_\lambda := \alpha(t_\infty)/\alpha(t_\lambda)\hat{u}_\lambda \circ \Psi_\lambda\,.$$

Then $\tilde{u}_\lambda = \alpha(t_\infty)\hat{g}$ in $\partial_D\widehat{\Omega}$, therefore $\tilde{u}_\lambda \in \widehat{\mathcal{U}}(t_\infty, \hat{\ell}_\infty)$. Moreover $D\tilde{u}_\lambda = \alpha(t_\infty)/\alpha(t_\lambda)(D\hat{u}_\lambda \circ \Psi_\lambda)D\Psi_\lambda$, therefore $\tilde{u}_\lambda$ is uniformly bounded in $H^1(\widehat{\Omega} \setminus K_{\hat{\ell}_\infty};\mathbb{R}^2)$. As $\tilde{u}_\lambda \to \hat{u}$ in $L^2(\Omega;\mathbb{R}^2)$ it follows that $\hat{u} \in \widehat{\mathcal{U}}(t_\infty, \hat{\ell}_\infty)$. ■

Proposition 6.2 *The sequence $\widehat{\mathcal{F}}_\lambda(t_\lambda, \hat{\ell}_\lambda, \cdot)$ Γ-converges to $F(t_\infty, \hat{\ell}_\infty, \cdot)$ with respect to the strong topology of $L^2(\widehat{\Omega} \setminus K_L;\mathbb{R}^2)$.*

Proof. Let $\hat{u}_\lambda \in \widehat{\mathcal{U}}(t_\lambda, \hat{\ell}_\lambda)$ be such that $\hat{u}_\lambda \to \hat{u}$ in $L^2(\widehat{\Omega} \setminus K_L;\mathbb{R}^2)$. It is not restrictive to assume that $\sup_{\lambda\geq 1} \widehat{\mathcal{F}}_\lambda(t_\lambda, \hat{\ell}_\lambda, \hat{u}_\lambda) \leq C < +\infty$. By Proposition 6.1 the limit $\hat{u}$ belongs to $\widehat{\mathcal{U}}(t_\infty, \hat{\ell}_\infty)$.

To prove the Γ-liminf and Γ-limsup inequalities it is enough to employ a change of variable and then apply the Γ-convergence result of [10]. To this end, let $\varepsilon_\lambda := \lambda^{-1/2}\alpha(t_\lambda)/\alpha(t_\infty)$. By continuity $\alpha(t_\lambda)/\alpha(t_\infty) \to 1$ and thus $\varepsilon_\lambda \to 0$ as $\lambda \to +\infty$. Moreover, for Ψ_λ as in the proof of Proposition 6.1, let $\tilde{u} := \alpha(t_\infty)/\alpha(t_\lambda)\hat{u} \circ \Psi_\lambda$. Then $\varepsilon_\lambda D\tilde{u} = \lambda^{-1/2}(D\hat{u} \circ \Psi_\lambda)D\Psi_\lambda$ and $\lambda = \varepsilon_\lambda^{-2}(\alpha(t_\lambda)/\alpha(t_\infty))^2$. Hence we can re-write the non-linear energy functional $\widehat{\mathcal{F}}_\lambda(t_\lambda, \hat{\ell}_\lambda, \hat{u})$ by means of $\tilde{u}$ as follows:

$$\begin{aligned}
\widehat{\mathcal{F}}_\lambda(t_\lambda, \hat{\ell}_\lambda, \hat{u}) &= \int_{\widehat{\Omega}\setminus K_{\hat{\ell}}} \lambda W(I + \lambda^{-1/2} D\hat{u})\, d\hat{x} \\
&= (\alpha(t_\lambda)/\alpha(t_\infty))^2 \int_{\widehat{\Omega}\setminus K_{\hat{\ell}_\infty}} \varepsilon_\lambda^{-2} W(I + \varepsilon_\lambda D\tilde{u} D\Psi_\lambda^{-1}) \det D\Psi_\lambda d\hat{x} \\
&:= (\alpha(t_\lambda)/\alpha(t_\infty))^2 \widetilde{\mathcal{F}}_\lambda(t_\infty, \hat{\ell}_\infty, \tilde{u})\,.
\end{aligned}$$

Instead of $\widehat{\mathcal{F}}_\lambda(t_\lambda, \hat{\ell}_\lambda, \cdot)$, we can consider $\widetilde{\mathcal{F}}_\lambda(t_\infty, \hat{\ell}_\infty, \cdot)$. Since $D\Psi_\lambda^{-1} \to I$ and $\det D\Psi_\lambda \to 1$ uniformly in $\widehat{\Omega}$, it is sufficient to invoke Proposition 4.1 and Proposition 4.4 in [10] to have respectively the liminf inequality and the limsup inequality. ■

The following strong convergence result is a consequence of [28, Theorem 3.1].

Proposition 6.3 *Let $\hat{u}_\lambda \in \operatorname{argmin} \widehat{\mathcal{F}}_\lambda(t_\lambda, \hat{\ell}_\lambda, \cdot)$. Then, up to subsequences, $\hat{u}_\lambda$ converges strongly in $H^1(\widehat{\Omega} \setminus K_L; \mathbb{R}^2)$ to the minimizer $\hat{u}$ of $F(t_\infty, \hat{\ell}_\infty, \cdot)$.*

Proof. We employ again a change of variable. For Ψ_λ as in the proof of Proposition 6.1, let $\tilde{u}_\lambda := \alpha(t_\infty)/\alpha(t_\lambda)\hat{u}_\lambda \circ \Psi_\lambda$ be the sequence of minimizers of $\widetilde{\mathcal{F}}_\lambda(t_\infty, \hat{\ell}_\infty, \cdot)$. By Γ-convergence $\widetilde{\mathcal{F}}_\lambda(t_\infty, \hat{\ell}_\infty, \tilde{u}_\lambda)$ converge to $F(t_\infty, \hat{\ell}_\infty, \hat{u})$ where $\hat{u}$ is the unique minimizer. Then by [28, Theorem 3.1] it follows that $\tilde{u}_\lambda$ converge to $\hat{u}$ strongly in $H^1(\widehat{\Omega} \setminus K_{\hat{\ell}_\infty}; \mathbb{R}^2)$. By an easy change of variable it follows that $\hat{u}_\lambda$ as well converge to $\hat{u}$ strongly in $H^1(\widehat{\Omega} \setminus K_L; \mathbb{R}^2)$. ■

7 Convergence of energy release

7.1 Pointwise convergence

With the scaling defined in §5.3, the energy release rate is "λ-invariant". Indeed, we introduce the reduced energies

$$\mathcal{F}_{\lambda,\min}(t,\ell) := \min\{\mathcal{F}_\lambda(t,\ell,u) : u \in \mathcal{U}(t,\ell)\}\,,$$
$$\widehat{\mathcal{F}}_{\lambda,\min}(t,\hat{\ell}) := \min\{\widehat{\mathcal{F}}_\lambda(t,\hat{\ell},u) : u \in \widehat{\mathcal{U}}(t,\hat{\ell})\}\,.$$

Since $\mathcal{F}_{\lambda,\min}(t,\ell) = \lambda \widehat{\mathcal{F}}_{\lambda,\min}(t,\hat{\ell})$ and $d_\ell \hat{\ell} = 1/\lambda$, we have

$$\mathcal{G}_\lambda(t,\ell) = -\partial^+_\ell \mathcal{F}_{\lambda,\min}(t,\ell) = -\lambda\, \partial^+_{\hat{\ell}}\, \widehat{\mathcal{F}}_{\lambda,\min}(t,\hat{\ell})\, d_\ell \hat{\ell} = -\partial^+_{\hat{\ell}}\, \widehat{\mathcal{F}}_{\lambda,\min}(t,\hat{\ell}) = \widehat{\mathcal{G}}_\lambda(t,\hat{\ell})\,.$$

This property will allow for an easy change of the variable in the set of conditions which define the quasi-static evolution (see §8.2). Actually, to prove the convergence of $\widehat{\mathcal{G}}_\lambda$ it will be necessary also to employ the representations of the energy release with the Eshelby tensor. Before proceeding, we state this (classical) result which follows from generalized dominated convergence.

Lemma 7.1 *Let $f_n, f : \widehat{\Omega} \to [0,+\infty)$ be L^1-functions such that $f_n \to f$ a.e. Then*

$$\|f_n\|_{L^1} \to \|f\|_{L^1} \implies \|f_n - f\|_{L^1} \to 0\,.$$

Now we can prove the main result. For notational convenience we rename the energy release rate from linear elasticity G by $\widehat{G}$.

Theorem 7.2 *$\widehat{\mathcal{G}}_\lambda \to \widehat{G}$ pointwise in $[0,T] \times [l_0, L]$, as $\lambda \to +\infty$.*

Proof. Step 1. We start by writing $\widehat{\mathcal{G}}_\lambda$ and $\widehat{G}$ in terms of the Eshelby tensor. Note that for $\widehat{\mathcal{G}}_\lambda$ it is not possible to apply directly [20] since the rescaled energy does not satisfy all the hypotheses, one for all, it is not orientation preserving. We will use instead the representation in Ω_λ and then apply a change of variable. Let $\hat{\psi} \in C^\infty_0(\mathbb{R}^2;\mathbb{R}^2)$ be such that $\hat{\psi} = \hat{e}_1$ in a neighborhood of the origin, and let $\hat{\psi}_{\hat{\ell}}(\hat{x}) := \hat{\psi}(\hat{x}_1 - \hat{\ell}, \hat{x}_2)$ be its translation to the crack tip.

Consider a map $\psi_{\lambda,\ell}$ of the form $\psi_{\lambda,\ell}(x) := \hat{\psi}_{\hat{\ell}}(x/\lambda)$ in the representation of the energy release $\mathcal{G}_\lambda(t,\ell)$ (i.e., (23) with Ω replaced by Ω_λ). Then $\mathcal{G}_\lambda$ reads

$$\mathcal{G}_\lambda(t,\ell) = \int_{\Omega_\lambda} \Big[(I + Du_\lambda)^T DW(I + Du_\lambda) - W(I + Du_\lambda)\boldsymbol{I}\Big] : D\psi_{\lambda,\ell}\, dx\,.$$

Next, we apply the change of variables of §5.3, that is: $x = \lambda\hat{x}$, $\ell = \lambda\hat{\ell}$, and $u_\lambda(x) = \lambda^{1/2}\hat{u}_\lambda(\hat{x})$. Hence $Du_\lambda(x) = \lambda^{-1/2}D\hat{u}_\lambda(\hat{x})$ and $Dv_\lambda = I + \lambda^{-1/2}D\hat{u}_\lambda$. Moreover $D\psi_{\lambda,\ell}(x) = D\hat{\psi}_{\hat{\ell}}(\hat{x})/\lambda$. Then,

$$\widehat{\mathcal{G}}_\lambda(t,\hat{\ell}\,) = \mathcal{G}_\lambda(t,\ell) = \int_{\widehat{\Omega}} (I + \lambda^{-1/2}D\hat{u}_\lambda)^T DW(I + \lambda^{-1/2}D\hat{u}_\lambda) : D\hat{\psi}_{\hat{\ell}}\lambda\, d\hat{x} + \tag{30}$$

$$- \int_{\widehat{\Omega}} W(I + \lambda^{-1/2}D\hat{u}_\lambda)\boldsymbol{I} : D\hat{\psi}_{\hat{\ell}}\lambda\, d\hat{x}\,. \tag{31}$$

Employing the map $\hat{\psi}_{\hat{\ell}}$ in (27) the linear energy release rate $\widehat{G}(t,\hat{\ell})$ is written as

$$\widehat{G}(t,\hat{\ell}\,) = \int_{\widehat{\Omega}} \Big[D\hat{u}^T\boldsymbol{\sigma}(\hat{u}) - W_{\mathrm{lin}}(D\hat{u})\boldsymbol{I}\Big] : D\hat{\psi}_{\hat{\ell}}\, d\hat{x}\,.$$

Step 2. Referring to the splitting (30)-(31) of $\widehat{\mathcal{G}}_\lambda(t,\hat{\ell}\,)$, we prove first that

$$\int_{\widehat{\Omega}} \lambda W(I + \lambda^{-1/2}D\hat{u}_\lambda)\boldsymbol{I} : D\hat{\psi}_{\hat{\ell}}\, d\hat{x} \quad\to\quad \int_{\widehat{\Omega}} W_{\mathrm{lin}}(D\hat{u})\boldsymbol{I} : D\hat{\psi}_{\hat{\ell}}\, d\hat{x}\,, \tag{32}$$

as $\lambda \to +\infty$. Indeed, by Proposition 6.3 $D\hat{u}_\lambda \to D\hat{u}$ (strongly in $L^2(\widehat{\Omega} \setminus K_L; \mathbb{R}^{2\times 2})$ and a.e. up to subsequences). Moreover, for λ large enough, a Taylor expansion around the identity (recalling that $W(I) = 0$ and $DW(I) = 0$, cf. Lemma 2.3) provides

$$W(I + \lambda^{-1/2}D\hat{u}_\lambda) = \lambda^{-1}\tfrac{1}{2}D\hat{u}_\lambda : D^2W(I)[D\hat{u}_\lambda] + o(\lambda^{-1}|D\hat{u}_\lambda|^2) \qquad \text{a.e. on } \widehat{\Omega}\,.$$

Hence

$$\lambda W(I + \lambda^{-1/2}D\hat{u}_\lambda) \to \tfrac{1}{2}D\hat{u} : \mathbf{C}[D\hat{u}] \qquad \text{a.e. on } \widehat{\Omega}\,.$$

Moreover by Proposition 6.2 and Proposition 6.3

$$\int_{\widehat{\Omega}} \lambda W(I + \lambda^{-1/2}D\hat{u}_\lambda)\, d\hat{x} = \widehat{\mathcal{F}}_\lambda(t,\hat{\ell},\hat{u}_\lambda) \to F(t,\hat{\ell},\hat{u})\,.$$

Then by generalized dominated convergence theorem, cf. Lemma 7.1, we get that

$$\lambda W(I + \lambda^{-1/2}D\hat{u}_\lambda) \to \tfrac{1}{2}D\hat{u} : \mathbf{C}[D\hat{u}] = W_{\mathrm{lin}}(D\hat{u}) \qquad \text{strongly in } L^1(\widehat{\Omega}).$$

As a consequence (32) holds, as $\lambda \to +\infty$.

Step 3. Now we take into account the term (30) and prove that

$$\int_{\widehat{\Omega}} (I + \lambda^{-1/2}D\hat{u}_\lambda)^T DW(I + \lambda^{-1/2}D\hat{u}_\lambda) : D\hat{\psi}_{\hat{\ell}}\lambda\, d\hat{x} \to \int_{\widehat{\Omega}} D\hat{u}^T\boldsymbol{\sigma}(\hat{u}) : D\hat{\psi}_{\hat{\ell}}\, d\hat{x}\,, \tag{33}$$

as $\lambda \to +\infty$. Let's try first with pointwise convergence: the term $(I+\lambda^{-1/2}D\hat{u}_\lambda) \to I$ strongly in $L^2(\widehat{\Omega} \setminus K_L; \mathbb{R}^{2\times 2})$ and a.e. (up to subsequences). Next, it seems natural to multiply the term $DW(I + \lambda^{-1/2}D\hat{u}_\lambda)$ by a factor $\lambda^{1/2}$; indeed, by Taylor expansion

$$\begin{aligned} \lambda^{1/2}DW(I + \lambda^{-1/2}D\hat{u}_\lambda) &= D^2W(I)D\hat{u}_\lambda + o(1)|D\hat{u}_\lambda| \\ &= \boldsymbol{\sigma}(\hat{u}_\lambda) + o(1)|D\hat{u}_\lambda| \to \boldsymbol{\sigma}(\hat{u})\,. \end{aligned} \tag{34}$$

With this (natural) choice the pointwise convergence of the remainder in (30) is not the right one, since $\lambda^{1/2}|D\hat{\psi}_{\hat{\ell}}| \to +\infty$. Therefore, we need to employ a different argument; the idea is to use the Euler-Lagrange equation (21). For $\phi = \psi_{\lambda,\ell}$, $v = v_\lambda = \mathrm{id} + u_\lambda$, and $\Omega = \Omega_\lambda$, (21) becomes

$$\int_{\Omega_\lambda} DW(Dv_\lambda)Dv_\lambda^T : D\psi_{\lambda,\ell}(v_\lambda)\,dx = 0\,.$$

Now, we apply the change of variables to pass from Ω_λ to the fixed domain $\widehat{\Omega}$. Since $D\psi_{\lambda,\ell}(x) = D\hat{\psi}_{\hat{\ell}}(x/\lambda)/\lambda$, in terms of $\hat{x}$ we get

$$\begin{aligned} D\psi_{\lambda,\ell}\big(v_\lambda(x)\big) &= D\hat{\psi}_{\hat{\ell}}\big(v_\lambda(x)/\lambda\big)/\lambda = D\hat{\psi}_{\hat{\ell}}\big((x+u_\lambda(x))/\lambda\big)/\lambda \\ &= D\hat{\psi}_{\hat{\ell}}\big(\hat{x} + \lambda^{-1/2}\hat{u}_\lambda(\hat{x})\big)/\lambda\,. \end{aligned}$$

Then the Euler-Lagrange equation reads

$$\int_{\widehat{\Omega}} DW(I + \lambda^{-1/2}D\hat{u}_\lambda)(I + \lambda^{-1/2}D\hat{u}_\lambda)^T : D\hat{\psi}_{\hat{\ell}}(\mathrm{id} + \lambda^{-1/2}\hat{u}_\lambda)\lambda\,d\hat{x} = 0\,. \tag{35}$$

At this point we add (35) to (30), which gives the integral

$$\begin{aligned} I_\lambda = \int_{\widehat{\Omega}} \Big(&(I + \lambda^{-1/2}D\hat{u}_\lambda)^T DW(I + \lambda^{-1/2}D\hat{u}_\lambda) : D\hat{\psi}_{\hat{\ell}} \\ &- DW(I + \lambda^{-1/2}D\hat{u}_\lambda)(I + \lambda^{-1/2}D\hat{u}_\lambda)^T : D\hat{\psi}_{\hat{\ell}}(\mathrm{id} + \lambda^{-1/2}\hat{u}_\lambda)\Big)\lambda\,d\hat{x}\,. \end{aligned} \tag{36}$$

To prove (33) we will actually prove that

$$I_\lambda \to \int_{\widehat{\Omega}} D\hat{u}^T \boldsymbol{\sigma}(\hat{u}) : D\hat{\psi}_{\hat{\ell}}\,d\hat{x}\,. \tag{37}$$

Step 4. To prove the above limit we will employ generalized dominated convergence. Let us look first for a dominant sequence which converges strongly in $L^1(\widehat{\Omega})$. Observe that it is not possible to use directly the growth estimate $|F^T DW(F)| \le C_1(W(F)+1)$ which follows from (8), since it should be multiplied by λ and the right-hand side would then diverge. Instead, it is convenient to re-write (36) in the following way

$$\begin{aligned} I_\lambda = &\int_{\widehat{\Omega}} \lambda\big[(I + \lambda^{-1/2}D\hat{u}_\lambda)^T DW(I + \lambda^{-1/2}D\hat{u}_\lambda) + \\ &\qquad - DW(I + \lambda^{-1/2}D\hat{u}_\lambda)(I + \lambda^{-1/2}D\hat{u}_\lambda)^T\big] : D\hat{\psi}_{\hat{\ell}}\,d\hat{x} + \\ &\int_{\widehat{\Omega}} \lambda^{1/2}\big[DW(I + \lambda^{-1/2}D\hat{u}_\lambda)(I + \lambda^{-1/2}D\hat{u}_\lambda)^T\big] : \lambda^{1/2}\big[D\hat{\psi}_{\hat{\ell}} - D\hat{\psi}_{\hat{\ell}}(\mathrm{id} + \lambda^{-1/2}\hat{u}_\lambda)\big]\,d\hat{x} \\ =: &\; I^1_\lambda + I^2_\lambda \end{aligned}$$

where I^i_λ denotes the first and second integral above.

Now we apply the enhanced estimates obtained in Lemma 2.5. Precisely, using (16) in Lemma 2.5 with $F = I + \lambda^{-1/2} D\hat{u}_\lambda$, we get for the integrand of I^1_λ that there exists a positive constant C such that

$$\begin{aligned}\lambda\big|(I + \lambda^{-1/2} D\hat{u}_\lambda)^T DW(I \,+\, \lambda^{-1/2} D\hat{u}_\lambda) - DW(I + \lambda^{-1/2} D\hat{u}_\lambda)(I + \lambda^{-1/2} D\hat{u}_\lambda)^T\big||D\hat{\psi}_{\hat{\ell}}| \\ \leq C\lambda W(I + \lambda^{-1/2} D\hat{u}_\lambda) + C\lambda|\lambda^{-1/2} D\hat{u}_\lambda|^2 .\end{aligned}$$

Thank to Propositions 6.2 and 6.3 together with Lemma 7.1 the right-hand side converges to $CW_{\mathrm{lin}}(D\hat{u}) + C|D\hat{u}|^2$ strongly in $L^1(\widehat{\Omega})$.

For I^2_λ we use (15) from Lemma 2.5 with $F = I + \lambda^{-1/2} D\hat{u}_\lambda$ and get by the regularity of $\hat{\psi}_{\hat{\ell}}$

$$\begin{aligned}\lambda^{1/2}\big|DW(I \,+\, \lambda^{-1/2} D\hat{u}_\lambda)(I + \lambda^{-1/2} D\hat{u}_\lambda)^T\big|\lambda^{1/2}\big|D\hat{\psi}_{\hat{\ell}} - D\hat{\psi}_{\hat{\ell}}(\mathrm{id} + \lambda^{-1/2}\hat{u}_\lambda)\big| \\ \leq c_3\big(\lambda^{1/2} W(I + \lambda^{-1/2} D\hat{u}_\lambda) + \lambda^{1/2}|\lambda^{-1/2} D\hat{u}_\lambda|\big)\lambda^{1/2}\big|D\hat{\psi}_{\hat{\ell}} - D\hat{\psi}_{\hat{\ell}}(\mathrm{id} + \lambda^{-1/2}\hat{u}_\lambda)\big| \\ \leq C\lambda W(I + \lambda^{-1/2} D\hat{u}_\lambda)\|D\hat{\psi}_{\hat{\ell}}\|_\infty + C|D\hat{u}_\lambda||\hat{u}_\lambda|\|D^2\hat{\psi}_{\hat{\ell}}\|_\infty \\ \leq C\lambda W(I + \lambda^{-1/2} D\hat{u}_\lambda) + |D\hat{u}_\lambda||\hat{u}_\lambda| ,\end{aligned}$$

where the value of the positive constant C may change even in the same line. Again by Propositions 6.2 and 6.3 and Lemma 7.1 the right-hand side converges to $CW_{\mathrm{lin}}(D\hat{u})+C|D\hat{u}||\hat{u}|$ strongly in $L^1(\widehat{\Omega})$. The above two estimates give the upper bound to apply the generalized dominated convergence.

Step 5. It remains to study the pointwise convergence of the integrand in (36). It is convenient to split the term $I + \lambda^{-1/2} D\hat{u}_\lambda$ writing the integrand as

$$\begin{aligned}\lambda^{1/2} DW(I \,+\, \lambda^{-1/2} D\hat{u}_\lambda) : \lambda^{1/2}\big(D\hat{\psi}_{\hat{\ell}} - D\hat{\psi}_{\hat{\ell}}(\mathrm{id} + \lambda^{-1/2}\hat{u}_\lambda)\big) + \\ + D\hat{u}^T_\lambda \lambda^{1/2} DW(I + \lambda^{-1/2} D\hat{u}_\lambda) : D\hat{\psi}_{\hat{\ell}} + \\ - \lambda^{1/2} DW(I + \lambda^{-1/2} D\hat{u}_\lambda) D\hat{u}^T_\lambda : D\hat{\psi}_{\hat{\ell}}(\mathrm{id} + \lambda^{-1/2}\hat{u}_\lambda) ,\end{aligned}$$

and consider pointwise convergence line by line. In the first line we get

$$\lambda^{1/2} DW(I + \lambda^{-1/2} D\hat{u}_\lambda) : \lambda^{1/2}\Big[D\hat{\psi}_{\hat{\ell}} - D\hat{\psi}_{\hat{\ell}}(\mathrm{id} + \lambda^{-1/2}\hat{u}_\lambda)\Big] \quad \to \quad -\boldsymbol{\sigma}(\hat{u}) : D^2\hat{\psi}_{\hat{\ell}}\hat{u} ,$$

for the second line we obtain

$$D\hat{u}^T_\lambda \lambda^{1/2} DW(I + \lambda^{-1/2} D\hat{u}_\lambda) : D\hat{\psi}_{\hat{\ell}} \quad \to \quad D\hat{u}^T \boldsymbol{\sigma}(\hat{u}) : D\hat{\psi}_{\hat{\ell}} ,$$

while for the third line

$$-\lambda^{1/2} DW(I + \lambda^{-1/2} D\hat{u}_\lambda) D\hat{u}^T_\lambda : D\hat{\psi}_{\hat{\ell}}(\mathrm{id} + \lambda^{-1/2}\hat{u}_\lambda) \quad \to \quad -\boldsymbol{\sigma}(\hat{u}) D\hat{u}^T : D\hat{\psi}_{\hat{\ell}} .$$

Step 6. Let us introduce the field $\tilde{u} = D\hat{\psi}_{\hat{\ell}}\hat{u}$ and note that $\tilde{u}$ is an admissible variation for $\hat{u}$. Then, by the previous steps we can write

$$
\begin{aligned}
I_\lambda \to I_\infty &= \int_{\widehat{\Omega}} -\boldsymbol{\sigma}(\hat{u}) : D^2\hat{\psi}_{\hat{\ell}}\hat{u} + D\hat{u}^T\boldsymbol{\sigma}(\hat{u}) : D\hat{\psi}_{\hat{\ell}} - \boldsymbol{\sigma}(\hat{u}) : D\hat{\psi}_{\hat{\ell}}D\hat{u}\, d\hat{x} \\
&= \int_{\widehat{\Omega}} -\boldsymbol{\sigma}(\hat{u}) : \left[D^2\hat{\psi}_{\hat{\ell}}\hat{u} + D\hat{\psi}_{\hat{\ell}}D\hat{u}\right] d\hat{x} + \int_{\widehat{\Omega}} D\hat{u}^T\boldsymbol{\sigma}(\hat{u}) : D\hat{\psi}_{\hat{\ell}}\, d\hat{x} \\
&= \int_{\widehat{\Omega}} -\boldsymbol{\sigma}(\hat{u}) : D\tilde{u}\; d\hat{x} + \int_{\widehat{\Omega}} D\hat{u}^T\boldsymbol{\sigma}(\hat{u}) : D\hat{\psi}_{\hat{\ell}}\, d\hat{x} \\
&= \int_{\widehat{\Omega}} D\hat{u}^T\boldsymbol{\sigma}(\hat{u}) : D\hat{\psi}_{\hat{\ell}}\, d\hat{x}\,,
\end{aligned}
$$

where the last equality holds thank to the fact that $\tilde{u}$ is an admissible variation. This yields (37) and the proof is concluded. ∎

7.2 Locally uniform convergence in time and space

To prove the local uniform convergence of the energy releases $\widehat{\mathcal{G}}_\lambda$ we use the next auxiliary lemma, for which we sketch a proof for the sake of completeness.

Lemma 7.3 *Let $a < b$ and $c < d$ be four real numbers. Let $f_\lambda, f : [a,b] \times [c,d] \to \mathbb{R}$ be such that $f_\lambda \to f$ pointwise, with f continuous. Then $f_\lambda \to f$ uniformly if and only if*

$$f_\lambda(t_\lambda, s_\lambda) \to f(t_\infty, s_\infty) \tag{38}$$

for every sequence $(t_\lambda, s_\lambda) \in [a,b] \times [c,d]$ with $(t_\lambda, s_\lambda) \to (t_\infty, s_\infty)$.

Proof. If $f_\lambda \to f$ uniformly then (38) follows by using the triangle inequality and the continuity of f.

Assume now (38) and suppose, by contradiction, that f_λ does not converge uniformly to f. Then there exist $\varepsilon > 0$ and $(t_\lambda, s_\lambda) \in [a,b] \times [c,d]$ such that

$$|f_\lambda(t_\lambda, s_\lambda) - f(t_\lambda, s_\lambda)| > \varepsilon$$

for every λ. Since (t_λ, s_λ) belongs to a compact set, up to a subsequence, there exist (t_∞, s_∞) such that $(t_\lambda, s_\lambda) \to (t_\infty, s_\infty)$. By (38) and continuity of f we get a contradiction with the previous inequality and the proof is concluded. ∎

Proposition 7.4 $\widehat{\mathcal{G}}_\lambda \to \widehat{G}$ *locally uniformly in* $[0,T] \times [l_0, L]$.

Proof. It is sufficient to apply previous Lemma with $f_\lambda = \widehat{\mathcal{G}}_\lambda$. To prove (38) that is

$$\widehat{\mathcal{G}}_\lambda(t_\lambda, \hat{\ell}_\lambda) \to \widehat{G}(t_\infty, \hat{\ell}_\infty)$$

for every $(t_\lambda, \hat{\ell}_\lambda) \in [0,T] \times [l_0, L]$ with $(t_\lambda, \hat{\ell}_\lambda) \to (t_\infty, \hat{\ell}_\infty)$, it is sufficient to re-use the proof of the pointwise convergence Theorem 7.2. The convergence of the energy release depends indeed only on the strong convergence of minimizers, which holds thanks to Proposition 6.3. ∎

8 Convergence of evolutions

The convergence of the release rates $\widehat{\mathcal{G}}_\lambda$ studied in the previous section together with the arguments of [27, Theorem 5.1] provide all the ingredients to get the convergence of the quasi-static evolutions.

8.1 Evolution of ℓ_λ

For $\lambda > 1$ consider the evolution ℓ_λ in Ω_λ. According to Definition 3.2, ℓ_λ satisfies the following properties:

(a_λ) ℓ_λ is non-decreasing,

(b_λ) if $t \notin J(\ell_\lambda)$ then $\mathcal{G}_\lambda(t, \ell_\lambda(t)) \le G_c$,

(c_λ) if $\mathcal{G}_\lambda(t, \ell_\lambda(t)) < G_c$ then $\ell'_\lambda(t) = 0$,

(d_λ) for every $t_* \in J(\ell_\lambda)$ and $\ell_* \in [\ell_\lambda(t_*^-), \ell_\lambda(t_*^+)]$ we have $\mathcal{G}_\lambda(t_*, \ell_*) \ge G_c$.

The following lemma is a consequence of properties (c_λ) and (d_λ).

Lemma 8.1 *Let ℓ_λ satisfy properties (c_λ)–(d_λ). If $\mathcal{G}_\lambda(\tau, s) < G_c$ for $\|(\tau, s) - (t^\star, l^\star)\|_\infty \le \delta$ and if $|\ell_\lambda(t) - l^\star| < \delta$ for some $|t - t^\star| < \delta$, then ℓ_λ is constant in $|t - t^\star| \le \delta$.*

8.2 Pull back

Using the change of variables of §5.3, we scale ℓ_λ back to the fixed domain $\widehat{\Omega}$. In particular $\hat{\ell}_\lambda(t) = \ell_\lambda(t)/\lambda$ and $\mathcal{G}_\lambda(t, \ell_\lambda(t)) = \widehat{\mathcal{G}}_\lambda(t, \hat{\ell}_\lambda(t))$. If $\ell_\lambda(t)$ satisfies (a_λ)–(d_λ), then the rescaled evolution $\hat{\ell}_\lambda(t)$ satisfies the following (rescaled) properties:

($\hat{a}_\lambda$) $\hat{\ell}_\lambda$ is non-decreasing,

($\hat{b}_\lambda$) if $t \notin J(\hat{\ell}_\lambda)$ then $\widehat{\mathcal{G}}_\lambda(t, \hat{\ell}_\lambda(t)) \le G_c$,

($\hat{c}_\lambda$) if $\widehat{\mathcal{G}}_\lambda(t, \hat{\ell}_\lambda(t)) < G_c$ then $\hat{\ell}'_\lambda(t) = 0$,

($\hat{d}_\lambda$) for every $t_* \in J(\hat{\ell}_\lambda)$ and $\hat{\ell}_* \in [\hat{\ell}_\lambda(t_*^-), \hat{\ell}_\lambda(t_*^+)]$ we have $\widehat{\mathcal{G}}_\lambda(t_*, \hat{\ell}_*) \ge G_c$.

8.3 Convergence

We are now in the position of proving the convergence result for the evolutions.

Theorem 8.2 *Let $\hat{\ell}_\lambda$ be a sequence of rescaled quasi-static evolutions satisfying ($\hat{a}_\lambda$)–($\hat{d}_\lambda$). Then (up to subsequences) $\hat{\ell}_\lambda \to \hat{\ell}$ pointwise in $[0, T]$ where the limit evolution $\hat{\ell}$ satisfies the following conditions:*

($\hat{a}$) *$\hat{\ell}$ is non-decreasing,*

($\hat{b}$) *if $t \notin J(\hat{\ell})$ then $\widehat{G}(t, \hat{\ell}(t)) \le G_c$,*

($\hat{c}$) *if* $\widehat{G}(t, \hat{\ell}(t)) < G_c$ *then* $\hat{\ell}'_\infty(t) = 0$,

($\hat{d}$) *for every* $t_* \in J(\hat{\ell})$ *and* $\hat{l}_* \in [\hat{\ell}(t_*^-), \hat{\ell}(t_*^+)]$ *we have* $\widehat{G}(t_*, \hat{l}_*) \geq G_c$.

Proof. By Helly's theorem there exists a subsequence (not relabelled) such that $\hat{\ell}_\lambda \to \hat{\ell}$ pointwise. We prove that $\hat{\ell}$ satisfies properties ($\hat{a}$)–($\hat{d}$).

($\hat{a}$) It follows by pointwise convergence.

The next properties rely also on the fact that $\widehat{G}$ is continuous in $[0, T] \times [l_0, L]$, cf. Proposition 4.1.

($\hat{b}$) Note that $\mathcal{H}^1(J(\hat{\ell}_\lambda)) = 0$. Thus, $t \notin J(\hat{\ell})$ implies that for every $\delta > 0$ there exists t_δ such that $|t_\delta - t| < \delta$ and $t_\delta \notin J(\hat{\ell}_\lambda) \cup J(\hat{\ell})$ for every λ. Hence, by ($\hat{b}_\lambda$), $\widehat{\mathcal{G}}_\lambda(t_\delta, \hat{\ell}_\lambda(t_\delta)) \leq G_c$. Then, by Proposition 7.4, $\widehat{\mathcal{G}}_\lambda(t_\delta, \hat{\ell}_\lambda(t_\delta)) \to \widehat{G}(t_\delta, \hat{\ell}(t_\delta))$ so that

$$\widehat{G}(t_\delta, \hat{\ell}(t_\delta)) \leq G_c\,.$$

By the arbitrariness of δ and the continuity of $\hat{\ell}$ and $\widehat{G}$, we get $\widehat{G}(t, \hat{\ell}(t)) \leq G_c$.

($\hat{c}$) If $\widehat{G}(t, \hat{\ell}(t)) < G_c$ then by continuity there exists $\delta > 0$ such that $\widehat{G}(\tau, s) < G_c$ for $\|(\tau, s) - (t, \hat{\ell}(t))\|_\infty \leq \delta$. By the locally uniform convergence of Proposition 7.4 there exists $0 < \delta' \leq \delta$ such that for λ large enough $\widehat{\mathcal{G}}_\lambda(\tau, s) < G_c$ for $\|(\tau, s) - (t, \hat{\ell}(t))\|_\infty \leq \delta'$. Then, by Lemma 8.1, $\hat{\ell}_\lambda$ is constant in $|\tau - t| \leq \delta'$. As a consequence (by pointwise convergence) $\hat{\ell}$ is constant in $|\tau - t| \leq \delta'$.

($\hat{d}$) Let $t_* \in J(\hat{\ell})$ and assume by contradiction that there exists $\hat{l}_* \in [\hat{\ell}(t_*^-), \hat{\ell}(t_*^+)]$ such that $\widehat{G}(t_*, \hat{l}_*) < G_c$. If $\hat{\ell}(t_*^-) < \hat{l}_* < \hat{\ell}(t_*^+)$ then, by continuity, there exists $\delta > 0$ such that $\widehat{G}(\tau, s) < G_c$ for $\|(\tau, s) - (t_*, \hat{l}_*)\|_\infty < \delta$. By locally uniform convergence there exists $0 < \delta' \leq \delta$ such that, for λ large enough,

$$\widehat{\mathcal{G}}_\lambda(\tau, s) < G_c \quad \text{for } \|(\tau, s) - (t_*, \hat{l}_*)\|_\infty < \delta'. \tag{39}$$

Assume, without loss of generality, that $\delta' \ll 1$ so that $\hat{\ell}(t_*^-) < \hat{l}_* - \delta' < \hat{l}_* + \delta' < \hat{\ell}(t_*^+)$. Let $t_1 < t_2$ be such that

$$t_* - \delta' < t_1 < t_* < t_2 < t_* + \delta'.$$

Note that (by pointwise convergence and monotonicity)

$$\hat{\ell}_\lambda(t_2) \to \hat{\ell}(t_2) \geq \hat{\ell}(t_*^+) > \hat{l}_* + \delta'$$
$$\hat{\ell}_\lambda(t_1) \to \hat{\ell}(t_1) \leq \hat{\ell}(t_*^-) < \hat{l}_* - \delta'\,.$$

If for every $\tau \in (t_1, t_2)$, for $\lambda \gg 1$,

$$\hat{l}_* - \delta' < \hat{\ell}_\lambda(\tau) < \hat{l}_* + \delta',$$

then $\widehat{\mathcal{G}}_\lambda(\tau, \hat{\ell}_\lambda(\tau)) < G_c$. Lemma 8.1 implies that $\hat{\ell}_\lambda$ is constant in $(t_* - \delta', t_* + \delta')$. As a consequence the limit $\hat{\ell}$ is constant, which is a contradiction. If, on the contrary, there exists $\tau \in (t_1, t_2)$ such that, for $\lambda \gg 1$,

$$\hat{\ell}_\lambda(\tau^-) \leq \hat{l}_* - \delta \quad \text{and} \quad \hat{\ell}_\lambda(\tau^+) \geq \hat{l}_* + \delta\,,$$

then, by (39) we have $\widehat{\mathcal{G}}_\lambda(\tau, s_*) < G_c$ for any $s_* \in (\hat{l}_* - \delta', \hat{l}_* + \delta') \subset (\hat{\ell}_\lambda(\tau^-), \hat{\ell}_\lambda(\tau^+))$, which is in contradiction with $(\hat{d}_\lambda)$. Thus, if $\widehat{G}(t_*, \hat{l}_*) < G_c$ for some $\hat{l}_* \in [\hat{\ell}(t_*^-), \hat{\ell}(t_*^+)]$ then necessarily $\hat{l}_* \in \{\hat{\ell}(t_*^-), \hat{\ell}(t_*^+)\}$. If $\hat{l}_* = \hat{\ell}(t_*^\pm)$ it is sufficient to follow the above reasoning using this time right or left neighborhoods of t_* to get again a contradiction. ■

Remark 8.3 First of all we remark that pointwise convergence of the energy release is not sufficient to have the (pointwise) convergence of the evolutions. Second, we can re-write the evolution in the following (weak) Kuhn-Tucker form. Since $\widehat{G}$ is continuous, for $t \in J(\hat{\ell})$ we have $\widehat{G}(t, \hat{\ell}^-(t)) = G_c$. Thus $(\hat{b})$ can be re-written in the following way: $\widehat{G}(t, \hat{\ell}^-(t)) \le G_c$ for every $t \in [0, T]$. Moreover, $(\hat{b})$ can be written in the sense of measures as

$$(\widehat{G}(t, \hat{\ell}^-(t)) - G_c)\, d\hat{\ell}(t) = 0\,.$$

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