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## COMPLEMENTING MAPS, CONTINUATION AND GLOBAL BIFURCATION

BY P. M. FITZPATRICK, I. MASSABÒ AND J. PEJSACHOWICZ

**ABSTRACT.** We state, and indicate some of the consequences of, a theorem whose sole assumption is the nonvanishing of the Leray-Schauder degree of a compact vector field, and whose conclusions yield multidimensional existence, continuation and bifurcation results.

**Complementing maps and the Theorem.** Let  $X$  be a Banach space,  $m$  be a positive integer, and  $O \subseteq \mathbf{R}^m \times X$  be open. Suppose  $f: O \rightarrow X$  is an  $m$ -parameter compact vector field: i.e.  $f(\lambda, x) = x - F(\lambda, x)$ , for  $(\lambda, x) \in O$ , where  $F$  is continuous and maps bounded sets into relatively compact sets. A continuous map  $g: O \rightarrow \mathbf{R}^m$ , which maps bounded sets into bounded sets, will be called a *complement* for  $f: O \rightarrow X$  provided that the Leray-Schauder degree,  $\deg((g, f), O, 0)$ , is defined and nonzero:  $(g, f)((\lambda, x)) \equiv (g(\lambda, x), f(\lambda, x))$ , for  $(\lambda, x) \in O$ , and since  $O$  is not assumed to be bounded, "defined" means  $(g, f)^{-1}(0)$  is compact.

By cohomology we will mean Čech cohomology with integral coefficients. By dimension of a topological space we mean the Čech-Lebesgue covering dimension, and if  $p \in A$ , the space  $A$  will be said to have dimension at least  $m$  at  $p$  provided that each neighborhood, in  $A$ , of  $p$  has dimension at least  $m$ .

**THEOREM.** Let  $X$  be a Banach space,  $m$  be a positive integer, and  $O \subseteq \mathbf{R}^m \times X$  be open. Suppose that  $f: O \rightarrow X$  is complemented by  $g: O \rightarrow \mathbf{R}^m$ . Then there exists a closed connected subset,  $C$ , of  $f^{-1}(0)$ , whose dimension at each point is at least  $m$ , and (\*) whenever  $K$  is a compact subset of  $C$ , with  $g^{-1}(0) \cap C \subseteq K$ , the map of pairs  $g: (C, C - K) \rightarrow (\mathbf{R}^m, \mathbf{R}^m - 0)$  induces a nontrivial map in the  $m$ th cohomology group. In particular,  $C \cap g^{-1}(0) \neq \emptyset$  and either  $C$  is unbounded or  $\overline{C} \cap \partial O \neq \emptyset$ . In the case when  $f$  and  $g$  are defined on  $\overline{O}$  with  $f^{-1}(0) \cap g^{-1}(0) \cap \partial O = \emptyset$ ,  $C$  also has the following properties: if  $C$  is bounded, then  $\dim(\overline{C} \cap \partial O) \geq m - 1$ , when  $m > 1$ , and  $\overline{C} \cap \partial O$  has at least two points, when  $m = 1$ ; if  $g: f^{-1}(0) \cap \overline{O} \rightarrow \mathbf{R}^m$  is proper and  $\dim(\overline{C} \cap \partial O) < m - 1$ , then  $g(\overline{C}) = \mathbf{R}^m$ .

**SKELETON OF THE PROOF.** Since  $\deg((g, f), O, 0) \neq 0$ , by using the cup-product in cohomology, it follows that whenever  $K$  is compact and  $g^{-1}(0) \subseteq K \subseteq f^{-1}(0)$  the map  $g: (f^{-1}(0), f^{-1}(0) - K) \rightarrow (\mathbf{R}^m, \mathbf{R}^m - 0)$  is cohomologically nontrivial. Passing to the limit over all such  $K$ 's we obtain a nontrivial class,  $\xi$ , in the  $m$ th Čech cohomology group with compact supports of  $f^{-1}(0)$ . The continuity of Čech theory enables us to choose a set,  $C$ , which is minimal

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among the closed subsets of  $f^{-1}(0)$  to which  $\xi$  restricts nontrivially. This  $C$  has the properties claimed.  $\square$

**Some consequences of the Theorem.** In what follows,  $f: O \subseteq \mathbf{R}^m \times X \rightarrow X$  is an  $m$ -parameter compact vector field.

1. *Continuation under global hypotheses.* Let  $\lambda_0 \in \mathbf{R}^m$  and let  $f_{\lambda_0}$  be the section of  $f$  over the slice  $O_{\lambda_0}$ . One shows that if  $\deg(f_{\lambda_0}, O_{\lambda_0}, 0) \neq 0$  then  $f: O \rightarrow X$  is complemented by  $g: O \rightarrow \mathbf{R}^m$  defined by  $g(\lambda, x) = \lambda - \lambda_0$ . Thus the theorem furnishes a description of how  $f^{-1}(0)$  emanates from  $O_{\lambda_0}$ . This is a multidimensional refinement of the Leray-Schauder continuation principle (see [4, 6 and 7]).

2. *Continuation under local hypotheses.* Let  $(\lambda_0, x_0) \in O$  and suppose that the map  $x \rightarrow f(\lambda_0, x)$  has a Fréchet derivative,  $L$ , at  $x = x_0$ . Assume  $L \in \mathcal{L}(X, X)$  is invertible. Then, letting  $U = O - \{(\lambda_0, x) \mid x \neq x_0, f(\lambda_0, x) = 0\}$ , one shows that  $f: U \rightarrow X$  is complemented by  $g: U \rightarrow \mathbf{R}^m$  defined by  $g(\lambda, x) = \lambda - \lambda_0$ . Thus, there is an  $m$ -dimensional connected subset,  $C$ , of  $f^{-1}(0) \cap U$ , which contains  $(\lambda_0, x_0)$ , and which is either unbounded or  $\overline{C} \cap \{\partial O \cup \{(\lambda_0, x) \mid x \neq x_0, f(\lambda_0, x) = 0\}\} \neq \emptyset$ . Another global version of the implicit function theorem was obtained in [3].

3. *Nonlinear perturbation of linear Fredholm operators.* Let  $\Omega \subseteq \mathbf{R}^2$  be simply connected, open and bounded, with  $\partial\Omega$  a smooth closed curve. Suppose  $\tau: \partial\Omega \rightarrow S^1$  is smooth and such that the winding number of  $\tau: \partial\Omega \rightarrow S^1$  equals  $-k < 0$ . Given  $\phi, \psi: \overline{\Omega} \times \mathbf{R}^2 \rightarrow \mathbf{R}$  we consider the following nonlinear Riemann-Hilbert problem: find  $u, v: \overline{\Omega} \rightarrow \mathbf{R}$  such that, if  $\tau = (\tau_1, \tau_2)$

$$(i) \quad \begin{cases} u_x - v_y = \phi(x, y, u, v), \\ v_x + u_y = \psi(x, y, u, v) \end{cases} \quad \text{in } \Omega,$$

(R-H)

$$(ii) \quad u\tau_1 - v\tau_2 = 0 \quad \text{on } \partial\Omega.$$

Let  $\alpha \in (0, 1)$  be such that  $\psi$  and  $\phi$  lie in  $C^{1+\alpha}(\overline{\Omega} \times A, \mathbf{R})$  for each bounded subset  $A$  of  $\mathbf{R}^2$  ( $C^{1+\alpha}$  denotes the usual Schauder space). Under the assumption that  $\psi(x, y, 0, 0) = \phi(x, y, 0, 0) = 0$ , for each  $(x, y) \in \overline{\Omega}$ , it follows that for each  $r > 0$ ,  $\{(u, v) \in C^{1+\alpha}(\overline{\Omega}, \mathbf{R}^2) \mid (u, v) \text{ solves R-H, } \|(u, v)\|_{1+\alpha} = r\}$  has dimension at least  $2k$ .

Let  $W = \{(u, v) \in C^{1+\alpha}(\overline{\Omega}, \mathbf{R}^2) \mid (u, v) \text{ satisfies (ii)}\}$ , and let  $L: W \rightarrow C^\alpha(\overline{\Omega}, \mathbf{R}^2)$  be the linear operator defined by the left-hand side of (i). Choose  $z_1, \dots, z_k$  in  $\Omega$  and define  $g: W \rightarrow \mathbf{R}^{2k+1}$  by

$$g((u, v)) = (u(z_1), v(z_1), \dots, u(z_k), v(z_k), \int_{\partial\Omega} [\tau_1 v + \tau_2 u] ds).$$

Letting  $X = g^{-1}(0)$ , the linear theory (see [10]) implies  $L: X \rightarrow C^\alpha(\overline{\Omega}, \mathbf{R}^2)$  has an inverse,  $T$ , and  $W = V \oplus X$ , with  $\dim(V) = 2k + 1$ .

If we rewrite (R-H) as  $f((u, v)) \equiv T(L - H)((u, v)) = 0$ , one shows that  $f: V \oplus X \rightarrow X$  is complemented by  $g$  on each ball about the origin in  $W$ , and so we can apply the Theorem.

4. *Global bifurcation.* For simplicity, we assume  $O = \mathbf{R}^m \times X$ . We assume  $\mathbf{R}^m \times \{0\} \subseteq f^{-1}(0)$ , and call  $\mathbf{R}^m \times \{0\}$  the trivial solutions of  $f$ . Suppose  $\alpha, \beta \in \mathbf{R}^m$  are such that  $(\alpha, 0)$  and  $(\beta, 0)$  are not bifurcation points of  $f^{-1}(0)$  and that

$\text{ind}(f_\alpha, 0) \neq \text{ind}(f_\beta, 0)$ , where "ind" denotes the Leray-Schauder index. Then, if  $\Gamma$  is any open curve (i.e. homeomorphic image of  $\mathbf{R}$ ) in  $\mathbf{R}^m \times \{0\}$  which passes through  $(\alpha, 0)$  and  $(\beta, 0)$ , there exists a connected set,  $C$ , of nontrivial zeros of  $f$ , whose dimension at each point is at least  $m$ , which intersects the segment,  $(\alpha, 0), (\beta, 0)$ , of  $\Gamma$ , determined by  $(\alpha, 0)$  and  $(\beta, 0)$ , and either  $C$  is unbounded or  $C$  intersects  $\Gamma - \{(\alpha, 0), (\beta, 0)\}$ .

When  $\alpha = 0$ ,  $\beta = (1, 0, \dots)$  and  $\Gamma$  is the line through  $\alpha$  and  $\beta$  the proof runs as follows. Choose  $r > 0$  such that  $f(\lambda, x) \neq 0$  when  $0 < \|x\| \leq r$  and either  $|\lambda| \leq 3r$  or  $|\lambda - \beta| \leq 3r$ . Let  $h: \mathbf{R} \rightarrow [0, r]$  be continuous, vanish outside of  $[-r, 1 + r]$ , and equal  $r$  on  $[r, 1 - r]$ . Then define  $g: \mathbf{R}^m \times X \rightarrow \mathbf{R}^m$  by  $g(\lambda_1, \dots, \lambda_m) = (\|x\|^2 - (h(\lambda_1))^2, \lambda_2, \dots, \lambda_m)$ .

One shows that if  $U = \mathbf{R}^m \times \{X - \{0\}\}$ , then  $\text{deg}((g, f), U, 0) = \text{ind}(f_\beta, 0) - \text{ind}(f_\alpha, 0)$ , and so  $g$  complements  $f$  on  $U$ . So we extract the subset,  $C$ , of  $f^{-1}(0) \cap U$ , having the properties in the conclusion of the Theorem. Conclusion (\*) implies our assertions regarding  $C \cap \Gamma$ .

This bifurcation result yields the principle abstract global bifurcation results of [9 and 1]. J. Ize (see [8]) has given a proof of the bifurcation theorem in [9] using a map similar to the above  $g$ .

REMARK. In the definition of complementing map if one replaces the Leray-Schauder degree by the Browder-Petryshyn degree for  $A$ -proper mappings (see [5]) the Theorem still holds. We believe that approximation results similar to those used in [2] will also yield the Theorem when  $F$  is assumed to be condensing.

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