POLITECNICO DI TORINO
Repository ISTITUZIONALE

A relativistic positioning system exploiting pulsating sources for navigation across the Solar System and beyond

A RELATIVISTIC POSITIONING SYSTEM EXPLOITING PULSATING SOURCES FOR NAVIGATION ACROSS THE SOLAR SYSTEM AND BEYOND

Emiliano Capolongo, Matteo Luca Ruggiero,[*] Angelo Tartaglia
DIFIS Politecnico di Torino, corso Duca degli Abruzzi 24, 10129 Torino, Italy
INFN, Sezione di Torino, Via Pietro Giuria 1, 10125 Torino, Italy

Key Words: relativistic positioning system, astrometry and reference systems, null frames.

Abstract

We introduce an operational approach to the use of pulsating sources, located at spatial infinity, for defining a relativistic positioning and navigation system, based on the use of null four-vectors in a flat Minkowskian spacetime. We describe our approach and discuss the validity of it and of the other approximations we have considered in actual physical situations. As a prototypical case, we show how pulsars can be used to define such a positioning system: the reception of the pulses for a set of different sources whose positions in the sky and periods are assumed to be known allows the determination of the user's coordinates and spacetime trajectory, in the reference frame where the sources are at rest. In order to confirm the viability of the method, we consider an application example reconstructing the world-line of an idealized Earth in the reference frame of distant pulsars: in particular we have simulated the arrival times of the signals from four pulsars at the location of the Parkes radiotelescope in Australia. After pointing out the simplifications we have made, we discuss the accuracy of the method. Eventually, we suggest that the method could actually be used for navigation across the Solar System and be based on artificial sources, rather than pulsars.

1. Introduction

The evocative and romantic image of an ancient carrack which, with a scanty crew and little else on board, approaches to cross an entire ocean, can indeed evoke, beyond the

[*]E-mail address: matteo.ruggiero@polito.it

greatness of the effort, the sense of confusion and littleness you feel when venturing into the exploration of the unknown. All along, man has been eager for knowledge and discovery, and all along he has tried hard to elaborate methods to design devices suited to facilitate his life and direct his steps. Imagine, for a moment, how vital it would be to know, in a hypothetical space travel, where you are and be able to locate yourself in a unique way in space and time. In this case, lost into the deep space, even more than on a fragile ship at the mercy of the waves, you would more dramatically feel the need for a positioning system that could guide you and light your way.

Actually, the problem of positioning on the Earth is today solved by systems like GPS and GLONASS (Ashby, 2003; Pascual-Sánchez, 2007); from a physical viewpoint, in these systems positioning takes place in a classical (Euclidean) space and absolute time, over which relativistic corrections are introduced (Ashby, 2003). Even in the space surrounding the Earth up to Low Earth Orbit (LEO) altitude, a complete navigation solution can be provided by the current GPS or, with better performance, by his Differential version DGPS (Parkinson et al., 1996; Kaplan et al., 2006).

For interplanetary and deep space missions, current navigation methods highly rely on Earth ground-based extensive operations for absolute/relative position determination (Emadzadeh et al., 2011; Jordan, 1987; Melbourne, 1976). Different approaches are used to this aim such as radar ranging, optical tracking and planet imaging (Wertz, 1978; Bate et al., 1971; Battin, 1999).

An important advantage of the radar ground-based system consists in the absence of active hardware on vehicles. By the way, many drawbacks hold: high costs in terms of ground operations; problematic reception of data corrupted by a strong, sometime critical, background noise; need for precise positional information about radar observation stations on the Earth and Solar System objects (Jordan, 1987); position estimation error increasing directly proportionally with respect to the distance from the Earth, with degrading factor given by the angular accuracy. Even if active transmitters were used on the space vehicles to send signals back to the Earth (Jordan, 1987), it would be possible to estimate the radial velocity by measuring the Doppler frequency of this received signals and to obtain some little improvements only, provided that errors always increase with distance; early experiments using these tracking systems on the Viking spacecraft showed accuracies up to about 50 km in position estimation error for missions to Mars and positional accuracies of the order of 100 km for the outer planets (Emadzadeh et al., 2011; Melbourne, 1976).

The optical tracking system is very similar in his basic principles to the radar tracking, except that it utilizes visible light reflected from a spacecraft (Bate et al., 1971). In some environmental conditions, where optical measurements are favored, this can be an appealing chance; the vehicle's position is calculated by the comparison of a taken photograph with respect to a fixed star background and, therefore, real-time measurements using such methods are typically not easily achieved.

For planetary observation missions in the neighborhood of the examined planet, it is possible to obtain the positioning of the vehicle relative to the planet itself by comparing video images of the planet, taken on board, to the known planetary parameters such as diameter and position relative to the other celestial objects (Battin, 1999).

In order to increase the performance of the navigation system, a combination of these techniques can be employed, still requiring tight human interaction and a not so easy interpretation of the data. Furthermore, as radar-ranging errors increase as the vehicle distance from the Earth increases, accurate navigation becomes more complex because of the required more refined pointing accuracy of ground antennas. Moreover, the imaging process on vehicles, sufficiently close to the planets, implies complex and expensive on board systems.

Most of actual spacecrafts employ the NASA Deep Space Network (DSN, 2010). This system allows to reach accurate radial position but, as mentioned above, the angular uncertainty increases with distance, leading to low performance in long range navigation. Position accuracies in the order of 10 km per astronomical unit of distance from Earth are achievable using interferometric measurements of the Very Long Baseline Interferometer (VLBI) through the DSN (Emadzadeh et al., 2011).

The requirement of higher accuracy, as well as of reduction of costs associated with the ground system, and also the possibility to augment and improve the current available navigation systems push toward different and alternative methods and more autonomous solutions (Folta et al., 1999; Gounley et al., 1984) without human assistance and communication with the Earth. The ultimate goal would be a complete, accurate, absolute, autonomous navigation solution working throughout the Solar System and, eventually, at galactic/intergalactic scales.

In this chapter we discuss how it is possible to define a relativistic positioning system, effective for the navigation in the Solar System and beyond, by using electromagnetic signals coming from pulsating sources, located at spatial infinity. Actually, a *fully relativistic* positioning system can be built by exploiting the worldlines of electromagnetic signals to define the so called *emission coordinates* (Coll, 2006; Coll et al., 2006a,b; Rovelli, 2002; Blagojevic et al., 2002; Ruggiero & Tartaglia, 2007; Bini et al., 2008). The simplest way of understanding what emission coordinates are is to consider four emitting clocks, in motion in spacetime, broadcasting their proper times: the intersection of the past lightcone of an event with the worldlines of the emitting clocks corresponds to the proper times of emission along the worldlines of the emitters; these proper times are the emission coordinates of the given event. A positioning system based on the use of emission coordinates can be effective for positioning on the Earth and, in this case, one may think of a set of satellites orbiting around the Earth and equipped with onboard clocks broadcasting their proper time. However, for positioning and navigation in the Solar System, a set of *pulsars* (or, more generally, suitable pulsating sources) could rather be used (Coll & Tarantola, 2004, 2009). In fact, known pulsars emit their signals at a highly regular rate (this is the case, in particular, of the *millisecond- pulsars*, see e.g. Kramer et al. (2004)), which makes them natural beacons for building a relativistic positioning system. What can be measured with great accuracy is the arrival time of the N-th pulse, so that counting these pulses can in principle allow to define something similar to the emission coordinates, even though in this case the actual proper emission time is unknown and inaccessible.

The idea of using pulsars as stellar beacons has been considered since the early years of the discovery of pulsars (Downs, 1974), and also more recently (Sala et al., 2004; Sheikh et al., 2006, 2007): indeed, in these approaches, positioning is not autonomous, but can be referred to a reference frame centered at the Solar System barycenter (SSB), because pulses arrival times or their phases have to be related to their expected values in the SSB.

What we want to show here is that it is possible to operationally define an autonomous positioning system, by building emission coordinates exploiting electromagnetic signals coming from periodic sources (assumed to be at rest at spatial infinity) such as pulsars (Tartaglia, 2009; Tartaglia et al., 2011; Ruggiero et al., 2011). In other words, we will show that, by counting pulses from a set of different (at least four) sources, whose positions in the sky and periods are assumed to be known, and measuring the proper time intervals between successive arrivals of the signals, it is possible to determine, in the reference frame where the sources are at rest, the user's coordinates and spacetime trajectory, within an accuracy controlled by the precision of the user clock. In doing so, the phases of the received pulses play the role of emission coordinates and we assume that the user worldline is a straight line during a proper time interval corresponding to the reception of a limited number of pulses, which means that the effects of the acceleration are negligibly small. This system can allow the autopositioning with respect to an arbitrary event in spacetime and three directions in space, so that it could be used for space navigation and positioning in the Solar System and even beyond. In practice, the initial event of the self-positioning process is used as the origin of the reference frame, and the axes are oriented according to the positions of the distant sources; all subsequent positions will be given in that precise frame. If one wants to further position the whole section of worldline of the receiver in some other external reference frame the location of the initial event in the external frame has to be known by other means. Our approach is based on the use of null frames in a flat Minkowskian spacetime, but we discuss the validity of this and other approximations we have considered for actual physical situations.

Moreover, we focus on a simple application of our method, in which we imagine that our sources are four millisecond pulsars and, simulating the arrival times of their signals, we show how the worldline of the receiver is reconstructed: in particular, we make use of the TEMPO2 software (Hobbs et al., 2006), a pulsar-timing package that simulates the times of arrival of pulses at a given location on the Earth, and we determine the trajectory of that location in spacetime, due to the combined motion of the Earth around the Sun and to its daily rotation.

The chapter is organized as follows: in Section 2. we show how to build the basic reference frame and describe the localization procedure; in Section 3. we discuss the sources of error; in Section 4. we present an application example to test our method, by using simulated data. Eventually, discussion and conclusion are in Section 5.

2. Definition of the Basic Null Frame Grid and corresponding Localization

In this Section we first focus on the theoretical framework which enables to define the basic (null) reference frame where positioning takes place and, then, we show how the localization procedure can be operationally implemented in this frame.

2.1. The Basic Null Frame

Let us consider a number of sources of periodic electromagnetic signals, at rest at spatial infinity, in a four-dimensional Minkowski spacetime. For our purposes, at least four sources are needed. Each of these sources is characterized by the frequency of its periodic signals and by their directions in space; since the sources are supposed to be far away (i.e. at spatial infinity), their signals can be seen as corresponding to plane waves. In the inertial frame where the sources are at rest, once Cartesian coordinates are chosen, we associate to each source a null four-vector[1] $\boldsymbol{f}$ whose Cartesian contravariant components are given by

$$f^{\mu} \doteq \frac{1}{cT}(1, \vec{\mathbf{n}}), \tag{1}$$

T being the (proper) signal period, and $\vec{\mathbf{n}}$ the unit vector describing the direction of propagation in the given frame. If in the same reference frame we consider the position four-vector

$$\boldsymbol{r} \doteq (ct, \vec{\mathbf{x}}), \tag{2}$$

with respect to an arbitrary and yet unspecified origin, then we can define the scalar function X at the spacetime event identified by $\boldsymbol{r}$, the position four-vector

$$X(\boldsymbol{r}) \doteq \boldsymbol{f} \cdot \boldsymbol{r}, \tag{3}$$

where dot stands for Minkowski scalar product. The scalar X might be thought of as the phase difference of the wave described by $\boldsymbol{f}$ with respect to its value at the origin of the coordinates. Four linearly independent four-vectors constitute a basis, or a frame: we may think of choosing four null four-vectors to serve as a basis (see e.g. Blagojevic et al. (2002)), so that the four wave four-vectors $\{\boldsymbol{f}_{(a)}, \boldsymbol{f}_{(b)}, \boldsymbol{f}_{(c)}, \boldsymbol{f}_{(d)}\}$ in the form (1) constitute our *null frame*, or *null tetrad*. Then, according to the general approach to coordinate systems and frames developed by Coll et. al (2009) in connection with positioning systems, the four phase differences

$$X_{(N)} \doteq \boldsymbol{f}_{(N)} \cdot \boldsymbol{r}, \quad N = a, b, c, d \tag{4}$$

obtained at any event $\boldsymbol{r}$ whose coordinates are defined by (2), with a, b, c, d labeling the sources, are *null coordinates*: in other words, they are spacetime functions with null spacetime gradient and, hence, they define a *null coordinate system*. Furthermore, the $\{X_{(N)}\}$ are *emission coordinates*, since they are physically related to the reception of electromagnetic signals emitted by the sources.

The tetrad formalism (see e.g. Chandrasekhar (1983)) allows to define the symmetric matrix

$$\eta_{(M)(N)} = \boldsymbol{f}_{(M)} \cdot \boldsymbol{f}_{(N)}, \tag{5}$$

which, in this case, has constant components, and whose inverse is determined by the relation

$$\eta_{(M)(P)}\eta^{(P)(N)} = \delta^{(N)}_{(M)}. \tag{6}$$

Tetrad indices $N = a, b, c, d$ are lowered and raised by means of the matrices $\eta_{(M)(P)}$ and $\eta^{(M)(P)}$. We can write the position four-vector $\boldsymbol{r}$ in the form

$$\boldsymbol{r} = X^{(N)}\boldsymbol{f}_{(N)} = X_{(N)}\boldsymbol{f}^{(N)}, \tag{7}$$

[1]Arrowed boldface letters like $\vec{\mathbf{x}}$ refer to spatial vectors, while boldface letters like $\boldsymbol{f}$ refer to four-vectors; Greek indices refer to spacetime components, while Latin letters label the sources.

and, as a consequence, we see that the phase differences $X_{(N)}$ are the components of the position four-vector with respect to the vectors

$$\boldsymbol{f}^{(N)} = \eta^{(N)(M)} \boldsymbol{f}_{(M)}, \tag{8}$$

or, differently speaking, the functions

$$X^{(N)} = \eta^{(N)(M)} X_{(M)} \tag{9}$$

are the components of the position four-vector with respect to the null tetrad vectors $\boldsymbol{f}_{(N)}$. It is useful to remark that while the frame $\{\boldsymbol{f}_{(a)}, \boldsymbol{f}_{(b)}, \boldsymbol{f}_{(c)}, \boldsymbol{f}_{(d)}\}$ is constituted by null vectors, the frame $\{\boldsymbol{f}^{(a)}, \boldsymbol{f}^{(b)}, \boldsymbol{f}^{(c)}, \boldsymbol{f}^{(d)}\}$ is constituted by space-like vectors.

In summary, if we consider the hyperplanes conjugated to the null frame $\{\boldsymbol{f}_{(a)}, \boldsymbol{f}_{(b)}, \boldsymbol{f}_{(c)}, \boldsymbol{f}_{(d)}\}$ vectors, we are able to define a spacetime grid (see Tartaglia (2009)), in which each event is identified by the relative phase of the electromagnetic signals with respect to an arbitrary origin and, in this frame, the coordinates of each event are given by the functions $\left\{X^{(N)}\right\}$; equivalently, the phases $\left\{X_{(N)}\right\}$ are the coordinates with respect to the space-like frame $\{\boldsymbol{f}^{(a)}, \boldsymbol{f}^{(b)}, \boldsymbol{f}^{(c)}, \boldsymbol{f}^{(d)}\}$.

2.2. Localization within the Grid

After having shown how to build a grid, we want to focus on how localization can be achieved within the grid. In particular, we suppose to deal with periodic signals in the form of electromagnetic pulses, such as those coming from pulsars, and that these signals can be thought of as plane waves locally. Furthermore, we suppose that the user is equipped with a receiver able to recognize and count the pulses coming from the various sources, and a clock, that can be used to measure the proper time span between the arrivals from each source.

Let us start with a toy model, where the emission from the sources is continuous and the phases of any signal can be determined with an arbitrary precision, at any event. We choose a starting event, from which the phases of each signal are measured, which is the origin of our coordinates (in other words, the event with $\boldsymbol{r} = \boldsymbol{0}$, according to what we have described above), and three directions in space, defining the Cartesian axes of the inertial frame of the sources. We point out that even though the starting event is arbitrary, in order to correctly define the null frame, the directions of the sources in the sky have to be known: in other words, we have to know the unit vectors $\vec{\mathbf{n}}$ for each source (and their proper frequency ν too), which also enable us to calculate the matrices $\eta_{(N)(M)}, \eta^{(N)(M)}$ of the given frame.

To a subsequent event $\boldsymbol{r}$, we associate the measured phases

$$X_{(N)} = \boldsymbol{f}_{(N)} \cdot \boldsymbol{r}, \tag{10}$$

and, according to eq. (7), it is then possible to obtain the coordinates of the event $\boldsymbol{r}$, in terms of the measured phases:

$$\boldsymbol{r} = X_{(a)} \boldsymbol{f}^{(a)} + X_{(b)} \boldsymbol{f}^{(b)} + X_{(c)} \boldsymbol{f}^{(c)} + X_{(d)} \boldsymbol{f}^{(d)}. \tag{11}$$

and to reconstruct the user's worldline.

Coming to a more realistic situation, such as the one in which the emitters are pulsars, we should consider that the received signals consist in a series of pulses and are not continuous. In this case, we may proceed as follows. First, we call "reception" the event corresponding to the arrival of a pulse from one of the sources. As a consequence, the position in spacetime of an arbitrary reception event can be written in the form

$$\boldsymbol{r} = X_{(a)}\boldsymbol{f}^{(a)} + X_{(b)}\boldsymbol{f}^{(b)} + X_{(c)}\boldsymbol{f}^{(c)} + X_{(d)}\boldsymbol{f}^{(d)}, \tag{12}$$

with

$$X_{(a)} = n_{(a)} + p, \tag{13}$$
$$X_{(b)} = n_{(b)} + q, \tag{14}$$
$$X_{(c)} = n_{(c)} + s, \tag{15}$$
$$X_{(d)} = n_{(d)} + w. \tag{16}$$

The $X_{(N)}$'s in the case of a continuous signal would be phases. Here they are given by an integer $n_{(N)}$, numbering the order of the successive pulses from a given source, and a fractional value: e.g. p means a fractional value of the cycle in $X^{(a)}$, and the same thing holds for q, s, w, where $0 < p, q, s, w$. In eqs. (13)-(16), only one of the p, q, s, w will in general be zero: when, for instance, a pulse from source "a"arrives, p will be zero, while q, s, w will not; when a pulse from source "b"comes, q will be zero and the other three fractions will not; and so on. Once we choose an arbitrary origin, we may count the pulses in order to measure the $n_{(N)}$, but we have no direct means to measure the fractional values p, q, s, w. By the way, a procedure to determine these values can be obtained, based on geometric considerations: we suppose that the acceleration of the user is small during a limited series of reception events, so that we may identify the user's worldline with a straight line; furthermore, we also suppose that by means of his own clock the user can measure the proper time interval τ_{ij} between the i-th and j-th arrivals. With these assumptions we can proceed as follows to determine the fractional values p, q, s, w. Let us consider two sequences[2] of arrival times from the sources; we have eight events, each of them in the form

$$\boldsymbol{r}_j = X_{(a)j}\boldsymbol{f}^{(a)} + X_{(b)j}\boldsymbol{f}^{(b)} + X_{(c)j}\boldsymbol{f}^{(c)} + X_{(d)j}\boldsymbol{f}^{(d)}, j = 1, .., 8, \tag{17}$$

where $X_{(N)j}$ are expressions like (13–16). The events are arranged in such a way that $\boldsymbol{r}_1$ is the generic arrival of the signal from pulsar "a", $\boldsymbol{r}_2$ is the arrival of the first signal of pulsar "b" after $\boldsymbol{r}_1$, $\boldsymbol{r}_3$ is the arrival of the first signal of pulsar "c" after $\boldsymbol{r}_1$, and $\boldsymbol{r}_4$ is the arrival of the first signal of pulsar "d" after $\boldsymbol{r}_1$ (the pulsars are ordered from largest ("a") to shortest ("d") period); $\boldsymbol{r}_5$ is the arrival of the next signal from pulsar "a", and so on. The flatness hypothesis allows us to write the displacement four-vector between two reception events in the form

$$\boldsymbol{r}_{ij} \doteq \boldsymbol{r}_i - \boldsymbol{r}_j = \left(X_{(N)i} - X_{(N)j}\right)\boldsymbol{f}^{(N)} \doteq \Delta X_{(N)ij}\boldsymbol{f}^{(N)}. \tag{18}$$

Indeed, the assumption that the worldline of the receiver is straight during a limited number of pitches of the signals can be used also to provide further information. In fact, let us

[2] They may be subsequent or not, provided the total time span does not spoil the hypothesis of linearity of the worldline.

consider three successive reception events i, j, k; we have

$$\boldsymbol{r}_{ji} = \Delta X_{(N)ji}\boldsymbol{f}^{(N)}, \qquad \boldsymbol{r}_{kj} = \Delta X_{(N)kj}\boldsymbol{f}^{(N)}. \tag{19}$$

The straight line hypothesis allows us to write

$$\frac{\tau_{ji}}{\tau_{kj}} = \frac{\Delta X_{(a)ji}}{\Delta X_{(a)kj}} = \frac{\Delta X_{(b)ji}}{\Delta X_{(b)kj}} = \frac{\Delta X_{(c)ji}}{\Delta X_{(c)kj}} = \frac{\Delta X_{(d)ji}}{\Delta X_{(d)kj}}, \tag{20}$$

where τ_{ji}, τ_{kj} are the proper times elapsed between the i-th and j-th, and j-th and k-th reception events, respectively. These relations enable us to obtain the values we are interested in: in fact, we may arrange the coefficients of eqs. (17) in an 8×4 matrix (8 events, 4 sources):

$$X_{(N)i} = \left(\begin{array}{c|c|c|c} n_1^{(a)} & n_1^{(b)} + q_1 & n_1^{(c)} + s_1 & n_1^{(d)} + w_1 \\ \hline n_2^{(a)} + p_2 & n_2^{(b)} & n_2^{(c)} + s_2 & n_2^{(d)} + w_2 \\ \hline n_3^{(a)} + p_3 & n_3^{(b)} + q_3 & n_3^{(c)} & n_3^{(d)} + w_3 \\ \hline n_4^{(a)} + p_4 & n_4^{(b)} + q_4 & n_4^{(c)} + s_4 & n_4^{(d)} \\ \hline n_5^{(a)} & n_5^{(b)} + q_5 & n_5^{(c)} + s_5 & n_5^{(d)} + w_5 \\ \hline n_6^{(a)} + p_6 & n_6^{(b)} & n_6^{(c)} + s_6 & n_6^{(d)} + w_6 \\ \hline n_7^{(a)} + p_7 & n_7^{(b)} + q_7 & n_7^{(c)} & n_7^{(d)} + w_7 \\ \hline n_8^{(a)} + p_8 & n_8^{(b)} + q_8 & n_8^{(c)} + s_8 & n_8^{(d)} \end{array} \right) \tag{21}$$

As it can be seen, the p, q, s, w are zero along the main diagonals of the upper and lower square half matrices forming the whole matrix. This happens in correspondence of the arrivals of the pulses from the various sources: on the arrival of a pulse from "a" the corresponding p is zero, from "b" q is zero, and so on. Then, on using relations like (20) we obtain the fractional values in terms of observed quantities, i.e. proper time intervals measured by the observer. For instance, we have

$$p_1 = 0, \quad q_1 = n_2^{(b)} - n_1^{(b)} - \left(n_6^{(b)} - n_2^{(b)}\right)\frac{\tau_{21}}{\tau_{62}}, \tag{22}$$

$$s_1 = n_3^{(c)} - n_1^{(c)} - \left(n_7^{(c)} - n_3^{(c)}\right)\frac{\tau_{31}}{\tau_{73}}, \tag{23}$$

$$w_1 = n_4^{(d)} - n_1^{(d)} - \left(n_8^{(d)} - n_4^{(d)}\right)\frac{\tau_{41}}{\tau_{84}}, \tag{24}$$

$$p_2 = n_1^{(a)} - n_2^{(a)} + \left(n_5^{(a)} - n_1^{(a)}\right)\frac{\tau_{21}}{\tau_{51}}, \quad q_2 = 0, \tag{25}$$

$$s_2 = n_3^{(c)} - n_2^{(c)} + \left(n_7^{(c)} - n_3^{(c)}\right)\frac{\tau_{23}}{\tau_{73}}, \tag{26}$$

$$w_2 = n_4^{(d)} - n_2^{(d)} + \left(n_8^{(d)} - n_4^{(d)}\right)\frac{\tau_{24}}{\tau_{84}}, \tag{27}$$

$$p_3 = n_1^{(a)} - n_3^{(a)} + \left(n_5^{(a)} - n_1^{(a)}\right)\frac{\tau_{31}}{\tau_{51}}, \tag{28}$$

$$q_3 = n_2^{(b)} - n_3^{(b)} - \left(n_6^{(b)} - n_2^{(b)}\right)\frac{\tau_{23}}{\tau_{62}}, \tag{29}$$

$$s_3 = 0, \quad w_3 = n_4^{(d)} - n_3^{(d)} + \left(n_8^{(d)} - n_4^{(d)}\right)\frac{\tau_{34}}{\tau_{84}}, \tag{30}$$

$$p_4 = n_1^{(a)} - n_4^{(a)} + \left(n_5^{(a)} - n_1^{(a)}\right)\frac{\tau_{41}}{\tau_{51}}, \tag{31}$$

$$q_4 = n_2^{(b)} - n_4^{(b)} - \left(n_6^{(b)} - n_2^{(b)}\right)\frac{\tau_{24}}{\tau_{62}}, \tag{32}$$

$$s_4 = n_3^{(c)} - n_4^{(c)} - \left(n_7^{(c)} - n_3^{(c)}\right)\frac{\tau_{34}}{\tau_{73}}, \quad w_4 = 0, \tag{33}$$

and so on. Moving the pair of sequences and repeating the elaboration step by step, we are able to reconstruct the whole worldline of the receiver, in terms of measured quantities, i.e. proper times.

3. Biases and uncertainties

In this Section we will discuss the sources of error and their importance in the positioning process that we have presented. Before going into details of our analysis, we would like to preliminarily discuss the nature of errors and inaccuracies that affect our positioning process, in order to clarify how they can be dealt with to improve accuracy.

Roughly speaking we can distinguish between *systematic errors* and *uncertainties* (or *fluctuations*). Systematic errors originate from mismodelling of the physical processes we are dealing with or from poor knowledge of the system parameters; they globally affect the positioning process and are either constant or time dependent. For instance, errors in the angular positions of the sources in the sky as well as in the periods of the pulses are systematic errors; furthermore, if we deal with pulsars, their angular positions may change because of their proper motion and their periods because of energy loss. As for the uncertainties, they are related to the stochastic variations of the quantities involved in the positioning process: for instance in the procedure of measuring the arrival times of the pulses, fluctuations are due to the detection device and to the user's clock as well as to the emission mechanism at the surface of the star; in principle also the turbulence of the interstellar plasma could have a role. Systematic errors produce global consequences with an unknown distortion of the re-built spacetime trajectory of the user; the uncertainties transform the worldline in an uncertainty stripe across spacetime. Systematics can be reduced by improving our model and the knowledge of its parameters; a statistical analysis and the best technologies can help to reduce the impact of random disturbances.

3.1. Model Limitations

Let us begin by discussing the validity of our model, which is based on the propagation of electromagnetic signals originating from pulsating sources at rest in a given reference frame, in Minkowski spacetime. The question we would like to address is: how realistic is this model? Can it be used for positioning in actual physical situations? We start with some general observations.

We explicitly work in flat spacetime, thus eliminating the effects of the gravitational field. It is however obvious that, for positioning in the Solar System, the gravitational field of the Sun (and of the other major bodies) influences the propagation of electromagnetic signals[3]. The dimensionless magnitude of the static gravitational field of the Sun is of the order of $\delta_\odot \simeq \frac{GM_\odot}{c^2 d} \simeq 10^{-8}\left(\frac{1\ \mathrm{A.U.}}{d}\right)$, and reaches its maximum value near the Sun[4], where $\delta_\odot \simeq 10^{-6}$. This field produces effects on the times of arrival of the pulses which are relevant for our purposes only if they change in a time comparable with the integration times used for our algorithm; only the radial component (with respect to the Sun) is important. The effects due to the motion of a user who travels with speed $\vec{\mathbf{v}}$ in the radial direction $\hat{\mathbf{r}}$ (and thus experiences a time varying gravitational field) in a time span δt are expressed in terms of apparent fractional change of the period of the sources[5] as $\delta_{\odot,v} \simeq \frac{GM_\odot}{c^2 d^2}\vec{\mathbf{v}}\cdot\hat{\mathbf{r}}\ \delta t \simeq 6\times 10^{-8}\left(\frac{1\ \mathrm{A.U.}}{d^2}\right)\left(\frac{v}{30\ \mathrm{km/s}}\right)\mathrm{y}^{-1}\delta t$. This fixes an upper limit to the acceptable δt in order the effect to be compatible with the required tolerance. One more problem is that when the line of sight to a source grazes the Sun the time of flight of the signals depends on the geometric curvature of the rays and on the Shapiro time delay, which is indeed huge depending on the apparent impact parameter (Straumann, 2004). However these disturbances can be dealt with either choosing sources which are located in the sky so that their lines of sight are far away from the gravitating bodies; or having recourse, which is appropriate for many other reasons also, to a redundant number of sources, so that only the best are used each time.

Another systematic error comes from considering the sources as being at rest, at spatial infinity, in a given reference frame, which is a very idealized situation. In fact, actual sources, such as galactic pulsars, have both a proper motion and a finite distance from the observer. Taking into account estimates of the proper motion of real pulsars (Hobbs et al., 2005) it is possible to see that the rate of change of the angular position[6] is of the order

[3] *En passant*, we notice that since in our procedure only ratios of proper times measured in practice at the same spatial position are involved, the effect of the gravitational field on the user's clock can be safely neglected.

[4] The distance d is here expressed in astronomic units (A.U.) so that a distance equal to 1 corresponds to the position of the Earth.

[5] The velocity is in units of the average speed of the Earth along its orbit (30 km/s) and the rate of change is per year (y^{-1}).

[6] Actually, proper motion affects any observed change in periodicity, see e.g. Straumann (2004),

$10^{-6}\left(\frac{100\ \mathrm{pc}}{d}\right)$ rad per year. In practice these figures tell us that we are allowed to keep the position nominally fixed for months before correcting the value of the direction cosines, which is possible because the behaviour of the sources is known. As for the radial proper motion of the pulsar, which is commonly far worse known than its transverse motion, it is indeed contained in the times of arrival of the pulses, but can reasonably be considered as fixed during very long times, practically not affecting the positioning process. Finally it is known that the periods of the pulses are generally increasing with time, due to rotation energy loss of the star. However the decay rate is in general known, so that we may deal with this kind of change in the same way as we do for the position in the sky: keep the period fixed for a time compatible with the required accuracy (which would in any case be rather long), then introduce a corrected new value (which is known once the pulsar is given). As for glitches (sudden and random jumps in the frequency) they can be cured thanks to the redundancy of the number of sources above the minimum of four, corresponding to the dimensions of spacetime.

In the case of pulsars there are many other effects that could affect the arrival times of the signal; a thorough analysis can be found in Kramer et al. (2004). Besides what we have already mentioned, we should, for instance, take into account the emission mechanism at the surface of the star and the propagation across the interstellar plasma: these effects are responsible for the fact that each single incoming pulse is in general different from any other. The method to deal with this is the same adopted at radiotelescopes: the acquisition of the data runs for a high number of pulses, then the signal is analyzed by folding, in order to extract the fiducial sequence with the desired accuracy. In the case of the positioning far shorter integration times are needed, because the pulsar is already known and the aim is simply to recognize it, rather than investigate on it. For completeness we should finally mention the problems related to the acquisition and elaboration chain in the receivers, however to discuss the technological aspects of the antenna and acquisition apparatus is beyond the scope of the present work.

Finally, we would like to focus on the approximation that is implicit in our method for the conversion of a sequence of times of arrival of discrete pulses into the coordinates of the receiver. We exposed the method in Section 2.2., where we limited the extension of the time series of the signals we used at each step of the process to an interval allowing to locally approximate the user's worldline with a straight line: how far is this hypothesis tenable? Given the user's clock accuracy[7] $\delta\tau$, we can define the maximum proper time interval $\Delta\tau_{\max}$ that can be considered in order to be self-consistent with the straight line hypothesis. Developing the worldline of the receiver in powers of its proper time up to the second order we see that, if a is the order of magnitude of the user's acceleration, and v his velocity, the following condition should be satisfied:

$$\Delta\tau_{\max} = \sqrt{2\frac{v}{a}\delta\tau}. \tag{34}$$

Damour & Taylor (1991), so that an estimate of the relative time variation of the period is of the order $10^{-10}\left(\frac{100\ \mathrm{pc}}{d}\right)\mathrm{y}^{-1}\delta t$.

[7]Into this $\delta\tau$ we should actually include also the drifts due to the proper motion and the period decay of the pulsar, which we keep constant during one step of the process; the latter are however expectedly far smaller effects than those due to the acceleration of the receiver.

For instance, if the user is moving in flat spacetime with $\delta\tau \simeq 10^{-10}$ s, $v = 5 \times 10^5$ m/s and an acceleration $a = 1$ m/s^2, we have $\Delta\tau_{\max} = 10^{-2}$ s, which corresponds to several periods of millisecond pulsars: enough both for the averaging away of the fluctuations of the single pulses, and for the piecewise reconstruction of the worldline. Actually, the deviation from the linearity of the user's worldline can also be due to the curvature of spacetime, i.e. to the presence of the gravitational field. We can give a similar estimate of the corresponding maximum proper time interval $\Delta\tau_{\max}$ by setting $a = |\nabla\Phi|$ in (34) where Φ is the gravitational potential. For instance, for $a = 10^{-3}$ m/s^2, which is the order of magnitude of the gravitational field of the Sun at 1 A.U., we get for $v = 10^3$ m/s, $\Delta\tau_{\max} \simeq 10^{-2}$ s. So, we see that there are actual physical situations where the hypothesis of linearity holds, and the procedure that we have described is meaningful.

3.2. Errors in the procedure of position determination

After having discussed the systematic errors related to the physical model underlying our procedure of position determination, we would like to turn to the analysis of the errors in the procedure itself which, as we are going to see, are connected both with our knowledge of the system parameters and with the measurement process.

Actually, the coordinates of an event $\boldsymbol{r}$ are determined by solving eq. (10). We notice that eq. (10) can be written in the form

$$A\overline{x} = \overline{y} \tag{35}$$

where[8]

$$A = \begin{pmatrix} f^0_{(a)} & -f^1_{(a)} & -f^2_{(a)} & -f^3_{(a)} \\ f^0_{(b)} & -f^1_{(b)} & -f^2_{(b)} & -f^3_{(b)} \\ f^0_{(c)} & -f^1_{(c)} & -f^2_{(c)} & -f^3_{(c)} \\ f^0_{(d)} & -f^1_{(d)} & -f^2_{(d)} & -f^3_{(d)} \end{pmatrix}, \quad \overline{x} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \quad \overline{y} = \begin{pmatrix} X_{(a)} \\ X_{(b)} \\ X_{(c)} \\ X_{(d)} \end{pmatrix} \tag{36}$$

with $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$.

Eq. (35) is a linear system where the unknown vector $\overline{x}$ is obtained in terms of the matrix A (which has to be non singular) and the vector $\overline{y}$:

$$\overline{x} = A^{-1}\overline{y}. \tag{37}$$

The entries of matrix A are related to the signal periods $T_{(N)} = (cf^0_{(N)})^{-1}$ and the direction cosines $n^i_{(N)} = cT_{(N)}f^i_{(N)}$ defining the angular positions of the sources; the fact that the $\vec{\mathbf{n}}_{(N)}$'s, $N = a, b, c, d$ are different ensures that the matrix is not singular. As for the vector $\overline{y}$, its entries are the measured phases determined (at each event) by the procedure described in Section 2.2..

The entries of the matrix A are affected by systematic errors, while phase measurements have random errors. It is then possible to evaluate the maximum relative error in the solutions $\overline{x}$ of the linear system (37) by estimating the maximum errors on the entries of A,

[8] We use $\overline{x}$ to refer to generic vectors belonging to $\mathbb{R}^4$, not to be confused with the four-vectors.

while a covariance analysis allows to quantify how random errors in phase measurements translate into errors in the determination of the coordinates of the spacetime event $\boldsymbol{r}$, on the basis of the geometric properties of matrix A.

In fact, after defining a suitable norm[9], the following relation holds (see e.g. Demidovic & Maron (1966))

$$\delta\overline{x} \leq k(A)^2 \delta A \tag{38}$$

where $\delta\overline{x} = \frac{\|\Delta\overline{x}\|}{\|\overline{x}\|}$, $\delta A = \frac{\|\Delta A\|}{\|A\|}$ and $k(A) = \|A\|\|A^{-1}\|$ is the condition number of the system (35). If we suppose that the relative errors in periods $\Delta T/T$, and direction cosines Δn^i are roughly the same for all sources in all directions, then eq. (38) allows to conservatively estimate the relative error in the form

$$\delta\overline{x} \leq k(A)^2 \left[\sqrt{\left(\frac{\Delta T}{T}\right)^2 + \frac{3}{2}\left(\Delta n\right)^2} \right] \tag{39}$$

We remember that it is always $k(A) \geq 1$; in our case, we may write

$$k(A) \propto \frac{1}{|\det(A)|^2} \sum_{N=a,b,c,d}^{4} \frac{1}{T^2_{(N)} c^2} \tag{40}$$

As a consequence, we see that the relative error is minimized when: (i) the determinant is maximized and (ii) periods are minimized; since the spatial components of A are the direction cosines, the determinant is maximized when the volume spanned by these directions is maximized. To fix ideas, for $\Delta T/T \simeq 10^{-4},\ \Delta n \simeq 10^{-8}$, it is $\delta x \lesssim k(A)^2 10^{-4}$.

Further insight on the accuracy achievable by our positioning procedure can be obtained by discussing the geometric properties of the system (37), which ultimately depends on the actual angular positions of the sources in the sky: we refer to the Geometric Dilution Of Precision (GDOP), which is based on the covariance matrix of the errors in position determination, and provides a measure of how fit the set of sources is: indeed, it is related to the geometric properties of the matrix A and a great accuracy can be obtained if sources are chosen that are sufficiently scattered in the sky (this is pretty much like what happens with GPS satellites, see e.g. Hofmann-Wellenhof et al. (2001)). The covariance matrix of the positioning errors, as determined by the phase measurements, can be expressed as

$$\mathrm{cov}\ \overline{x} = \left[A^T \left(\mathrm{cov}\ \overline{y}\right) A\right]^{-1} \tag{41}$$

If the phase measurements errors from each source are uncorrelated and Gaussian distributed, with zero mean and variances $\sigma^2_{(N)}$, the position covariance matrix turns out to be

$$\mathrm{cov}\ \overline{x} = \left[A^T \mathrm{diag}(\sigma^2_{(a)}, \sigma^2_{(b)}, \sigma^2_{(c)}, \sigma^2_{(d)}) A\right]^{-1} \tag{42}$$

The GDOP is defined by

$$\mathrm{GDOP} = \sqrt{\mathrm{Tr}(\mathrm{cov}\ \overline{x})} \tag{43}$$

[9]For instance: $\|A\| = \sqrt{\sum_{i,j} |a_{ij}|^2}, \quad \|\overline{x}\| = \sqrt{\sum_i |x_i|^2}$.

To understand the meaning of GDOP, let us consider the case when all phase measurements have the same variance σ^2, then $\mathrm{cov}\ \overline{y} = \mathrm{diag}(\sigma^2, \sigma^2, \sigma^2, \sigma^2)$; furthermore it is $(A^T A)^{-1} = \frac{1}{|\mathrm{det} A|^2} (\mathrm{cof} A)^T (\mathrm{cof} A)$ so that we can write

$$\mathrm{GDOP} = \frac{\sigma}{|\mathrm{det} A|} \sqrt{\mathrm{Tr}\,(\mathrm{cofA})^{\mathrm{T}}\,(\mathrm{cofA})} \tag{44}$$

Again, we see the role of $|\mathrm{det}(A)|$ in minimizing the errors: GDOP is minimized when the solid angle spanned by the directions of the sources is maximized.

The error in position determination can be minimized by using a number n of sources greater than four, which will result in an over-determined linear system in the form (35), $B\overline{x} = \overline{y}$, provided that now B is an $n \times 4$ matrix and $\overline{y}$ is a vector of n measured phases. Redundant equations allow to improve the accuracy of the solutions via least squares estimation techniques (see e.g. Stark & Woods (1986)), so that Eq. (41) is generalized to[10]

$$\mathrm{cov}\ \overline{x} = (B^T B)^{-1} B^T\,(\mathrm{cov}\ \overline{y})\, B (B^T B)^{-1} \tag{45}$$

If, as before, we consider the phase measurements errors to be identically Gaussian distributed with the same variance and independent, we obtain equation (41) back, also in the case $n > 4$: in summary, we may say that the components of $(B^T B)^{-1}$ determine how phase errors translate into errors of the computed spacetime position.

4. Numerical Simulation

In order to test the procedure described above, it is necessary to define the null frame, that is to say the basis four-vectors in the form (1) for each source. In other words, we need to know the positions of the sources and their periods. Then, in order to apply the procedure, we need the arrival times of the pulses, as measured by the receiver. Our purpose is to demonstrate how the system works in practice, but we have no actual device at hands so we may follow two strategies: a) simulate the sources, giving them an arbitrary position in the sky and an arbitrary periodicity, then somehow mimicking the uncertainties associated with real sources; b) choose, as an example, four real millisecond pulsars with the data we find in the literature. In practice the difference between the two approaches is not really important, since the second choice is only nominally different from the first, so we decided to use the parameters of four real pulsars as they are listed in Table 1.

Next we use a software which simulates the arrival times of the pulses received at a given terrestrial location, emitted by our set of pulsars; by this way we try and reconstruct the worldline of the receiver at the chosen position on Earth. The simulator generates sequences of arrival times as they would be obtained at an antenna and we use them, applying our algorithm in order to rebuild the motion of the Earth or, more correctly, the trajectory of a terrestrial location where the pulses would be received, which moves because of the daily rotation and the motion of the Earth around the Sun: this trajectory is then compared to the one obtained by the ephemerides. More precisely we reconstruct the motion of the receiver with respect to the ”fixed” stars, assuming as the origin the event where the reception has started: in a sense we produce a self-positioning. The position of the initial event with respect to any given reference frame must be known by other means.

[10]Notice that if $n = 4$ it reduces to Eq. (41).

Pulsar	T (ms)	Elong (°)	Elat (°)
J1730-2304	8.123	263.19	0.19
J2322+2057	4.808	0.14	22.88
B0021-72N	3.054	311.27	−62.35
B1937+21	1.558	301.97	42.30

Table 1. The parameters of the four pulsars we chose are listed as they were taken from the ATNF Pulsar Catalogue. The basis four-vectors are obtained after computing the direction cosines from the ecliptic coordinates; then use is made of the formula $\boldsymbol{f}_{(N)} = \frac{1}{cT_{(N)}}(1, \vec{\mathbf{n}}_{(N)})$, for $N = a,\ b,\ c,\ d$. Both the periods and the direction cosines are assumed to be known with an accuracy limited by the numerical precision only.

For generating the sequence of arrival times we have used TEMPO2 (Hobbs et al., 2006), a specific software environment, widely used nowadays by the astronomers and astrophysicists studying pulsars. In particular, the TEMPO2 plug-in "*fake*" enables to simulate the time residuals expected from a given pulsar observation session. In fact, this code automatically generates a set of times of arrival for a specific pulsar at a predefined location on the Earth surface (corresponding, for instance, to a radiotelescope site), in a time window defined by the user, and starting from the transit time of the given pulsar through the local meridian (superior culmination point). It takes into account the contribution to timing of the gravitational field in the Solar System due to the Sun and the other bodies, and other kinematical effects (see e.g. (Straumann, 2004)). The possibility to add various types of error, in particular the Gaussian one or the red noise one (a timing noise that is actually negligible for most millisecond pulsars), to the times of arrival is also allowed. Hence, we have simulated the signals coming from the real pulsars described in Table 1 and we have introduced a Gaussian 1 μs uncertainty: this is a conservative estimate of the error in the timing procedure, due both to the detection process and to the fluctuations of the sources. Then TEMPO2 has been for us the equivalent of an antenna where the sequences of pulses from our quartet of sources are received. The arrival times have been simulated during a time interval of about three days, at a given position on the surface of the Earth, that is the one of the Parkes observatory in Australia. In particular, we considered for each pulsar a set of about 28000 pulses, sampled out of the continuous sequence each 10 seconds. The duration of the simulation allows to evidence the actual motion of the observatory, due to the combined motion of the Earth around the Sun and of its daily rotation. The chosen pulsars define the null frame, and they are supposed to be at rest in the International Celestial Reference System (ICRS) (where, in turn, the barycenter of the Solar System is at rest).

By applying the procedure described above, we have rebuilt the trajectory of the observatory. To make a comparison, we consider ζ, that is the trajectory of the observatory, as determined by the ICRS ephemerides having components t, x, y, z, while, as before, the reconstructed trajectory $\bar{\zeta}$ has been obtained according to eq. (12):

$$\boldsymbol{r}[i] = X[i]_{(a)}\boldsymbol{f}^{(a)} + X[i]_{(b)}\boldsymbol{f}^{(b)} + X[i]_{(c)}\boldsymbol{f}^{(c)} + X[i]_{(d)}\boldsymbol{f}^{(d)}. \tag{46}$$

where $[i]$ is an index labeling the i-th reception event. In particular, from (46) we obtain the

Cartesian components $\bar{t}, \bar{x}, \bar{y}, \bar{z}$.

The results are shown in Figure 1 where the reconstructed spatial trajectory is compared with the one determined by the ICRS Ephemerides of the chosen observatory. The scale of the figure does not permit to appreciate the differences between the two trajectories. Actually this application of the method is purely indicative. TEMPO2 has of course not been designed for our purposes, so the sampling of the data each 10 seconds may introduce some additional uncertainty; moreover, as we stressed in Section 3. referring to the Geometric Dilution Of Precision (GDOP), a crucial role in minimizing the uncertainty is played by the geometry of the sources: the uncertainty is minimized when the volume spanned by the sources directions is maximized.

We stress the demonstrative purpose of our work, which has led us to disregard a series of aspects that should be taken into account for a real positioning system (see e.g. Ruggiero et al. (2011))

5. Discussion and Conclusion

We have described an operational approach to the use of pulsating signals for positioning purposes; in particular, pulsars signals can be used. Our procedure is based on the definition of a null frame in flat spacetime, by means of the four-vectors associated to the signals in the inertial reference frame where the sources are at rest (so that the emission directions and the frequencies of the pulsating signals have to be known) and far away (so that their signals can be dealt with as plane waves). The procedure is fully relativistic and allows position determination with respect to an arbitrary event in flat spacetime. Once a null frame has been defined, it turns out that the phases of the electromagnetic signals can be used to label an arbitrary event in spacetime. If the sources emit continuously and the phases can be determined with arbitrary precision at any event, it is straightforward to obtain the coordinates of the user and his worldline. However, actual sources emit signals which consist of a series of pulses: so, we have developed a simple method that can be used to determine the user's worldline by assuming that the worldline is a straight line during a proper time interval corresponding to the reception of a limited number of pulses, which means that the effects of the acceleration are negligibly small. This is indeed true for any solid system when the time span is only a fraction of a second, of the order of, say, one hundredth or less. We have discussed the source of errors which affect the positioning process, taking into account model limitations (including the hypothesis of dealing with a flat spacetime) and uncertainties, due to the stochastic changes of the quantities involved in the procedure.

Then, we have focused on an application of our approach, by which we have reconstructed the trajectory of a given terrestrial location, due to the combined motion of the Earth around the Sun and to its daily rotation: on using a simulating plug-in of the TEMPO2 software, we sampled the times of arrival of the signals from a given set of pulsars, expected from an observation session at a specific location on the Earth, which in our case is the Parkes observatory in Australia. By collecting data which simulate an observation of about three days, we determined the trajectory of the observatory due to the combined rotation and revolution motion of the Earth. Then, we compared the reconstructed world-line with the one obtained by the ephemerides. The comparison was only for qualitative purposes since the use of TEMPO2 intermittently and for such a long (from the view point of our method) time may introduce some additional uncertainty; furthermore the choice and the use we made of pulsars corresponds to an idealized situation. However, the results seem promising for the testing of the reliability of our approach.

What we want to discuss, now, is the actual possibility of using our method for a positioning system effective for the navigation in the Solar System.

First, we focus on the use of pulsars, as basic "clocks" for the definition of the reference system: actually, we mostly refer to pulsars because they represent natural sources of regular pulses and they are suggestive of the old navigation at sea using the stars; an advantage of using pulsars is the fact that they can practically be considered as being at infinite distance and occupying fixed positions in the sky, however a major drawback is the weakness of their signals. Indeed, this is a severe limitation for the design of a positioning system, but we would like to point out that our method can perfectly work with any other artificial source of pulses provided that one knows the law of motion of the source in a given reference frame. As a consequence, for the use in the Solar System, one could for instance think to lay down regular pulse emitters on the surface of some celestial bodies: let us say the Earth, the Moon, Mars etc. The behaviour of the most relevant bodies is indeed pretty well known, so that we have at hands the time dependence of the direction cosines of the pulses: this is enough to apply the method and algorithm we have described and the final issue in this case would be the position within the Solar system. In principle the same can be done in the terrestrial environment: here the sources of pulses would be onboard satellites, just as it happens for GPS, but without the need of continuous intervention from the ground: again the key point is a very good knowledge of the motion of the sources in the reference frame one wants to use.

Another important issue for the practical implementation of our system, is the number of independent sources. In the previous sections we have many times mentioned the fact that the number of independent sources to be used is at least four. The reason for that number is that the spacetime coordinates needed to localize the user are of course four (three space components plus time). There are however many reasons for using a bigger number of emitters. If we have $n > 4$ pulsars (or equivalent sources) we may apply our method to all possible quadruples contained in n: the position will be determined as an average of the results obtained from all quadruples. This would be one more way to average the effect of random disturbances at the emission of the pulses out. Furthermore if one of the sources fails or disappears for any reason (e.g. because it is eclipsed) the localization process is not interrupted; if one new source comes in, provided that its position in the sky and proper recursion time are known, the sequence of its pulses is hooked to the main sequence of the arrival times and may be used for further positioning. What matters is that all arrival times from all visible sources are arranged in one single sequence identified by the onboard clock and that the main sequence is not interrupted. To say better, the maximum tolerable total blackout of the sequence should not last more than what can be reconstructed by extrapolation from the last portion of the world-line of the observer. With reasonable accelerations at the typical speeds of an interplanetary travel this may be as long as a few seconds.

Eventually, our procedure is based on the measurement of signals times of arrival, by means of the user's clock: it is then implicit that everything works if the clock used is a good and reliable clock. We are considering atomic clocks fit for being carried on board a spacecraft. The problems with the clocks arise from both random and systematic instabilities. The former contribute to the general uncertainty in the positioning, whilst the latter introduce an apparent shortening of the period of the received pulses. A strategy to minimize the effects of the drift in the frequency of the clock is similar to the one adopted for GPS. It is redundancy: more than one atomic clocks, and of different types, should be carried by the observer (GPS satellites have four clocks on board): the arrival times would be given by an average of the different readings. The redundancy would partly reduce also the effect of the drift of the frequency of the clocks, but this would emerge in long time. The accuracy of portable present day atomic clocks is in the order of 1×10^{-14} (see e.g. Ashby (2003)) which means that the measured repetition time of a millisecond pulsar would be significantly affected by the frequency drift of the clock over times of the order of months, if not years. It would in any case be advisable to periodically check the accuracy of the measurement from some reference station on the Earth, and the best strategy would be to check the position and compare with the one obtained by self-positioning, rather than to try to directly verify the clock, which would be possible but it is a very delicate task indeed and presupposes, any way, a good knowledge of the relative motion between the space observer and the ground station. A periodic check up (after weeks or months intervals) would in any case be cheaper than a continuous guidance from the ground.

In summary, the practical implementation of our method in the real world requires a careful analysis of these issues, together with several technological aspects pertaining to the signals detection and elaboration chain. However, our work suggests that investigation in this direction seems worthwhile, for the definition of a positioning system effective for the navigation across the Solar System and beyond.

References

Ashby, N., Relativity in the Global Positioning System Living Rev. Rel. 6, 1, [`http://www.livingreviews.org/lrr-2003-1`], 2003.

Bate, R. R., Muller, D. D., White, J. E., *Fundamentals of Astrodynamics*, New York: Dover, 1971.

Battin, R. H., *An Introduction to the Mathematics and Methods of Astrodynamics*, Washington D.C.: American Institute of Aeronautics and Astronautics, Revised Edition, 1999.

Bini, D., Geralico, A., Ruggiero, M.L., & Tartaglia, A., Emission versus Fermi coordinates: applications to relativistic positioning systems, Class. Quantum Grav. 25, 205011, 2008.

Blagojevic, M., Garecki J. F., Hehl, W., & Obukhov, Yu. N, Real null coframes in general relativity and GPS type coordinates, Phys. Rev. D 65, 044018, 2002.

Chandrasekhar, S., *The Mathematical Theory of Black Holes*, Oxford University Press (Oxford, UK, 1983).

Coll, B., & Tarantola, A., Galactic Positioning System, in "Journées Systèmes de Référence Spatio-Temporels, St. Petersburg", edited by Finkelstein, A., & Capitaine, N., Institute of Applied Astronomy of Russian Academy of Science, 333-334 (St. Petersburg, Russia, 2004).

Coll, B., Relativistic Positioning Systems, in "A century of Relativity Physics ERE 2005; XXVIII Spanish Relativity Meeting", AIP Conf. Proc. 841, 277-284, 2006.

Coll, B., Ferrando, J. J., & Morales, J. A., Two-dimensional approach to relativistic positioning systems, Phys. Rev. D 73, 084017, 2006.

Coll, B., Ferrando, J. J., & Morales J. A., Positioning with stationary emitters in a two-dimensional space-time, Phys. Rev. D 74, 104003, 2006.

Coll B., & Tartantola A., Using pulsars to define space-time coordinates, [arXiv:0905.4121], 2009.

Coll, B., Ferrando, J. J., & Morales J. A., Four Causal Classes of Newtonian Frames, Found. of Phys. 39, 1280-1295, 2009.

Damour, T., & Taylor, J.H., On the orbital period change of the binary pulsar PSR 1913 + 16, Ap. J. 366, 501-511, 1991.

Demidovic, B.P, & Maron, I.A., *Foundations of Numerical Mathematics*, SNTL (Praha, Czech Republic, 1966).

Downs, G.S., NASA Technical Report 32-1594, 1974.

About The Deep Space Network (2010). Jet Propulsion Lab (JPL), California Institute of Technology, Pasadena, CA., see online `http://deepspace.jpl.nasa.gov/dsn/`

Emadzadeh, A. A., Speyer, J. L., *Navigation in Space by X-ray Pulsars*, Springer Science+Business Media, pp. 3-5, 2011.

Folta, D., Gramling, C., Long, A., Leung, D., Belur, S., Autonomous Navigation Using Celestial Objects, in Americal Astronautical Society (AAS), Girdwood, Alaska, USA, Aug. 1999, pp. 2161-2177.

Gounley, R., White, R., Gai, E., Autonomous Satellite Navigation by Stellar Refraction, Journal of Guidance, Control and Dynamics, vol. 7, no. 2, pp. 129-134, 1984.

Hobbs, G., Lorimer, D. R., Lyne, A. G., & Kramer, M., A statistical study of 233 pulsar proper motions, Mon. Not. Roy. Astron. Soc. 360, 963-973, 2005.

Hobbs, G.B., Edwards, R.T., Manchester, R.N., *Monthly Notices of the Royal Astronomical Society* **369**, 655 (2006), see also the web site `http://www.atnf.csiro.au/research/pulsar/tempo2`

Hofmann-Wellenhof, B., Lichteneeger, H., & Collins, J., *Global Positioning System: Theory and Practice*, Springer (Wien, Austria, 2001)

Jordan, J. F., Navigation of Spacecraft on Deep Space Missions, Journal of Navigation, vol. 40, no. 1, pp. 19-29, Jan. 1987.

Kaplan, E. D., Hegarty, C. J., *Understanding GPS: principles and applications*, 2nd ed., Artech House, Inc., 2006.

Lorimer, D., Kramer, M. *Handbook of Pulsar Astronomy*, Cambridge University Press (Cambridge, UK, 2004).

Melbourne, W. G., Navigation Between the Planets, Scientific American, vol. 234, no. 6, pp. 58-74, Mar. 1976.

Parkinson, B. W., Spiker Jr., J. J., *Global Positioning System: Theory and Applications*, Washington D.C.: American Institute of Aeronautics and Astronautics, Inc., 1996, vols. I and II.

Pascual-Sánchez, J. F., Introducing relativity in global navigation satellite systems, Ann. Phys. (Leipzig), 16, 258-273, 2007.

Rovelli, C., GPS observables in general relativity, Phys. Rev. D 65, 044017, 2002.

Ruggiero, M.L., & Tartaglia, A., Mapping Cartesian coordinates into emission coordinates: Some toy models, Int. J. Mod. Phys. D17, 311-326, 2008.

Ruggiero M.L., Capolongo E., Tartaglia A., (2011), *Int. J. of Mod. Phys. D* , at press (2011), [arXiv:1011.0065]

Sala, J., Urruela, A., Villares, X., Estalella, R., & Paredes, J. M. Feasibility Study for Spacecraft Navigation System relying on Pulsar Timing Information, ESA Advanced Concept Team. ARIADNA Study 03/4202 (Final Report), June 23, 2004. Available on the ACT website [http://www.esa.int/gsp/ACT/phy/pp/pulsar-navigation.htm], 2004.

Sheikh, S.I., Pines, D.J., Ray, P.S., Wood, K.S., Lovellette, M.N., & Wolff M.T., Spacecraft navigation using X-ray pulsars, J. Guid. Control Dynam. 29, 49-63, 2006.

Sheikh, S.I., Hellings, R.W., & Matzner, R.A., High-Order Pulsar Timing For Navigation, in Proceedings of the 63rd Annual Meeting of The Institute of Navigation, Cambridge, MA, April 2007, 432-443, 2007.

Smits, R., Kramer, M., Stappers, B., Lorimer, D. R., Cordes, J., Faulkner, A., *Astronomy and Astrophysics* **493**, 1161 (2009)

Stark, H., & Woods, J.W., *Probability, Random Processes and Estimation Theory for Engineers*, Prentice Hall (New Jersey, USA, 1986)

Straumann N., *General Relativity*, Springer (Berlin, Germany, 2004).

Tartaglia, A., Emission coordinates for the navigation in space, arXiv:0910.2758, 2009; Acta Astronaut. 67, 539-545, 2010.

Tartaglia, A., Ruggiero, M.L., Capolongo, E., (2010), *Advances in Space Research* **47**, 645 (2011), [arXiv:1001.1068]

Wertz, J. R., *Spacecraft Attitude Determination and Control*, Dordrecht: Kluwer, 1978.

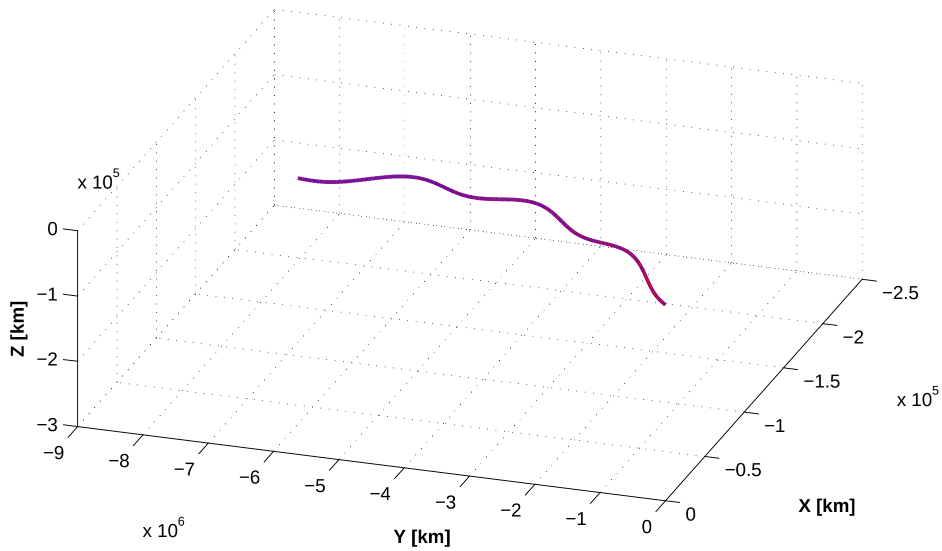

Figure 1. Space trajectory of the Parkes observatory on Earth with respect to the pulsars during three days. At this scale the ideal and the reconstructed curves are indistinguishable.