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REDUCTION FOR CONSTRAINED VARIATIONAL PROBLEMS ON 3-DIMENSIONAL NULL CURVES*

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Abstract. We consider the optimal control problem for null curves in de Sitter 3-space defined by a functional which is linear in the curvature of the trajectory. We show how techniques based on the method of moving frames and exterior differential systems, coupled with the reduction procedure for systems with a Lie group of symmetries, lead to the integration by quadratures of the extremals. Explicit solutions are found in terms of elliptic functions and integrals.

Key words. null curves, invariant variational problems, extremal trajectories, optimal control systems, moving frames, Lax formulation, Marsden–Weinstein reduction

AMS subject classifications. 49F05, 58E10, 58A17

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1. Introduction.

Let M^3 be a 3-dimensional Lorentz space form and $\gamma \subset M^3$ a null curve parametrized by the natural (pseudo-arc) parameter s which normalizes the derivative of its tangent vector field. It is known that, in general, γ admits a curvature $k_\gamma(s)$ that is a Lorentz invariant and that uniquely determines γ up to Lorentz transformations. We consider the variational problem on null curves defined by the Lorentz invariant functional

$$\mathcal{L}(\gamma) = \int_\gamma (m + k_\gamma) ds, \quad m \in \mathbb{R}, \tag{1.1}$$

and ask how to determine the explicit form for the extremal trajectories. Motivations are provided by optimal control theory and recent work on relativistic particle models associated with action functionals of the type above (cf. [19], [18], [17], [7], and references therein).

From the Euler–Lagrange equation of the action it follows that the curvature of an extremal trajectory either is constant or is an elliptic function (possibly degenerate) of the natural parameter. In the first case, the extremals are orbits of 1-parameter subgroups of the group of Lorentz transformations and can be described in terms of elementary functions [6]. In the second case, we are led to a linear system of ODEs whose coefficients are doubly periodic functions. By the Fuchsian theory of ODEs, and in particular the results of Picard [20], the trajectories are then expressible in terms of the Weierstrass elliptic functions $\wp$, σ, and ζ. Alternatively, we follow a general scheme for the reduction of constrained variational problems on homogeneous spaces. We will use techniques from optimal control theory based on the method of moving frames and on Cartan's exterior differential systems [4], [11], [8], [9], coupled with the reduction procedure for systems admitting a Lie group of symmetries extended to

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this setting [2]. For other applications of this general scheme of integration we refer to [10], [15], [16].

In this article, we determine the explicit form of the extremal curves when the target manifold is de Sitter 3-space. In this case, the functional (1.1) is invariant under the group $\mathrm{SL}(2,\mathbb{C})$, which doubly covers the identity component of the isometry group of de Sitter 3-space. The starting point of our study is the replacement of the original variational problem on null curves in de Sitter 3-space by an $\mathrm{SL}(2,\mathbb{C})$-invariant variational problem for integral curves of a control system on $M \cong \mathrm{SL}(2,\mathbb{C})\times\mathbb{R}$ defined by a suitable Pfaffian differential ideal $(\mathcal{I},\omega)$ with an independence condition. This is accomplished by proving the existence of a preferred $\mathrm{SL}(2,\mathbb{C})$-invariant frame along null curves without flex points (cf. section 2). We then follow a general construction due to Griffiths [11] and carry out a calculation to associate to the variational problem a Pfaffian differential system $\mathcal{J}$, the *Euler–Lagrange system*, whose integral curves are stationary for the associated functional. The Euler–Lagrange system is defined on the *momentum space* $Y \cong \mathrm{SL}(2,\mathbb{C}) \times \mathbb{R}^3$, which turns out to carry a contact structure, whose characteristic curves coincide with the integral curves of $\mathcal{J}$. As a matter of fact, in the case at hand all extremal trajectories arise as projections of integral curves of the Euler–Lagrange system. The theoretical reason for this is that all the derived systems of $(\mathcal{I},\omega)$ have constant rank (cf. [1]). Further, we show that the characteristic flow factors over a flow in an affine 3-dimensional subspace of $\mathfrak{sl}(2,\mathbb{C})$ and find a Lax formulation of its defining differential equation. This implies that the momentum map induced by the Hamiltonian action of $\mathrm{SL}(2,\mathbb{C})$ on Y is constant on solution curves of the Euler–Lagrange system, which leads to the integration by quadratures of the extremals (cf. section 4).

The paper is organized as follows. Section 2 gives the details of the construction of the canonical frame along null curves with no flex points by the method of moving frames, and defines the Pfaffian differential system of such frames. Section 3 studies the action functional (1.1), introduces the corresponding Euler–Lagrange system, and proves the constancy of the momentum map on its integral curves. Section 4 focuses on the integration procedure. It first outlines some facts from the theory of elliptic functions and then carries out the explicit integration of the extremals in terms of elliptic functions and elliptic integrals of the third kind.

2. Preliminaries.

2.1. The geometry of de Sitter 3-space. Let $\mathrm{Herm}(2)$ be the 4-dimensional space of 2×2 Hermitian complex matrices endowed with the Lorentz metric given by the quadratic form $\langle X,X\rangle = -\det X$ for all $X \in \mathrm{Herm}(2)$. De Sitter 3-space, $\mathbb{S}^3_1$, can be viewed as the set of 2×2 Hermitian matrices of determinant -1,

$$\mathbb{S}^3_1 = \{X \in \mathrm{Herm}(2) \,|\, \det X = -1\}\,, \tag{2.1}$$

with the induced metric g. The special linear group $\mathrm{SL}(2,\mathbb{C})$ acts transitively by isometries on $\mathbb{S}^3_1$ via the action

$$A\cdot X = AXA^*,$$

where A^* stands for the conjugate transpose of A. The stability subgroup at

$$J = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

is the group $\mathrm{SL}(2,\mathbb{R})$, and $\mathbb{S}^3_1$ may be described as a Lorentzian symmetric space

$$\mathbb{S}^3_1 \cong \mathrm{SL}(2,\mathbb{C})/\mathrm{SL}(2,\mathbb{R}).$$

The projection

$$\pi : \mathrm{SL}(2,\mathbb{C}) \ni A \mapsto AJA^* \in \mathbb{S}^3_1$$

makes $\mathrm{SL}(2,\mathbb{C})$ into a principal bundle with structure group $\mathrm{SL}(2,\mathbb{R})$.

Let $\Omega = \alpha + i\beta$ be the Maurer–Cartan form of $\mathrm{SL}(2,\mathbb{C})$, where

$$\alpha = \begin{pmatrix} \alpha^1_1 & \alpha^1_2 \\ \alpha^2_1 & -\alpha^1_1 \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta^1_1 & \beta^1_2 \\ \beta^2_1 & -\beta^1_1 \end{pmatrix}. \tag{2.2}$$

Note that the matrix of 1-forms β is semibasic[1] for the projection π, and that the Lorentz metric g on $\mathbb{S}^3_1$ is given by

$$g = \left(\beta^1_1\right)^2 - \beta^2_1\beta^1_2.$$

The matrix α amounts to the Levi–Civita (spinor) connection of g. The Maurer–Cartan equations of $\mathrm{SL}(2,\mathbb{C})$, or the structure equations, are given by

$$\begin{cases} d\alpha^1_1 = -\alpha^1_2 \wedge \alpha^2_1 + \beta^1_2 \wedge \beta^2_1, \\ d\alpha^2_1 = 2\alpha^1_1 \wedge \alpha^2_1 - 2\beta^1_1 \wedge \beta^2_1, \\ d\alpha^1_2 = -2\alpha^1_1 \wedge \alpha^1_2 + 2\beta^1_1 \wedge \beta^1_2, \end{cases}$$

$$\begin{cases} d\beta^1_1 = -\beta^1_2 \wedge \alpha^2_1 + \beta^2_1 \wedge \alpha^1_2, \\ d\beta^2_1 = 2\beta^1_1 \wedge \alpha^2_1 - 2\beta^2_1 \wedge \alpha^1_1, \\ d\beta^1_2 = -2\beta^1_1 \wedge \alpha^1_2 + 2\beta^1_2 \wedge \alpha^1_1. \end{cases}$$

2.2. The canonical frame along a null curve. A smooth parametrized curve

$$\gamma : I \to \mathbb{S}^3_1,$$

where I denotes any open interval of real numbers, is *null* (or *light-like*) if the velocity vector field γ' is null along γ, i.e., $g(\gamma'(t),\gamma'(t)) = 0$, for each $t \in I$. We will assume throughout that γ has no flex points, i.e., $\gamma'(t)$ and $\gamma''(t)$ are linearly independent, for each $t \in I$, where γ'' denotes the covariant derivative of γ' along the curve.

A frame field along γ is a smooth map $\Gamma : I \to \mathrm{SL}(2,\mathbb{C})$ such that $\gamma = \pi \circ \Gamma$. For any such frame, let $\Theta = \Gamma^*\Omega$ denote the pullback of the Maurer–Cartan form of $\mathrm{SL}(2,\mathbb{C})$ and write $\Theta = \phi + i\theta$. Given a frame field along γ, any other is given by

$$\tilde{\Gamma} = \Gamma X,$$

where $X : I \to \mathrm{SL}(2,\mathbb{R})$ is a smooth map. If $\tilde{\Theta} = \tilde{\Gamma}^*\Omega = \tilde{\phi} + i\tilde{\theta}$, then

$$\tilde{\Theta} = X^{-1}\Theta X + X^{-1}dX. \tag{2.3}$$

[1] We recall that a differential form φ on the total space of a fiber bundle $\pi : P \to B$ is said to be *semibasic* if its contraction with any vector field tangent to the fibers of π vanishes, or equivalently, if its value at each point $p \in P$ is the pullback via π^*_p of some form at $\pi(p) \in B$. Some authors call such a form *horizontal*. A stronger condition is that φ is *basic*, meaning that it is locally the pullback via π^* of a form on the base B.

A frame field $\Gamma : I \to \mathrm{SL}(2,\mathbb{C})$ along γ is said to be of *first order* if

$$\theta_1^1 = \theta_2^1 = 0, \quad \theta_1^2 \neq 0. \tag{2.4}$$

It is easily seen that first-order frame fields exist locally. If $\Gamma : I \to \mathrm{SL}(2,\mathbb{C})$ is a first-order frame along γ, then any other is given by $\tilde{\Gamma} = \Gamma X$, where $X : I \to G_1 \subset \mathrm{SL}(2,\mathbb{R})$ is a smooth map, and

$$G_1 = \left\{ \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} : a \neq 0, c \in \mathbb{R} \right\}.$$

According to (2.3), one computes

$$\tilde{\phi}_2^1 = a^2 \phi_2^1, \quad \tilde{\theta}_1^2 = \frac{1}{a^2}\theta_1^2. \tag{2.5}$$

Moreover, for first-order frames the form ϕ_2^1 is semibasic. If the curve γ has no flex points, then $\phi_2^1 \neq 0$. We say that the curve has *positive* or *negative spin* according to whether ϕ_2^1 is a positive or negative multiple of θ_1^2.

Under our assumption, it follows from the transformation formula (2.3) that there always exist local first-order frames along γ such that

$$\phi_2^1 = \varepsilon \theta_1^2, \tag{2.6}$$

where $\varepsilon = \pm 1$, according to whether γ has positive or negative spin. A first-order frame field is said to be of *second order* if it satisfies (2.6) on I.

A second-order frame field along γ is said to be a *canonical* frame if

$$\phi_1^1 = 0. \tag{2.7}$$

Note that canonical frame fields exist on I, and that if Γ is a canonical frame, then any other is given by $\pm\Gamma$.

Summarizing, we have proved the following.

PROPOSITION 2.1. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{S}_1^3$ be a null curve with no flex points. Then there exists a frame along γ, the* canonical frame,

$$\Gamma : I \to \mathrm{SL}(2,\mathbb{C}),$$

such that

$$\Gamma^{-1} d\Gamma = \begin{pmatrix} 0 & \varepsilon \\ k+i & 0 \end{pmatrix} \omega, \tag{2.8}$$

where $\varepsilon = \pm 1$, ω is a nowhere vanishing 1*-form, the* canonical pseudo-arc element, *and $k : I \to \mathbb{R}$ is a smooth function, the* curvature *of γ. Moreover, if Γ is a canonical frame field along γ, then any other canonical frame field is given by $\pm\Gamma$.*

Remark 1. Henceforth, we abuse the terminology and refer to the $\mathbb{Z}_2$-class $[\Gamma] = \{\pm\Gamma\}$ as the canonical frame Γ of a null curve γ.

Remark 2. Conversely, for a smooth function $k : I \to \mathbb{R}$, let $H(k) : I \to \mathfrak{sl}(2,\mathbb{C})$ be

$$H(k) = \begin{pmatrix} 0 & \varepsilon \\ k+i & 0 \end{pmatrix}. \tag{2.9}$$

Then by solving a linear system of ODEs, we see that there exists a unique (up to left multiplication)

$$\Gamma : I \to \mathrm{SL}(2,\mathbb{C})$$

such that

$$\Gamma^{-1}\Gamma' = H(k). \tag{2.10}$$

In particular, $\gamma = \Gamma J \Gamma^* : I \to \mathbb{S}^3_1$ is a null curve without flex points and with curvature k.

Remark 3 (null helices). The simplest examples are null helices, that is, null curves with constant curvature. Such curves are orbits of 1-parameter subgroups of $\mathrm{SL}(2,\mathbb{C})$ (cf. Remark 9) and have been described by elementary functions in [6].

2.3. The Pfaffian system of canonical frames. Let $(\mathcal{I},\omega)$ be the Pfaffian differential system on $M := \mathrm{SL}(2,\mathbb{C}) \times \mathbb{R}$ defined by the differential ideal $\mathcal{I}$ generated by the linearly independent 1-forms

$$\begin{cases} \eta^1 = \beta^1_1, \quad \eta^2 = \beta^1_2, \quad \eta^3 = \alpha^1_1 - \varepsilon\omega, \\ \eta^4 = \alpha^1_1, \quad \eta^5 = \alpha^2_1 - k\omega, \end{cases}$$

where

$$\omega := \beta^2_1$$

gives the independence condition $\omega \neq 0$.

Now, let $\gamma : I \to \mathbb{S}^3_1$ be a null curve without flex points. Then, by Proposition 2.1, the curve $g = (\Gamma_\gamma, k_\gamma) : I \to M$, whose components are, respectively, the canonical frame field along γ and the curvature of γ, is an integral curve of the Pfaffian system $(\mathcal{I},\omega)$. Conversely, if $g = (\Gamma, k) : I \to M$ is an integral curve of the Pfaffian system $(\mathcal{I},\omega)$, then $\gamma = \Gamma J \Gamma^* : I \to \mathbb{S}^3_1$ defines a null curve with no flex points, Γ is the canonical frame field along γ, and k is the curvature of γ. For this reason, null curves without flex points in $\mathbb{S}^3_1$ can be identified with the integral curves of the Pfaffian system $(\mathcal{I},\omega)$.

DEFINITION 2.2. *The Pfaffian differential system* $(\mathcal{I},\omega)$ *will be referred to as the* canonical system.

Remark 4. A smooth curve $g = (\Gamma, k) : I \to M$ is an integral curve of the canonical system if and only if $\Gamma : I \to \mathrm{SL}(2,\mathbb{C})$ is a solution of the linear system

$$\Gamma^{-1}(t)\Gamma'(t) = H(k(t)).$$

The function k plays the role of a control. Note that if we assign a smooth map $k : I \to \mathbb{R}$ and a point $A_0 \in \mathrm{SL}(2,\mathbb{C})$, then there exists a unique integral curve $g = (\Gamma, k)$ of the control system satisfying the initial condition $\Gamma(t_0) = A_0$ for $t_0 \in I$.

Exterior differentiation and use of the Maurer–Cartan equations give, modulo the algebraic ideal generated by $\eta^1, \dots, \eta^5$, the *quadratic equations* of $(\mathcal{I},\omega)$:

$$\begin{cases} d\omega \equiv 2(k\eta^1 + \eta^4) \wedge \omega, \\ d\eta^1 \equiv -(k\eta^2 + \eta^3) \wedge \omega, \\ d\eta^2 \equiv -2\varepsilon\eta^1 \wedge \omega, \\ d\eta^3 \equiv -2\varepsilon(k\eta^1 + 2\eta^4) \wedge \omega, \\ d\eta^4 \equiv (\eta^2 - k\eta^3 + \varepsilon\eta^5) \wedge \omega, \\ d\eta^5 \equiv -\big(dk + 2(1+k^2)\eta^1\big) \wedge \omega. \end{cases} \tag{2.11}$$

3. The variational problem and the Euler–Lagrange system.

3.1. The constrained variational problem. Let $\mathcal{N}$ be the space of null curves in $\mathbb{S}^3_1$ without flex points. We consider the action functional

$$\mathcal{L}_m : \gamma \in \mathcal{N} \mapsto \int_{I_\gamma} (m + k_\gamma)\omega_\gamma, \quad m \in \mathbb{R}, \tag{3.1}$$

where I_γ is the domain of definition of the curve, k_γ is its curvature, and ω_γ the canonical pseudo-arc element (cf. section 2). We refer to [18], [19], [17], [7], and the references therein for a discussion on the particle model associated with this action functional.

DEFINITION 3.1. *A curve $\gamma \in \mathcal{N}$ is said to be an* extremal trajectory *(or simply a* trajectory*) in $\mathbb{S}^3_1$ if it is a critical point of the action functional $\mathcal{L}_m$ when one considers compactly supported variations. The constant m is called the Lagrange multiplier of the trajectory.*

Remark 5. As usual, by a compactly supported variation of $\gamma \in \mathcal{N}$ we mean a mapping $V : I \times (-\epsilon, \epsilon) \to \mathbb{S}^3_1$ such that (1) for all $u \in (-\epsilon, \epsilon)$, the map $\gamma_u := V(t, u) : I \to \mathbb{S}^3_1$ is a null curve without flex points; (2) $\gamma_0 = \gamma(t)$ for all $t \in I$; and (3) there exists a closed interval $[a, b] \subset I$ such that

$$V(t, u) = \gamma(t) \quad \forall t \in I \setminus [a, b], \forall u \in (-\epsilon, \epsilon). \tag{3.2}$$

Accordingly, a curve $\gamma \in \mathcal{N}$ is an extremal trajectory if, for every compactly supported variation V, we have that

$$\frac{d}{du}\left(\int_{a_V}^{b_V} (m + k_{\gamma_u})\, ds_u\right)\bigg|_{u=0} = 0,$$

where $[a_V, b_V]$ is the support of the variation, i.e., the smallest closed interval for which (3.2) holds, and ds_u is the canonical pseudo-arc element of the curve γ_u.

In [7], the authors derive the Euler–Lagrange equation associated with (3.1) for null curves with prescribed endpoints and the same canonical frame at each end.

By the preceding discussion (cf. Proposition 2.1 and section 2.3), a curve $\gamma \in \mathcal{N}$ is an extremal trajectory if and only if the pair $g = (\Gamma_\gamma, k_\gamma)$ of its canonical frame field and curvature function is a critical point of the variational problem on the space $\mathcal{V}(\mathcal{I}, \omega)$ of all integral curves of $(\mathcal{I}, \omega)$ defined by the functional,

$$\widehat{\mathcal{L}} : g \in \mathcal{V}(\mathcal{I}, \omega) \mapsto \int_{I_g} g^*((m + k)\omega), \tag{3.3}$$

when one considers compactly supported variations through integral curves of $(\mathcal{I}, \omega)$.

Remark 6. The replacement of the original functional by the functional (3.3) is the starting point in the application of the Griffiths formalism. This approach to constrained variational problems with one independent variable provides conditions for criticality in terms of Pfaffian differential systems and is particularly well suited when one considers compactly supported variations among constrained curves. More importantly, it furnishes the appropriate setting for the explicit integration of the extremals (cf. [11], [1], [2], [12], and below).

3.2. The Euler–Lagrange system. Associated to the functional $\widehat{\mathcal{L}}$ we will introduce, following Griffiths [11], the Euler–Lagrange system $(\mathcal{J}, \omega)$ on a new manifold Y, which will be made explicit below.

For this, let $Z \subset T^*M$ be the affine subbundle defined by

$$Z = (m+k)\omega + I \subset T^*M,$$

where I is the subbundle of T^*M associated to the differential ideal $\mathcal{I}$. The 1-forms $(\eta^1, \dots, \eta^5, \omega)$ induce a global affine trivialization of Z, which may be identified with $M \times \mathbb{R}^5$ by setting

$$M \times \mathbb{R}^5 \ni ((\Gamma, k); x_1, \dots, x_5) \mapsto \omega_{|(\Gamma,k)} + x_j \eta^j{}_{|(\Gamma,k)} \in Z$$

(throughout we use summation convention). Thus, the Liouville (canonical) 1-form of T^*M restricted to Z is given by

$$\mu = (m+k)\omega + x_j \eta^j.$$

Exterior differentiation and use of the quadratic equations (2.11) give

$$\begin{aligned} d\mu \equiv\ & dk \wedge \omega + 2(m+k)(k\eta^1 + \eta^4) \wedge \omega + dx_j \wedge \eta^j \\ & - x_1(k\eta^2 + \eta^3) \wedge \omega - 2\varepsilon x_2 \eta^1 \wedge \omega \\ & - 2\varepsilon x_3(k\eta^1 + 2\eta^4) \wedge \omega + x_4(\eta^2 - k\eta^3 + \varepsilon\eta^5) \wedge \omega \\ & - x_5(dk + 2(1+k^2)\eta^1) \wedge \omega \qquad \text{mod } \{\eta^i \wedge \eta^j\}. \end{aligned}$$

Next, we compute the Cartan system $\mathcal{C}(d\mu) \subset T^*Z$ determined by the 2-form $d\mu$, i.e., the Pfaffian system generated by the 1-forms

$$\{i_\xi d\mu \,|\, \xi \in \mathfrak{X}(Z)\} \subset \Omega^1(Z).$$

Contracting $d\mu$ with the vector fields of the tangent frame

$$\left(\frac{\partial}{\partial\omega}, \frac{\partial}{\partial k}, \frac{\partial}{\partial\eta^1}, \dots, \frac{\partial}{\partial\eta^5}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_5} \right)$$

on Z, dual to the coframe

$$\left(\omega, dk, \eta^1, \dots, \eta^5, dx_1, \dots, dx_5\right),$$

we find the 1-forms

$$\eta^1, \dots, \eta^5, \tag{3.4}$$

$$\pi_1 = (x_5 - 1)dk, \tag{3.5}$$

$$\pi_2 = (1 - x_5)\omega, \tag{3.6}$$

$$\beta_1 = dx_1 - 2\left\{km + k^2 - \varepsilon x_2 - \varepsilon k x_3 - x_5(1+k^2)\right\}\omega, \tag{3.7}$$

$$\beta_2 = dx_2 + (kx_1 - x_4)\omega, \tag{3.8}$$

$$\beta_3 = dx_3 + (x_1 + kx_4)\omega, \tag{3.9}$$

$$\beta_4 = dx_4 - \{2(m+k) - 4\varepsilon x_3\}\omega, \tag{3.10}$$

$$\beta_5 = dx_5 - \varepsilon x_4 \omega. \tag{3.11}$$

We have proven the following.

LEMMA 3.2. *The Cartan system* $(\mathcal{C}(d\mu), \omega)$ *associated to* $(\mathcal{I}, \omega)$ *is the differential ideal on* $Z \cong M \times \mathbb{R}^5$ *generated by*

$$\left\{\eta^1, \dots, \eta^5, \pi_1, \pi_2, \beta_1, \dots, \beta_5\right\}$$

and with independence condition ω.

DEFINITION 3.3. *The involutive prolongation of* $(\mathcal{C}(d\mu), \omega)$ *on* Z *gives rise to a Pfaffian differential system* $(\mathcal{J}, \omega)$ *on a submanifold* $Y \subset Z$, *which is called the* Euler–Lagrange differential system *associated to the variational problem. The submanifold* Y *is called the* momentum space. *We refer the reader to the book of Griffiths* [11] *for a discussion of how this system is derived and for more details on Pfaffian systems.*

LEMMA 3.4. *The momentum space* Y *is the* 9*-dimensional submanifold of* Z *defined by the equations*

$$x_5 = 1, \quad x_4 = 0, \quad x_3 = \frac{\varepsilon}{2}(m + k).$$

The Euler–Lagrange system $(\mathcal{J}, \omega)$ *is the Pfaffian differential system on* Y *with independence condition* ω *generated by the* 1*-forms*

$$\begin{cases} \eta^1{}_{|Y}, \dots, \eta^5{}_{|Y}, \\ \sigma_1 = dx_1 + \left(k^2 - mk + 2\varepsilon x_2 + 2\right)\omega, \\ \sigma_2 = dx_2 + kx_1\omega, \\ \sigma_3 = dk + 2\varepsilon x_1\omega. \end{cases}$$

Moreover,

$$\begin{aligned} \mu_{|Y} = & \frac{1}{2}(m - k)\beta_1^2 - \frac{\varepsilon}{2}k'\beta_1^1 + \frac{1}{2}\left(\frac{k''}{2} - \varepsilon k(k - m) - 2\varepsilon\right)\beta_2^1 \\ & + \frac{\varepsilon}{2}(m + k)\alpha_2^1 + \alpha_1^2. \end{aligned}$$

Proof. Let $V_1(d\mu) \hookrightarrow \mathbb{P}[T(Z)] \to Z$ be the totality of 1-dimensional integral elements of $\mathcal{C}(d\mu)$. In view of (3.5) and (3.6), we find that

$$V_1(d\mu)_{|((\Gamma,k);x)} \neq \emptyset \iff x_5 = 1.$$

Thus, the first involutive prolongation of $(\mathcal{C}(d\mu), \omega)$, i.e., the image $Z_1 \subset Z$ of $V_1(d\mu)$ with respect to the natural projection $V_1(d\mu) \to Z$, is given by

$$Z_1 = \{((\Gamma, k); x) \in Z : x_5 = 1\}.$$

Next, the restriction of β_5 to Z_1 takes the form $-\varepsilon x_4\omega$. Thus, the second involutive prolongation Z_2 is characterized by the equations

$$x_5 = 1, \quad x_4 = 0.$$

Considering then the restriction of β_4 to Z_2 yields the equations

$$x_5 = 1, \quad x_4 = 0, \quad x_3 = \frac{\varepsilon}{2}(m + k),$$

which define the third involutive prolongation Z_3. Now, the restriction $\mathcal{C}_3(d\mu)$ to Z_3 of $\mathcal{C}(d\mu)$ is generated by the 1-forms $\eta^1, \dots, \eta^5$ and

$$\begin{aligned} \sigma_1 &= dx_1 + \left(k^2 - mk + 2\varepsilon x_2 + 2\right)\omega, \\ \sigma_2 &= dx_2 + kx_1\omega, \\ \sigma_3 &= dk + 2\varepsilon x_1\omega. \end{aligned}$$

This implies that there exists an integral element of $V_1(d\mu)$ over each point of Z_3, i.e., $V_1(d\mu)_p \neq \emptyset$, for each $p \in Z_3$. Hence $Y := Z_3$ and $(\mathcal{J}, \omega) := (\mathcal{C}_3(d\mu), \omega)$ is the involutive prolongation of the Cartan system $(\mathcal{C}(d\mu), \omega)$. □

Remark 7. The importance of this construction is that the natural projection $\pi_Y : Y \to M$ maps integral curves of the Euler–Lagrange system to extremals of the variational problem associated to $(M, \mathcal{I})$. The converse is not true in general. However, it is known to be true if all the derived systems of $(\mathcal{I}, \omega)$ are of constant rank (cf. [1], [12]). In our case, one can easily check, using (2.11), that all the derived systems of $(\mathcal{I}, \omega)$ have indeed constant rank, so that all the extremals do arise as projections of integral curves of the Euler–Lagrange system (see also section 3.3).

Remark 8. A direct calculation shows that

$$\mu_{|Y} \wedge (d\mu_{|Y})^4 \neq 0 \tag{3.12}$$

on Y, i.e., the variational problem is nondegenerate.[2] This implies that $\mu_{|Y}$ is a contact form and that there exists a unique vector field $\zeta \in \mathfrak{X}(Y)$, the *characteristic vector field* of the contact structure, such that $\mu_{|Y}(\zeta) = 1$ and $i_\zeta\, d\mu_{|Y} = 0$. In particular, the integral curves of the Euler–Lagrange system coincide with the characteristic curves of ζ.

3.3. The natural equation of integral curves. Let $\mathcal{V}(\mathcal{J}, \omega)$ be the set of integral curves of the Euler–Lagrange Pfaffian system $(\mathcal{J}, \omega)$. If $y = ((\Gamma, k); x_1, x_2) : I \to Y$ is in $\mathcal{V}(\mathcal{J}, \omega)$, then equations

$$\eta^1 = \eta^2 = \cdots = \eta^5 = 0$$

and the independence condition $\omega \neq 0$ tell us that Γ defines a canonical frame along the null curve $\gamma = \Gamma J \Gamma^*$ and that k is the curvature of γ.

Next, for the smooth function $k : I \to \mathbb{R}$, let k', k'', and k''' be defined by

$$dk = k'\omega, \quad dk' = k''\omega, \quad dk'' = k'''\omega.$$

Equation $\sigma_3 = 0$ implies

$$x_1 = -\frac{\varepsilon}{2} k'.$$

Further, $\sigma_1 = 0$ gives

$$x_2 = \frac{1}{4} k'' - \frac{\varepsilon}{2}(k^2 - mk + 2).$$

Finally, $\sigma_2 = 0$ yields

$$k''' - 6\varepsilon k k' + 2\varepsilon m k' = 0. \tag{3.13}$$

[2] A variational problem is said to be *nondegenerate* in the case when

$$\dim Y = 2m + 1 \quad \text{and} \quad \mu_{|Y} \wedge (d\mu_{|Y})^m \neq 0.$$

Let $V(\mathcal{J}, \omega)$ and $V(\mathcal{C}(d\mu|Y), \omega)$ denote the set of integral elements of the Euler–Lagrange system and of the Cartan system. For nondegenerate problems we have $V(\mathcal{J}, \omega) = V(\mathcal{C}(d\mu|Y), \omega)$, whereas in general we have only inclusion $V(\mathcal{J}, \omega) \subset V(\mathcal{C}(d\mu|Y, \omega)$ (cf. [11, p. 84]). For a discussion on the relation between the classical Legendre transform and the construction of the Euler–Lagrange system on the momentum space, with special attention to the nondegeneracy condition, we refer the reader to [11, Chapter I, section e]).

This is the Euler–Lagrange equation of the extremals of (3.1). It has been computed, for example, in [7]. Thus, an integral curve of the Euler–Lagrange system projects to an extremal trajectory in $\mathbb{S}^3_1$.

Conversely, let $\gamma : I \to \mathbb{S}^3_1$ be a null curve without flex points, Γ_γ its canonical frame, and k_γ its curvature. Define the lift $y_\gamma : I \to Y$ of γ to the momentum space Y by

$$y_\gamma(t) = \left((\Gamma_\gamma, k_\gamma); -\frac{\varepsilon}{2}k'_\gamma, \frac{1}{4}k''_\gamma - \frac{\varepsilon}{2}(k^2_\gamma - mk + 2)\right).$$

Then, y_γ is an integral curve of the Euler–Lagrange system if and only if k_γ satisfies (3.13) if and only if γ is an extremal trajectory. Thus, the integral curves of the Euler–Lagrange system arise as lifts of trajectories in $\mathbb{S}^3_1$.

3.4. The Lax formulation. Introduce the *reduced curvature*

$$h := \frac{\varepsilon}{2}\left(k - \frac{m}{3}\right)$$

and identify $Y \cong \mathrm{SL}(2,\mathbb{C}) \times \mathbb{R}^3$, where $\mathbb{R}^3$ has coordinates (h, h', h''). Then, the Pfaffian equations defining the Euler–Lagrange system $\mathcal{J}$ are given by

$$\begin{cases} \eta^j = 0, \quad (j = 1, \dots, 5), \\ dh = h'\omega, \\ dh' = h''\omega, \\ dh'' = 12hh'\omega, \end{cases} \tag{3.14}$$

where $\omega \neq 0$ is the independence condition. Equation (3.13) becomes

$$h''' - 12hh' = 0, \tag{3.15}$$

$$H(h) = \begin{pmatrix} 0 & \varepsilon \\ 2\varepsilon h + \frac{m}{3} + i & 0 \end{pmatrix} \tag{3.16}$$

and we also have

$$\begin{aligned}\mu = &-\left(\varepsilon h - \frac{m}{3}\right)\beta^2_1 - h'\beta^1_1 + \frac{\varepsilon}{2}\left(h'' - 4h^2 + \frac{2}{3}\varepsilon m h + \frac{2}{9}m^2 - 2\right)\beta^1_2 \\ &+ \left(h + \frac{2}{3}\varepsilon m\right)\alpha^1_2 + \alpha^2_1.\end{aligned}$$

Next, define the *momentum* associated with h, $U(h) \in \mathfrak{sl}(2,\mathbb{C})$, by

$$\begin{pmatrix} ih' & 2i\varepsilon\left(h - \varepsilon\left(\frac{m}{3} + i\right)\right) \\ 2\left(h + \frac{2\varepsilon m}{3}\right) - i\varepsilon\left(h'' - 4h^2 + \frac{2\varepsilon m h}{3} + \frac{2m^2}{9} - 2\right) & -ih' \end{pmatrix}. \tag{3.17}$$

A direct computation shows that (3.15) is equivalent to

$$U(h)' = [U(h), H(h)].$$

The above discussion yields the following result.

PROPOSITION 3.5. *A map $(A; h, h', h'') : I \subset \mathbb{R} \to Y$ is an integral curve of the Euler–Lagrange system $(\mathcal{J}, \omega)$ if and only if*

$$\begin{cases} A^{-1}A' = H(h), \\ U(h)' = [U(h), H(h)]. \end{cases} \tag{3.18}$$

As a consequence, we have the following.

COROLLARY 3.6. *The momentum map*

$$\Phi : Y \to \mathfrak{sl}(2,\mathbb{C}), \quad (A; h, h', h'') \mapsto AU(h)A^{-1}$$

is constant on integral curves of the Euler–Lagrange system.

Remark 9. The momentum space Y may be identified with $\mathrm{SL}(2,\mathbb{C}) \times \mathfrak{a}$, where $\mathfrak{a} = \mathrm{span}\,\{U(h)\}$ is an affine subspace of $\mathfrak{sl}(2,\mathbb{C})$. The group $\mathrm{SL}(2,\mathbb{C})$ acts on (Y,μ) by

$$g \cdot (A; U(h)) = (gA; U(h)), \quad \text{for each } \; g \in \mathrm{SL}(2,\mathbb{C}),\, U(h) \in \mathfrak{a},$$

in a Hamiltonian way. Using the isomorphism of $\mathfrak{sl}(2,\mathbb{C})$ with its dual Lie algebra induced by the Killing form, one sees that the momentum map associated with this action is given by Φ. Moreover, if $y = (A(t), U(h)(t))$ is an integral curve of the characteristic vector field ζ, then $U(h)(t)$ is an integral curve of the vector field

$$X_\zeta : U(h) \mapsto [U(h), H(h)]$$

and ζ can be written

$$\zeta_{|y} = H(h)_{|A} + X_\zeta(U(h)),$$

for all $y = (A, U(h)) \in Y$. If $\mathfrak{a}_s$ denotes the singular set of X_ζ, then the integral curves through $(A, U(h)) \in \mathrm{SL}(2,\mathbb{C}) \times \mathfrak{a}_s$ are orbits of the 1-parameter subgroups generated by $H(h)$. By (3.17), these project to curves with constant curvature (null helices). Next, consider $\Phi : \mathrm{SL}(2,\mathbb{C}) \times \mathfrak{a}_r \to \mathfrak{sl}(2,\mathbb{C})$, where $\mathfrak{a}_r$ denotes the complement of $\mathfrak{a}_s$ in $\mathfrak{a}$. For each regular value $\ell \in \mathfrak{sl}(2,\mathbb{C})$ of Φ, the isotropy subgroup at ℓ, $\mathrm{SL}(2,\mathbb{C})_\ell$, is abelian and $\dim \mathrm{SL}(2,\mathbb{C})_\ell = \mathrm{rank}\, \mathrm{SL}(2,\mathbb{C})_\ell = 2$. The reduced space $Y_\ell = \Phi^{-1}(\ell)/\mathrm{SL}(2,\mathbb{C})_\ell$ is then 1-dimensional. This implies that an integral curve y with momentum ℓ (i.e., $\Phi \circ y = \ell$) can be found by quadratures. Any other integral curve with momentum ℓ is given by $b \cdot y$ for some $b \in \mathrm{SL}(2,\mathbb{C})_\ell$.

Note that when the action of the symmetry group on the momentum space is co-isotropic (as in the present case), the equation governing the flow of X_ζ can always be written in Lax form. See, for instance, [10].

4. Integration of the trajectories.

4.1. Preparatory material.

From (3.15), it follows that the reduced curvature h satisfies

$$(h')^2 = 4h^3 - g_2 h - g_3 \tag{4.1}$$

for real constants g_2 and g_3. Hence h is expressed by the real values of either a Weierstrass $\wp$-function with invariants g_2, g_3, or one of its degenerate forms.

We call a solution to (4.1) a *potential* with analytic invariants g_2, g_3. Two potentials are considered equivalent if they differ by a reparametrization of the form $s \mapsto s + c$, where c is a constant.[3] For real g_2 and g_3, let $\Delta(g_2, g_3) = 27g_3^2 - g_2^3$ be the discriminant of the cubic polynomial

$$P(t; g_2, g_3) = 4t^3 - g_2 t - g_3.$$

[3] When invariants g_2 and g_3 are given, such that $27g_3^2 \neq g_2^3$, the general solution of the differential equation $(\frac{dy}{dz})^2 = 4y^3 - g_2 y - g_3$ can be written in the form $\wp(z + \alpha; g_2, g_3)$, where α is a constant of integration.

The study of the real values of the Weierstrass $\wp$-function with real invariants g_2, g_3 (and its degenerate forms) leads to primitive half-periods ω_1, ω_3 such that (see for instance [14])

- $\Delta(g_2, g_3) < 0$: $\omega_1 > 0$, $\omega_3 = i\nu\omega_1$, $\nu > 0$.
- $\Delta(g_2, g_3) > 0$: $\omega_1 > 0$, $\omega_3 = \frac{1}{2}(1 + i\nu)\omega_1$, $\nu > 0$.
- $\Delta(g_2, g_3) = 0$ and $g_3 > 0$: $\omega_1 > 0$, $\omega_3 = +i\infty$.
- $\Delta(g_2, g_3) = 0$ and $g_3 < 0$: $\omega_1 = +\infty$, $-i\omega_3 > 0$.
- $g_2 = g_3 = 0$: $\omega_1 = +\infty$, $\omega_3 = +i\infty$.

Accordingly, denoting by $\mathcal{D}(g_2, g_3)$ the fundamental period-parallelogram spanned by $2\omega_1$ and $2\omega_3$, the only possible cases for the potential function $h : I \to \mathbb{R}$ are

- $\Delta < 0$: $h(s) = \wp(s; g_2, g_3)$, $I = (0, 2\omega_1)$.
- $\Delta < 0$: $h(s) = \wp_3(s; g_2, g_3) = \wp(s + \omega_3; g_2, g_3)$, $I = \mathbb{R}$.
- $\Delta > 0$: $h(s) = \wp(s; g_2, g_3)$, $I = (0, 2\omega_1)$.
- $\Delta = 0$, $g_3 = -8a^3 > 0$:

$$h(s) = -3a\tan^2\left(\sqrt{-3a}s\right) - 2a, \quad I = \left(-\frac{\pi}{\sqrt{-12a}}, \frac{\pi}{\sqrt{-12a}}\right).$$

- $\Delta = 0$, $g_3 = -8a^3 < 0$:

$$h(s) = 3a\tanh^2\left(\sqrt{3a}s\right) - 2a, \quad I = \mathbb{R}.$$

- $g_2 = g_3 = 0$: $h(s) = s^{-2}$, $I = (-\infty, 0)$, or $I = (0, +\infty)$.

Let h be a Weierstrass potential with real invariants g_2, g_3, and let $U(h)$ be the corresponding momentum as given by (3.17). Then

$$\begin{aligned}\det U(h) &= \left(\frac{4}{27}m^3 - 4m - \frac{m}{3}g_2 - \varepsilon g_3\right) + i\varepsilon\left(\frac{4}{3}m^2 - g_2 - 4\right)\\ &= P\left(\varepsilon\left(\frac{m}{3} + i\right); g_2, g_3\right).\end{aligned}$$

Let

$$\nu(m, h) := \sqrt{P\left(\varepsilon\left(\frac{m}{3} + i\right); g_2, g_3\right)},$$

chosen once for all. Then $\pm\nu(m, h)$ are the eigenvalues of the momentum $U(h)$.

Next, define

$$\phi(m, h) := \begin{cases}\displaystyle\int \frac{\nu(m, h)}{h - \varepsilon\left(\frac{m}{3} + i\right)}ds, & \nu(m, h) \neq 0,\\ \displaystyle\int \frac{1}{h - \varepsilon\left(\frac{m}{3} + i\right)}ds, & \nu(m, h) = 0.\end{cases} \tag{4.2}$$

These are elliptic integrals of the third kind. Let $w(m, h)$ be the unique point in the period-parallelogram $\mathcal{D}(g_2, g_3)$ such that

$$h(w) = \varepsilon\left(\frac{m}{3} + i\right) \quad \text{and} \quad h'(w) = \nu(m, h).$$

Denote by σ_h and ζ_h, respectively, the sigma and zeta Weierstrassian functions corresponding to the potential h, i.e., the unique analytic odd functions whose meromorphic extensions satisfy $\zeta_h' = -h$ and $\sigma_h'/\sigma_h = \zeta_h$. Under the above assumptions, we now compute the elliptic integrals (4.2). Three cases are considered.

Case I. $\nu(m,h) \neq 0$. In this case,

$$\phi(m,h) = \int \frac{h'(w)}{h(s)-h(w)} ds = \log \frac{\sigma_h(s-w)}{\sigma_h(s+w)} + 2s\zeta_h(w) + \text{const.}$$

Case II. $\nu(m,h) = 0$ and $g_2^2 + g_3^2 \neq 0$. In this case, $h(w) = \varepsilon\left(\frac{m}{3}+i\right)$ is a root of the cubic polynomial P, say e_3. If e_1, e_2 denote the other two roots, we have

$$\begin{aligned}\phi(m,h) &= \int \frac{ds}{h(s)-e_3} = \int \frac{h(s+w)-e_3}{(e_3-e_1)(e_3-e_2)} ds \\ &= \frac{1}{\frac{g_2}{4} - 3\left(\frac{m}{3}+i\right)^2}\left\{\zeta_h(s+w) + \varepsilon\left(\frac{m}{3}+i\right)s\right\} + \text{const.}\end{aligned}$$

Case III. $\nu(m,h) = 0$ and $g_2 = g_3 = 0$. In this case,

$$\phi(m,h) = \frac{1}{3}s^3 + \text{const.}$$

4.2. Explicit integration. We are now in a position to explicitly integrate the extremal trajectories. This amounts to integrating by quadratures the reduced system associated to the Hamiltonian action of $\mathrm{SL}(2,\mathbb{C})$ on Y (cf. Remark 9). The key to explicit integration is the conservation of the momentum map along integral curves of the Euler–Lagrange system.

THEOREM 4.1. *Let $\gamma : I \to \mathbb{S}_1^3$ be an extremal trajectory with Lagrange multiplier m and reduced curvature h with real invariants g_2, g_3. Let $U_\gamma(h)$ be the momentum of h given by (3.17), and assume that γ is parametrized by the canonical parameter s, i.e., $\omega = ds$. According to whether $\det U_\gamma(h)$ is zero or different from zero, we distinguish two cases.*

Case I. *If $\det U(h) \neq 0$, then the canonical frame field $\Gamma : I \to \mathrm{SL}(2,\mathbb{C})$ along γ is given by*

$$\Gamma(s) = A \cdot M(s),$$

where $A \in \mathrm{SL}(2,\mathbb{C})$ and $M(s)$ takes the form

$$\frac{1}{\sqrt{-4i\varepsilon\nu}}\begin{pmatrix} e^{\phi(m,h)} & 0 \\ 0 & e^{-\phi(m,h)}\end{pmatrix}\begin{pmatrix} \dfrac{ih'+\nu}{\sqrt{h-\varepsilon\left(\frac{m}{3}+i\right)}} & 2i\varepsilon\sqrt{h-\varepsilon\left(\frac{m}{3}+i\right)} \\ \dfrac{-ih'+\nu}{\sqrt{h-\varepsilon\left(\frac{m}{3}+i\right)}} & -2i\varepsilon\sqrt{h-\varepsilon\left(\frac{m}{3}+i\right)}\end{pmatrix}$$

Case II. *If $\det U(h) = 0$, then the canonical frame field $\Gamma : I \to \mathrm{SL}(2,\mathbb{C})$ along γ is given by*

$$\Gamma(s) = A \cdot M(s),$$

where $A \in \mathrm{SL}(2,\mathbb{C})$ and $M(s)$ takes the form

$$\frac{1}{\sqrt{-2i\varepsilon}}\begin{pmatrix} \dfrac{1}{\sqrt{h-\varepsilon(\frac{m}{3}+i)}} & \dfrac{-\phi(m,h)}{2i} \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ \dfrac{-ih'}{\sqrt{h-\varepsilon(\frac{m}{3}+i)}} & -2i\varepsilon\sqrt{h-\varepsilon(\frac{m}{3}+i)}\end{pmatrix}$$

Proof of Case I. Let $\Gamma = (C_1, C_2) : I \to \mathrm{SL}(2,\mathbb{C})$ be a canonical frame along γ and $U_\gamma(h)$ be the momentum of γ given by (3.17). Consider the eigenvalues $\pm\nu(m,h)$ of $U_\gamma(h)$ and denote by $\mathbf{L}_\pm$ the corresponding eigenspaces. From the definition of $U_\gamma(h)$, it follows that

$$\begin{aligned} L_+ &= -2i\varepsilon\left(h - \varepsilon\left(\frac{m}{3} + i\right)\right) C_1 + (ih' - \nu(m,h))\, C_2 : I \to \mathbf{L}_+, \\ L_- &= -2i\varepsilon\left(h - \varepsilon\left(\frac{m}{3} + i\right)\right) C_1 + (ih' + \nu(m,h)) C_2 : I \to \mathbf{L}_- \end{aligned}$$

are eigenvectors of $U_\gamma(h)$ corresponding to $\nu(m,h)$ and $-\nu(m,h)$, respectively. Thus, we must have

$$L'_+ = \rho_1 L_+, \quad L'_- = \rho_2 L_-$$

for analytic functions ρ_1, ρ_2. Using the Maurer–Cartan equation $\Gamma' = \Gamma H(m,h)$, we compute

$$L'_+ = \frac{h' + \nu(m,h)}{2\left(h - \varepsilon\left(\frac{m}{3} + i\right)\right)} L_+, \quad L'_- = \frac{h' - \nu(m,h)}{2\left(h - \varepsilon\left(\frac{m}{3} + i\right)\right)} L_-.$$

We thus see that the two vectors

$$\begin{aligned} \Lambda_1 &:= \exp\left(-\int \frac{h' + \nu(m,h)}{2\left(h - \varepsilon\left(\frac{m}{3} + i\right)\right)} ds\right) L_+, \\ \Lambda_2 &:= \exp\left(-\int \frac{h' - \nu(m,h)}{2\left(h - \varepsilon\left(\frac{m}{3} + i\right)\right)} ds\right) L_- \end{aligned}$$

are constant along γ. By (4.2), they become

$$\Lambda_1 = \frac{\exp(-\phi(m,h))}{\sqrt{h - \varepsilon\left(\frac{m}{3} + i\right)}} L_+, \qquad \Lambda_2 = \frac{\exp(\phi(m,h))}{\sqrt{h - \varepsilon\left(\frac{m}{3} + i\right)}} L_-.$$

Hence

$$\Gamma \cdot R(m,h) \cdot S(m,h) = \Lambda = (\Lambda_1, \Lambda_2),$$

where

$$R(m,h) = \begin{pmatrix} -2i\varepsilon\left(h - \varepsilon\left(\frac{m}{3} + i\right)\right) & -2i\varepsilon\left(h - \varepsilon\left(\frac{m}{3} + i\right)\right) \\ ih' - \nu(m,h) & ih' + \nu(m,h) \end{pmatrix}$$

and

$$S(m,h) = \begin{pmatrix} \frac{1}{\sqrt{h - \varepsilon(\frac{m}{3} + i)}} & 0 \\ 0 & \frac{1}{\sqrt{h - \varepsilon(\frac{m}{3} + i)}} \end{pmatrix} \begin{pmatrix} \exp(-\phi(m,h)) & 0 \\ 0 & \exp(\phi(m,h)) \end{pmatrix}.$$

From this, we obtain

$$\Gamma \begin{pmatrix} -2i\varepsilon\sqrt{h - \varepsilon(\frac{m}{3} + i)} & -2i\varepsilon\sqrt{h - \varepsilon(\frac{m}{3} + i)} \\ \frac{ih' - \nu}{\sqrt{h - \varepsilon(\frac{m}{3} + i)}} & \frac{ih' + \nu}{\sqrt{h - \varepsilon(\frac{m}{3} + i)}} \end{pmatrix} \begin{pmatrix} e^{-\phi(m,h)} & 0 \\ 0 & e^{\phi(m,h)} \end{pmatrix} = \Lambda,$$

and hence

$$\Gamma = \tilde{\Lambda} \begin{pmatrix} e^{\phi(m,h)} & 0 \\ 0 & e^{-\phi(m,h)} \end{pmatrix} \begin{pmatrix} \dfrac{ih' + \nu}{\sqrt{h - \varepsilon\left(\frac{m}{3} + i\right)}} & 2i\varepsilon\sqrt{h - \varepsilon\left(\frac{m}{3} + i\right)} \\ \dfrac{-ih' + \nu}{\sqrt{h - \varepsilon\left(\frac{m}{3} + i\right)}} & -2i\varepsilon\sqrt{h - \varepsilon\left(\frac{m}{3} + i\right)} \end{pmatrix}. \qquad \square$$

Proof of Case II. Again, let $\Gamma = (C_1, C_2) : I \to \mathrm{SL}(2, \mathbb{C})$ be a canonical frame along γ and $U_\gamma(h)$ be the momentum of γ. If $\nu(m, h) = 0$, then

$$L_1 = -2i\varepsilon\left(h - \varepsilon\left(\frac{m}{3} + i\right)\right) C_1 + ih' C_2$$

belongs to the kernel of $U_\gamma(h)$, and proceeding as in Case I, we see that the vector

$$\Lambda_1 = \frac{1}{\sqrt{h - \varepsilon\left(\frac{m}{3} + i\right)}} L_1 \tag{4.3}$$

is a first integral. In order to find another first integral, we look for analytic functions f and g such that

$$\Lambda_2 := gC_2 + fL_1 \tag{4.4}$$

is a constant vector. Differentiating and using the Maurer–Cartan equation $\Gamma' = \Gamma H(m, h)$, we obtain

$$g' C_2 + g\varepsilon C_1 + f' L_1 = 0,$$

from which we compute

$$g = \frac{1}{\sqrt{h - \varepsilon\left(\frac{m}{3} + i\right)}}, \qquad f = \frac{1}{2i}\int \frac{ds}{h - \varepsilon\left(\frac{m}{3} + i\right)} = \frac{1}{2i}\phi(m, h).$$

Now, from (4.3) and (4.4), we obtain

$$\Gamma \begin{pmatrix} -2i\varepsilon\sqrt{h - \varepsilon(\frac{m}{3} + i)} & 0 \\ \dfrac{ih'}{\sqrt{h - \varepsilon(\frac{m}{3} + i)}} & 1 \end{pmatrix} \begin{pmatrix} 1 & \dfrac{1}{2i}\phi(m, h) \\ 0 & \dfrac{1}{\sqrt{h - \varepsilon(\frac{m}{3} + i)}} \end{pmatrix} = \Lambda = (\Lambda_1, \Lambda_2),$$

and hence the required result. $\square$

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