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Developing a Contextualization of Students’ Mathematical Problem Solving

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Abstract

This paper investigates how students contextualize mathematical problem solving, not the actual problems. When students attempt to solve problems, what contexts (situational, cultural, or conceptual) do they evoke to describe their experiences with problem solving? The Common Core State Standards for Mathematical Practice emphasize contextualizing and decontextualizing problems, but what does this mean in practice? Middle and high school students were asked to attempt ability-appropriate problems during semi-structured interviews in this qualitative study. Situational contexts were analyzed using representation analysis (symbolic and nonsymbolic) while cultural contexts were analyzed using linguistic analysis (metaphors). The synergy of these two analyses developed a coherent and consistent conceptual contextualization for mathematical problem solving. Secondary students conceptualized problems as containers with the given information within the problem and solutions outside the problem. Thus students' representations are a means to travel from within the problem to outside of the problem.

Keywords:

Problem Solving; Reasoning; Metaphors; Context Analysis; and Representation Analysis

1. Introduction

Reasoning abstractly, making sense of problems, and persevering during problem solving are critical elements of the Standards for Mathematical Practice (National Governors Association, Council of Chief State School Officers, 2010). Moreover, problem solving is central to mathematics and instruction should give students daily experiences with it (Kilpatrick, Swafford, & Findell, 2001). Despite this agreed upon significance, problem solving is difficult to define experientially (Stanic & Kilpatrick, 1989). In the last 60 years, mathematics educators have perceived mathematical problem solving as a heuristic process (Pólya, 1945), a logic-based program (Newell & Simon, 1972), a means of inductive and deductive discovery (Lakatos, 1976), a framework for goal-oriented decision making (Schoenfeld, 1985, 2011), methodologies with multiple variables (Kilpatrick, 2004), a standard (NCTM, 1989), and a model-eliciting activity (Lesh & Zawojewski, 2007). Each contextualization of mathematical problem solving affects one's perception of what defines its purpose (Schoenfeld, 1992). For example, perceiving problem solving as discovery is epistemologically and pedagogically different from perceiving problem solving as a process (Silver, 1985). This study was interested in the population that matter the most for understanding, students. Thus, the research question for this study is:
How do students' contextualize mathematical problem solving?

Students' contextualization of mathematical problem solving is difficult to organize, analyze, and conceptualize (Nilsson, 2009). Nilsson and Ryve (2010) offer two significant factors that can aid researchers in understanding contextualization: coherence and consistency (these terms are described in the Framework section). Using these two factors as criteria, this study interviewed three high-school students and three middle-school students. Participants were asked to solve mathematics problems that were challenging, but ability-appropriate. Two researchers were involved and each researcher focused on specific aspects of the students' responses. Reseracher1 analyzed students' contextualization of mathematical problem solving through the linguistic tool of metaphors. Reseracher2 analyzed students' contextualization of mathematical problem solving through their use of symbolic and nonsymbolic mathematical representations, which were provided during the think aloud. This collaboration was unique because it allowed identical data to be triangulated via distinct vantage points.

To clarify the interpretation of our findings, we will first discuss the overarching theoretical framework, participants, and procedures shared by both researchers. Second, we will discuss the framework, method, analysis, and results of linguistic metaphors students used to solve mathematics problems. Third, we will discuss the framework, method, analysis, and results of students' mathematical representations employed during problem solving. Finally, this study will synthesize both sets of results to identify how students contextualize mathematical problem solving.

2. Framework

2.1. Problem Solving

As a framework for problem solving, we define a problem as a developmentally appropriate challenge for which a problem solver has a goal but the means for achieving it are not immediately apparent (Pólya, 1945; Schoenfeld, 2011). Often when solving a problem, the existence of a solution is uncertain because the means to attain a solution is unknown (Lesh & Zawojewski, 2007; Pólya, 2004; Schoenfeld, 2011). Problem solving requires making sense of the problem situation and the means necessary for making decisions, which directs an individual's understanding (Schoenfeld, 2011). Research on students' problem solving indicates that prior experiences and knowledge, beliefs and dispositions, and culture play a huge role in how individuals approach problem solving (Lesh & Zawojewski, 2007; Schoenfeld, 2011). Students experiencing rich problem-solving instruction have better problem-solving outcomes than peers in exercise-laden learning environments (Bostic, 2011; Lesh & Zawojewski, 2007). Thus, prior experiences (including but not limited to past instruction) influence students' problem-solving performance and approaches.

The theoretical framework for this study stems from an embodied cognition perspective (Lakoff & Núñez, 2000; Núñez, Edwards, & Matos, 1999). One's cognition and behavior are greatly influenced by the connections "within biological and experiential contexts, contexts which have shaped, in a non-arbitrary way, our characteristic ways of making sense of the world" (Núñez et al., 1999, p. 46). Students' problem solving is influenced by the cognitive network (i.e., beliefs and academic knowledge), external relationships with the environment, and other individuals (Lesh & Zawojewski, 2007). Fundamentally, mathematics and mathematics education are not mind-free (Lakoff & Nunez, 1997). Mathematics education is embodied because it relies on the prior experiences of both the teacher and the student (Lakoff & Núñez, 2000). This embodied cognition perspective lends itself to the notions of coherence and consistency upon which our study rests.

2.2. Coherence and consistency

Interpreting perceptions of students attempting to solve mathematical problems is the purpose and crucial theory within this study. When interpreting data from student interviews, it is vital to understand students' representations of how they would solve a problem. Constructivist epistemology challenges the means in which one interprets representations because the interviews only demonstrate re-presentations (Steffe, 1991; von Glasersfeld, 1995). Von Glasersfeld (1995) clarifies representation and re-presentation within radical constructivism by stating that a representation within a student's work is the image-like icon of the object and not the cognitive object itself of the student. However, if the image-like icon is consistent and coherent with the student's interpretation of the problem, epistemological disequilibrium is avoided (Piaget, 1970). The term re-presentation refers to the understanding that if the student has constructed their own knowledge, the image-like icon is their attempt to present again (re-present) their cognitive image of the object despite the possibility that it may have been modified from its initial construction cognitively. Hence a representation is an iconic re-presentation of the cognitive image within the student's mind.

The distinction between re-presentation and representation is significant because this study is not attempting to justify one re-presentation as correct for problem solving. If this were even possible, such justification would require enormous prior knowledge of the participant's cognitive schema and an absolute definition of the object, limiting generalizability. Similar to Nilsson and Ryve (2010), we define objects as *coherent* if significant traits of the objects similarly coalesce for a specific purpose. Additionally, we define objects as *consistent* if the significant traits reoccur frequently. Epistemologically, this study is looking for multiple coherent representations that are consistent with students' problem solving to identify how students contextualize problem solving.

The same framework of coherence and consistency is employed by Researcher1 to examine students' language while problem solving. Independent of Von Glasersfeld's (1995) perspective of re-presentation, Lakoff and Johnson (1980) recognized the value of listening for coherence when interpreting experiences through the linguistic tool of metaphor. Metaphors denote one figure of speech as another figure of speech (Merriam-Webster, 2011). The purpose of metaphors in discourse is to relate a personal experience through a presumed, shared experience (Sfard, 2000, 2009; Yee, 2012, 2013). These shared experiences are considered shared by multiple people within a cultural context. However, these shared experiences are presumed by the speaker, which are similar to re-presentations of iconic images. Thus a "correct" metaphor of the shared experience is not possible because shared experiences depend upon cultural contexts. Similar to the aforementioned representational epistemology, this study will identify coherence between the shared experiences (metaphors) students used consistently to identify the cultural context within mathematical problem solving.

The absolutist paradigm does not align with embodied cognition (Lakoff & Johnson, 1980; Steffe, 1991; von Glasersfeld, 1995). Nilsson and Ryve (2010) drive home this point when stating:

> A prerequisite for, and an unavoidable consequence of, the aspiration of coherency and consistency is to restrict the ways we look at and experience things in the world. We cannot describe or understand even the simplest thing in a completely exhaustive way. We always experience a phenomenon in a certain way, from a certain set of premises and assumptions, whereby some aspects of the phenomenon become activated and are made available for reflection, but many other aspects are left out (Säljö, Riesbeck, & Wyndhamn, 2003). From such a perspective, talking about students' contextualization is a way of organizing and conceptualizing such principles of experiencing and understanding (p. 244)

To this end, our study claims that consistency and coherence with representations and metaphors are sufficient to identify student contextualization. Thus, to identify coherence and consistency in students' mathematical problem-solving experiences, the focus must shift to students' contextualization.

2.3. Contextualization

Contextualization is a complex term despite its significance with the Common Core State Standards (National Governors Association, Council of Chief State School Officers, 2010). Indeed, the second standard for mathematical practice (SMP) states

> Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. (NGAC,CCSSO, 2010, p.6)

The CCSSM associate abstraction to the opposite of contextualizing. However, it is not clear what "to abstract a given situation" (NGAC,CCSSO, 2010, p.6) would mean in practice. Systemic Functional Linguistics may argue that abstraction has a discursive use for semiotic mediation by clustering similar situations and offering a general representation (Shreyar, Zolkower, & Perez, 2010), while cognitive theorists may prefer to look at abstraction as a reification of interiorization (Piaget, 1970) offering a purely mathematical representation void of any situational elements. Hence, there is some ambiguity in this language of the SMP. Nonetheless, significant research has been done on contextualization (Hallden, 1999; Nilsson & Ryve, 2010). Hallden (1999) states "To contextualize a problem can mean to relate it to a specific physical situation, but it can also mean to relate the problem to other ideas" (p. 60).

Through Hallden's (1999) work with conceptual change and how context can alter one's perspective of mathematical reasoning, Nilsson (2009) found that there were three specific contexts to analyze for mathematical understanding: situational context, cultural context, and conceptual context. The *situational context* refers to the interaction of the individual with the materials, environment, sensations, and actions involved. The *cultural context* refers to discursive rules, conventions, and patterns of behavior. The *conceptual context* is involved with personal constructions of concepts of the situation. Since this study's research question stems from an embodied cognition to generate theory on how students contextualize mathematical problem solving, our analysis of contextualization aims to identify how the situational context (i.e., representations employed by the student) and cultural context (i.e., metaphors used by the student) inform their conceptual context of mathematical problem solving.

The metaphorical analysis focused on determining students' conceptual metaphors of problem solving for cultural context. The representational analysis focused on determining students' representations of problem solving for situational context. As different analyses were used, coherence and consistency were dependent upon the form of analysis and will be explained in the linguistic and representation sections. After the analyses were completed independently, the authors discussed their results and a conceptual context of problem solving emerged that was consistent and coherent with both analytical methods. The results were synthesized with the aim of expressing students' contextualization of problem solving in a coherent manner that is consistent with our representational and metaphorical findings.

2.4. Participants

The participants for this study (for both Reseracher1 and Reseracher2) were three high-school students from Ohio chosen by Reseracher1 and three middle-school students in Florida chosen by Reseracher2. The choice of three participants was made so that each researcher could select students of varying performance for the given problem set (i.e., above average, average, below average). That is, one middle- and high-school participant of each performance-level was representatively sampled from larger samples gathered during previous investigations (Bostic, 2011, 2013). Gender was not a consideration when selecting the participants. However, three females and three males were chosen. Two male and one female high-school student and one male and two female middle-school students responded to problems during the think aloud. The participants and data collection procedures were identical for both researchers.

The study was kept to only six students because both representational and linguistic methods of qualitative analysis are extremely time-consuming. Participants' pseudonyms, grade level, and performance levels are listed in Table 1. Participants may be referenced by their performance on the given problems (i.e., above average students) and this naming applies only to their outcomes on these problems. This naming conventionality has nothing to do with any other aspect of the participants.

Insert Table 1 approximately here.

The prerequisite for each high-school participant was the completion of an algebra 1 and geometry course and enrollment in an algebra 2 course. Thus grade level was not a determining factor for high-school participants. Middle-school participants were enrolled in a sixth-grade mathematics course taught by the same instructor. Each middle-school student had an equivalently performing high-school participant with respect to performance on the problems administered during the interview.

2.5. Data collection

Students were asked to participate in a semi-structured interview. Table 2 lists the target problems administered during the semi-structured interview. Problems were chosen at each grade level that would evoke discussion on how students solved the problems. Problems were deemed developmentally appropriate for the participants after review by an expert panel consisting of a middle- or high-school mathematics teacher and mathematics education university faculty members.

Both researchers conducted pilot studies to determine an appropriate number of problems that could be completed during the time frame and determine how long participants might be able to maintain focus on the target problems. The middle-school students completed four target problems whereas the high-school students responded to three target problems. Practice was not included with high school students due to time constraints. The three or four target problems were intended to address a variety of mathematics concepts during the 40-minute interview. The problems are referenced within this paper by their grade level (i.e., MS and HS) and number in task sequence (i.e., 1, 2, 3, and 4). For example, HS2 is High School Question 2.

Insert Table 2 approximately here.

All participants were asked to solve the problems using materials (e.g., manipulatives and markers) provided during the interview and voice their thinking aloud while problem solving. Students' problem solving was recorded during their semi-structured interview, which lasted approximately 40 minutes. The researchers did not provide any assistance during the interviews and asked only clarification questions to better understand students' thinking (e.g., Can you describe what you mean?). Both high-school and middle-school interviews were video recorded. The interviews were transcribed and images were captured from the video data as needed. Students' responses were initially scored as correct or incorrect/no response prior to the present study. Correct responses had (a) solutions that answered the problem and (b) representation(s) that supported the solution. Participants' performance was compared to the larger samples from which they came to determine their performance-level descriptor (e.g., above average). Analyses focused on the artifacts (e.g., scratchwork on the problems and use of manipulatives) and language students used to describe problem solving.

3. Examining Students' Language and Metaphors

3.1. Linguistic framework

Language and cognition overlap when solving problems (Lakatos, 1976; Polya, 1945; Presmeg, 1997). When a student uses language to communicate their thoughts about how they solved a set of problems (as was the case for all of the students of this study), they presume a set of experiences (representations) shared by listener(s) and attempt to discuss their personal experiences (cognitive re-presentations) using the shared experiences (Lakoff & Nunez, 1997; Sfard, 2000; Steffe, 1991; von Glasersfeld, 1995). The researcher's dilemma arises in determining what experiences are perceived by the student as shared and what experiences they presume are personal (Steffe, 1991; Yee, 2013). For example, consider a student explaining his/her problem solving by stating, "I was trying to find a shortcut so I wouldn't have to do the calculations". The student presumes the listener understands the purpose and cultural context of a shortcut. Moreover, the student also presumes that the listener has experienced calculations so that they may relate shortcuts to calculations. Hence the complexity in interpreting the context of a student's description is difficult without the appropriate linguistic tool.

Lakoff and Johnson (1980) constructed the conceptual metaphor as a means to identify shared experiences methodically, succinctly, and concretely. Metaphors embody experiences and support individual transfer through vivid imagery (Presmeg, 1997). Lakoff and Johnson (1980) built upon Reddy's (1979) notion of the conduit metaphor to describe conceptual metaphors, which characterizes the conceptual mapping between linguistic expressions (Kövecses & Benczes, 2010).

Conceptual metaphor theory (CMT) includes the literal component and conceptual component (Lakoff & Johnson, 1980). The literal component is the actual literal expression, while the conceptual metaphor is a mapping between two objects: the source and the target domain. The source domain is the experientially-known domain and the related concept is the target domain. For example, “Your theoretical framework has a solid foundation” involves the conceptual metaphor of “THEORIES ARE BUILDINGS”. The target domain is theories and the source domain is building. Within a conceptual metaphor, variations of being (is, am, are, was, were) indicate unidirectional flow from the target domain to source domain. Hence a conceptual metaphor is interpreted as TARGET DOMAIN → SOURCE DOMAIN.

Conceptual metaphors can be classified in one of three hierarchical categories: structural, ontological, and orientational (Kövecses & Benczes, 2010; Lakoff & Johnson, 1980). Structural metaphors tend to describe a complex concept in terms of a concrete experiential object. For example, the literal metaphor “Don’t waste my time” corresponds to the conceptual metaphor TIME IS A LIMITED RESOURCE. Ontological metaphors employ less structured target domains and necessitate a newly-defined reality to understand the shared experience. Personifications are regularly ontological. For example, two students from the present study repeatedly stated (Betty and George) “I don’t know what this problem is saying,” personifying the problem as a person able to speak to the students. This generates the ontological conceptual metaphor of WRITTEN WORDS ARE SPOKEN WORDS. Orientational metaphors broadly conceptualize a specific direction inherent in human development. Orientational metaphors are broad because the concepts they represent are so complex and difficult to concretize, they can only add a direction, an orientation, nothing more to their meaning. The literal expression, “Things are looking up” or “rise and shine” demonstrates the conceptual metaphor of GOOD IS

UP. Conceptual metaphors are used to map how individuals' cognitive domains are related to expression of their experiences (Lakoff & Johnson, 2003). Teachers and students use conceptual metaphors during mathematics instruction (i.e. NUMBERS ARE LINEAR and LIMITS ARE APPROACHING) and these metaphors are understood because of coherence between the teacher and students' cognitive frameworks.

The interplay of student and teacher experiences is vital to mathematics education. Teachers and students share an experiential set: solving mathematics problems. However, the students and teacher's perspectives of what constitutes mathematical problems and/or solutions are complex in structure (Lakatos, 1976; Pólya, 2004). The linguistic tool of conceptual metaphor can be rich with representational elements to clarify the complexity of the structure (Danesi, 2007; Kövecses & Benczes, 2010). Conceptual metaphors are culturally designed to articulate implicit perspectives that encourage and incite cognition (Lakoff & Núñez, 2000; Sfard, 1997). Hence by using conceptual metaphors to analyze high-school and middle-school students' language for problem solving, cultural context will be identified through language.

3.2. Linguistic method

To analyze how students contextualized problem solving, the linguistic tool of conceptual metaphor theory was applied in analyzing the interviews of all six students. Language that referred to problem-solving experiences and cultural context were identified. The student's conceptual metaphors were analyzed for coherency and consistency throughout the entire conversation. If a student used the literal metaphor, "I seem to be lost, I don't know where I am going", Reseracher1 interpreted this metaphor as PROBLEM SOLVING IS A JOURNEY. If a student stated other literal metaphors such as "I think I'm missing some information, what am I trying to find?" this would be interpreted by the conceptual metaphor PROBLEM SOLVING IS

SEARCHING. The source and target domains were not categorically predesigned, but identified using the naturalistic paradigm relevant to the context of students’ conversations. The conversations were analyzed coherently (Did the student mix metaphors? Were the experiences closely related in traits and use?) and consistently (Did the conceptual mapping demonstrate itself frequently?) to corroborate the interpretations. The conceptual metaphors used by students were tallied and classified so that their problem-solving language could be analyzed.

Consistency was also determined by the number of times a student demonstrated metaphors that were structural, ontological, and orientational (Kövecses & Benczes, 2010; Lakoff & Johnson, 1980). Table 3 quantitatively summarizes the number of times a student referenced a conceptual metaphor. These data were validated by transcriptions and observational analysis done repeatedly with the same interview. Reseracher1 then conferred with Reseracher2 for internal reliability by explaining Reseracher1’s analysis and having Reseracher2 corroborate the methods and conceptual metaphors identified. Students’ conceptual metaphors were transcribed, qualitatively coded openly using CMT, and counted for each participant. The total number of each conceptual metaphors and categorical type was determined by multiple reviews of all transcripts and videos.

3.3. Linguistic results

Participants’ metaphor use offered insight into their contextualization of problem solving, which is tallied in Table 3.

Insert Table 3 approximately here.

Students who scored above average (Nicole and Theo) used more conceptual metaphors than students who did not perform as well at their developmental level. Nicole and Theo tended to use action verbs more often than their peers. For instance, Nicole used “equals” more often than David and Alexandra, who tended to use variations of “to be” or “is”.

As a whole, middle-school participants also spoke simply with a smaller variety of verbs and employed diverse metaphors far less than their high school counterparts. Nicole, David, and Alexandra said “got” and variations of “to be” frequently whereas high school students’ language was more complex in vocabulary and grammar structure. For example, George stated that he was “going in the other direction” and “getting off track”. These literal metaphors align with the structural conceptual metaphor of PROBLEM SOLVING IS A JOURNEY. As a result, the total number of metaphors used by high school students (103, 62, 64) were much greater than those used by middle-school students (24, 3, 7). It is interesting to note that the below-average students from the middle- and high-school groups used more total conceptual metaphors than the corresponding average students. Concomitantly, Theo had the literal metaphor, “my mind hit a wall” indicating a similar conceptual metaphor as George, but slightly different as the mind was the object on the journey. The use of the mind as a separate entity capable to travel classified this conceptual metaphor as an ontological metaphor.

Students regularly said “(verb) out” more often than their peers. Nicole frequently made the comment “figure out this problem”, “take him [number] out”, and “draw it [re-presentation] out”. These types of ontological metaphors indicated that Nicole perceived the solution as outside of the problem’s context. Thus students contextually suggested that problems are ontologically identified as containers while solutions lie outside of the container (problem). When looking for consistency, the word “out” occurred many times relative to the student. Table 4 demonstrates students’ use of the word “out”.

Insert Table 4 approximately here.

Table 4 captures one result of this metaphorical analysis: Every student used the word "out" while solving mathematical problems. This consistency with the literal metaphors generated a coherent ontological metaphor of MATHEMATICS PROBLEMS ARE CONTAINERS.

If the metaphorical analysis suggests the word "out" is of significance, how does the inverse ontology of "in" support the conceptual metaphor? George buttresses this concept of the container metaphor when he states, "Instead of being vague, I should have gotten into it." George added, "Generalizing at the beginning of the problem instead of jumping right into making triangles would have been better". George's remarks also offer insight into what is inside the container. When George refers to getting "into" the problem, he is referring to working with the given information, the conditions under which the problem exists. This result gave a novel means to interpret problem solving from a conceptual context. An examination of the situational context (representation used by students) is shared. A synthesis of the cultural context (metaphorical analysis) and situational context are discussed later in order to offer insight into students' conceptual contextualization of mathematical problems solving.

4. Examining Students' Problem-Solving Representations

4.1. Representation framework

Representations are central to doing and learning mathematics as they characterize both the means (e.g., problem solving) and the end (e.g., a solution) (Goldin, 2002; NCTM, 2000). As discussed earlier, there is a difference between a re-presentation and representation (von Glasersfeld, 1995). This distinction is critically linked to a contextualized understanding of mathematics (Goldin, 2002). The ways that learners make sense of their world through a mathematical lens can be examined via the representations used while problem solving.

Learners encode situational contexts as internal representations such as beliefs, competencies, and expectations (Goldin, 2002; Goldin & Kaput, 1996). These internal representations are (a) based on everyday experiences, (b) shared by many, (c) extensively linked within one's cognition, (d) developed prior to learning mathematics in a context, and (e) supported by one's culture (Goldin, 2002; Goldin & Kaput, 1996). Internal representations manifest themselves as external representations during students' mathematical problem solving (Ainsworth, 1999; Goldin, 2002; Goldin & Kaput, 1996). These external representations include various types such as (a) manipulative (i.e., concrete) models, (b) pictures, diagrams, and graphs (c) written words, and (d) tables (Lesh & Doerr, 2003).

These representations can be grouped into two broad representation categories: symbolic and nonsymbolic (Goldin & Kaput, 1996). Symbolic representations include abstract, symbol-driven ways of expressing oneself that include forms such as expressions, equations, and inequalities (Goldin & Kaput, 1996; Herman, 2007). Nonsymbolic representations characterize all other types of representations: diagrams, graphs, models formed with manipulatives or drawings, and tables (Goldin & Kaput, 1996; Lesh & Doerr, 2003). Symbolic and nonsymbolic representations that students use while problem solving is greatly influenced by prior problem-solving experiences (Schoenfeld, 2011) and provide a window into students' contextualization of problem solving

An aim of this study is to examine students' contextualization of problem solving through their use of symbolic and nonsymbolic representations. Research investigating students' representations during problem solving has indicated that students tend to employ symbolic representations over nonsymbolic representations (Bostic & Pape, 2010; Preston & Garner, 2003; Santos-Trigo, 1996). This perception begins during elementary years (Perry & Atkins, 2002), continues into middle school (Bostic, 2011; Preston & Garner, 2003), into high school (Bostic & Pape, 2010; Santos-Trigo, 1996), and later into students' college-age years (Herman, 2007). There is a common impression after examining the results from these studies: Students perceive symbolic representations as more mathematically appropriate than nonsymbolic representations. This impression is evident from studies involving interviews focused on students' problem solving. For example, students from Santos-Trigo's (1996) study were asked whether they could use nonsymbolic approaches after initially solving a problem. The students indicated they could not imagine another way to solve the task albeit a graphical or tabular approach might have led to a correct solution. Nonsymbolic representations are just as effective (Bostic & Pape, 2010; Herman, 2007) and sometimes, more efficient than symbolic-oriented strategies (Herman, 2007; Santos-Trigo, 1996). These findings are likely due to the ways that a problem solver interacts with the tasks and learning environment.

Prior research has largely examined students' representation use during effective problem solving. To build coherence around students' contextualization of problem solving, all representations employed during problem solving were examined, regardless of students' success with them on a given task. Thus, Reseracher2 examined the representations that middle- and high-school students employed on mathematical problem-solving tasks to coherently understand students' contextualization of problem solving through the lens of a situational context. These findings were compared to findings related to the cultural context (metaphors). Taken together, the representation and metaphor results synthesized a model of students' contextualization of problem solving that was consistent with the data.

4.2. Representation method

The two representation categories were symbolic and nonsymbolic. Symbolic representations included any representations using abstract symbols (i.e. variables) and numerical expressions. Nonsymbolic representations were coded as concrete model, pictorial, tabular, and mixed. The concrete model category characterized students employing manipulatives as the exclusive means for solving the problem. Pictorial approaches include diagrams and graphs while tabular representations characterized tables and charts. Mixed representation use indicated that a participant used multiple representations during problem solving, which was further coded as the appropriate combination (e.g. pictorial-symbolic).

Analysis of students' representations was conducted to make sense of students' situational context and identify a coherent contextualization of problem solving. It was necessary to count and categorize all student representations because the aim of the study was to investigate students' contextualization of problem solving, not merely those who arrive at the correct solution. First, both authors coded the representations on each student's worksheet. There was perfect agreement between their codes. After examining students' work on the worksheets, Reseracher2 watched the video data from the interviews to corroborate the results. The interviews were reviewed for additional evidence and/or representations that were erased or illegible on the worksheets. After watching the interviews, the initial codes were reexamined and confirmed. Additional coding of representations not seen on the worksheets (e.g., manipulative use and erased representations) was also made at this time to ensure that all representations were coded. Researcher1 shared the additional coding of representations with Researcher2 and discussed them to build internal consistency. Again, this resulted in ideal agreement between coders. This suggested that we were consistently coding students' representations, which supports our aims of building a coherent understanding of students' contextualization. Analyses benefited from the video data for two reasons. First, it was helpful to understand the situational context surrounding the ways participants employed a representation. Second, images of participants' representations from the video provided concrete evidence of their mathematical representations, which were not always evident from the scratch work on the worksheets. For instance, coding an approach as a concrete representation required seeing students use the manipulative.

4.3. Representation results

Impressions of participants' representations lead into a close examination of relationships between their representations to illuminate the situational contexts. The first impression was that high school students employed more diverse approaches than middle-school students. Table 5 summarizes the representations participants employed to successfully solve problems. Table 6 links the type of representation with the frequency of successful attempts with it.

Insert Table 5 approximately here.

Insert Table 6 approximately here.

The middle-school participants overwhelmingly employed symbolic approaches, despite success only two out of nine times. High school participants employed multiple (i.e., mixed) representations two times yet successfully only once. High school students have had more time and experience with mathematical ideas thus it is reasonable that they have developed more connected mathematical knowledge than their sixth-grade peers, or at least more experience in expressing their mathematical knowledge. In short, there was a difference in the number of types of representations used when comparing a middle- and high-school student having similar performance on the think-aloud problems.

A second impression was that facility with nonsymbolic representations was connected to problem-solving performance. This consistent pattern emerged from the frequency of students' use of different types of representations. Across the six participants, symbolic representations were employed 11 times but successfully only twice. On the other hand nonsymbolic representations (not including mixed representations) were employed only seven times and four of these occasions were successful. Participants had a 57% problem-solving success rate with nonsymbolic representations but an 18% problem-solving success rate with symbolic representations. Put another way, participants employing a nonsymbolic representation were more than three times more likely to reach the solution than their peers using symbolic representations. This impression about facility with nonsymbolic representations suggests that middle- and high-school students tend to *perceive* problem solving as employing a symbolic representation, performing procedures and manipulate symbols, and arriving at a result.

This finding lends evidence regarding participants' situational contextualization of problem solving. Participants interacted with the task and problem-solving environment (i.e., tools and physical space) in such a way that this situation led them to problem solve more frequently with symbolic representations than nonsymbolic ones. This notion is connected to our second impression because those demonstrating greater facility with nonsymbolic representations also tended to have greater problem-solving performance compared to their peers using only symbolic representations. By addressing each problem's situational context with multiple means (i.e., symbolic and nonsymbolic representations), the students had more ways to solve problems, which offered them more opportunities to reach the solution.

Those who solved more problems than their peers tended to employ a greater variety of representations while problem solving. The above-average participants used a total of four unique approaches compared to the below average participants who employed three representation types. The number of representations compared to performance was not statistically significant, yet the representations were consistent with students' sense-making of their situational context while problem solving. This finding also aligns with previous research (e.g., Bostic & Pape, 2010; Herman, 2007) indicating that greater use of representations is associated with greater problem-solving performance. The results from the present study suggest that effective problem solvers perceive problem solving as an opportunity to employ various representations whereas less successful peers stick to fewer representations and usually rely on symbolic ones.

There is a corollary related to this finding stemming from a common impression garnered from analyzing those who solved problems and their use of nonsymbolic or mixed representations. Nonsymbolic representations are often just as effective and sometimes more efficient than symbolic representations while problem solving (Bostic & Pape, 2010; Herman, 2007; Santos-Trigo, 1996). The way Alexandra and others stated that they solved the problem using symbolic representations suggested a lack of certainty compared to their peers using nonsymbolic representations.

Alexandra combined numbers from the problems' text without a clear direction in her calculations. She and her peers seemed either (a) uncertain whether their answer was correct and/or (b) unclear on the task within the problem. In either case, the students did not answer the problem except for two out of 11 occasions. In Alexandra's case, her voice hesitated when she said "yes" when responding to whether she solved MS1. This is in stark contrast to those students who employed nonsymbolic representations during problem solving. They seemed convinced that they were correct and there was no question about their certainty. For example, Betty exclaimed "Oh yeah, I figured it out!" after employing concrete representations to solve HS3. This finding suggests that nonsymbolic representations appeared to provide problem solvers a greater sense of validation that they were correct.

A third and final impression stemmed from corroborating students' representation use with the video data. The focus of this analysis was on students' coherent use of representations while problem solving. Transcript data and still images from the interviews are provided to keep the method of analysis transparent and demonstrate coherence. Students who used symbolic representations tended to employ them without meaningfully attending to the situation within the problems. Alexandra's work on the problem MS1 portrays this:

Alexandra: She's making 12 muffins per batch and there's 52 students in the fifth grade. So we have to do 52 times 12. times 2 is 4, 2 times 5 is 10, 1,2, 2 times 1 is 2 and 1 times 5 is 5. 4, 10, and 6. [See figure 1 for her initial symbolic representation and result.] I don't think that's right.

Reseracher2: Why don't you think that's right?

A: That's 624 and that's over.

R: Oh ok.

A: So how many times says adding. So 4 and 6… 64 batches sounds better. [See figure 2 for her second symbolic representation and result.]

R: OK. Is that your final answer?

A: Yes.

Insert Figure 1 approximately here.

Insert Figure 2 approximately here.

This act of combining of numbers from the task was common among those who employed symbolic representations and not just the below average performers.

This use of representations differs from the ways participants employed nonsymbolic representations while problem solving. For example, Betty was an average performer compared to her peers from the greater sample. She used molding clay to represent her thinking while problem solving during task HS3. When presented with the task, she examined her available tools and proceeded to grab the modeling clay. Betty formed the clay into various representations of cake until she settled on a cylindrical form (see Figure 3). She negotiated her ideas within her environment to represent her thinking then examined the situation within the task to represent a cake as a cylinder. After representing her idea, she proceeded to solve the task (see Figure 4).

Insert Figure 3 approximately here.

Insert Figure 4 approximately here.

Betty's sense-making of the problem provides evidence that she, like her high school and middle-school peers who used nonsymbolic approaches, sought to reason about the task while problem solving in a coherent manner. This is in sharp contrast to participants' use of symbolic approaches, which was fairly incoherent and/or inconsistent with the problem.

In comparison, George spent 11 of the 40 minutes during his interview working with a pictorial representation to solve this same problem. His representation focused on drawing a circle on paper and then attempting to cut it into multiple pieces using straight lines (Figure 5). While George did use a nonsymbolic representation, he did so without attending to the situational elements of his environment or the situation within the task. He was so focused on using a single representation (i.e., pictorial), he appeared to ignore that the cake was a three dimensional cylinder. George's attempts were inconsistent with the problem. This manner of problem solving was consistent with applications of symbolic representations like Alexandra's, which supports our third impression.

Insert Figure 5 approximately here.

Three impressions arose after examining participants' representation use. The first was expected: high-school students employed a larger variety of representations than their middle-school counterparts. The second was that multiple, nonsymbolic representations were consistently more successful when solving problems than those employing a symbolic representation. The third and final impression was that participants employing nonsymbolic representations tended to employ them coherently to align with the situational context of their environment and the context within the task. Less effective problem solvers represented their ideas without necessarily seeking to make sense of the situation, typically using symbolic representations.

The above impressions and the metaphorical analysis of students' language during the interview provide a window into the participants' contextualization of problem solving.

Coherently and consistently, students interacted with their environment to construct representations for solving the problem.

5. Synthesis

The metaphorical analysis used conceptual metaphors to identify structural, orientational, and ontological conceptual metaphors to determine the experiences students associated with mathematical problem solving. Using conceptual metaphor theory, students offered many experiences embedded within their cultural context related to problem solving. A conclusion that emerged from the linguistic results suggested that students regularly used the conceptual metaphor PROBLEMS ARE CONTAINERS as represented by their use of the words “in” and “out” with certain verbs. The natural question arose from this finding, what is inside and outside of this container context and is this context helpful in generating a cohesive theory related to students’ contextualization of mathematical problem solving? Synthesizing the results from the analysis of students’ metaphors and representations led to answering the research question: Students contextualize problem solving, not the problems, as the transfer from inside to outside of a container, which is identified as the problem. The representations (not the procedures) that students employed during the problem solving sessions were the means to facilitate students’ movement from inside the problem (i.e., container) to the solution (i.e., space outside of the container). Representations, not necessarily the procedures applied to them, were ways to “get out”, “find out”, or “figure out” the solution. We offer a representation to adequately capture this contextualization of problem solving, which is shown in Figure 6.

Insert Figure 6 approximately here.

To better understand the significance of this contextualization of problem solving with PROBLEMS ARE CONTAINERS, INFORMATION IS INSIDE, and SOLUTIONS ARE OUTSIDE, Theo's method of solving HS1 is shared. His use of multiple representations is discussed as a way to better understand the way problem solvers employ multiple representations as tools to move from the problem to the solution and demonstrate this contextualization of problem solving. During this task and others he examined his environment, considered his means to determine the solution, and tried multiple representations to reach the area outside of the container.

Initially, Theo began problem solving with a concrete representation by forming an equilateral triangle (see Figure 7) since that was his hunch. "My first thought would be an equilateral triangle." Next, Theo manipulated the triangle in Figure 7 in multiple ways.

> As you approach making one angle as large and that angle as small as possible it [the small angle] would approach zero. If you go the other direction and the one angle as large as possible and the other as small as possible then it would also approach zero. So logically…having these two angles as close as possible to each other…45 degrees… then it [the triangle] would be the greatest area. (Theo)

Theo *transformed* his initial concrete representations into other concrete representations (see Figure 8 for one example), which showed that he was still problem solving through one representational context.

Insert Figure 7 approximately here.

Insert Figure 8 approximately here.

After a few moments, he *translated* from the concrete representations to a pictorial one (see Figure 9) to show that the area would approach zero units. In less than one minute after that statement, Theo recognized an issue. “Those variables aren’t a very good way of describing it so that’s not the best way of looking at it.” His language suggests that he realized the representation’s weaknesses and/or his limitations with it. Theo completed a second translation (i.e., from pictorial to concrete representation) and returned to his triangle in Figure 8 after realizing that the graph was not helping him approach the solution. While continuing to transform the concrete representation, Theo said “...in between the two [areas of triangles with zero area]… directly in between the two - the area would grow, grow grow until the midway point [a point at which the area of a fixed perimeter triangle is maximized]”. Thus he determined that on any single orientation, an isosceles triangle was necessary. Thus after rotating the concrete representation he argued that the triangle must be isosceles on all sides, forcing the triangle to be equilateral. Hence iterative uses of the concrete representations helped him pass through the problem to the solution space.

Insert Figure 9 approximately here.

Using these representations, Theo attempted to move from within the container towards the solution space outside of the problem. Theo used multiple representations, developed representational structures, and translated between representations to successfully navigate outside of the problem. This problem-solving behavior was consistent among those employing multiple representations meaningfully. Some students succeeded by persevering with only a single representation (Betty with the concrete representation) while others were incapable

(Alexandra with the symbolic representation). However, as discussed in the representation results, those who used only symbolic representations were less likely to succeed.

We acknowledge that Theo and others could have gone farther with a single representation, but the findings from our sample suggest that the students found it more advantageous to employ multiple representations and navigate between these representations. Participants began with one representation, manipulated it as much as they could and either (a) arrived at the solution after employing a second representation or (b) returned to the original representation. Theo, identified as an above average problem solver, recognized ways to employ multiple representations, which supported him to move from the problem to the solution, which lay outside of the container's frame. This problem-solving behavior also demonstrates coherence in the students' awareness of their situational and cultural context during problem solving.

Theo began with a concrete representation that he believed gave him the best opportunity to move from inside the problem (container) to the solution space (outside). When he saw an opportunity to potentially get out of the problem, he took it (i.e., translating to the pictorial representation). Drawing on the container contextualization, Theo attempted to pass through the container but was unsuccessful. Not reaching the solution space, Theo drew the graph as a new means to travel through the container (pictorial representation). He manipulated the representation until he realized the limitations of that representation as an explanatory problem-solving tool to traverse the container. Thus Theo sensed that his original representation provided him greater opportunity to reach the solution hence he returned to that representation (approach). After thinking with the representation more as a problem-solving tool, used the representation dynamically until, he passed through the container's surface and arrived outside of the container (i.e., solution). Our data consistently corroborates that students contextualize problems as containers and problem solver representations are means to get out of the container.

6. Conclusions

The aim of this study was to examine students' situational, cultural, and conceptual contextualization of problem solving using the notions of coherence and. We drew on prior studies of linguistics and mathematical representations and analyzed interview data from these perspectives in a search for coherent and consistent traits among these students' problem solving. This sample was chosen to begin understanding relationships that might highlight students' perceptions of problem solving while further analyses may use larger data sets for corroboration and application of this contextualization. We offer a theory based on the findings of our sample of middle and high school students, which is held tentatively as an associative relationship rather than a causal explanation for students' outcomes. When focusing on contextualization of problem solving, three important points of discussion emerged using linguistics and representations as a frame for this study that will now be discussed.

6.1. Comparing high-school and middle-school results

Researcher1 identified that high-school students coherently used more cultural context linguistically than middle-school students as demonstrated by the diversity of their verb use and the frequency of their conceptual metaphors. Additionally, Table 3 demonstrates that high-school students consistently used more structural, ontological, and orientational metaphors than middle-school students. Naturally, one may be concerned that the research design and questions were such that the research methods were diverse, which led to diverse results. However, both researchers observed the videos and agreed the middle-school students had a greater opportunity to share their thoughts because they had more problems available.

Researcher2 identified in Table 5 that high-school students consistently used a larger variety of representations than middle-school students. In fact, most middle-school students stuck primarily with symbolic representations, which demonstrated misunderstandings of situational context as shown by Researcher2's analysis of Alexandra's problem solving. The perception of mathematics as abstract due to its highly symbolic nature may be encouraging middle-school students to abstract mathematics problems prematurely. It is surprising that high-school students did not continue the middle-school trend to decontextualize the problems using symbolic representations as seen in the findings from Santos-Trigo (1996). One may argue that the questions available to the high-school students encouraged such representations, but the questions for middle-school students could have been easily represented using nonsymbolic representations (MS1, MS3, and MS4 could have used pictorial or concrete representations of handshakes or cups to demonstrate a pattern). Thus a dangerous result of decontextualization occurred in our study with middle-school students primarily using symbolic representations.

6.2. Performance

When looking at performance at separate grade levels as seen in Tables 3 and 5, similarities and differences occurred when comparing high, middle, and low performers of comparable grades. Researcher1 found that the high-performing, high-school student (Theo) had more structural, ontological, and orientation conceptual metaphors used than the average performing or below average performing, high-school student (Betty and George). In fact, Theo used nearly as many conceptual metaphors as Betty and George combined. Similarly, the high-performing middle-school student (Nicole) used more structural, ontological, and orientational conceptual metaphors than the middle-performing or low-performing middle-school students (David and Alexandra). Nicole used more than twice David and Alexandra's total number of conceptual metaphors combined. Consistently through both middle- and high-school, the students who performed better used significantly more conceptual metaphors. This suggests that with respect to cultural context, those with a more diverse means of communicating concepts (structurally, ontologically, and orientationally) were able to perform better than those who did not use as many conceptual metaphors.

Researcher2 echoed the significance of diversity in performance when comparing the above average performing high-school student to the average and below average performing high-school students. As demonstrated in Table 5, the above-average performing students used more of both representations (symbolic and nonsymbolic) than the average and below-average performing middle- and high-school students used. Thus the better performing students approached the problems using multiple representations. Moreover, when looking at the average and below-average performing students of middle school and high school, there is a focus on the symbolic and concrete approaches respectively. Qualitatively, clarifying conceptual understanding through contextualization can be clearly seen with Theo and Alexandra's methods of problem solving. As stated in the representation results, students (like Theo) using nonsymbolic representations were more successful because they employed nonsymbolic representations coherently to align with the situational context.

6.3. Students' contextualization of problem solving

A question emerged across all grade levels, why do effective problem solvers tend to use more nonsymbolic representations and more conceptual metaphors to support their problem solving? The model generated by the students' language of PROBLEMS ARE CONTAINERS offers such a conclusion. The conceptual metaphor of PROBLEMS ARE CONTAINERS is an ontological metaphor itself, which is why it illuminates how students contextualize problem solving. Within this ontological metaphor lies a means to interpret representations as demonstrated with Theo. Representations are means to move from inside the container, through the problem, to outside the container. Thus PROBLEMS ARE CONTAINERS entails (Lakoff and Johnson, 1980) the contextualization of REPRESENTATIONS ARE APPROACHES from the inside of the problem to the outside of the problem demonstrates the significance between symbolic and nonsymbolic representations. If symbolic representations are contextualized as a single means to escape the container then it is natural to apply multiple representations to improve one's chances.

Reflecting upon our overarching theoretical framework and results, the following question arose: how are sharing experiences and concepts more significant to students than exercise-laden instruction? By focusing on representations and the linguistic tool of metaphors, it was concluded that contextualization is qualitatively the determining factor. The experiences through which students described their problem solving indicated that students saw problems ontologically as containers. The representation analysis supported the ontological conceptual metaphor PROBLEMS ARE CONTAINERS. Successful problem solvers employed a variety of representations, and more importantly they selected the one that gave them the greatest chance to move from inside of the container to outside of it. On the other hand, less successful problem solvers tended to clutch to symbolic representations that mimicked exercise-based assignments and seemed unable or unwilling to consider alternate representations. Probabilistically, this limited their options and thus their means if the first attempt did not make sense contextually (Alexandra). Put a different way, less successful problem solvers typically remained within the container because their representations did not facilitate a means to solve the problem and thus leave the container. It is clear that the ways individuals interact with a task and their problem-solving environment (i.e., situational context) must be considered to understand how students contextualize problems.

Among all grade levels studied, students consistently used the concept of “in” and “out” because it was fundamental to their perceptions and experiences of solving problems (Table 4). This result is natural as Kövecses and Benczes (2010) argue the experiential understanding of “in” and “out” is inherent with human existence. From birth there are objects inside of our bodies and objects outside of our bodies, thus making the container a student-perceived shared experience. To this end, the ontological metaphor of container is powerful and intimately involved with our perception of the world. Thus, students’ cultural context (discursive rules, conventions, and patterns of behavior) of mathematical problem solving is a natural shared experience to relate using the ontological conceptual metaphor of container. The cultural and situational contexts of students led us to our conceptual context that problem solving is the means to move from inside to outside of container.

Students referred to the mathematical resources they worked with as residing within the container. Some specific examples are given in Figure 6. The container (i.e., problem) held all knowledge needed to solve the problem. Therefore, the action of solving the problem is to use the knowledge in a manner that will move one’s understanding from inside to outside the container. This is not linear, nor clearly algorithmic, yet the students’ language suggests this is how they have contextualized mathematical problem solving.

6.4. Implications

The container metaphor for mathematical problem solving illuminates a significant gap in mathematics education: What would represent moving from outside to inside of the container? This opens new perspectives on how to better describe and understand the underlying perceptions of this contextualization of problem solving. Conjecturing solutions and moving towards the inside of the problem suggests a viable practical implication: problem posing with a genuine opportunity for a long period of deep reflection. Problem posing (e.g., Brown & Walter, 2005; Silver, 1994, 1997) and reflection have been explored in mathematics problem-solving research (e.g., Hamilton, Lesh, Lester, & Yoon, 2007; Silver, 1997). However, this study is the first contextualization to offer the container model as a means for common discussion.

NCTM (2000) and various scholars (Brown & Walter, 2005; Silver, 1994, 1997) have suggested that students should have opportunities to create new problems and modify old ones. The present study corroborated prior research (see Bostic & Pape, 2010; Herman, 2007) indicating that facility with multiple representations was associated with greater problem-solving performance. Our findings also go one step further by augmenting the field's understanding of students' use of representations as it pertains to their contextualization of problem solving. Facility with multiple representations entails some creativity while problem solving. For example, abstracting mathematical elements from a problem took students like Theo and Betty multiple representations until they reached the solution. They showed some creativity in their approaches and were successful. This is in contrast to Alexandra and David who pushed forward with symbolic representations and joining numbers in without clear purpose or understanding.

Silver (1997) reminded the mathematics education field "creativity-enriched instruction might be appropriate for a broad range of students, and not merely a few exceptional individuals (e.g., above average problem solvers)" (p. 76). Silver (1997) shared that creativity-enriched instruction, which may be developed through problem posing and solving, supports building flexible content knowledge. Mathematics students engaged in problem solving might benefit from determining a solution and working backwards to create a problem with that answer (i.e., problem posing). The challenge of considering viable representations to reach the solution while designing a problem may help students reach the area outside of the container during future problem solving. Thus, a viable instructional opportunity for enhancing students’ opportunities to reach the outside of the container is problem posing with opportunities to reflect on their problem.

A necessary requirement for retentive problem posing and problem solving is meaningful reflection. A means to scaffold students’ reflection while problem posing could be an adaptation of reflection tools (Hamilton et al., 2007). A reflection tool helps problem solvers by giving them space to record “significant aspects about what they have done” (Hamilton et al., 2007, p. 347) and to use these aspects to foster and sustain discussions that are centrally focused on developing students as mathematical problem solvers. Hamilton et al. (2007) suggested that reflection tools enhance students’ problem solving because they highlight specific roles and cognitive shifts while problem solving. By starting with the solution space and working into the information space (problem posing), the problem container is more fully understood (reflection), thus informing students of the critical design of what makes the container a problem.

6.5. Summary

The aim of this study was to begin to understand how middle-school and high-school students contextualize problem solving. By using conceptual metaphors and representations, this study determined the cultural and situational contexts respectively using CMT and analysis of students’ mathematical representations. Synthesizing these results, a conceptual context emerged across all participants; Problem solving is the process of moving from the given information (inside the container) through the problem (the container) to the solution (outside the container). The representations are the means of moving from moving inside to outside of the container. The ontological metaphor of PROBLEMS ARE CONTAINERS has potential for informing mathematics educators of perceived, shared experiences generated by students so teachers and students can discuss problem solving with a common framework.

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