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Bayesian Analysis of Inverse Lomax Distribution Using Approximation Techniques

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Abstract: The main aim of the present paper is to study the behavior of the shape parameter of Inverse Lomax Distribution by using various approximation techniques like Normal approximation and Tierney and Kadane (T-K) approximation. Different informative and non informative priors have been considered to obtain the Bayes' estimate of the parameter of Inverse Lomax Distribution under these approximation techniques. Moreover, the estimates obtained under these priors have been compared using the simulation technique as well as real life data set.
Keywords: Bayesian estimation, Prior distribution, Normal approximation, T-K approximation.

1. Introduction:

Inverse Lomax distribution is a special case of the Generalized Beta distribution of the second kind. It is one of the notable lifetime models in statistical applications. The inverse Lomax distribution is used in various fields like stochastic modeling, economics and actuarial sciences and life testing as discussed by Kleiber and Kotz (2003). Kleiber (2004) used this Inverse Lomax distribution to get Lorenz ordering relationship among ordered statistics. McKenzie (2011) applied this life time distribution on geophysical data especially on the sizes of land fibres in California State of United States. The estimated and predicted values calculated through Bayesian approach using various loss functions have been studied in detail by Rahman et. al.(2013). Further, Rahman and Aslam (2014) used two component mixture Inverse Lomax model for the prediction of future ordered observations in Bayesian framework using predictive models. Singh et. al. (2016) considered the said model and obtained its reliability estimates under Type II censoring using Markov Chain Monte Carlo method. In addition to this, hybrid censored Inverse Lomax distribution was applied to the survival data by Yadav et.al. (2016).

If a random variable Y has Lomax distribution then $X = \frac{1}{Y}$ has an Inverse Lomax distribution. The probability density function of the Inverse Lomax distribution is given as:

$$f(x) = \frac{\alpha x^{\alpha-1}}{\beta^{\alpha}\left(1+\frac{x}{\beta}\right)^{\alpha+1}} \qquad x>0,\ \alpha>0, \beta>0 \qquad (1.1)$$

The likelihood function for a random sample $(x_1, x_2, ...x_n)$ which is taken from the Inverse Lomax distribution (1.1) is given as:

$$L(x\,|\,\alpha) \propto \alpha^n e^{-\alpha\sum_{i=1}^{n}\ln\left(1+\frac{\beta}{x_i}\right)} \qquad (1.2)$$

2. Methods and Materials:

The practitioners of Bayesian statistics study the various characteristics of posterior and predictive distributions especially their densities, posterior means and posterior variances. When the problem under consideration does not involve a conjugate prior-likelihood pair, the evaluation of posterior density and simultaneously characterizing it, is tedious in closed form and thus analytical or numerical approximation methods are required. In such situations, it is often desirable to use approximations that are more accurate and not computationally difficult as numerical integration methods.

2.1 Normal approximation:

In Bayesian statistics, large sample approximations are mostly based on normal approximation to the posterior distribution. The posterior distribution tends to normality under certain regularity conditions as the sample size n increases. This posterior distribution concentrates in the neighbourhood of the posterior mode. The normal approximation method is expected to be accurate if the distribution has a single sharp peak and is not too skewed. Thus, when the posterior distribution $P(\alpha \mid x)$ is unimodal and roughly symmetric, the convenient procedure is to approximate it by a normal distribution centered at the mode, yielding the approximation

$$P(\alpha \mid x) \sim N\left[\hat{\alpha}, \{I(\hat{\alpha})\}^{-1}\right],$$

where $I(\hat{\alpha}) = -\frac{\partial^2}{\partial \alpha^2} \log P(\alpha \mid x)$ (2.1)

If the mode, $\hat{\alpha}$ is in the interior parameter space, then $I(\alpha)$ is positive; if $\hat{\alpha}$ is a vector parameter, then $I(\alpha)$ is a matrix.

This topic has been reviewed by Ahmed et.al (2007, 2011). They discussed Bayesian analysis of exponential and gamma distribution using normal and Laplace approximations. Further, Sultan et.al (2015) obtained the Bayes' estimates of shape parameter of Topp-leone and Kumaraswamy distributions under different priors using Bayesian approximation techniques.

In our study normal approximations for the shape parameter α of Inverse Lomax distribution will be obtained using different priors.

2.1.1 Jeffery's Prior: Using Jeffery's prior $P(\alpha) \propto \alpha^{-1}$, the posterior distribution for α is as

$$P(\alpha \mid x) \propto \alpha^{n-1} e^{-\alpha \sum_{i=1}^{n} \ln\left(1+\frac{\beta}{x_i}\right)}$$

$$\propto \alpha^{n-1} e^{-\alpha T} \text{ where } T = \sum_{i=1}^{n} \ln\left(1+\frac{\beta}{x_i}\right). \tag{2.2}$$

The log posterior is $\log P(\alpha \mid x) = (n-1)\log \alpha - \alpha T$

The first derivative is

$$\frac{\partial}{\partial \alpha} \log P(\alpha \mid x) = \frac{n-1}{\alpha} - T.$$

and the posterior mode is obtained as

$$\hat{\alpha} = \frac{n-1}{T}.$$

The second order derivative of log posterior density is given as:

$$\frac{\partial^2}{\partial \alpha^2} \log P(\alpha \mid x) = -\frac{(n-1)}{\alpha^2}.$$

Therefore, the negative of Hessian

$$I(\hat{\alpha}) = -\frac{\partial^2}{\partial \alpha^2} \log P(\alpha \mid x) \quad = \frac{T^2}{(n-1)},$$

$$\Rightarrow \qquad [I(\hat{\alpha})]^{-1} = \frac{(n-1)}{T^2}.$$

Thus, the posterior distribution can be approximated as:

$$P(\alpha \mid x) \sim N\left[\frac{(n-1)}{T}, \frac{(n-1)}{T^2}\right]. \quad (2.3)$$

2.1.2 Modified Jeffery's Prior: Using modified Jeffrey's prior $P(\alpha) = \alpha^{-\frac{3}{2}}$, the posterior distribution of α for the given data $(x_1, x_2, ...x_n)$ using is given as:

$$P(\alpha \mid x) \propto \alpha^{n-\frac{3}{2}} e^{-\alpha T} \quad \text{where } T = \sum_{i=1}^{n} \ln\left(1 + \frac{\beta}{x_i}\right). \quad (2.4)$$

The log posterior is $\log P(\alpha \mid x) = \left(n - \frac{3}{2}\right)\log \alpha - \alpha T$.

The first derivative is:

$$\frac{\partial}{\partial \alpha} \log P(\alpha \mid x) = \frac{\left(n - \frac{3}{2}\right)}{\alpha} - T$$

$\therefore$ The posterior mode is obtained as:

$$\hat{\alpha} = \frac{\left(n - \frac{3}{2}\right)}{T}.$$

The second order derivative of log posterior density is given as:

$$I(\hat{\alpha}) = \frac{\partial^2}{\partial \alpha^2} \log P(\alpha \mid x) = \frac{T^2}{\left(n - \frac{3}{2}\right)},$$

$$\Rightarrow \quad [I(\hat{\alpha})]^{-1} = \frac{\left(n - \frac{3}{2}\right)}{T^2}.$$

Thus, the posterior distribution can be approximated as:

$$P(\alpha \mid x) \sim N\left[\frac{\left(n - \frac{3}{2}\right)}{T}, \frac{\left(n - \frac{3}{2}\right)}{T^2}\right]. \quad (2.5)$$

2.1.3 Gamma Prior: Under gamma prior $P(\alpha) \propto \alpha^{a-1} e^{-b\alpha}$; $a, b, \alpha > 0$ where a,b are the known hyper parameters, the posterior distribution of α for the given data is given as:

$$P(\alpha \mid x) \propto \alpha^{n+a-1} e^{-\alpha[b+T]}. \quad (2.6)$$

The log posterior is given as $\log P(\alpha \mid x) = (n + a - 1)\log \alpha - \alpha[b + T]$.

The fist derivative is:

$$\frac{\partial}{\partial \alpha} \log P(\alpha \mid x) = \frac{(n + a - 1)}{\alpha} - (b + T).$$

Therefore, posterior mode is obtained as $\alpha = \frac{n + a - 1}{b + T}$.

Then, the negative hessian $I(\alpha) = -\frac{\partial^2}{\partial \alpha^2} \log P(\alpha \mid x) = \frac{(T + b)^2}{(n + a - 1)}$.

$$\therefore \quad [I(\hat{\alpha})]^{-1} = \frac{(n + a - 1)}{(T + b)^2}.$$

Thus, the posterior distribution can be approximated as:

$$P(\alpha \mid x) \sim N\left[\frac{n+a-1}{b+T}, \frac{(n+a-1)}{(b+T)^2}\right]. \qquad (2.7)$$

2.1.4 Inverse Levy Prior: Using inverse levy prior, an informative prior with hyper parameter c, $P(\alpha) = \sqrt{\frac{c}{2\pi}}\ \alpha^{\frac{-1}{2}} e^{-\frac{c\alpha}{2}}$ $c, \alpha > 0$ we obtain the posterior distribution of α for the given data $(x = x_{1,} x_2, \ldots x_n)$ as:

$$P(\alpha \mid x) \propto \alpha^{n-\frac{1}{2}} e^{-\alpha\left[\frac{c}{2}+T\right]}. \qquad (2.8)$$

The log posterior is given by $\log P(\alpha \mid x) = \left(n - \frac{1}{2}\right)\log \alpha - \alpha\left[\frac{c}{2} + T\right]$

and the posterior mode is obtained as:

$$\hat{\alpha} = \frac{\left(n-\frac{1}{2}\right)}{\frac{c}{2}+T}.$$

The negative of hessian is $I(\hat{\alpha}) = -\frac{\partial^2}{\partial \alpha^2}\log P(\alpha \mid x) = \frac{\left(T+\frac{c}{2}\right)^2}{\left(n-\frac{1}{2}\right)}$.

$$\therefore\ [I(\hat{\alpha})]^{-1} = \frac{\left(n-\frac{1}{2}\right)}{\left(\frac{c}{2}+T\right)^2}.$$

Thus, the posterior distribution can be approximated as:

$$P(\alpha \mid x) \sim N\left[\frac{\left(n-\frac{1}{2}\right)}{\frac{c}{2}+T}, \frac{\left(n-\frac{1}{2}\right)}{\left(\frac{c}{2}+T\right)^2}\right]. \qquad (2.9)$$

2.2 T- K Approximation:

Laplace's method is an efficient technique for approximating certain integrals arising in mathematics. It is a tool for approximating the integrals when the integrand has a sharp maximum in the interior of the domain of integration. In Bayesian paradigm, Laplace's method is used to calculate expected values of functions of parameters and marginal densities. This method can be easily implemented and is faster as it requires fewer simulations than the Monte Carlo methods, such as Gibbs sampling (Gelfand and Smith 1990). Furthermore, Laplace approximations technique can provide better insight to the problem. Some good sources which give a detailed description of Laplace approximation technique includes Lindley (1980), Tierney and Kadane (1986), Tierney, Kass and Kadane (1989) and Leonard, Huss and Tsui (1989). Tierney and Kadane (1986) gave Laplace method to evaluate $E[h(\alpha) \mid x]$ as

$$E[h(\alpha) \mid x] \cong \frac{\hat{\phi}^* \exp\left[-nh^*(\hat{\alpha}^*)\right]}{\phi \exp\left[-nh(\hat{\alpha})\right]}. \qquad (2.10)$$

where $-nh(\hat{\alpha}) = \log P(\alpha \mid x)$; $-nh^*(\hat{\alpha}^*) = \log P(\alpha \mid x) + \log h(\alpha)$.

$\hat{\phi}^2 = -[-nh''(\hat{\alpha})]^{-1}$; $\phi^{*2} = -[-nh''^*(\hat{\alpha}^*)]^{-1}$.

Thus for Inverse Lomax, T-K approximation for shape parameter α under different priors is obtained as:

2.2.1 Jeffery's Prior: Under Jeffrey's prior $P(\alpha) \propto \alpha^{-1}$**,** the posterior distribution for shape parameter α is calculated in (2.3)

$$-nh(\alpha) = (n-1)\log\alpha - \alpha T \,;\; -nh'(\alpha) = \frac{(n-1)}{\alpha} - T\,.$$

$$\Rightarrow \qquad \hat{\alpha} = \frac{(n-1)}{T}.$$

Also, $-nh''(\hat{\alpha}) = -\dfrac{T^2}{(n-1)}$.

Therefore, $\hat{\phi}^2 = \dfrac{(n-1)}{T^2} \;\Rightarrow \hat{\phi} = \dfrac{\sqrt{(n-1)}}{T}$.

Now,

$$-nh^*(\alpha^*) = -nh(\alpha) + \log h(\alpha) \;= n\log\alpha - \alpha T\,.$$

Also,

$$-nh'^*(\alpha^*) = \frac{n}{\alpha} - T \;\Rightarrow \qquad \hat{\alpha}^* = \frac{n}{T}.$$

Further, $-nh''^*(\hat{\alpha}^*) = \dfrac{-T^2}{n} \Rightarrow \hat{\phi}^* = \dfrac{\sqrt{n}}{T}$.

Using the values in (2.10) we have

$$E(\alpha \mid x) = \left(\frac{n}{n-1}\right)^{\frac{1}{2}} \frac{\exp\left[n\log\hat{\alpha}^* - \hat{\alpha}^* T\right]}{\exp\left[(n-1)\log\hat{\alpha} - \hat{\alpha}T\right]} = \left(\frac{n}{n-1}\right)^{n+\frac{1}{2}}\left(\frac{n-1}{T}\right)e^{-1}. \qquad (2.11)$$

Similarly, $E(\alpha^2 \mid x) = \dfrac{\hat{\phi}^* \exp\left[-nh^*(\hat{\alpha}^*)\right]}{\hat{\phi}\exp\left[-nh(\hat{\alpha})\right]}$; Here $-nh^*(\hat{\alpha}^*) = \log\alpha^2 - nh(\alpha)$.

$-nh^*(\hat{\alpha}^*) = (n+1)\log\alpha - \alpha T$ and $-nh'^*(\hat{\alpha}^*) = \dfrac{(n+1)}{\alpha} - T$

$\Rightarrow \hat{\alpha}^* = \dfrac{(n+1)}{T}$.

Now, $-nh''^*(\hat{\alpha}^*) = -\dfrac{(n+1)}{\alpha^2} = -\dfrac{T^2}{(n+1)}$

Then, $\hat{\phi}^* = \dfrac{\sqrt{n+1}}{T}$

Therefore, $E[\alpha^2 \mid x] = \left(\dfrac{n+1}{n-1}\right)^{\frac{1}{2}} \dfrac{\exp\left[(n+1)\log\hat{\alpha}^* - \hat{\alpha}^* T\right]}{\exp\left[(n-1)\log\hat{\alpha} - \hat{\alpha}T\right]}$.

$$E[\alpha^2 \mid x] = \left(\frac{n+1}{n-1}\right)^{n+\frac{1}{2}} \frac{(n+1)(n-1)}{T^2} e^{-2}.$$

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Hence, $V(\alpha\,|\,x)=E[\alpha^2\,|\,x]-[E(\alpha\,|\,x)]^2$

$$=\left(\frac{n+1}{n-1}\right)^{n+\frac{1}{2}}\frac{(n+1)(n-1)}{T^2}e^{-2}-\left[\left(\frac{n}{n-1}\right)^{n+\frac{1}{2}}\left(\frac{n-1}{T}\right)e^{-1}\right]^2.$$

2.2.2 Modified Jefferys Prior: Using modified Jeffreys prior $P(\alpha)\propto\alpha^{\frac{-3}{2}}$, the posterior distribution for α is given by the equation (2.4)

$$\therefore -nh(\alpha)=\left(n-\frac{3}{2}\right)\log\alpha-\alpha T\ ,\ -nh'(\alpha)=\frac{\left(n-\frac{3}{2}\right)}{\alpha}-\alpha$$

$$\Rightarrow \hat{\alpha}=\frac{\left(n-\frac{3}{2}\right)}{T}.$$

Now, $-nh''(\alpha)=\dfrac{-T^2}{\left(n-\frac{3}{2}\right)} \quad\Rightarrow\quad \hat{\phi}=\dfrac{\sqrt{\left(n-\frac{3}{2}\right)}}{T}$

Also, $-nh^*(\alpha^*)=-nh(\alpha)+\log h(\alpha)=\left(n-\frac{1}{2}\right)\log\alpha-\alpha T$.

$$-nh'^*(\alpha^*)=\frac{\left(n-\frac{1}{2}\right)}{\alpha}-T \quad\Rightarrow\quad \alpha^*=\frac{\left(n-\frac{1}{2}\right)}{T}.$$

Then, $-nh''(\hat{\alpha}^*)=\dfrac{-T^2}{\left(n-\frac{1}{2}\right)} \quad\Rightarrow\quad \hat{\phi}^*=\dfrac{\sqrt{\left(n-\frac{1}{2}\right)}}{T}.$

Therefore, $$E[\alpha\,|\,x]=\frac{\hat{\phi}^*\exp[-nh^*(\hat{\alpha}^*)]}{\hat{\phi}\exp[-nh(\hat{\alpha})]}=\left(\frac{n-\frac{1}{2}}{n-\frac{3}{2}}\right)^{\left(n+\frac{1}{2}\right)}\sqrt{\frac{\left(n-\frac{3}{2}\right)^3}{\left(n-\frac{1}{2}\right)}}\,\frac{e^{-1}}{T}. \qquad (2.12)$$

Now, $E(\alpha^2\,|\,x)=\dfrac{\hat{\phi}^*\exp\left[-nh^*(\hat{\alpha}^*)\right]}{\hat{\phi}\exp\left[-nh(\hat{\alpha})\right]}$; Here $-nh^*(\hat{\alpha}^*)=\log\alpha^2-nh(\alpha)$

$$\Rightarrow\ E[\alpha^2\,|\,x]=\left(\frac{n+\frac{1}{2}}{n-\frac{3}{2}}\right)^{n+\frac{1}{2}}\frac{\sqrt{\left(n+\frac{1}{2}\right)\left(n-\frac{3}{2}\right)^3}}{T^2}e^{-2}.$$

Also, $V[\alpha \mid x] = \left(\frac{n+\frac{1}{2}}{n-\frac{3}{2}}\right)^{n+\frac{1}{2}} \frac{\sqrt{\left(n+\frac{1}{2}\right)\left(n-\frac{3}{2}\right)^3}}{T^2} e^{-2} - \left[\left(\frac{n-\frac{1}{2}}{n-\frac{3}{2}}\right)^{\left(n+\frac{1}{2}\right)} \sqrt{\frac{\left(n-\frac{3}{2}\right)^3}{\left(n-\frac{1}{2}\right)}} \frac{e^{-1}}{T}\right]^2$.

2.2.3 Gamma Prior: Under the Gamma prior $P(\alpha) \propto \alpha^{a-1} e^{-b\alpha}$, $a, b, \alpha > 0$, the posterior distribution of α for the given data $(x = x_1, x_2, \ldots . x_n)$ is given in equation (2.6)

Then, $-nh(\alpha) = \log P(\alpha \mid x) = (n+a-1)\log \alpha - \alpha[b+T]$

$$-nh'(\alpha) = \frac{(n+a-1)}{\alpha} - (b+T) \Rightarrow \hat{\alpha} = \frac{n+a-1}{b+T}$$

Now, $-nh''(\alpha) = -\frac{(n+a-1)}{\alpha^2} \Rightarrow \hat{\phi} = \frac{\sqrt{(n+a-1)}}{(b+T)}$

Moreover, $-nh^*(\alpha^*) = -nh(\alpha) + \log h(\alpha)$

Then, $-nh'^*(\alpha^*) = \frac{(n+\alpha)}{\alpha} - (b+T) \Rightarrow \hat{\alpha}^* = \frac{n+\alpha}{b+T}$

Also, $-nh''^*(\alpha^*) = \frac{(b+T)^2}{(n+\alpha)} \Rightarrow \hat{\phi}^* = \frac{\sqrt{(n+\alpha)}}{b+T}$

Therefore, $E[\alpha \mid x] = \frac{\hat{\phi}^* \exp\left[-nh^*(\hat{\alpha}^*)\right]}{\hat{\phi} \exp\left[-nh(\hat{\alpha})\right]} = \left(\frac{n+\alpha}{n+\alpha-1}\right)^{n+\alpha+\frac{1}{2}} \frac{(n+\alpha-1)}{(b+T)} e^{-1}$. (2.13)

Similarly, $E[\alpha^2 \mid x] = \frac{\hat{\phi}^* \exp\left[-nh^*(\hat{\alpha}^*)\right]}{\hat{\phi} \exp\left[-nh(\hat{\alpha})\right]}$; Here $-nh^*(\alpha^*) = \log \alpha^2 - nh(\alpha)$

$$E[\alpha^2 \mid x] = \left(\frac{n+\alpha+1}{n+\alpha-1}\right)^{n+\alpha+\frac{1}{2}} \frac{(n+\alpha+1)(n+\alpha-1)}{(b+T)^2} e^{-2}.$$

Also, $V(\alpha \mid x) = \left(\frac{n+\alpha+1}{n+\alpha-1}\right)^{n+\alpha+\frac{1}{2}} \frac{(n+\alpha+1)(n+\alpha-1)}{(b+T)^2} e^{-2} - \left[\left(\frac{n+\alpha}{n+\alpha-1}\right)^{n+\alpha+\frac{1}{2}} \frac{(n+\alpha-1)}{(b+T)} e^{-1}\right]^2$.

2.2.4 Inverse Levy Prior: Using Inverse Levy prior $P(\alpha) = \sqrt{\frac{c}{2\pi}}\, \alpha^{\frac{-1}{2}} e^{-\frac{c\alpha}{2}}$ $c, \alpha > 0$, the posterior distribution for the parameter α is given in (2.8)

$$\therefore -nh(\alpha) = \log P(\alpha \mid x) = \left(n - \frac{1}{2}\right)\log \alpha - \alpha\left[\frac{c}{2} + T\right].$$

and $-nh'(\alpha) = \frac{\left(n-\frac{1}{2}\right)}{\alpha} - \left(\frac{c}{2} + T\right) \Rightarrow \hat{\alpha} = \frac{\left(n-\frac{1}{2}\right)}{\left(\frac{c}{2}+T\right)}$.

Also, $-nh''(\alpha)=-\dfrac{\left(\dfrac{c}{2}+T\right)^2}{\left(n-\dfrac{1}{2}\right)} \Rightarrow \hat{\phi}=\dfrac{\sqrt{\left(n-\dfrac{1}{2}\right)}}{\left(\dfrac{c}{2}+T\right)}$.

Now, $-nh^*(\alpha^*)=-nh(\alpha)+\log h(\alpha)=\left(n+\dfrac{1}{2}\right)\log\alpha-\left(\dfrac{c}{2}+T\right)\alpha$.

Then, $-nh'^*(\lambda^*)=\dfrac{\left(n+\dfrac{1}{2}\right)}{\alpha}-\left(\dfrac{c}{2}+T\right) \Rightarrow \hat{\alpha}^*=\dfrac{\left(n+\dfrac{1}{2}\right)}{\left(\dfrac{c}{2}+T\right)}$.

Further, $-nh''(\hat{\alpha}^*)=-\dfrac{\left(\dfrac{c}{2}+T\right)^2}{\left(n+\dfrac{1}{2}\right)} \Rightarrow \hat{\phi}^*=\dfrac{\sqrt{n+\dfrac{1}{2}}}{\left(\dfrac{c}{2}+T\right)}$.

$$\therefore\ E(\alpha\mid x)=\frac{\hat{\phi}^*\exp\left[-nh^*(\hat{\alpha}^*)\right]}{\hat{\phi}\exp\left[-nh(\hat{\alpha})\right]}=\left(\frac{n+\frac{1}{2}}{n-\frac{1}{2}}\right)^{n+\frac{1}{2}}\frac{\sqrt{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}}{\frac{c}{2}+T}e^{-1}. \qquad (2.14)$$

Now, $E(\alpha^2\mid x)=\dfrac{\hat{\phi}^*\exp\left[-nh^*(a^*)\right]}{\hat{\phi}\exp\left[-nh(\alpha)\right]}$; where $-nh^*(\alpha^*)=\log\alpha^2-nh(\alpha)$.

$$\Rightarrow E(\alpha^2\mid x)=\left(\frac{n+\frac{3}{2}}{n-\frac{1}{2}}\right)^{n+\frac{1}{2}}\frac{\sqrt{\left(n+\frac{3}{2}\right)^3\left(n-\frac{1}{2}\right)}}{\left(\frac{c}{2}+T\right)^2}e^{-2}$$

$$\therefore V[\alpha\mid x]=\left(\frac{n+\frac{3}{2}}{n-\frac{1}{2}}\right)^{n+\frac{1}{2}}\frac{\sqrt{\left(n+\frac{3}{2}\right)^3\left(n-\frac{1}{2}\right)}}{\left(\frac{c}{2}+T\right)^2}e^{-2}-\left[\left(\frac{n+\frac{1}{2}}{n-\frac{1}{2}}\right)^{n+\frac{1}{2}}\frac{\sqrt{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}}{\frac{c}{2}+T}e^{-1}\right]^2.$$

3. Simulation study:

In our simulation study, we have generated samples of size 25, 50 and 100 to represent small, medium and large data sets respectively using the R software to compare the behavior of the estimate of shape parameter of Inverse Lomax distribution. The comparison of the estimates is made by using the various informative and non informative priors under the two Bayesian approximation techniques. The values of the hyper parameter are taken as 0.5, 1, 2. Similarly, the values of the shape parameter α have been chosen as 0.5, 1, 2. The experiment has been iterated 5000 times to observe the performance of the estimates of shape parameter of Inverse Lomax distribution. The results obtained have been presented in the tables 1 and 2 as given below with posterior variances enclosed in brackets.

Table 1: Posterior estimates and posterior variance of the shape parameter of Inverse Lomax distribution using the Normal approximation

n	α	Jeffery Prior	Modified Jeffery Prior	Gamma Prior			Inverse Levy Prior		
				a=b=0.5	a=b=1	a=b=2	c=0.5	c=1	c=2
25	0.5	0.3491769 0.0050801	0.3419024 **0.00497435**	0.3538771 0.00511138	0.3585100 0.00514117	0.3675791 0.00519670	0.3551596 0.00514850	0.3538771 0.00511138	0.3513398 0.00503835
	1	0.6743607 0.0189484	0.6603115 0.01855367	0.6788723 0.01881092	0.6832605 0.01867380	0.6916869 0.01840118	0.6836078 0.01907427	0.6788723 0.01881092	0.6695953 **0.01830032**
	2	2.75985 0.3173656	2.702354 0.3107538	2.664166 0.2897053	2.578350 0.2659156	2.430787 **0.2272587**	2.738616 0.3061233	2.664166 0.2897053	2.526783 0.2605972
50	0.5	0.5153314 0.0054197	0.5100729 0.00536442	0.5178667 0.00541789	0.5203756 0.00541581	0.5253159 0.00541091	0.5192247 0.00544635	0.5178667 0.00541789	0.5151718 **0.00536165**
	1	1.081895 0.0238876	1.070855 0.02364394	1.081001 0.02360732	1.080126 0.02333344	1.078431 **0.02280421**	1.086935 0.02386722	1.081001 0.02360732	1.069325 0.02310010
	2	1.682181 0.0577496	1.665016 0.05716037	1.670669 0.05638656	1.659539 0.05508139	1.638352 **0.05263129**	1.684886 0.05735029	1.670669 0.05638656	1.642944 0.05453057
100	0.5	0.4885727 0.0024111	0.4861051 **0.00239896**	0.4898315 0.00241140	0.4910842 0.00241163	0.4935712 0.00241200	0.4904351 0.00241735	0.4898315 0.00241140	0.4886288 0.00239957
	1	0.8798114 0.0078188	0.8753679 0.0077793	0.8803431 0.0077889	0.8808701 0.0077593	0.8819103 **0.0077006**	0.8822947 0.0078235	0.8803431 0.0077889	0.8764658 0.00772052
	2	1.624415 0.02665377	1.616211 0.02651915	1.619334 0.02635418	1.614335 0.02606076	1.604575 **0.02549168**	1.625949 0.02656995	1.619334 0.02635418	1.606263 0.02593046

Table 2: Posterior estimates and posterior variances of shape parameter of Inverse Lomax of Laplace approximation

n	α	Jeffery Prior	Modified Jeffery Prior	Gamma Prior			Inverse Levy Prior		
				a=b=0.5	a=b=1	a=b=2	c=0.5	c=1	c=2
25	0.5	0.4485067 0.00804409	0.4395392 **0.00788321**	0.4534078 0.00805975	0.4582225 0.00807361	0.4676020 0.00809628	0.4554320 0.00813188	0.4534078 0.00805975	0.4494129 0.00791835
	1	1.096852 0.04811001	1.074921 0.04714781	1.094770 0.04698832	1.092776 0.04591744	1.089029 **0.04391493**	1.106646 0.04801331	1.094770 0.04698832	1.071767 0.04503441
	2	1.795949 0.1289815	1.76004 0.1264019	1.768349 0.1225971	1.742600 0.1167644	1.695952 **0.1065026**	1.799544 0.1269605	1.768349 0.1225971	1.709097 0.1145190
50	0.5	0.6475023 0.008384613	0.6410277 0.008300767	0.6497697 0.008359852	0.6520082 0.008335037	0.6564004 0.008285276	0.6518665 0.008413893	0.6497697 0.00835985	0.6456164 **0.0082533**
	1	1.070788 0.02293018	1.060081 0.02270088	1.070038 0.02267138	1.069303 0.02241833	1.067879 **0.02192874**	1.075736 0.02291348	1.070038 0.02267138	1.058821 0.0221985
	2	1.890866 0.0715026	1.871958 0.07078757	1.874333 0.06956219	1.858403 0.06771443	1.828223 **0.06427287**	1.891887 0.07087126	1.874333 0.06956219	1.840185 0.0670505

100	0.5	0.4556102 0.00207577	0.4533322 **0.00206539**	0.4568475 0.00207667	0.4580792 0.00207755	0.4605259 0.00207922	0.4573673 0.00208140	0.4568475 0.00207667	0.4558115 0.0020672
	1	1.161718 0.01349566	1.155909 0.01342818	1.160784 0.01340693	1.159861 0.01331935	1.158046 **0.01314753**	1.164145 0.01348470	1.160784 0.01340693	1.154119 0.0132534
	2	2.091094 0.04372602	2.080639 0.04350739	2.079805 0.04303995	2.068746 0.04237266	2.047295 **0.04109164**	2.090621 0.04348877	2.079805 0.04303995	2.058505 0.0421629

4. Real Life Example:

The following data represent the tensile strength measured in GPa, of 69 carbon fibres tested under tension at gauge length of 20mm (Bader and Priest, 1982):

1.312 1.314 1.479 1.552 1.700 1.803 1.861 1.865 1.944 1.958 1.966 1.997
2.006 2.021 2.027 2.063 2.098 2.140 2.179 2.224 2.240 2.253 2.270 2.274
2.301 2.301 2.359 2.382 2.382 2.426 2.434 2.435 2.478 2.490 2.511 2.514
2.535 2.554 2.566 2.570 2.586 2.629 2.633 2.642 2.648 2.684 2.697 2.726
2.770 2.773 2.800 2.809 2.818 2.821 2.848 2.880 2.954 3.012 3.067 3.084
3.090 3.096 3.128 3.233 3.433 3.585 3.585

Table 3: Posterior estimates and Posterior variances of shape parameter of Inverse Lomax distribution using Normal approximation

n	Jefferys Prior	Modified Jefferys Prior	Gamma Prior			Inverse Levy Prior		
			a=b=0.5	a=b=1	a=b=2	c=0.5	c=1	c=2
Posterior mean	1.412313	1.401614	1.407949	1.403675	1.395392	1.415441	1.407949	1.393200
Posterior variance	0.03022165	0.0299927	0.02980931	0.02940753	**0.02863409**	0.03012740	0.02980931	0.02918807

Table 4: Posterior estimates and Posterior variances of shape parameter of Inverse Lomax distribution using T-K approximation

n	Jefferys Prior	Modified Jefferys Prior	Gamma Prior			Inverse Levy Prior		
			a=b=0.5	a=b=1	a=b=2	c=0.5	c=1	c=2
Posterior mean	1.433739	1.42304	1.429147	1.424652	1.415937	1.436752	1.429147	1.414177
Posterior variance	0.03067956	0.03045061	0.03025757	0.02984645	**0.02905518**	0.03058044	0.03025757	0.02962699

Conclusion:
While comparing the estimates of the posterior variances of the shape parameter of Inverse Lomax distribution using both informative and non informative priors under the two approximation techniques, it is clearly evident that gamma prior is the best prior for the estimation of shape parameter especially when the value of the hyper parameters is taken as 2. This is because it has the minimum value of posterior variance in the simulation study which is apparent in the tables 1 and 2.Further, this prior has least value in the real life data set as well which confirms the efficiency of the Gamma prior as observed in the tables 3 and 4. However, modified Jeffreys prior proved to be better among the non informative priors. In addition to this, the variability of the estimates in the tables 1 and 2 goes on decreasing as the sample size increases.

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