Geometrical, Fresnel and Fraunhofer regime in single slit diffraction with decreasing slit width

Recently, Panuski and Mungan presented the results of their study of light diffraction from a variable-width single slit by observing and measuring the diffraction pattern on an observation screen placed at a fixed distance. In this way, the authors were able to demonstrate to students the important near-field effects for single slit diffraction which most often have been neglected in introductory courses. In this paper, we show that there exists very good agreement between observed data and theoretical calculations by plotting intensity from single slits using a method with Fresnel-Kirchhoff integrals (Fig. 2). In addition to these comparisons, and with the evaluation of Fresnel numbers, we argue that Panuski and Mungan demonstrated not only the near-field Fresnel, but also the far-field Fraunhofer regimes as well as the transition between the two. Furthermore, with a greater increase in the slit width, the transition to the geometric regime can be demonstrated (Fig. 3).

The application of the Fresnel-Kirchhoff integral
to evaluate the light field behind a slit

In their experiment, Panuski and Mungan[1] measured normalized intensity of light at the screen situated at the fixed distance from the single slit, for nine selected values of the slit width a. The slit was illuminated with light from red diode laser. Our aim in this paper is to compare these experimentally measured light intensities and intensities evaluated using the Fresnel-Kirchhoff integral.

Light is an electromagnetic wave, with electric ($\vec{E}(\vec{r},t)$) and magnetic field ($\vec{B}(\vec{r},t)$) vectors both satisfying the homogeneous wave equation, which in vacuum reads[2a,3a,4a,5a,6a]:

$$\nabla^2 \vec{E}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2} = 0, \quad \nabla^2 \vec{B}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \vec{B}(\vec{r},t)}{\partial t^2} = 0 \tag{1}$$

Here, $\vec{r}$ is a position vector, t is time and c velocity of light in vacuum.   This means that each rectangular component $V(\vec{r},t)$ of the field vectors satisfies the homogeneous wave equation:

$$\nabla^2 V(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 V(\vec{r},t)}{\partial t^2} = 0$$

In the case of a laser, we consider a monochromatic incident wave described by the complex function:

$$V(\vec{r},t) = \Psi(\vec{r}) \cdot e^{-i\omega t}, \tag{2}$$

where ω is angular frequency. It follows that the space dependent part of the   field component $\Psi(\vec{r})$ satisfies the Helmholtz equation[4b]

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Psi(\vec{r}) + k^2\Psi(\vec{r}) = \nabla^2\Psi(\vec{r}) + k^2\Psi(\vec{r}) = 0 \tag{3}$$

Here λ denotes the wavelength and $k = \omega/c = 2\pi/\lambda$ is a wave number. The problem of diffraction is then reduced to the problem of finding the solution of the Helmholtz equation behind a grating/slit (Fig. 1), which satisfies specified initial and boundary conditions, corresponding to the physical situation.
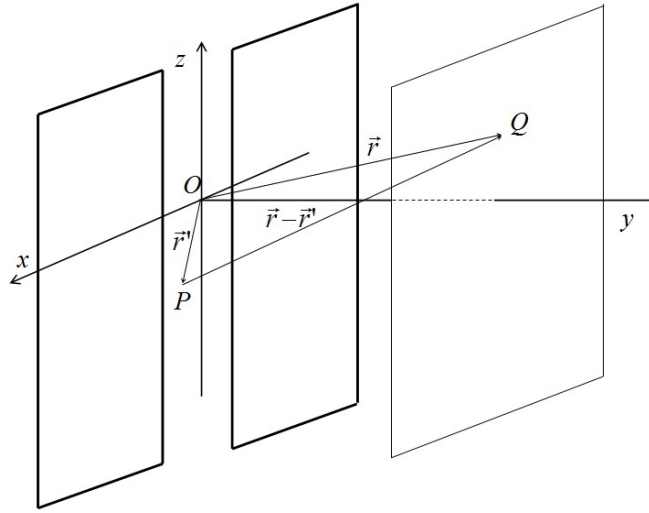
Fig. 1. Scheme of the slit and screen.

We shall consider the simplified case of an infinite vertical slit of finite width so that the electric and magnetic fields do not depend on a vertical z coordinate. In that case, independently of polarization of incident light, electric and magnetic fields are expressed through a scalar function $\Psi(x, y)$ which satisfies the Helmholtz equation, too. The solution behind the slit of width a is written in the form derived (Appendix 1) from the Fresnel-Kirchhoff integral[3b,4b]:

$$\Psi(x, y) = \frac{A}{\sqrt{y\lambda}} e^{-i\pi/4} e^{iky} \int_{-a/2}^{a/2} e^{ik(x-x')^2/2y} dx' \tag{4}$$

.

Here, A is a constant determined by the incident wave. Assuming that waves spreading from various points of the slit are cylindrical[7,8]

$$\Psi^{cyl}(\rho, t) = \frac{A_0}{\sqrt{\rho}} e^{i(k\rho - \omega t)} = e^{-i\omega t}\Psi^{cyl}(\rho) \tag{5}$$

where $\rho = \sqrt{(x-x')^2 + (y-y')^2}$ and using the principle of superposition eq. (4), up to the constant and phase factor, follows directly:

$$\Psi(x, y) = \int \frac{A_0}{\sqrt{\rho}} e^{ik\rho} dx' \approx \frac{A_0}{\sqrt{y}} e^{iky} \int_{-a/2}^{a/2} e^{ik(x-x')^2/2y} dx' \tag{6}$$

.

Light intensity at point (x,y) is proportional to the modulus square of $\Psi(x, y)$:

$$I(x, y) \propto |\Psi(x, y)|^2 \tag{7}$$

.

Comparison between experimental and theoretical curves

In the experiment of Panuski and Mungan[1], wavelength of laser beam was $\lambda = 660$ nm, slit width a was gradually reduced from $a = 3.30$ mm to 0.100 mm. The image was observed and registered at the fixed distance $L = 0.656$ m.

By numerical integration of the integral in (4), normalized intensity curves, $I(x,y)/I_0(y)$, were evaluated and presented with full lines at Fig. 2. Here $I_0(y)$ is the maximal value of the intensity at the distance y. Experimental data of Panuski and Mungan, extracted using WebPlotDigitizer[9] are given at the same figure with crosses. The program for the numerical evaluation of the integral in (4), written using Matlab[10], is given in the Appendix 2. At Fig. 3 are presented theoretical curves for values of a which are greater than the values of a in the experiment[1]. By comparing theoretical graphs and experimentally measured graphs at Fig. 2, we see very good agreement of two sets of curves. We are going to analyze and discuss essential features of these curves.
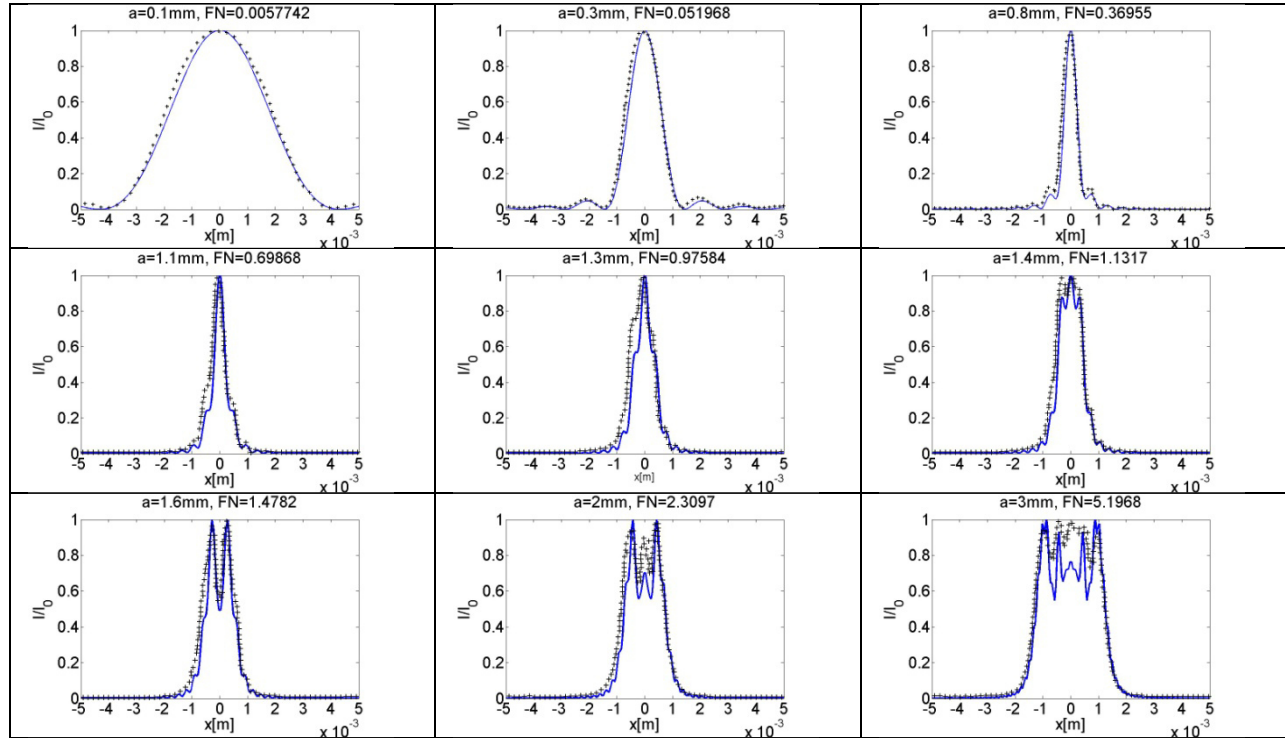

Fig. 2. Evaluated light intensities (blue solid line) and measured values[1] (crosses) at the distance $L = 0.656$ m from the slit, for nine values of the slit width. Wavelenght of light is $\lambda = 660$ nm.

The application of the Fresnel number in the characterization of the regions

In our analysis we are going to use the known criteria to characterize geometric, Fresnel and Fraunhofer region. These criteria are based on the Fresnel number[2b,3c,4b]:

$$FN = a^2 / 4\lambda L \qquad (8)$$

This criteria are:

$$FN \ll 1 \qquad \text{Fraunhofer diffraction} \tag{9a}$$

$$FN \approx 1 \qquad \text{Fresnel diffraction} \tag{9b}$$

$$FN \gg 1 \qquad \text{geometrical optics (either the slit width is very large or} \tag{9c}$$
the observation screen is very near the grating)

The "center" of the Fresnel region is for $FN = 1$. By fixing the distance of the screen L and varying slit width one determines from (9b) the slit width associated with the center of the Fresnel region on a-scale:

$$a_c = 2\sqrt{\lambda L} \tag{10}$$

For parameters in the experiment[1] one finds a_c = 1.316 mm, in agreement with the forms of intensity curves shown at Fig. 2.

For a = 0.1 mm and 0.3 mm we see the typical Fraunhofer pattern, in agreement with the criteria (9a), since FN = 0.0058<<1 and FN = 0.0520<<1, respectively. This pattern is characterised by the width which is larger than the slit width and increases with the distance of the screen from the slit. In addition, there are oscillations with decreasing amplitudes, from both sides of the central maximum.

For a = 0.8, 1.10, 1.3, 1.4, 1.6, 2.0, 3.00 mm Fresnel number is of the order of unity and the curves are typical for the Fresnel region, in agreement with the criteria (9b). The width of these curves at the bottom is very close to the width of the slits, but this width decreases towards the top of the curve which consists of several oscillations. As the slit width decreases (Fresnel number decreases), number of oscillations at the top and their amplitudes decrease and the curve becomes nerrower at the top.

According to criteria (9c) geometric regime is associated with $FN \gg 1$, (in practice for $FN > 10$) i.e. for $a > 2\sqrt{10 \cdot \lambda \cdot L}$. It follows that in this case geometric region is for a > 4,16 mm. Therefore, in order to demonstrate the geometric regime in the experiment[1] it would be necessary to increase the slit width beyond a = 3 mm. At Fig. 3 are shown evaluated intensity curves for three values of slit widths which correspond to $FN > 14 \gg 1$. The diffraction pattern looks like a shadow of the slit with many oscillations at the top.
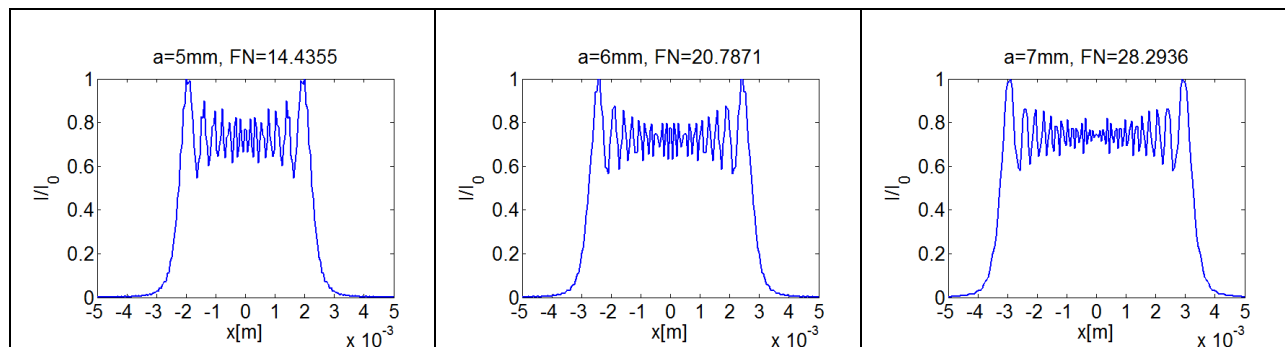
Fig. 3. Evaluated light intensities at the distance L = 0.656 m for three values of the slit width which are larger than the values in[1].

Characteristic features of three regimes

Let us summarize qualitatively features of curves at Figs. 2 and 3. For larger values of a, each intensity curve (Fig. 3) has steep edges and many oscillations at the top (approximately it looks like a

rectangle with oscillating plateau). Each curve looks like a hat with oscillations at the top. The width of the "hat" is very close to the slit width. So, for this large values of *a* for which $FN \gg 1$ it looks like that light propagates along a straight line (no diffraction) and this regime is called geometric regime. With decreasing *a*, edges become less steep and number of oscillations at the top decreases. With further decrease of *a,* the central narrow maximum with many small oscillations along the wings emerges. This region is called the Fresnel region. With further decrease of *a* the typical Fraunhofer intensity pattern appears, consisting of a large central maximum and several small decreasing maxima from its two sides (first two graphs at Fig. 2).

Harris *et al.*[11] studied experimentally and theoretically, with 435.8 nm and 546.1 nm light, single slit diffraction pattern at the distance 32,632 m from the slit, varying slit width from 0.5 to 32 mm. Based on the form of diffraction patterns authors interpreted their results as demonstrating Fresnel regime and the transition from Fresnel to Fraunhofer regime (see Fig. 2 in[11]) at slit width *a* = 2.5 mm. By evaluating the Fresnel number for their parameters we find that authors implicitly took $FN = 0.1$ for the value of Fresnel number marking the Fresnel-Fraunhofer transition, in agreement with the form of their graphs and the criteria (9a). Applying the same criteria to the parameters of Panuski and Mungan one finds the Fresnel-Fraunhofer transition at slit width $a = 0.390$ mm. The value $a = 0.390$ mm is in between the values $a = 0.3$ mm and $a = 0.8$ mm at Fig. 2 in[1] for which typical Fraunhofer and Fresnel type curves, respectively were measured. Analogous set of theoretical curves Hecht[6b] also interpreted as the Fresnel-Fraunhofer transition.

For large values of slit width Harris *et al.* measured[11] intensity graphs (see Fig. 3 in[11]) similar to our graphs at Fig. 3. One such graph, belonging to the geometric region, is presented in the book by Möller[12], where $L = 4000$ mm, $a = 10$ mm, $\lambda = 500$ nm and the corresponding $FN = 12.5$.

It follows from the above analysis that Fresnel-Kirchhoff integral is a very powerful tool to describe the single slit diffraction in the near and far field. Taking into account that programs for numerical integration are available to students, we consider that it is appropriate and useful to teach students to use and evaluate Fresnel-Kirchhoff integral. To support our view we draw attention to the collection of numerical programs in optics by Möller[12], showing how students could learn optics by computing.

Single slit diffraction of matter waves

Fresnel-Kirchhoff integral will be useful to students when they start to learn quantum mechanics because in that course they will have to solve the Helmholtz equation, too. For example, single slit diffraction of a matter wave is described[13] by the same Fresnel-Kirchhoff integral as in (4). The only difference is in the expression for the wave vector *k*. In the case of matter waves *k* is determined by particle momentum, $p = mv = \hbar k$ ($\hbar$ is Planck's constant, *m* is particle's mass), and angular frequency ω is determined by particle's kinetic energy, $E = \hbar^2 k^2 / 2m = \hbar \omega$. Probability distributions of particles at the screen for various distances of the screen from the slit were presented by Vušković *et al.*[13] and look similar to the graphs of light intensity behind the slit.

Appendix 1: The solution of the Helmholtz equation in the Fresnel-Kirchhoff form

Here. we are going to present the simplified derivation of the Fresnel-Kirchhoff integral from which the expression (4) for the field behind the single slit was obtained.

The simplest solution of the Helmholtz equation (3) is a plane wave solution, such as:

$$\Psi(y) = Ae^{iky} \tag{1.1}$$

It describes a wave propagating along y-axis (A is a constant).

It is easy to verify that the following solution also satisfies eq. (3)

$$\Psi(\vec{r};\vec{r}') = \frac{A}{|\vec{r}-\vec{r}'|} \cdot e^{ik|\vec{r}-\vec{r}'|} \tag{1.2}$$

This solution describes the component of the spherical wave field at point Q with position vector $\vec{r} = (x, y, z)$, which propagates from the point P (Fig. 1) with position vector $\vec{r}' = (x', y', z')$.

To describe the wave propagating after the incident wave (1.1) encountered the slit, we shall use the superposition principle which is justified by the fact the wave equation is a linear equation. The superposition principle is the generalization of the Huygens principle. According to the Huygens principle every point on a wave front is a source of secondary spherical waves. This means that the solution of (3) behind the grating may be written as a superposition of spherical waves (1.2) propagating from points $\vec{r}'$ at the slit:

$$\Psi(\vec{r}) = \frac{1}{i\lambda} \int_{slit} dx' dz' \Psi(\vec{r},\vec{r}') \cos\theta = \frac{1}{i\lambda} \int_{slit} dx' dz' \Psi(x',0^+,z') \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \cos\theta \tag{1.3}$$

The factor $(1/i\lambda)$ in front of the integral and the so called obliquity factor $\cos\theta$ were introduced by Fresnel and Kirchhoff. Because of that the solution of the wave equation in the form (1.3) is called the Fresnel-Kirchhoff integral.

In a more rigorous way one derives the expression (1.3) using Green's theorem and Green's functions[4b]. Here, we generalized the derivation for the electric field component, given in the standard Introductory physics textbooks which is valid only for points very far from the slit (Fraunhofer region). The advantage of the above derivation is that it leads to the expression (1.3) which is valid in the far field and in the near field, as well.

In order to evaluate the integral in (1.3) one has to express the absolute value of the vector $\vec{r} - \vec{r}'$ in terms of x, y, z and x',y',z':

$$|\vec{r}-\vec{r}'| = \sqrt{(x-x')^2 + (z-z')^2 + y^2} \tag{1.4}$$

For points which are not very near the grating, the approximation $|x-x'| \ll y$, $|z-z'| \ll y$ is valid. Using this approximation, the square root in the integral is approximated by:

$$\sqrt{(x-x')^2 + (z-z')^2 + y^2} \cong y + (x-x')^2/2y + (z-z')^2/2y \tag{1.5}$$

and the obliquity factor is approximated by:

$$\cos\theta \cong 1 \tag{1.6}$$

We shall assume also that the incident wave to a single slit grating is a plane wave:

$$\Psi_{in}(y) = Ae^{iky} \tag{1.7}$$

This implies that the incident field does not depend on x' and z' coordinates. Using the approximation (1.5) in the exponent of the integral (1.3), and by approximating the nominator by the distance y of the screen, the integral takes the form:

$$\Psi(\vec{r}) = \frac{1}{i\lambda y} e^{iky} \int_{slit} dx' dz' \Psi(x',0^+,z') e^{ik(x-x')^2/2y} e^{ik(z-z')^2/2y} \tag{1.8}$$

Assuming that the walls are completely absorbing and the slit completely transparent, field component just behind the slit is:

$$\Psi(x',0^+,z) = \begin{cases} A, & -a/2 < x' < a/2 \\ 0, & |x'| \geq a/2 \end{cases}$$

(1.9)

One usually assumes that the slit is very long compared to its width. So, we write the integral in (1.8) as:

$$\Psi(\vec{r}) = \frac{A}{i\lambda y} e^{iky} \int_{-\infty}^{+\infty} dz' e^{ik(z-z')^2/2y} \int_{-a/2}^{a/2} dx' e^{ik(x-x')^2/2y}$$

(1.10)

The value of the first integral was determined using the complex analysis[3c,13]:

$$\int_{-\infty}^{+\infty} dz' e^{ik(z-z')^2/2y} = \sqrt{\frac{2\pi y}{k}} e^{-i\pi/4}$$

(1.11)

So, the field behind the single slit is determined by the following expression:

$$\Psi(x,y,z) = \frac{A}{\sqrt{y\lambda}} e^{-i\pi/4} e^{iky} \int_{-a/2}^{+a/2} dx' e^{ik(x-x')^2/2y}$$

(1.12)

Very far from the grating, the quadratic term $ikx'^2/2y$ in the exponent in (1.12) may be also neglected, so (1.12) is further simplified

$$\Psi(x,y,z) = \frac{A}{\sqrt{y\lambda}} e^{-i\pi/4} e^{iky} e^{ikx^2/2y} \int_{-a/2}^{a/2} dx' e^{ikxx'/y}$$

(1.13)

It is easy to determine the integral in (1.13)

$$\int_{-a/2}^{a/2} dx' e^{-ikxx'/y} = \frac{\lambda y}{\pi x} \sin\frac{xa\pi}{\lambda y}$$

(1.14)

By substituting this result into (1.13) we find:

$$\Psi_{ff}(x,y,z) = \frac{Aa}{\sqrt{\lambda y}} e^{-i\pi/4} e^{iky} e^{ikx^2/2} \frac{2y}{kxa} \sin\frac{kxa}{2y}$$

(1.15)

Here the index *ff* denotes far field. It is convenient to express $\Psi_{ff}(x,y,z)$ using the value of this function at $x=0$

$$\Psi_{ff}(0,y,z) = \lim_{x\to 0} \Psi(x,y,z) = \frac{aA}{\sqrt{\lambda y}} e^{iky} e^{-i\pi/4}$$

(1.16)

Using (1.16) one writes $\Psi_{ff}(x,y,z)$ in the form familiar from the Introductory physics courses[5b]:

$$\Psi_{ff}(x,y,z) = \Psi_{ff}(0,y,z) e^{ikx^2/2y} \frac{\sin(xa\pi/\lambda y)}{(xa\pi/\lambda y)}$$

(1.17)

So, the Fresnel-Kirchhoff integral (1.12) for the field behind a single slit grating encompasses as a special case the expression valid in the Fraunhofer region, which has been widely present in the textbook of Introductory physics courses. The advantage of the expression (1.12) is in the fact that it gives the values of the field in the whole space behind the grating, very near it (geometrical region), in the Fresnel region and finally in the Fraunhofer region.

From (1.17) and (7) follows the expression for the intensity of light in the far field:

$$I_{ff}(x,y,z) = I_{ff}(0,y,z) \frac{\sin^2(xa\pi/\lambda y)}{(xa\pi/\lambda y)^2}$$

(1.18)

It is in agreement with the expression for the intensity of light in the Fraunhofer region derived in the Introductory physics courses. The latter derivation uses the geometrical method[5b] to evaluate the expression which is equivalent to the integral in (1.13), except that the sum is used instead of the integral. The application of the geometrical method is possible, because the difference of phases of the exponential function in the sum/integral at points $x'+\Delta x'$ and x', $\varphi(x'+\Delta x') - \varphi(x') = -ikx\Delta x'/y$, does not depend on x'.

Appendix 2: MATLAB program for evaluation of intensity distribution behind a single slit

```
clear all; close all; format long e;
a=3e-3; % slit width in m
lambda=660e-9; % wavelength in m
k=2*pi/lambda; % wave number
y=0.656; %distance from the slit in m
x=linspace(-5e-3,5e-3,200); %x coordinates of points at the observation screen in m
for ix=1:length(x)
fun=@(xp)exp(i*k*sqrt(y^2+(x(ix)-xp).^2))./sqrt(sqrt(y^2+(x(ix)-xp).^2)); %cylindrical wave emitted
from a line source (integrand in eq.(4))
psi(ix)=integral(fun,-a/2,a/2); %evaluation of the integral in eq. (4) (superposition of cylindrical waves)
end
I=(abs(psi)).^2; % intensity from          eq.(5)
plot(x,I/max(I), 'Linewidth', 2); % plot of the normalized intensity
xlabel('x[m]','FontSize',20); % size of x label
ylabel('I/I_0','FontSize',20); % size of y label
set(gca,'FontSize',20); % size of tick marks on both axes
title(['y=',num2str(y)]);
```

At the Wayant[15] and Wolfram[16] sites students may get intensity curves choosing various values of λ, a and y. We believe that it would be useful for students either to write by themselves the numerical program for the evaluation of the Fresnel-Kirchhoff integral (4), to use the above written program or to use program written by Möller[12]. Then they could compare the results of their evaluation with ready results at the Wayant and Wolfram websites.

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