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Getting More Out of Two-Asset Portfolios

Tom Arnold, Lance A. Nail, and Terry D. Nixon

Two-asset portfolio mathematics is a fixture in many introductory finance and investment courses. However, the actual development of the efficient frontier and capital market line are generally left to a heuristic discussion with diagrams. In this article, the mathematics for calculating these attributes of two-asset portfolios are introduced in a framework intended for the undergraduate classroom. [G10, G11]

■The use of two-asset portfolios in the classroom is very convenient, as the instructor is able to demonstrate the benefits of risk diversification without introducing much in the way of mathematics. By varying portfolio weights, it is simple to demonstrate that some portfolio weight combinations result in better risk-return tradeoffs than others (i.e., the efficient frontier). Although the portfolio in question is small, the basic lessons it demonstrates are applicable to much larger portfolios. However, more can be demonstrated with the two-asset portfolio than portfolio mean and portfolio variance calculations. In this article, a framework is developed that allows the student to calculate the minimum variance portfolio weights and the weights of a tangency portfolio when a third "riskless" security is added to the portfolio. This method allows the student to demonstrate how a capital allocation line dominates the efficient frontier, naturally leading to a discussion of the Capital Asset Pricing Model (Sharpe, 1964).

The remainder of this article is organized as follows: in Section I, the efficient frontier is developed for a portfolio consisting of two risky securities. Pursuant to that goal, formulae for determining the minimum variance portfolio weights are given and an example is developed for a two-stock portfolio consisting of McDonald's and Pepsico. In Section II, the framework is expanded to include a risk-free security, and formulae are generated to find the portfolio weights of the two risky assets that comprise the portfolio tangent with the efficient frontier developed in Section I (using McDonald's and Pepsico). Further, a capital allocation line consisting of the risk-free security and the tangency portfolio containing the two risky securities is developed. The dominance of the capital allocation line relative to the efficient frontier becomes apparent. Section III concludes the article. An appendix is then provided to clarify the derivation of the tangency portfolio formulae.

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I. A Simple Framework for Determining the Efficient Frontier for a Portfolio of Two Risky Stocks

The equations for calculating a two-asset portfolio's mean and standard deviation are available in any number of basic finance and investment textbooks. Let W_A be the proportion of the portfolio invested in Security A and W_B be the proportion of the portfolio invested in Security B. (Note: $W_A + W_B = 100\%$.) Securities A and B have associated expected returns (μ_A and μ_B, respectively), associated standard deviations (σ_A and σ_B, respectively), and an associated covariance (σ_{AB}). Equations (1) and (2) demonstrate the calculation of the two-asset portfolio mean and standard deviation denoted with P as a subscript:

$$\mu_P = W_A * \mu_A + W_B * \mu_B \quad (1)$$

$$\sigma_P = \sqrt{(W_A)^2 * \sigma_A^2 + (W_B)^2 * \sigma_B^2 + 2 * W_A * W_B * \sigma_{AB}} \quad (2)$$

Many texts and instructors prefer to use correlation rather than covariance. Because the correlation coefficient is a "standardized" measure ranging between negative one and positive one, it is usually preferred, as it provides information on both the magnitude and the direction of co-movement provided by covariance. Correlation (ρ_{AB}) is derived from covariance: $\rho_{AB} = \sigma_{AB} \div [\sigma_A * \sigma_B]$, and is easily inserted as the substitute for covariance ($\sigma_{AB} = \rho_{AB} * \sigma_A * \sigma_B$) into Equation (2):

$$\sigma_P = \sqrt{(W_A)^2 * \sigma_A^2 + (W_B)^2 * \sigma_B^2 + 2 * W_A * W_B * \rho_{AB} * \sigma_A * \sigma_B} \quad (3)$$

Let us demonstrate these equations through an investor who invests 50% of the portfolio funds in the shares of McDonald's and the other 50% in Pepsico (see Exhibit 1 for the latest eleven years of annual stock prices and returns for these companies).[1] The investor's mean annual return is 0.50 * 0.1551 + 0.50 * 0.0872 = 12.11% and the annual standard deviation of the portfolio is $(0.50^2 * 0.4229^2 + 0.50^2 * 0.2591^2 + 2 * 0.50 * 0.50 * 0.3037 * 0.4229 * 0.2591)^{1/2} = 27.95\%$.

Given these basic formulae as a starting place, we will now graph the efficient frontier for an investor who holds these two risky securities. Our first goal is to determine the minimum variance portfolio (which is also the portfolio that minimizes the standard deviation). In order to minimize the portfolio variance (standard deviation), substitute $(1 - W_A)$ for W_B, take the derivative of the square of Equation (3) relative to W_A, set the derivative equal to zero (in calculus terms this is solving for a local minima, which, in this case is also shown to result in a global minima), then solve for W_A (call it $W_{A\text{-}MIN}$).[2]

$$\frac{d}{dW_A}\left\{(W_A)^2 * \sigma_A^2 + (1-W_A)^2 * \sigma_B^2 + 2 * W_A * (1-W_A) * \rho_{AB} * \sigma_A * \sigma_B\right\} = 0 \quad (4)$$

$$W_{A-MIN} = \frac{\left(\sigma_B^2 - \rho_{AB} * \sigma_A * \sigma_B\right)}{\left(\sigma_A^2 - \rho_{AB} * \sigma_A * \sigma_B\right) + \left(\sigma_B^2 - \rho_{AB} * \sigma_A * \sigma_B\right)} \quad (5)$$

Equation (5) is simplified further by defining F:

$$F = \frac{\left(\sigma_A^2 - \rho_{AB} * \sigma_A * \sigma_B\right)}{\left(\sigma_B^2 - \rho_{AB} * \sigma_A * \sigma_B\right)} \quad (6)$$

$$\therefore W_{A-MIN} = \frac{1}{(1+F)} \text{ and } W_{B-MIN} = \left(1 - W_{A-MIN}\right) \quad (7)$$

$W_{A\text{-}MIN}$ and $W_{B\text{-}MIN}$ are the appropriate security weights, resulting in the minimum variance or the minimum standard deviation for a two-security portfolio.

Let us now further develop the example of McDonald's and Pepsico. The minimum variance portfolio is found at $W_{McDonald's}$ = 18.88% and $W_{Pepsico}$ = 81.12%. These weights result in a mean annual return of 10.00% and an annual portfolio return standard deviation of 24.65%.[3]

To demonstrate that these weights do, in fact, generate the minimum variance portfolio, increase the portfolio weight in the security (relative to the minimum variance portfolio weight) with the higher expected return and calculate the portfolio mean and standard deviation. For example, a portfolio with 30% of its wealth invested in McDonald's and the remaining 70% in Pepsico has an expected mean annual return of 10.75% and standard deviation of 25.09%. This portfolio's mean and standard deviation are larger than the minimum variance portfolio's mean and standard deviation. Next, increase the portfolio weight (relative to the minimum variance portfolio weight) in the security with the lower expected return and calculate the portfolio mean and standard deviation. A portfolio with 10% of its wealth invested in McDonald's and the remaining 90% in Pepsico has an expected annual return of 9.40%, which is lower than the mean of the minimum variance portfolio, and a standard deviation of 24.93%, which is greater than the standard deviation of the minimum variance portfolio. Clearly, this portfolio is inefficient; thus, portfolios that provide

[1]Data for this analysis were obtained at Yahoo! Finance using Yahoo's closing prices adjusted for dividends and splits.

[2]It is more convenient to solve for weights resulting in the minimum variance portfolio knowing that these weights also lead to the minimum portfolio standard deviation.

[3]Rounding results in this minimum variance portfolio having a standard deviation equal to that of a portfolio with 20% of the investor's wealth in McDonald's shares and 80% in Pepsico shares (see Exhibit 2). The difference between the two portfolios' standard deviations is 0.000046.

Exhibit 1. Price and Return Data for McDonald's and Pepsico

	McDonald's (A)		Pepsico (B)	
Date	**Adj-Close**	**Annual Return**	**Adj-Close**	**Annual Return**
3-Jan-05	$32.39	0.2818	$40.96	-0.1386
2-Jan-04	$25.27	0.8351	$47.55	0.2418
2-Jan-03	$13.77	-0.4685	$38.29	-0.0599
2-Jan-02	$25.91	-0.0653	$40.73	-0.2373
2-Jan-01	$27.72	-0.2096	$53.40	0.0230
3-Jan-00	$35.07	-0.0423	$52.20	-0.1113
4-Jan-99	$36.62	0.6915	$58.74	0.0171
2-Jan-98	$21.65	0.0494	$57.75	0.1286
2-Jan-97	$20.63	-0.0835	$51.17	0.5525
2-Jan-96	$22.51	0.5621	$32.96	0.4558
3-Jan-95	$14.41		$22.64	

	Annualized Statistics		
	McDonald's (A)		**Pepsico (B)**
Mean Return:	0.1551		0.0872
Standard Deviation of Returns:	0.4229		0.2591
Covariance of Return:		0.0333	
Correlation of Returns:		0.3037	

higher expected returns relative to those of the minimum variance portfolio are optimal choices. Exhibit 2 graphs the return-standard deviation relationship (also known as the feasible set) based upon different portfolio weightings of McDonald's and Pepsico stock.

The efficient frontier for a two-stock portfolio is just a subset of the feasible set. If one assumes that $\mu_A > \mu_B$, the entire efficient frontier is mapped out by portfolios that have weights in Security A that are greater than $W_{A\text{-}MIN}$. For the optimal weights that generate a return (k) greater than the minimum variance portfolio return, the portfolio weights are a function of the security means.

$$W_{A-k} = \frac{(k-\mu_B)}{(\mu_A-\mu_B)} \text{ and } W_{B-k} = (1-W_{A-k}) = \frac{(\mu_A-k)}{(\mu_A-\mu_B)} \quad (8)$$

An investor in McDonald's and Pepsico who desires a 12% annual return can use these formulae to determine the stock weights necessary to achieve this return:

$$W_{A-k} = \frac{(0.1200-0.0872)}{(0.1551-0.0872)} = 48.31\%$$

and $W_{B-k} = (1-0.4831) = 51.69\%$

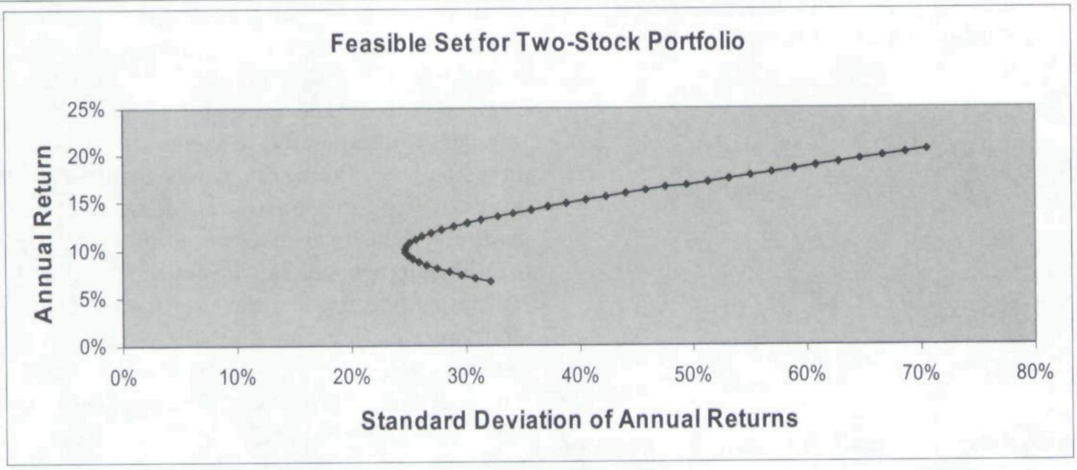
This framework allows the student to execute portfolio mathematics without calculus. In fact, the framework can easily be implemented into an Excel worksheet (as shown in Exhibit 2) to produce a mapping of the feasible set including the efficient frontier.[4]

II. Extending the Framework to Determine the Capital Allocation Line for a Two-Stock Portfolio

To generate a capital allocation line, a risk-free security (mean = R_F and standard deviation = 0) is introduced as a third portfolio security and its portfolio weight is denoted as W_R. Because the security is risk-free, its correlation and covariance with any risky security is undefined and treated as zero. This portfolio's standard deviation equation is actually the same as Equations (2) and (3) because the standard deviation of the risk-free security is zero; however: $W_R + W_A + W_B = 100\%$ and not $W_A + W_B = 100\%$ as was assumed in the first section of the article. This scenario leads to an interesting optimization problem. The minimum variance (minimum standard deviation) portfolio is attained by simply investing the entire portfolio in the risk-free security. Consequently, performing the variance minimization in the same manner as was done in the previous section is too simple in order to develop any additional conclusions. However, by introducing two additional constraints to the portfolio variance minimization technique (described in the Appendix), a framework emerges that is workable. We are able to develop a unique capital allocation line that is defined by two portfolios (or "funds"): the risk-free security (first portfolio) and a

[4]A weight less than 0% indicates that the negatively weighted security is being shorted with the proceeds invested into the security with the weight exceeding 100%.

Exhibit 2. The Feasible Set for a Two Stock Portfolio (Graph: Return vs. Standard Deviation)



Applicable Data

Weight- A (MCD)	Weight - B (PEP)	Return (Portfolio)	Std. Dev. (Portfolio)	Weight- A (MCD)	Weight - B (PEP)	Return (Portfolio)	Std. Dev. (Portfolio)
-30%	130%	0.0668	0.3219	75%	25%	0.1381	0.3424
-25%	125%	0.0702	0.3087	80%	20%	0.1415	0.3575
-20%	120%	0.0736	0.2964	85%	15%	0.1449	0.3731
-15%	115%	0.0770	0.2852	90%	10%	0.1483	0.3892
-10%	110%	0.0804	0.2752	95%	5%	0.1517	0.4059
-5%	105%	0.0838	0.2664	100%	0%	0.1551	0.4229
0%	100%	0.0872	0.2591	105%	-5%	0.1585	0.4403
5%	95%	0.0906	0.2534	110%	-10%	0.1619	0.4580
10%	90%	0.0940	0.2493	115%	-15%	0.1652	0.4759
15%	85%	0.0974	0.2470	120%	-20%	0.1686	0.4942
20%	80%	0.1008	0.2465	125%	-25%	0.1720	0.5127
25%	75%	0.1041	0.2478	130%	-30%	0.1754	0.5313
30%	70%	0.1075	0.2509	135%	-35%	0.1788	0.5502
35%	65%	0.1109	0.2558	140%	-40%	0.1822	0.5692
40%	60%	0.1143	0.2622	145%	-45%	0.1856	0.5883
45%	55%	0.1177	0.2702	150%	-50%	0.1890	0.6076
50%	50%	0.1211	0.2795	155%	-55%	0.1924	0.6271
55%	45%	0.1245	0.2901	160%	-60%	0.1958	0.6466
60%	40%	0.1279	0.3018	165%	-65%	0.1992	0.6662
65%	35%	0.1313	0.3145	170%	-70%	0.2026	0.6859
70%	30%	0.1347	0.3281	175%	-75%	0.2060	0.7057

tangency portfolio composed of securities A and B (second portfolio). This particular capital allocation line is unique in that it dominates or is at least equivalent to the efficient frontier defined in the first section of the article.

Solving for the weights of the two securities that comprise the tangency portfolio is somewhat complex and may not be appropriate for an undergraduate classroom. (We describe the necessary mathematics in the appendix for interested readers.) However, the final results of the calculation are important and can be implemented in the undergraduate classroom quite easily. $W_{A\text{-}TAN}$ and $W_{B\text{-}TAN}$ are the weights of securities A and B, respectively, in the tangency portfolio. Recall that the tangency portfolio contains only securities A and B; consequently, the sum of $W_{A\text{-}TAN}$ and $W_{B\text{-}TAN}$ must equal one. As demonstrated in the Appendix:

$$W_{A-TAN} = \frac{1}{1+G} \text{ and } W_{B-TAN} = 1 - W_{A-TAN} \quad (9)$$

where:

$$G = \frac{\sigma_A^2 * (\mu_B - R_F) - \rho_{AB} * \sigma_A * \sigma_B * (\mu_A - R_F)}{\sigma_B^2 * (\mu_A - R_F) - \rho_{AB} * \sigma_A * \sigma_B * (\mu_B - R_F)} \quad (10)$$

Given the efficient frontier described in Section I; a risk-free security; and the tangency portfolio found using the previous formulae—we can now develop the capital allocation line for our example from Section I with Pepsico and McDonald's (see Exhibit 1 for statistical information on the two stocks).

We assume an annual risk-free rate of return of 3%. Using the above formulae:

$$G=\frac{0.1788*(0.0872-0.0300)-(0.3037*0.4229*0.2591)*(0.1551-0.0300)}{0.0672*(0.1551-0.0300)-(0.3037*0.4229*0.2591)*(0.0872-0.0300)}=0.9325$$

Therefore:

$$W_{A-TAN}=\frac{1}{1+0.9325}=51.75\% \text{ and } W_{B-TAN}=1-0.5175=48.25\%$$

The annual expected return and standard deviation for this tangency portfolio are 12.23% and 28.31%, respectively.[5] In essence, the tangency portfolio can be considered a single security. At this point, drawing the capital allocation line is as simple an exercise as drawing a straight line that connects our risk-free investment with the tangency portfolio. Using Excel, we generate the capital allocation line for our portfolio of the risk-free security, McDonald's and Pepsico stocks—demonstrating its relation to the previously developed efficient frontier (see Exhibit 3).

By placing the dominant capital allocation line on the same graph as the efficient frontier, a student can see that the capital allocation line always produces the given portfolio expected return with less portfolio risk (i.e., less standard deviation). The only time the efficient frontier and the capital allocation line are equivalent is when 100% of the capital allocation line portfolio is invested in the tangency portfolio. Consequently, a rational investor will always invest in a combination of the risk-free security and the tangency portfolio in order to be on the capital allocation line—no matter the investor's risk preferences. In other words, the capital allocation line dominates the efficient frontier composed of risky assets.

As shown in Exhibit 3, investors can earn a greater rate of return than what is offered by investing 100% in the tangency portfolio. Investors who desire a higher expected return than the tangency portfolio expected return will short the risk-free security (i.e., borrow money which is equivalent to a negative portfolio weight) and invest the borrowed money in the tangency portfolio. An investor with -25% of their wealth in the risk-free security and the remaining 125% in the tangency portfolio has an expected annual return of 14.54% and corresponding standard deviation of 35.38%. More conservative investors invest in a combination of positive weights in the risk-free security and the tangency portfolio. Further, assuming the more conservative investor has a non-zero positive weight in the risk-free security, this investor is "saving" money and earning the risk-free rate on the savings. Consequently, in theory, the second type of investor (i.e., the saver) lends to the first type of investor (i.e., the borrower). This observation will naturally lead to a discussion about the necessary equilibrium between borrowers and savers.

The method used to develop the efficient frontier and capital allocation line for our small portfolio provides an opportunity to discuss the capital market line and the Capital Asset Pricing Model (CAPM). Instead of using two individual securities to generate an efficient frontier, we select and use a stock index fund (Vanguard 500 Index Fund, ticker symbol: VFINX) and a bond index fund (Vanguard Total Bond Market Index Fund, ticker symbol: VBMFX). Per the CAPM (Sharpe 1964), investors are assumed to expand their portfolios and invest in all risky securities (these risky securities are contained in the efficient frontier). This is not practical—in fact it is impossible—to represent in a classroom setting aside from heuristically, but our two-index fund portfolio funds provides a close approximation. Exhibit 4 displays the efficient frontier and capital allocation line for our portfolio of two index funds as well as the efficient frontier generated by our portfolio containing only McDonald's and Pepsico.[6] As shown, the efficient frontier from our previous two-stock example is dominated by the efficient frontier generated by our portfolio of two-index funds.[7]

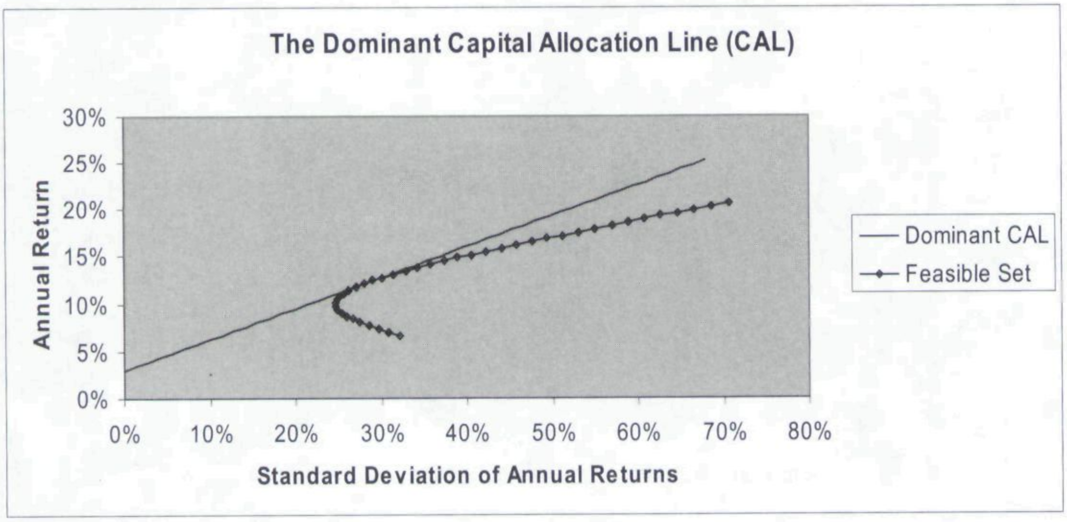
The new tangency portfolio that emerges contains hundreds of risky securities and can be viewed as analogous to a portfolio that contains *all* possible risky securities and is renamed the "market portfolio" in the CAPM. Further, our capital allocation line is now analogous to the "capital market line." Consistent with what is determined to be true above, any rational investor will invest in a combination of the risk-free security and the market portfolio in order to be on the capital market line. Consequently, an investment for achieving a particular expected return (k) consists of an investment in the risk-free security (W_R) and an investment in the market portfolio (W_{MKT}). Because the sum of the two portfolio weights must equal one,

[5]Due to a small amount of rounding error, it can be shown that the actual weights are $W_{A\text{-}TAN}$ = 51.72% and $W_{B\text{-}TAN}$ = 48.28%.

[6]For brevity, the data required to graph the dominant allocation line are not provided, but calculations are performed in a manner similar to the two-stock portfolio case.

[7]Some care must be taken when presenting this pedagogy. Per Roll (1977), the student must be made aware that it is impossible to duplicate the actual market portfolio. As a result, it is possible to find individual firms whose risk-return profiles fall outside our approximate efficient frontier.

Exhibit 3. The Feasible Set for a Two Stock Portfolio with the Dominant Capital Allocation Line (Graph: Return vs. Standard Deviation)



Applicable Data

Weight (RF)	Weight (MCD)	Weight (PEP)	Port. Return	Port. Std. Dev.	Weight (RF)	Weight (MCD)	Weight (PEP)	Port. Return	Port. Std. Dev.
-30%	67.24%	62.76%	0.1500	0.3679	75%	12.93%	12.07%	0.0531	0.0708
-25%	64.65%	60.35%	0.1454	0.3538	80%	10.34%	9.66%	0.0485	0.0566
-20%	62.07%	57.93%	0.1407	0.3396	85%	7.76%	7.24%	0.0438	0.0425
-15%	59.48%	55.52%	0.1361	0.3255	90%	5.17%	4.83%	0.0392	0.0283
-10%	56.89%	53.11%	0.1315	0.3113	95%	2.59%	2.41%	0.0346	0.0142
-5%	54.31%	50.69%	0.1269	0.2972	100%	0.00%	0.00%	0.0300	0.0000
0%	51.72%	48.28%	0.1223	0.2830	105%	-2.59%	-2.41%	0.0254	0.0142
5%	49.14%	45.86%	0.1177	0.2689	110%	-5.17%	-4.83%	0.0208	0.0283
10%	46.55%	43.45%	0.1131	0.2547	115%	-7.76%	-7.24%	0.0162	0.0425
15%	43.96%	41.04%	0.1084	0.2406	120%	-10.34%	-9.66%	0.0115	0.0566
20%	41.38%	38.62%	0.1038	0.2264	125%	-12.93%	-12.07%	0.0069	0.0708
25%	38.79%	36.21%	0.0992	0.2123	130%	-15.52%	-14.48%	0.0023	0.0849
30%	36.21%	33.79%	0.0946	0.1981	135%	-18.10%	-16.90%	-0.0023	0.0991
35%	33.62%	31.38%	0.0900	0.1840	140%	-20.69%	-19.31%	-0.0069	0.1132
40%	31.03%	28.97%	0.0854	0.1698	145%	-23.27%	-21.73%	-0.0115	0.1274
45%	28.45%	26.55%	0.0808	0.1557	150%	-25.86%	-24.14%	-0.0161	0.1415
50%	25.86%	24.14%	0.0761	0.1415	155%	-28.45%	-26.55%	-0.0208	0.1557
55%	23.27%	21.73%	0.0715	0.1274	160%	-31.03%	-28.97%	-0.0254	0.1698
60%	20.69%	19.31%	0.0669	0.1132	165%	-33.62%	-31.38%	-0.0300	0.1840
65%	18.10%	16.90%	0.0623	0.0991	170%	-36.21%	-33.79%	-0.0346	0.1981
70%	15.52%	14.48%	0.0577	0.0849	175%	-38.79%	-36.21%	-0.0392	0.2123

Note: Graph actually incorporates more data than displayed in the columns above

replace the weight for the risk-free security with (1 - W_{MKT}) and reduce the portfolio return equation to capture all of the terms associated with the weight on the market portfolio. Thus:

$$k = (1 - W_{MKT}) * R_F + W_{MKT} * \mu_{MKT} = R_F + W_{MKT} * (\mu_{MKT} - R_F) \quad (11)$$

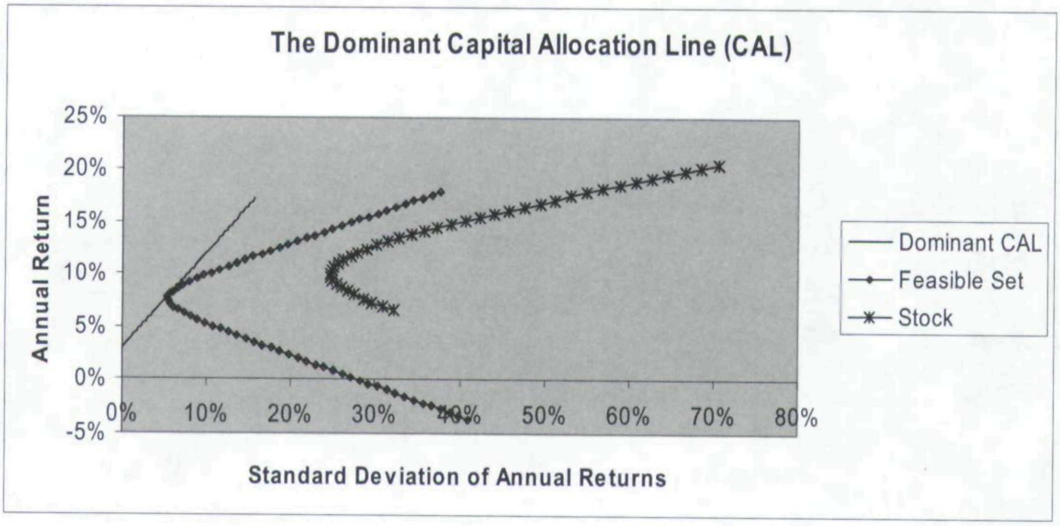
The classic CAPM equation for an expected security return of *k* is:

$$k = R_F + \beta * (\mu_{MKT} - R_F) \quad (12)$$

Notice, because Equations (11) and (12) both define *k*, the equations can be set equal to each other and reduced to find that the weight in the market portfolio from Equation (11) is the CAPM beta from Equation (12) (i.e., $\beta = W_{MKT}$) for the given expected return.

Because the CAPM beta is equivalent to W_{MKT} for a portfolio on the capital market line, the beta can be

Exhibit 4. The Feasible Set for a Portfolio of Two Index Funds with the Dominant Capital Allocation Line (Graph: Return vs. Standard Deviation)



Applicable Data

Weight- A (VFINX)	Weight - B (VBMFX)	Return (Portfolio)	Std. Dev. (Portfolio)	Weight- A (VFINX)	Weight - B (VBMFX)	Return (Portfolio)	Std. Dev. (Portfolio)
-30%	130%	0.0566	0.0888	75%	25%	0.1196	0.1655
-25%	125%	0.0596	0.0804	80%	20%	0.1226	0.1758
-20%	120%	0.0626	0.0727	85%	15%	0.1256	0.1862
-15%	115%	0.0656	0.0658	90%	10%	0.1287	0.1966
-10%	110%	0.0686	0.0601	95%	5%	0.1317	0.2070
-5%	105%	0.0716	0.0560	100%	0%	0.1347	0.2175
0%	100%	0.0746	0.0537	105%	-5%	0.1377	0.2280
5%	95%	0.0776	0.0536	110%	-10%	0.1407	0.2386
10%	90%	0.0806	0.0557	115%	-15%	0.1437	0.2492
15%	85%	0.0836	0.0597	120%	-20%	0.1467	0.2598
20%	80%	0.0866	0.0652	125%	-25%	0.1497	0.2704
25%	75%	0.0896	0.0720	130%	-30%	0.1527	0.2810
30%	70%	0.0926	0.0797	135%	-35%	0.1557	0.2917
35%	65%	0.0956	0.0880	140%	-40%	0.1587	0.3023
40%	60%	0.0986	0.0968	145%	-45%	0.1617	0.3130
45%	55%	0.1016	0.1061	150%	-50%	0.1647	0.3237
50%	50%	0.1046	0.1155	155%	-55%	0.1677	0.3344
55%	45%	0.1076	0.1253	160%	-60%	0.1707	0.3451
60%	40%	0.1106	0.1351	165%	-65%	0.1737	0.3558
65%	35%	0.1136	0.1452	170%	-70%	0.1767	0.3665
70%	30%	0.1166	0.1553	175%	-75%	0.1797	0.3772

"Stock" refers to the efficient frontier generated with a portfolio of Pepsico and McDonald's stocks. All other graphs refer to portfolios associated with the bond and stock index funds.

treated like a portfolio weight in a portfolio variance/standard deviation calculation. An individual security that generates an expected return of *k* has an associated β. The variance or risk for the security is at a minimum equal to $\beta^2 * \sigma^2_{MKT}$ (or $W^2_{MKT-k} * \sigma^2_{MKT}$; this would be the variance calculation associated with a capital market line portfolio that has a return of *k*), because the equivalent portfolio on the capital market line has, by design, the minimum possible level of risk. Generally, the variance for the security is higher than $\beta^2 * \sigma^2_{MKT}$ because an individual security is not a diversified portfolio. The variance in excess of $\beta^2 * \sigma^2_{MKT}$ (or the standard deviation in excess of $\beta * \sigma_{MKT}$) is "diversifiable" or "unsystematic" risk. This portion of the security risk is not compensated by additional return. Correspondingly, the "undiversifiable" risk or "systematic" risk is the portion of security risk that is compensated by additional return on the security and is measured by $\beta^2 * \sigma_{MKT}^2$ or $\beta * \sigma_{MKT}$.

Further, only when considering systematic risk, the single measure of risk that is of consequence is β. Because systematic risk is measured as $\beta^2 * \sigma_{MKT}^2$ or $\beta * \sigma_{MKT}$, the market variance (market standard deviation) is a common factor for all securities, making the individual β for the security the only distinguishing characteristic between securities. This is logical when considering that increasing β is the same as increasing W_{MKT} on the capital market line (i.e., the investor is accepting more risk because he or she is increasing the investment in the only source of risk available on the capital market line: the market portfolio). Consequently, the assessment of systematic risk is reduced to a simple consideration of a security's β rather than the security's variance or standard deviation. The latter two measures are measures of total risk, which equals a combination of unsystematic and systematic risk.

From this example, the ability to segue beyond the topics discussed above holds many possibilities. The mathematics for larger portfolios are not as convenient as computing *F* and *G* and require matrix algebra (see Arnold, 2002), but the intuition is still the same as in this framework.

III. Conclusion

This article produces a simple framework for constructing the efficient frontier and the capital allocation line associated with a portfolio of two risky securities. An instructor can avoid the mathematics associated with minimizing risk and produce the portfolio weights for the minimum variance/standard deviation portfolio and the tangency portfolio quite easily by using *F* and *G*. The result is a presentation in which the risk characteristics of the efficient frontier and the capital allocation line can be directly compared through calculation rather than with a graph, heuristically.

The presentation also lends itself to a discussion of the Capital Asset Pricing Model (CAPM) and can be extended in many directions—including a discussion of systematic and unsystematic risk. Consequently, the mathematics are simple enough for the undergraduate classroom, yet the intuition provided is at a level worthy of the graduate classroom.■

Appendix

In this appendix, we demonstrate that minimizing the portfolio variance with two additional constraints yields the security weights for the two risky securities that comprise the tangency portfolio. The additional constraints are equations that equal zero when the given constraint is satisfied. The specifics of the constraints are that the portfolio weights need to equal one and the portfolio mean is equal to a particular expected return, *k*.

$$1 - W_R - W_A - W_B = 0 \qquad \text{(A.1-A.2)}$$

$$k - W_R * R_F - W_A * \mu_A - W_B * \mu_B = (k - R_F) - W_A * (\mu_A - R_F) - W_B * (\mu_B - R_F) = 0$$

The variance minimization technique introduces the constraints into the portfolio variance calculation with Lagrange multipliers (symbolized by Greek letters multiplied by the given constraint).

$$\eta_P^2 = W_A^2 * \sigma_A^2 + W_B^2 * \sigma_B^2 + 2 * W_A * W_B * \rho_{AB} * \sigma_A * \sigma_B$$
$$+ \lambda * \left[(1 - W_R) - W_A - W_B\right] + \theta * \left[(k - R_F) - W_A * (\mu_A - R_F) - W_B * (\mu_B - R_F)\right] \qquad \text{(A.3)}$$

The first (partial) derivatives are taken with respect to W_A, W_B, λ, and θ, the derivatives are set to zero, and then the equations are solved for W_A and W_B.

$$\frac{\partial \eta_P^2}{\partial W_A} = 2 * W_A * \sigma_A^2 + 2 * W_B * \rho_{AB} * \sigma_A * \sigma_B - \lambda - \theta * (\mu_A - R_F) = 0 \qquad \text{(A.4)}$$

$$\frac{\partial \eta_P^2}{\partial W_B} = 2 * W_B * \sigma_B^2 + 2 * W_A * \rho_{AB} * \sigma_A * \sigma_B - \lambda - \theta * (\mu_B - R_F) = 0 \qquad \text{(A.5)}$$

$$\frac{\partial \eta_P^2}{\partial \lambda} = (1 - W_R) - W_A - W_B = 0 \qquad \text{(A.6)}$$

$$\frac{\partial \eta_P^2}{\partial \theta} = (k - R_F) - W_A * (\mu_A - R_F) - W_B * (\mu_B - R_F) = 0 \qquad \text{(A.7)}$$

$$W_{A-MIN} = \frac{(1 - W_R) * \left[\sigma_B^2 * (\mu_A - R_F) - \rho_{AB} * \sigma_A * \sigma_B * (\mu_B - R_F)\right]}{\left\{\begin{array}{l}\left[\sigma_B^2 * (\mu_A - R_F) - \rho_{AB} * \sigma_A * \sigma_B * (\mu_B - R_F)\right] \\ + \left[\sigma_A^2 * (\mu_B - R_F) - \rho_{AB} * \sigma_A * \sigma_B * (\mu_A - R_F)\right]\end{array}\right\}} \qquad \text{(A.8)}$$

Similar to the framework for finding the weights for the minimum variance portfolio for two risky assets, Equation (A.8) is simplified by defining a factor that we term *G*.

$$G = \frac{\sigma_A^2 * (\mu_B - R_F) - \rho_{AB} * \sigma_A * \sigma_B * (\mu_A - R_F)}{\sigma_B^2 * (\mu_A - R_F) - \rho_{AB} * \sigma_A * \sigma_B * (\mu_B - R_F)} \qquad \text{(A.9)}$$

$$\therefore W_{A-MIN} = (1 - W_R) * \left[\frac{1}{1+G}\right] \text{ and } W_{B-MIN} = (1 - W_R) * \left[1 - \frac{1}{1+G}\right] \qquad \text{(A.10)}$$

Although Equation (A.8) is rather complicated, Equations (A.9) and (A.10) greatly simplify the calculation. The calculation becomes even more simplified by defining the tangency portfolio weights of the capital allocation line.

$$W_{A-TAN} = \frac{1}{1+G}$$

and $W_{B-TAN} = 1 - W_{A-TAN}$ (A.11)

In order to provide further clarity, there is a need for a reinterpretation of the capital allocation line based on expected return rather than on risk. At the moment, the minimization of risk supplies a construct that is circular: given that W_R, W_A, and W_B emerge from Equation (A.8), the question remains as to how W_R is initially determined.

View the capital allocation line as a portfolio that consists of a risky asset (Security C with an expected return of μ_C) and a risk-free asset. For a given expected return (k), the mean of the portfolio is:

$$k = (1 - W_C) * R_F + W_C * \mu_C = R_F + W_C * (\mu_C - R_F) \quad \text{(A.12)}$$

where W_C is the proportion of the portfolio invested in Security C. Consequently, the weight of the portfolio to be invested in the risky security is driven by the desired expected return on the portfolio.

$$W_C = \frac{(k - R_F)}{(\mu_C - R_F)}$$

$$\therefore W_R = 1 - W_C = \frac{(\mu_C - k)}{(\mu_C - R_F)} \quad \text{(A.13)}$$

For the capital allocation line, the risky security (Security C in Equations (A.12) and (A.13)) is the tangency portfolio defined by the portfolio weights for Securities A and B in Equation (A.11). The expected return for the tangency portfolio is:

$$\mu_{TAN} = \left[\frac{1}{1+G}\right] * \mu_A + \left[\frac{G}{1+G}\right] * \mu_B \quad \text{(A.14)}$$

The variance of the tangency portfolio is:

$$\sigma_{TAN}^2 = \left[\frac{1}{1+G}\right]^2 * \sigma_A^2 + \left[\frac{G}{1+G}\right]^2 * \sigma_B^2 + 2 * \left[\frac{1}{1+G}\right] * \left[\frac{G}{1+G}\right] * \rho_{AB} * \sigma_A * \sigma_B \quad \text{(A.15)}$$

Putting it all together, a given expected return k on the capital allocation line consists of an investment in two portfolios (funds): the risk-free security and the tangency portfolio.

$$W_R = \frac{(\mu_{TAN} - k)}{(\mu_{TAN} - R_F)} \text{ and } W_{TAN} = \frac{(k - R_F)}{(\mu_{TAN} - R_F)} \quad \text{(A.16)}$$

The portfolio investment can then be expanded into an investment into component securities: the risk-free security and the component securities of the tangency portfolio (Securities A and B).

$$W_R = \frac{\left(\left[\frac{1}{1+G}\right] * \mu_A + \left[\frac{G}{1+G}\right] * \mu_B - k\right)}{\left(\left[\frac{1}{1+G}\right] * \mu_A + \left[\frac{G}{1+G}\right] * \mu_B - R_F\right)} \quad \text{(A.17)}$$

$$W_A = (1 - W_R) * \left[\frac{1}{1+G}\right] \quad \text{(A.18)}$$

$$W_B = (1 - W_R) * \left[\frac{G}{1+G}\right] \quad \text{(A.19)}$$

Although the latter interpretation is correct, the intuition becomes lost as all three portfolio weights change, adjusting to any given expected return on the capital allocation line. It becomes difficult to see that there are only two portfolios composing the capital allocation line. By defining the tangency portfolio first (which is easy by using G) and then viewing the capital allocation line as an investment in the risk-free security and the tangency portfolio using Equation (A.16), the intuition becomes clear. In fact, the portfolio variance for any portfolio with a particular expected return on the capital allocation line is simply:

$$W_{TAN}^2 * \sigma_{TAN}^2 = \left[\frac{k - R_F}{\mu_{TAN} - R_F}\right]^2 * \sigma_{TAN}^2 \quad \text{(A.20)}$$

This is very similar to the CAPM discussion toward the end of Section II.

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