A numerical study of fractional relaxation-oscillation equations involving ψ-Caputo fractional derivative

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Abstract

We provide a numerical method to solve a certain class of fractional differential equations involving ψ-Caputo fractional derivative. The considered class includes as particular case fractional relaxation-oscillation equations. Our approach is based on operational matrix of fractional integration of a new type of orthogonal polynomials. More precisely, we introduce ψ-shifted Legendre polynomial basis, and we derive an explicit formula for the ψ-fractional integral of ψ-shifted Legendre polynomials. Next, via an orthogonal projection on this polynomial basis, the problem is reduced to an algebraic equation that can be easily solved. The convergence of the method is justified rigorously and confirmed by some numerical experiments.

2010 Mathematics Subject Classification. 34A08; 34K28; 42C10.
Key words and phrases: ψ-Caputo fractional derivative; ψ-shifted Legendre polynomials; fractional relaxation-oscillation equation; convergence.

1 Introduction

A relaxation oscillator is an oscillator based upon the performance of a physical system's resending to equilibrium after being disturbed. The relaxation-oscillation equation is the primary equation of relaxation and oscillation processes. The standard form of a relaxation equation is given by

$$y'(t) + \lambda y(t) = f(t), \tag{1}$$

where $\lambda \in \mathbb{R}$ is a constant and f is a given function. Eq.(1) models several physical phenomena, such as the Maxwell model, which describes the behavior of a viscoelastic material using a spring and a dashpot in series. In this case, $\lambda = \frac{E}{\eta}$, where E is the elastic modulus, η is the viscosity coefficient, and $f(t)$ denotes E multiplying the strain rate. The standard oscillation equation describes a simple physical process with a controlled phase shift. In the simplest linear case, the equation describes oscillation y of a system responding to an external forcing f:

$$y''(t) + \lambda y(t) = f(t), \tag{2}$$

where $\lambda = \omega_0^2$ with ω_0 is the natural (resonant) frequency of the oscillator. Note that in (2), it is supposed that the damping coefficient is zero.

In order to represent slow relaxation and damped oscillation, fractional derivatives are employed in the relaxation and oscillation models (1) and (2) (see [19, 20]). Fractional relaxation-oscillation equation has the following form

$$D^\alpha y(t) + \lambda y(t) = f(t), \quad t > 0,\ \alpha \in (0,2) \setminus \{1\}, \tag{3}$$

under the initial conditions

$$y(0) = y_0, \quad \text{if} \quad 0 < \alpha < 1 \tag{4}$$

or

$$y(0) = y_0,\ y'(0) = y_1, \quad \text{if} \quad 1 < \alpha < 2, \tag{5}$$

where D^α is a certain fractional derivative operator of order α. In the case $0 < \alpha < 1$, (3)-(4) describe the relaxation with the power law attenuation. In the case $1 < \alpha < 2$, (3)-(5) represent the damped oscillation with viscoelastic intrinsic damping of oscillator (see [5, 26]).

Recently, the numerical study of fractional relaxation-oscillation equations has attracted much attention. In [5], the authors studied the numerical solution of (3) (with $f(t) = 0$) by considering positive fractional derivative and fractal derivative. In [11], the authors used a Taylor matrix method in order to obtain the numerical solution of (3) by considering Caputo fractional derivative. This method is based on a fractional version of Taylor's formula established in [21]. In [12], the numerical solution of (3) in which the fractional derivative is given in the Caputo sense, is obtained by the optimal homotopy asymptotic method. In [7], the numerical solution of (3) with Caputo fractional derivative, is computed using a trapezoidal approximation of the fractional integral. In [25], a generalized wavelet collocation operational matrix method based on Haar wavelets is proposed to solve (3) in which the fractional derivative is given in the Caputo sense.

In this paper, motivated by the above cited works, we are concerned with the numerical solution of the fractional differential equation

$$D_a^{\alpha,\psi} y(t) + \lambda y(t) = f(t), \quad a < t < b, \tag{6}$$

under the initial conditions

$$(\delta_\psi)^i y(a) = y_i, \quad i = 0, 1, \ldots, m-1, \tag{7}$$

where $\lambda \in \mathbb{R}$ is a constant, $\max\left\{m-1, \frac{1}{2}\right\} < \alpha < m$, $m \in \mathbb{N}$, $\psi : I = [a, b] \to [0, 1]$ is an increasing function that belongs to $C^m(I)$ satisfying $\psi'(t) > 0$, $t \in I$, and $\psi(I) = [0, 1]$, $f : I \to \mathbb{R}$ is a given function, $D_a^{\alpha,\psi}$ is the ψ-Caputo fractional derivative of order α (see Definition 2.3), and for $i = 0, 1, \ldots, m-1$,

$$(\delta_\psi)^i y(t) = \begin{cases} y(t), & \text{if} \quad i = 0, \\ \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^i y(t), & \text{if} \quad i = 1, 2, \ldots, m-1. \end{cases}$$

Note that in the case $\psi(t) = t$, (6)-(7) includes as particular cases (3)-(4) for $m = 1$, and (3)-(5) for $m = 2$, where D^α is the Caputo fractional derivative of order α. Our approach is based on operational matrix of fractional integration of a new type of orthogonal polynomials. More precisely, we introduce ψ-shifted Legendre polynomial basis, and we derive an explicit formula for the ψ-fractional integral of ψ-shifted Legendre polynomials. Next, by projecting the problem on this polynomial basis, we obtain an algebraic equation that can be easily solved. We present a rigorous proof of the convergence of the proposed method. Moreover, some numerical experiments are given to show the convergence of such procedure, comparing the exact solution with numerical approximation.

The operational matrix of integer integration has been used with different types of orthogonal polynomials such as Chebyshev polynomials [22], Legendre polynomials [23], Laguerre and Hermite polynomials [8], etc. Next, it was extended to the fractional case by many authors, see for example [3, 4, 9, 10, 13, 15, 24], and the references therein. Note that in all these cited works, only Caputo or Riemann-Liouville fractional derivatives were considered.

The paper is organized as follows. In Section 2, we present some preliminaries on fractional calculus and Hilbertian analysis, and we fix some notations. In Section 3, we introduce ψ-shifted Legendre polynomials, and study their properties. In Section 4, an explicit formula for the ψ-fractional integral of ψ-shifted Legendre polynomials is derived. The numerical scheme and the convergence of the method are discussed in Section 5. Finally, in Section 6, some numerical experiments are presented to demonstrate the efficiency of the proposed approach.

2 Preliminaries

Let $I=[a,b]$, $(a,b)\in\mathbb{R}^2$, $a<b$, and $\psi\in C^1(I;[0,1])$ be an increasing function such that $\psi'(t)>0$, $t\in I$ and $\psi(I)=[0,1]$.

First, let us define some functional spaces that will be used later.

Given a finite interval $J\subset\mathbb{R}$, let

$$L^2(J;\mathbb{R})=\left\{f:J\to\mathbb{R}:\ f \text{ is measurable } \& \int_J |f(t)|^2\,dt<\infty\right\}$$

be the Hilbert space with respect to the scalar product

$$(f,g)_{L^2(J)}=\int_J f(t)g(t)\,dt,\quad f,g\in L^2(J;\mathbb{R}).$$

We denote by $\|\cdot\|_{L^2(J)}$ the norm in $L^2(J;\mathbb{R})$ induced by the scalar product $(\cdot,\cdot)_{L^2(J)}$, i.e.,

$$\|f\|_{L^2(J)}=\sqrt{(f,f)_{L^2(J)}}=\left(\int_J |f(t)|^2\,dt\right)^{\frac{1}{2}},\quad f\in L^2(J;\mathbb{R}).$$

Let $L^1_\psi(I;\mathbb{R})$ be the Banach space defined by

$$L^1_\psi(I;\mathbb{R})=\left\{f:I\to\mathbb{R}:\ f \text{ is measurable } \& \int_I |f(t)|\psi'(t)\,dt<\infty\right\}$$

with respect to the norm

$$\|f\|_{L^1_\psi(I)}=\int_I |f(t)|\psi'(t)\,dt,\quad f\in L^1_\psi(I;\mathbb{R}).$$

Let $L^2_\psi(I;\mathbb{R})$ be the functional space defined by

$$L^2_\psi(I;\mathbb{R})=\left\{f:I\to\mathbb{R}:\ f \text{ is measurable } \& \int_I |f(t)|^2\psi'(t)\,dt<\infty\right\}.$$

It can be easily seen that $L^2_\psi(I;\mathbb{R})$ is a Hilbert space with respect to the scalar product

$$(f,g)_{L^2_\psi(I)}=\int_I f(t)g(t)\psi'(t)\,dt,\quad f,g\in L^2_\psi(I;\mathbb{R}).$$

We denote by $\|\cdot\|_{L^2_\psi(I)}$ the norm in $L^2_\psi(I;\mathbb{R})$ induced by the scalar product $(\cdot,\cdot)_{L^2_\psi(I)}$, i.e.,

$$\|f\|_{L^2_\psi(I)} = \sqrt{(f,f)_{L^2_\psi(I)}} = \left(\int_I |f(t)|^2\psi'(t)\,dt\right)^{\frac{1}{2}}, \quad f \in L^2_\psi(I;\mathbb{R}).$$

Further, given a function $f : I \to \mathbb{R}$, we define the function $\widetilde{f} : [0,1] \to \mathbb{R}$ by

$$\widetilde{f}(s) = f\left(\psi^{-1}(s)\right), \quad 0 \le s \le 1.$$

Lemma 2.1. *Let $f \in L^2_\psi(I;\mathbb{R})$. Then $\widetilde{f} \in L^2([0,1];\mathbb{R})$. Moreover, we have*

$$(f,g)_{L^2_\psi(I)} = (\widetilde{f},\widetilde{g})_{L^2([0,1])}, \quad f,g \in L^2_\psi(I;\mathbb{R}).$$

Proof. Let $f \in L^2_\psi(I;\mathbb{R})$. Then

$$\|f\|^2_{L^2_\psi(I)} = \int_a^b |f(t)|^2\psi'(t)\,dt.$$

Using the change of variable $s = \psi(t)$, we obtain

$$\begin{aligned}\int_a^b |f(t)|^2\psi'(t)\,dt &= \int_0^1 \left|f\left(\psi^{-1}(s)\right)\right|^2 ds \\ &= \int_0^1 |\widetilde{f}(s)|^2\,ds \\ &= \|\widetilde{f}\|^2_{L^2([0,1])}.\end{aligned}$$

Hence, we have

$$\|f\|_{L^2_\psi(I)} = \|\widetilde{f}\|_{L^2([0,1])},$$

which yields $\widetilde{f} \in L^2([0,1];\mathbb{R})$. Next, let $f,g \in L^2_\psi(I;\mathbb{R})$. We have

$$(f,g)_{L^2_\psi(I)} = \int_a^b f(t)g(t)\psi'(t)\,dt.$$

Again, using the change of variable $s = \psi(t)$, we obtain

$$\begin{aligned}(f,g)_{L^2_\psi(I)} &= \int_0^1 f\left(\psi^{-1}(s)\right)g\left(\psi^{-1}(s)\right)\,ds \\ &= \int_0^1 \widetilde{f}(s)\widetilde{g}(s)\,ds \\ &= (\widetilde{f},\widetilde{g})_{L^2([0,1])}.\end{aligned}$$

□

Definition 2.1 (see [16]). *Let $f \in L^1_\psi(I;\mathbb{R})$. The ψ-fractional integral of order $\alpha > 0$ of the function f is given by*

$$(I_a^{\alpha,\psi}f)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(x)(\psi(t)-\psi(x))^{\alpha-1}f(x)\,dx,$$

where Γ is the Gamma function.

Note that in the particular case $\psi(t)=t$, $I_a^{\alpha,\psi}$ reduces to Riemann-Liouville fractional integral of order α. In the case $\psi(t)=\ln t$ $(a>0)$, $I_a^{\alpha,\psi}$ reduces to Hadamard fractional integral of order α (see [16] for more details).

Lemma 2.2. *Let $\alpha>\frac{1}{2}$. Then $I_a^{\alpha,\psi}: L^2_\psi(I;\mathbb{R})\to L^2_\psi(I;\mathbb{R})$ is a linear and continuous operator. Moreover, we have*

$$\|I_a^{\alpha,\psi}f\|_{L^2_\psi(I)}\leq \frac{1}{\Gamma(\alpha)\sqrt{2\alpha-1}}\|f\|_{L^2_\psi(I)},\quad f\in L^2_\psi(I;\mathbb{R}).$$

Proof. Let $f\in L^2_\psi(I;\mathbb{R})$. We have

$$\begin{aligned}(I_a^{\alpha,\psi}f)(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(x)(\psi(t)-\psi(x))^{\alpha-1}f(x)\,dx\\ &= \frac{1}{\Gamma(\alpha)}\int_a^t \sqrt{\psi'(x)}(\psi(t)-\psi(x))^{\alpha-1}\sqrt{\psi'(x)}f(x)\,dx,\quad t\in I.\end{aligned}$$

Using Hölder's inequality, for all $t\in I$, we have

$$\begin{aligned}\left|(I_a^{\alpha,\psi}f)(t)\right|^2 &\leq \left(\frac{1}{\Gamma(\alpha)}\right)^2\left(\int_a^t \psi'(x)(\psi(t)-\psi(x))^{2\alpha-2}\,dx\right)\left(\int_a^t |f(x)|^2\psi'(x)\,dx\right)\\ &\leq \left(\frac{1}{\Gamma(\alpha)}\right)^2\left(\int_a^t \psi'(x)(\psi(t)-\psi(x))^{2\alpha-2}\,dx\right)\left(\int_a^b |f(x)|^2\psi'(x)\,dx\right)\\ &= \left(\frac{1}{\Gamma(\alpha)}\right)^2\frac{1}{2\alpha-1}\left(\psi(t)\right)^{2\alpha-1}\|f\|^2_{L^2_\psi(I)}\\ &\leq \left(\frac{1}{\Gamma(\alpha)}\right)^2\frac{1}{2\alpha-1}\|f\|^2_{L^2_\psi(I)}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\|I_a^{\alpha,\psi}f\|^2_{L^2_\psi(I)} &= \int_a^b \left|(I_a^{\alpha,\psi}f)(t)\right|^2\psi'(t)\,dt\\ &\leq \left(\frac{1}{\Gamma(\alpha)}\right)^2\frac{1}{2\alpha-1}\|f\|^2_{L^2_\psi(I)}\int_a^b \psi'(t)\,dt\\ &= \left(\frac{1}{\Gamma(\alpha)}\right)^2\frac{1}{2\alpha-1}\|f\|^2_{L^2_\psi(I)},\end{aligned}$$

which yields the desired result. □

Definition 2.1 can be extended to a vector function as follows.

Definition 2.2. *Let $F: I\to\mathbb{R}^N$, $N\in\mathbb{N}$, be a vector function given by*

$$F(t)=(F_1(t),F_2(t),\cdots,F_N(t))^T,\quad t\in I.$$

Suppose that $F_i\in L^1_\psi(I;\mathbb{R})$, $i=1,2,\cdots,N$. The ψ-fractional integral of order $\alpha>0$ of the vector function F is given by

$$(I_a^{\alpha,\psi}F)(t)=\left((I_a^{\alpha,\psi}F_1)(t),(I_a^{\alpha,\psi}F_2)(t),\cdots,(I_a^{\alpha,\psi}F_N)(t)\right)^T.$$

The ψ-fractional integral operator has the following properties (see [16]):

$$I_a^{\alpha,\psi}(\psi(t))^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}(\psi(t))^{\alpha+\gamma}, \quad \alpha > 0,\, \gamma > -1 \tag{8}$$

and

$$I_a^{\alpha,\psi} I_a^{\beta,\psi} f(t) = I_a^{\beta,\psi} I_a^{\alpha,\psi} f(t) = I_a^{\alpha+\beta,\psi} f(t), \quad \alpha, \beta > 0.$$

For $n \in \mathbb{N} \cup \{0\}$, we define the differential operator $(\delta_\psi)^n$ by

$$(\delta_\psi)^n f(t) = \begin{cases} f(t) & \text{if} \quad n = 0, \\ \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n f(t) & \text{if} \quad n \in \mathbb{N}. \end{cases}$$

Recently, Almeida et al [1, 2] introduced the concept of ψ-Caputo fractional derivative as follows.

Definition 2.3. *Let $n-1 < \alpha < n$, $n \in \mathbb{N}$, $\psi \in C^n(I;\mathbb{R})$ be an increasing function such that $\psi'(t) > 0$, $t \in I$ and $\psi(I) = [0,1]$, and $f \in C^{n-1}(I;\mathbb{R})$. The ψ-Caputo fractional derivative of order α of f is given by*

$$(D_a^{\alpha,\psi} f)(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_a^{n-\alpha,\psi}\left[f(t) - \sum_{i=0}^{n-1}\frac{(\delta_\psi)^i f(a)}{i!}(\psi(t))^i\right]. \tag{9}$$

For some special cases of ψ, we obtain from (9) different Caputo-type fractional derivatives. For example, if $\psi(t) = t$, then (9) reduces to the Caputo fractional derivative of order α (see [16]). In the case $\psi(t) = \ln t$, (9) reduces to the Caputo-Hadamard fractional derivative (see [14]). When function f is of class C^n, the ψ-Caputo fractional derivative of f can be represented by the expression (cf. [1, Theorem 3])

$$(D_a^{\alpha,\psi} f)(t) = I_a^{n-\alpha,\psi}(\delta_\psi)^n f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1}(\delta_\psi)^n f(s)\, ds.$$

For example, (see [1])

$$D_a^{\alpha,\psi}(\psi(t))^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}(\psi(t))^{\gamma-\alpha}, \quad \alpha > 0,\, \gamma > n-1.$$

Lemma 2.3 (see [2])**.** *Let $n-1 < \alpha < n$, $n \in \mathbb{N}$. Then:*

1. *If $f \in C(I;\mathbb{R})$, then*
$$D_a^{\alpha,\psi} I_a^{\alpha,\psi} f(t) = f(t).$$

2. *If $f \in C^{n-1}(I;\mathbb{R})$, then*
$$I_a^{\alpha,\psi} D_a^{\alpha,\psi} f(t) = f(t) - \sum_{i=0}^{n-1}\frac{(\delta_\psi)^i f(a)}{i!}(\psi(t))^i. \tag{10}$$

Further, let H be a Hilbert space with respect to a certain scalar product $(\cdot,\cdot)_H$. We denote by $\|\cdot\|_H$ the norm on H induced by $(\cdot,\cdot)_H$. Let $(e_i)_{i\in\mathbb{N}\cup\{0\}}$ be a Hilbertian basis of H. For $K\in\mathbb{N}$, we denote by

$$\Pi_K : H \to \text{span}\{e_i : i = 0, 1, \cdots, K-1\}$$

the orthogonal projection operator on $\text{span}\{e_i : i = 0, 1, \cdots, K-1\}$ defined by

$$\Pi_K(x) = \sum_{i=0}^{K-1} (x, e_i)_H e_i, \quad x \in H.$$

We recall the following standard result from functional analysis (see, for example [6]).

Lemma 2.4. *For $K\in\mathbb{N}$, Π_K is a linear and continuous operator on H, satisfying*

$$\|\Pi_K x\|_H \le \|x\|_H, \quad x \in H.$$

Moreover, we have

$$\lim_{K\to\infty} \|x - \Pi_K(x)\|_H = 0.$$

Lemma 2.5. *Let $\{x_K\}\subset H$ be the sequence defined by*

$$x_K = \sum_{i=0}^{K-1} a_i e_i, \quad K \in \mathbb{N},$$

where $\{a_n\}\subset\mathbb{R}$ is a certain real sequence. Suppose that there exists $x\in H$ such that

$$\lim_{K\to\infty} \|x_K - x\|_H = 0.$$

Then

$$x_K = \Pi_K(x), \quad K \in \mathbb{N}.$$

Proof. For all $K\in\mathbb{N}$, we have

$$\begin{aligned} \|x - x_K\|_H^2 &= \|\Pi_K(x) - x_K\|_H^2 + \|x - \Pi_K(x)\|_H^2 \\ &= \sum_{i=0}^{K-1} \left((x, e_i)_H - a_i\right)^2 + \|x - \Pi_K(x)\|_H^2 . \end{aligned}$$

Passing to the limit as $K\to\infty$, we obtain

$$\sum_{i=0}^{\infty} \left((x, e_i)_H - a_i\right)^2 = 0,$$

which yields

$$(x, e_i)_H = a_i, \quad i \in \mathbb{N}.$$

Then

$$x_K = \sum_{i=0}^{K-1} (x, e_i)_H e_i = \Pi_K(x), \quad K \in \mathbb{N}.$$

$\square$

3 ψ-shifted Legendre polynomials

The analytic form of the normalized shifted Legendre polynomials $L_i(s)$ of degree $i \in \mathbb{N} \cup \{0\}$, defined on $[0,1]$, is given by (see, for example [18]):

$$L_i(s) = \sqrt{2i+1} \sum_{n=0}^{i} \frac{(-1)^{i-n}(i+n)!}{(i-n)!(n!)^2} s^n, \quad s \in [0,1].$$

Lemma 3.1 (see, for example [17])**.** *The set $\{L_i : i \in \mathbb{N} \cup \{0\}\}$ is a Hilbertian basis of the Hilbert space $L^2([0,1];\mathbb{R})$.*

We introduce ψ-shifted Legendre polynomials as follows.
The ψ-shifted Legendre polynomials $P_{i,\psi}(t)$ of degree $i \in \mathbb{N} \cup \{0\}$, defined on I, are given by

$$P_{i,\psi}(t) = L_i(\psi(t)), \quad t \in I,$$

that is,

$$P_{i,\psi}(t) = \sqrt{2i+1} \sum_{n=0}^{i} \frac{(-1)^{i-n}(i+n)!}{(i-n)!(n!)^2} (\psi(t))^n, \quad t \in I.$$

Lemma 3.2. *The set $\{P_{i,\psi} : i \in \mathbb{N} \cup \{0\}\}$ is a Hilbertian basis of the Hilbert space $L^2_\psi(I;\mathbb{R})$.*

Proof. First let us prove that $\{P_{i,\psi} : i \in \mathbb{N} \cup \{0\}\}$ is an orthonormal basis of $L^2_\psi(I;\mathbb{R})$. Let $i, j \in \mathbb{N} \cup \{0\}$. Using Lemma 2.1, we have

$$\begin{aligned}
(P_{i,\psi}, P_{j,\psi})_{L^2_\psi(I)} &= \left(\widetilde{P_{i,\psi}}, \widetilde{P_{j,\psi}}\right)_{L^2([0,1])} \\
&= \int_0^1 \widetilde{P_{i,\psi}}(s)\widetilde{P_{j,\psi}}(s)\,ds \\
&= \int_0^1 P_{i,\psi}\left(\psi^{-1}(s)\right) P_{j,\psi}\left(\psi^{-1}(s)\right)\,ds \\
&= \int_0^1 L_i(s)L_j(s)\,ds \\
&= (L_i, L_j)_{L^2([0,1])}.
\end{aligned}$$

Therefore, by Lemma 3.1, $\{P_{i,\psi} : i \in \mathbb{N} \cup \{0\}\}$ is an orthonormal basis of $L^2_\psi(I;\mathbb{R})$.

Next, we have to prove that for any $f \in L^2_\psi(I;\mathbb{R})$, we have

$$f = \sum_{i=0}^{\infty} (f, P_{i,\psi})_{L^2_\psi(I)} P_{i,\psi}. \tag{11}$$

Let $f \in L^2_\psi(I;\mathbb{R})$. Then, by Lemma 2.1, we know that $\widetilde{f} \in L^2([0,1];\mathbb{R})$. Hence, by Lemma 3.1, we have

$$\widetilde{f}(s) = \sum_{i=0}^{\infty} \left(\widetilde{f}, L_i\right)_{L^2([0,1])} L_i(s), \quad s \in [0,1].$$

Then, for all $t \in I$, we have

$$\begin{aligned} f(t) &= f\left(\psi^{-1}\left(\psi(t)\right)\right) \\ &= \widetilde{f}\left(\psi(t)\right) \\ &= \sum_{i=0}^{\infty} \left(\widetilde{f}, L_i\right)_{L^2([0,1])} L_i(\psi(t)) \\ &= \sum_{i=0}^{\infty} \left(\widetilde{f}, L_i\right)_{L^2([0,1])} P_{i,\psi}(t). \end{aligned}$$

Again, using Lemma 2.1, we obtain

$$\left(\widetilde{f}, L_i\right)_{L^2([0,1])} = \left(\widetilde{f}, \widetilde{P_{i,\psi}}\right)_{L^2([0,1])} = (f, P_{i,\psi})_{L^2_\psi(I)}, \quad i \in \mathbb{N} \cup \{0\}.$$

Therefore, we obtain

$$f(t) = \sum_{i=0}^{\infty} (f, P_{i,\psi})_{L^2_\psi(I)} P_{i,\psi}(t), \quad t \in I,$$

which proves (11). □

Further, given a function $f \in L^2_\psi(I;\mathbb{R})$ and $K \in \mathbb{N}$, let $\Pi_K(f)$ be the orthogonal projection of f on

$$\mathrm{span}\{P_{i,\psi} : i = 0, 1, \cdots, K-1\},$$

i.e.,

$$\Pi_K(f)(t) = \sum_{i=0}^{K-1} (f, P_{i,\psi})_{L^2_\psi(I)} P_{i,\psi}(t), \quad t \in I. \tag{12}$$

By Lemmas 2.4 and 3.2, we deduce immediately the following fact.

Lemma 3.3. *Let $f \in L^2_\psi(I;\mathbb{R})$. Then*

$$\lim_{K\to\infty} \|f - \Pi_K(f)\|_{L^2\psi(I)} = 0.$$

4 Operational matrices of integrations

Let $F : I \to \mathbb{R}^K$, $K \in \mathbb{N}$, be a vector function given by

$$F(t) = (F_0(t), F_1(t), \cdots, F_{K-1}(t))^T, \quad t \in I.$$

Suppose that $F \in L^2_\psi(I;\mathbb{R}^K)$, i.e., $F_i \in L^2_\psi(I;\mathbb{R})$, $i = 0, 1, \cdots, K-1$. We shall use the notation

$$(\mathbf{\Pi_K} F)(t) = ((\Pi_K F_0)(t), \Pi_K(F_1)(t), \cdots, \Pi_K(F_{K-1})(t))^T, \quad t \in I, \tag{13}$$

where Π_K is the orthogonal projection operator defined by (12).

We define the binary relation $\simeq$ on $L^2_\psi(I;\mathbb{R}^K)$ by

$$U \simeq V \iff V = \mathbf{\Pi_K} U, \quad U, V \in L^2_\psi(I;\mathbb{R}^K).$$

We note that $\simeq$ is not symmetric.

For $K \in \mathbb{N}$ (supposed to be large enough), let $\phi_{K,\psi} : I \to \mathbb{R}^K$ be the vector function defined by

$$\phi_{K,\psi}(t) = (P_{0,\psi}(t), P_{1,\psi}(t), \cdots, P_{K-1,\psi}(t))^T, \quad t \in I. \tag{14}$$

Lemma 4.1. *Let $\alpha > \frac{1}{2}$. Then*

$$I_a^{\alpha,\psi}\phi_{K,\psi} \in L^2_\psi(I;\mathbb{R}^K).$$

Proof. Let $i \in \{0, 1, \cdots, K-1\}$. Since $P_{i,\psi} \in L^2_\psi(I;\mathbb{R})$ and $\alpha > \frac{1}{2}$, by Lemma 2.2, we have

$$I_a^{\alpha,\psi}P_{i,\psi} \in L^2_\psi(I;\mathbb{R}),$$

which yields the desired result. □

For $\alpha > 0$ and $i, \tau \in \{0, 1, \ldots, K-1\}$, let

$$\Omega(\tau, i) = \sum_{n=0}^{\tau} \Delta_{\tau,n,\alpha} G(i, n),$$

where

$$\Delta_{\tau,n,\alpha} = \frac{(-1)^{\tau-n}(\tau+n)!\sqrt{2\tau+1}}{n!(\tau-n)!\Gamma(\alpha+n+1)}$$

and

$$G(i, n) = \sum_{l=0}^{i} \frac{(-1)^{i-l}(i+l)!\sqrt{2i+1}}{(i-l)!(l!)^2(\alpha+n+l+1)}.$$

Let $M^\alpha_{K\times K}$ be the square matrix of size K, given by

$$M^\alpha_{K\times K} = (M^\alpha_{i,j})_{1\le i,j\le K},$$

where

$$M^\alpha_{i,j} = \Omega(i-1, j-1), \quad 1 \le i, j \le K.$$

We have the following result.

Theorem 4.1. *Let $\alpha > \frac{1}{2}$. Then*

$$I_a^{\alpha,\psi}\phi_{K,\psi} \simeq M^\alpha_{K\times K}\phi_{K,\psi}.$$

Proof. First, it is obvious that $M^\alpha_{K\times K}\phi_{K,\psi} \in L^2_\psi(I;\mathbb{R}^K)$. On the other hand, by Lemma 4.1, we know that $I_a^{\alpha,\psi}\phi_{K,\psi} \in L^2_\psi(I;\mathbb{R}^K)$. Now, we have to prove that

$$\mathbf{\Pi_K}\left(I_a^{\alpha,\psi}\phi_{K,\psi}\right) = M^\alpha_{K\times K}\phi_{K,\psi}. \tag{15}$$

For $t \in I$, we have

$$\mathbf{\Pi_K}\left(I_a^{\alpha,\psi}\phi_{K,\psi}\right)(t) = \left(\Pi_K\left(I_a^{\alpha,\psi}P_{0,\psi}\right)(t), \Pi_K\left(I_a^{\alpha,\psi}P_{1,\psi}\right)(t), \cdots, \Pi_K\left(I_a^{\alpha,\psi}P_{K-1,\psi}\right)(t)\right)^T$$

and

$$M^\alpha_{K\times K}\phi_{K,\psi}(t)$$
$$= \left(\sum_{j=1}^{K}\Omega(0, j-1)P_{j-1,\psi}(t), \sum_{j=1}^{K}\Omega(1, j-1)P_{j-1,\psi}(t), \cdots, \sum_{j=1}^{K}\Omega(K-1, j-1)P_{j-1,\psi}(t)\right)^T.$$

Let $\tau \in \{0, 1, \cdots, K-1\}$ be fixed. We have

$$\Pi_K\left(I_a^{\alpha,\psi}P_{\tau,\psi}\right)(t) = \sum_{i=0}^{K-1}\left(I_a^{\alpha,\psi}P_{\tau,\psi}, P_{i,\psi}\right)_{L^2_\psi(I)} P_{i,\psi}(t). \tag{16}$$

On the other hand, for all $i = 0, 1, \cdots, K-1$, we have

$$\left(I_a^{\alpha,\psi} P_{\tau,\psi}, P_{i,\psi}\right)_{L^2_\psi(I)} = \sqrt{2\tau+1} \sum_{n=0}^{\tau} \frac{(-1)^{\tau-n}(\tau+n)!}{(\tau-n)!(n!)^2} \left(I_a^{\alpha,\psi}(\psi(t))^n, P_{i,\psi}\right)_{L^2_\psi(I)}.$$

Using the property (8), for all $n = 0, 1, \cdots, \tau$, we have

$$I_a^{\alpha,\psi}(\psi(t))^n = \frac{n!}{\Gamma(\alpha+n+1)}(\psi(t))^{\alpha+n}.$$

Therefore, we get

$$\left(I_a^{\alpha,\psi} P_{\tau,\psi}, P_{i,\psi}\right)_{L^2_\psi(I)} = \sum_{n=0}^{\tau} \frac{(-1)^{\tau-n}(\tau+n)!\sqrt{2\tau+1}}{(\tau-n)!n!\Gamma(\alpha+n+1)} \left(\psi^{\alpha+n}, P_{i,\psi}\right)_{L^2_\psi(I)},$$

i.e.,

$$\left(I_a^{\alpha,\psi} P_{\tau,\psi}, P_{i,\psi}\right)_{L^2_\psi(I)} = \sum_{n=0}^{\tau} \Delta_{\tau,n,\alpha} \left(\psi^{\alpha+n}, P_{i,\psi}\right)_{L^2_\psi(I)}.$$

By Lemma 2.1, for all $n = 0, 1, \cdots, \tau$, we have

$$\left(\psi^{\alpha+n}, P_{i,\psi}\right)_{L^2_\psi(I)} = \left(\widetilde{\psi^{\alpha+n}}, \widetilde{P_{i,\psi}}\right)_{L^2([0,1])},$$

which yields

$$\begin{aligned}
\left(\psi^{\alpha+n}, P_{i,\psi}\right)_{L^2_\psi(I)} &= \int_0^1 s^{\alpha+n} L_i(s)\, ds \\
&= \sqrt{2i+1} \sum_{l=0}^{i} \frac{(-1)^{i-l}(i+l)!}{(i-l)!(l!)^2} \int_0^1 s^{\alpha+n+l}\, ds \\
&= \sum_{l=0}^{i} \frac{(-1)^{i-l}(i+l)!\sqrt{2i+1}}{(i-l)!(l!)^2(\alpha+n+l+1)} \\
&= G(i,n).
\end{aligned}$$

Hence, we obtain

$$\left(I_a^{\alpha,\psi} P_{\tau,\psi}, P_{i,\psi}\right)_{L^2_\psi(I)} = \sum_{n=0}^{\tau} \Delta_{\tau,n,\alpha} G(i,n), \tag{17}$$

for all $i = 0, 1, \cdots, K-1$. Combining (16) with (17), we get

$$\begin{aligned}
\Pi_K \left(I_a^{\alpha,\psi} P_{\tau,\psi}\right)(t) &= \sum_{i=0}^{K-1} \left(\sum_{n=0}^{\tau} \Delta_{\tau,n,\alpha} G(i,n)\right) P_{i,\psi}(t) \\
&= \sum_{i=0}^{K-1} \Omega(\tau,i) P_{i,\psi}(t) \\
&= \sum_{j=1}^{K} \Omega(\tau,j-1) P_{j-1,\psi}(t) \\
&= \left(M^{\alpha}_{K\times K}\phi_{K,\psi}\right)_\tau (t)
\end{aligned}$$

Hence, we proved (15), and the desired result follows. □

5 Numerical scheme and convergence

Let us consider the fractional boundary value problem (6)-(7). We suppose that $f \in L^2_\psi(I;\mathbb{R})$ and (6)-(7) admits a unique solution $y \in L^2_\psi(I;\mathbb{R})$ (for existence and uniqueness of solution results, we suggest the reader the work [2]). Under the considered assumptions, from (6), we have $D^{\alpha,\psi}_a y \in L^2_\psi(I;\mathbb{R})$.

For $K \in \mathbb{N}$ (K is supposed to be large enough), we have

$$\Pi_K\left(D^{\alpha,\psi}_a y\right) = \sum_{i=0}^{K-1} \left(D^{\alpha,\psi}_a y, P_{i,\psi}\right)_{L^2_\psi(I)} P_{i,\psi}(t), \quad t \in I,$$

where Π_K is the orthogonal projection operator given by (12). Let $H_{K,\alpha} \in \mathbb{R}^K$ be the vector defined by

$$H_{K,\alpha} = \begin{pmatrix} \left(D^{\alpha,\psi}_a y, P_{0,\psi}\right)_{L^2_\psi(I)} \\ \left(D^{\alpha,\psi}_a y, P_{1,\psi}\right)_{L^2_\psi(I)} \\ \vdots \\ \left(D^{\alpha,\psi}_a y, P_{K-1,\psi}\right)_{L^2_\psi(I)} \end{pmatrix}.$$

Then, we have

$$\Pi_K\left(D^{\alpha,\psi}_a y\right) = H^T_{K,\alpha}\phi_{K,\psi}(t), \quad t \in I,$$

where $\phi_{K,\psi}$ is the vector function defined by (14). Using Lemmas 2.4 and 3.2, we deduce immediately the following convergence result.

Lemma 5.1. *We have*

$$\lim_{K\to\infty} \left\|H^T_{K,\alpha}\phi_{K,\psi} - D^{\alpha,\psi}_a y\right\|_{L^2_\psi(I)} = 0.$$

On the other hand, since $\alpha > \frac{1}{2}$, by Lemma 2.2, we know that

$$I^{\alpha,\psi}_a : L^2_\psi(I;\mathbb{R}) \to L^2_\psi(I;\mathbb{R})$$

is a linear and continuous operator. Therefore, by Lemma 5.1, we deduce the following result.

Lemma 5.2. *We have*

$$\lim_{K\to\infty} \left\|H^T_{K,\alpha} I^{\alpha,\psi}_a \phi_{K,\psi} - I^{\alpha,\psi}_a D^{\alpha,\psi}_a y\right\|_{L^2_\psi(I)} = 0.$$

Next, we shall prove the following convergence result.

Lemma 5.3. *We have*

$$\lim_{K\to\infty} \left\|H^T_{K,\alpha}\mathbf{\Pi_K}\left(I^{\alpha,\psi}_a\phi_{K,\psi}\right) - I^{\alpha,\psi}_a D^{\alpha,\psi}_a y\right\|_{L^2_\psi(I)} = 0,$$

where $\mathbf{\Pi_K}$ is the operator defined by (13).

Proof. We have

$$\begin{aligned}\left\|H^T_{K,\alpha}\mathbf{\Pi_K}\left(I^{\alpha,\psi}_a\phi_{K,\psi}\right) - I^{\alpha,\psi}_a D^{\alpha,\psi}_a y\right\|_{L^2_\psi(I)} \leq\ & \left\|H^T_{K,\alpha}\mathbf{\Pi_K}\left(I^{\alpha,\psi}_a\phi_{K,\psi}\right) - \Pi_K\left(I^{\alpha,\psi}_a D^{\alpha,\psi}_a y\right)\right\|_{L^2_\psi(I)} \\ & + \left\|I^{\alpha,\psi}_a D^{\alpha,\psi}_a y - \Pi_K\left(I^{\alpha,\psi}_a D^{\alpha,\psi}_a y\right)\right\|_{L^2_\psi(I)}.\end{aligned}$$

On the other hand, observe that

$$H_{K,\alpha}^T \mathbf{\Pi_K} \left(I_a^{\alpha,\psi} \phi_{K,\psi} \right) = \Pi_K \left(H_{K,\alpha}^T I_a^{\alpha,\psi} \phi_{K,\psi} \right).$$

Hence, we obtain

$$\begin{aligned} \left\| H_{K,\alpha}^T \mathbf{\Pi_K} \left(I_a^{\alpha,\psi} \phi_{K,\psi} \right) - I_a^{\alpha,\psi} D_a^{\alpha,\psi} y \right\|_{L^2_\psi(I)} \leq & \left\| \Pi_K \left(H_{K,\alpha}^T I_a^{\alpha,\psi} \phi_{K,\psi} \right) - \Pi_K \left(I_a^{\alpha,\psi} D_a^{\alpha,\psi} y \right) \right\|_{L^2_\psi(I)} \\ & + \left\| I_a^{\alpha,\psi} D_a^{\alpha,\psi} y - \Pi_K \left(I_a^{\alpha,\psi} D_a^{\alpha,\psi} y \right) \right\|_{L^2_\psi(I)}. \end{aligned} \tag{18}$$

Next, using Lemma 2.4, we obtain

$$\begin{aligned} \left\| \Pi_K \left(H_{K,\alpha}^T I_a^{\alpha,\psi} \phi_{K,\psi} \right) - \Pi_K \left(I_a^{\alpha,\psi} D_a^{\alpha,\psi} y \right) \right\|_{L^2_\psi(I)} = & \left\| \Pi_K \left(H_{K,\alpha}^T I_a^{\alpha,\psi} \phi_{K,\psi} - I_a^{\alpha,\psi} D_a^{\alpha,\psi} y \right) \right\|_{L^2_\psi(I)} \\ \leq & \left\| H_{K,\alpha}^T I_a^{\alpha,\psi} \phi_{K,\psi} - I_a^{\alpha,\psi} D_a^{\alpha,\psi} y \right\|_{L^2_\psi(I)}. \end{aligned}$$

Therefore, by Lemma 5.2, we deduce that

$$\lim_{K\to\infty} \left\| \Pi_K \left(H_{K,\alpha}^T I_a^{\alpha,\psi} \phi_{K,\psi} \right) - \Pi_K \left(I_a^{\alpha,\psi} D_a^{\alpha,\psi} y \right) \right\|_{L^2_\psi(I)} = 0. \tag{19}$$

Again, by Lemma 2.4, we have

$$\lim_{K\to\infty} \left\| I_a^{\alpha,\psi} D_a^{\alpha,\psi} y - \Pi_K \left(I_a^{\alpha,\psi} D_a^{\alpha,\psi} y \right) \right\|_{L^2_\psi(I)} = 0. \tag{20}$$

Finally, combining (18), (19) and (20), the desired result follows. □

Now, using Theorem 4.1 and Lemma 5.3, we deduce the following result.

Lemma 5.4. *We have*

$$\lim_{K\to\infty} \left\| H_{K,\alpha}^T M_{K\times K}^\alpha \phi_{K,\psi} - I_a^{\alpha,\psi} D_a^{\alpha,\psi} y \right\|_{L^2_\psi(I)} = 0.$$

Next, using the property (10) and the initial conditions (7), we obtain

$$I_a^{\alpha,\psi} D_a^{\alpha,\psi} y(t) = y(t) - \sum_{i=0}^{m-1} \frac{y_i}{i!} (\psi(t))^i. \tag{21}$$

Let $Z_K \in \mathbb{R}^K$ be the (known) vector satisfying

$$\Pi_K \left(\sum_{i=0}^{m-1} \frac{y_i}{i!} (\psi(t))^i \right) = Z_K^T \phi_{K,\psi}(t), \quad t \in I. \tag{22}$$

Let $\{y_K\} \subset L^2_\psi(I;\mathbb{R})$ be the sequence defined by

$$y_K(t) = \left(H_{K,\alpha}^T M_{K\times K}^\alpha + Z_K^T \right) \phi_{K,\psi}(t), \quad t \in I. \tag{23}$$

Theorem 5.1. *We have*

$$\lim_{K\to\infty} \|y_K - y\|_{L^2_\psi(I)} = 0.$$

Proof. We have

$$\|y_K - y\|_{L^2_\psi(I)} = \left\| \left(H^T_{K,\alpha} M^\alpha_{K\times K} + Z^T_K \right) \phi_{K,\psi} - y \right\|_{L^2_\psi(I)}.$$

Using (21) and (22), we obtain

$$\begin{aligned} \|y_K - y\|_{L^2_\psi(I)} &= \left\| \left(H^T_{K,\alpha} M^\alpha_{K\times K} + Z^T_K \right) \phi_{K,\psi} - I^{\alpha,\psi}_a D^{\alpha,\psi}_a y - \sum_{i=0}^{m-1} \frac{y_i}{i!} \psi^i \right\|_{L^2_\psi(I)} \\ &\le \left\| H^T_{K,\alpha} M^\alpha_{K\times K} \phi_{K,\psi} - I^{\alpha,\psi}_a D^{\alpha,\psi}_a y \right\|_{L^2_\psi(I)} + \left\| \Pi_K \left(\sum_{i=0}^{m-1} \frac{y_i}{i!} \psi^i \right) - \sum_{i=0}^{m-1} \frac{y_i}{i!} \psi^i \right\|_{L^2_\psi(I)}. \end{aligned}$$

Using Lemma 5.4 and passing to the limit as $K \to \infty$, we obtain the desired result. □

From Theorem 5.1, the solution y to (6)-(7) can be approximated by the sequence $\{y_K\}$ defined by (23). However, this sequence depends on the unknown vector $H_{K,\alpha} \in \mathbb{R}^K$. Therefore, this vector should be computed before using the approximation given by Theorem 5.1.

Lemma 5.5. *For all $K \in \mathbb{N}$, we have*

$$y_K = \Pi_K(y).$$

Proof. The result follows immediately by Lemma 2.5 and Theorem 5.1. □

Let $Q_K \in \mathbb{R}^K$ be the (known) vector satisfying

$$\Pi_K(f)(t) = Q^T_K \phi_{K,\psi}(t), \quad t \in I.$$

Using (6), for all $K \in \mathbb{N}$, we have

$$\Pi_K \left(D^{\alpha,\psi}_a y \right) + \lambda \Pi_K(y) = \Pi_K(f).$$

Hence, by Lemma 5.5, we obtain

$$H^T_{K,\alpha} + \lambda \left(H^T_{K,\alpha} M^\alpha_{K\times K} + Z^T_K \right) = Q^T_K, \quad K \in \mathbb{N},$$

that is,

$$\mathbf{H}^T_{K,\alpha} = \mathcal{B}_K \mathcal{A}^{-1}_K, \tag{24}$$

where

$$\mathcal{A}_K = I_{K\times K} + \lambda M^\alpha_{K\times K}, \; \mathcal{B}_K = Q^T_K - \lambda Z^T_K$$

and $I_{K\times K}$ denotes the identity matrix of size K.

Remark 5.1. *Note that it is supposed in* (24) *that the matrix $\mathcal{A}_K$ is invertible. If it is not the case, then one may increase, iteratively, the number of the ψ-shifted Legendre coefficients by one, until $\mathcal{A}_K$ becomes invertible.*

Finally, after solving (24), the desired solution can be approximated via the sequence $\{y_K\}$ given by (23).

6 Numerical results

In this section, we illustrate the proposed method with some numerical experiments.

Let us consider the fractional oscillator equation

$$D_a^{3/2,\psi} y(t) + \frac{2}{\Gamma(3/2)} y(t) = \frac{2}{\Gamma(3/2)} \sqrt{\psi(t)} \left(1 + (\psi(t))^{3/2}\right), \quad t \in I, \tag{25}$$

under the initial conditions

$$y(a) = y'(a) = 0, \tag{26}$$

where $I = [a, b]$ and $\psi : I \to [0, 1]$ is an increasing function that belongs to $C^2(I)$ such that $\psi'(t) > 0$, $t \in I$ and $\psi(I) = [0, 1]$. We remark that problem (25)-(26) has a unique solution (cf [2]). We denote by y^* the exact solution of (25)-(26). It can be easily seen that

$$y^*(t) = (\psi(t))^2, \quad t \in I.$$

The numerical solution of (25)-(26) is denoted by y. We denote by $E(t)$ the absolute error at the point $t \in I$, that is,

$$E(t) = |y^*(t) - y(t)|, \quad t \in I.$$

Different choices of the function ψ are considered in this example.

The case $\psi(t) = t$, $t \in I = [0, 1]$.

In this case, (25)-(26) reduces to

$${}^C D_0^{3/2} y(t) + \frac{2}{\Gamma(3/2)} y(t) = \frac{2}{\Gamma(3/2)} \sqrt{t} \left(1 + t^{3/2}\right), \quad 0 < t < 1, \tag{27}$$

under the initial conditions

$$y(0) = y'(0) = 0, \tag{28}$$

where ${}^C D_0^{3/2}$ is the standard Caputo fractional derivative of order $\alpha = 3/2$.

The approximate solution of (27)-(28) and the absolute error at different points $t \in [0, 1]$, in the case $K = 6$, are shown in Table 1.

The case $\psi(t) = \frac{t}{2}(t + 1)$, $t \in I = [0, 1]$

In this case, (25)-(26) reduces to

$$D_0^{3/2,\psi} y(t) + \frac{2}{\Gamma(3/2)} y(t) = \frac{2}{\Gamma(3/2)} \sqrt{\frac{t}{2}(t+1)} \left(1 + \left(\frac{t}{2}(t+1)\right)^{3/2}\right), \quad 0 < t < 1, \tag{29}$$

under the initial conditions (28).

The approximate solution of (29)-(28) and the absolute error at different points $t \in [0, 1]$, in the case $K = 6$, are shown in Table 2.

t	$y^*(t)$	$y(t)$	$E(t)$
0	0	0.003400	0.003400
0.1	0.01	0.009870	0.000130
0.2	0.04	0.039930	0.000070
0.3	0.09	0.090070	0.000070
0.4	0.16	0.016011	0.000011
0.5	0.25	0.250030	0.000030
0.6	0.36	0.359930	0.000070
0.7	0.49	0.489910	0.000090
0.8	0.64	0.640000	0
0.9	0.81	0.810090	0.000089
1	1	0.999860	0.000140

Table 1: Comparison of exact solution and numerical solution of (27)-(28) and their errors for $K = 6$

t	$y^*(t)$	$y(t)$	$E(t)$
0	0	0.0006	0.0006
0.1	0.0030	0.0037	0.0006
0.2	0.0144	0.0152	0.0008
0.3	0.0380	0.0389	0.0009
0.4	0.0784	0.0795	0.0011
0.5	0.1406	0.1418	0.0011
0.6	0.2304	0.2316	0.0012
0.7	0.3540	0.3551	0.0011
0.8	0.5184	0.5194	0.0010
0.9	0.7310	0.7319	0.0009
1	1	1.0007	0.0007

Table 2: Comparison of exact solution and numerical solution of (29)-(28) and their errors for $K = 6$

The case $\psi(t) = \ln((e-1)t+1)$, $t \in I = [0,1]$

In this case, (25)-(26) reduces to

$$D_0^{3/2,\psi} y(t) + \frac{2}{\Gamma(3/2)} y(t) = \frac{2}{\Gamma(3/2)} \sqrt{\ln((e-1)t+1)} \left(1 + (\ln((e-1)t+1))^{3/2}\right), \quad 0 < t < 1, \tag{30}$$

under the initial conditions (28).

The approximate solution of (30)-(28) and the absolute error at different points $t \in [0,1]$, in the case $K = 6$, are shown in Table 3.

The case $\psi(t) = \tan\left(\frac{\pi t}{4}\right)$, $t \in I = [0,1]$

In this case, (25)-(26) reduces to

$$D_0^{3/2,\psi} y(t) + \frac{2}{\Gamma(3/2)} y(t) = \frac{2}{\Gamma(3/2)} \sqrt{\tan\left(\frac{\pi t}{4}\right)} \left(1 + \left(\tan\left(\frac{\pi t}{4}\right)\right)^{3/2}\right), \quad 0 < t < 1, \tag{31}$$

t	$y^*(t)$	$y(t)$	$E(t)$
0	0	0.0004	0.0004
0.1	0.0251	0.0258	0.0007
0.2	0.0873	0.0882	0.0009
0.3	0.1728	0.1738	0.0010
0.4	0.2737	0.2746	0.0009
0.5	0.3845	0.3854	0.0009
0.6	0.5020	0.5028	0.0008
0.7	0.6237	0.6245	0.0008
0.8	0.7479	0.7487	0.0007
0.9	0.8737	0.8743	0.0006
1	1	1.0005	0.0005

Table 3: Comparison of exact solution and numerical solution of (30)-(28) and their errors for $K = 6$

under the initial conditions (28).

The approximate solution of (31)-(28) and the absolute error at different points $t \in [0, 1]$, in the case $K = 6$, are shown in Table 4.

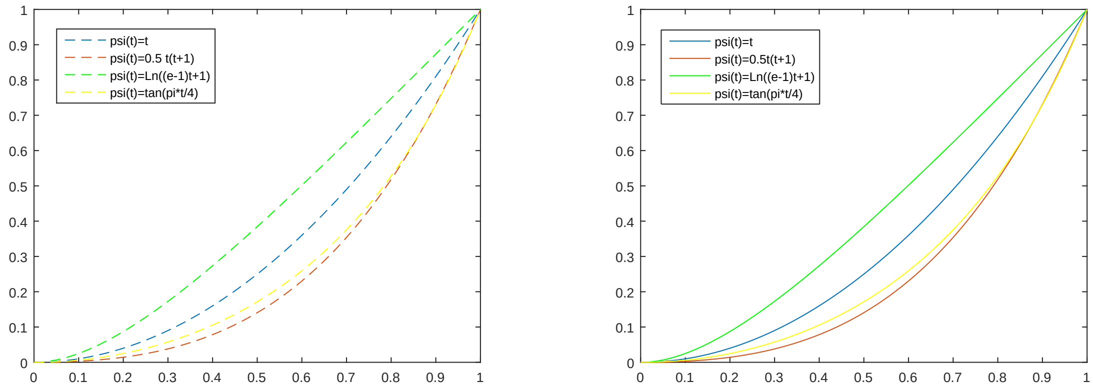
t	$y^*(t)$	$y(t)$	$E(t)$
0	0	0.0004	0.0004
0.1	0.0062	0.0067	0.0005
0.2	0.0251	0.0258	0.0007
0.3	0.0576	0.0585	0.0009
0.4	0.1056	0.1065	0.0009
0.5	0.1716	0.1725	0.0009
0.6	0.2596	0.2606	0.0010
0.7	0.3755	0.3764	0.0009
0.8	0.5279	0.5287	0.0008
0.9	0.7295	0.7302	0.0007
1	1	1.0005	0.0005

Table 4: Comparison of exact solution and numerical solution of (31)-(28) and their errors for $K = 6$

Comparison of exact solutions and numerical solutions of (25)-(26) for all the considered cases are shown in Figure 1, in the case $K = 6$.

7 Conclusion

A numerical approach based on operational matrix of fractional integration of a new type of orthogonal polynomials is introduced in this paper for solving a certain class of fractional differential equations involving ψ-Caputo fractional derivative. The convergence of the method is proved and the numerical experiments presented in Section 6 confirm the efficiency of this approach.



(a) Exact solutions for different choices of ψ (b) Approximate solutions for different choices of ψ

Figure 1: Comparison between exact and approximate solutions for different functions ψ, in the case $K = 6$

Acknowledgments

Ricardo Almeida was supported by Portuguese funds through the CIDMA - Center for Research and Development in Mathematics and Applications, and the Portuguese Foundation for Science and Technology (FCT-Fundação para a Ciência e a Tecnologia), within project UID/MAT/04106/2013. Bessem Samet extends his appreciation to the Deanship of Scientific Research at King Saud University for funding this work through research group No RGP–237.

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