An adjustable sample average approximation algorithm for the stochastic production-inventory-routing problem*†

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Abstract

We consider a stochastic single item production-inventory-routing problem with a single producer, multiple clients and multiple vehicles. At the clients, demand is allowed to be backlogged incurring a penalty cost. Demands are considered uncertain.

A recourse model is presented and valid inequalities are introduced to enhance the model.

A new general approach that explores the sample average approximation (SAA) method is introduced. In the sample average approximation method, several sample sets are generated and solved independently in order to obtain a set of candidate solutions. Then the candidate solutions are tested on a larger sample and the best solution is selected among the candidates. In contrast to this approach, called static, we propose an adjustable approach that explores the candidate solutions in order to identify common structures. Using that information, part of the first-stage decision variables are fixed and the resulting restricted problem is solved for a larger size sample. Several heuristic algorithms based on the mathematical model are considered within each approach.

Computational tests based on randomly generated instances are conducted to test several variants of the two approaches. The results show that the new adjustable SAA heuristic performs better than the static one for most of the instances.

Keywords: Inventory routing; Stochastic programming; Sample average approximation algorithm; Hybrid heuristic; Demand uncertainty; Iterated local search; Adaptive heuristic.

1 Introduction

We consider a single item stochastic production-inventory-routing (SPIR) problem with a single supplier/producer, multiple retailers/clients and multiple vehicles.

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A vendor managed inventory approach is followed where the supplier monitors the inventory at the retailers and decides on the replenishment policy for each retailer. Inventory aspects are considered at both the supplier and the retailers. Demand is allowed to be backlogged at the retailers and in that case a penalty cost is incurred. Backlog is not allowed at the supplier. Demands are considered uncertain, following a known probability distribution. A constant production capacity at the supplier is assumed. For the distribution plan, multiple vehicles are considered. The decision maker has to decide the production and the distribution plans for a finite time horizon. The production plan consists in defining the production periods and the amount to produce in each one of those periods. The distribution plan defines the retailers that should be visited in each time period, the quantities to deliver to each visited retailer, and the corresponding route for each used vehicle in each time period. Bounds on the delivered quantities and on client inventories are considered. The goal is to minimize the production and the routing cost plus the expected inventory and the penalty costs for backlogged demand.

We assume the production plan and the choice of which clients to visit in each time period (and consequently the routing) are first-stage decisions, that is, decisions that must be taken before the scenario is revealed. The quantities to deliver to each client in each time period as well as the inventory levels can be adjusted to the scenario (known as second-stage decisions). Such assumptions may hold for short planning horizons.

Complex problems combining production, inventory and routing decisions have received much attention in recent years [6, 9, 11, 14, 16, 17, 19, 20, 25, 29, 33, 38, 40]. For a survey on previous publications see [4]. Several such studies have been motivated by real applications [5, 7, 9, 29]. The advantages of coordination and integration of different supply chain decisions are reported by several authors. Hein and Almeder [25] study the benefits of integrated planning for a complex supply production network embedded in a dynamic discrete-type system, and conclude that substantial cost savings are possible with an integrated planning approach compared to a classical sequential approach. Darvish *et al.* [19] observe that making the right lot-sizing decisions becomes even more significant in interconnected logistic networks and show the benefits of an integrated approach to supply chain decision-making. Integrated approaches for a production-routing problem are also compared against uncoordinated ones in [37].

Most of the solution approaches for such complex problems use heuristics. Absi *et al.* [1] and Chitsaz *et al.* [17] propose two-phase iterative methods where the original inventory-routing problem is split into two subproblems, an inventory problem and a routing problem. Agra *et al.* [6] also propose the decomposition of the production-inventory-routing problem into a production-inventory subproblem and a routing subproblem using a new Lagrangian decomposition enhanced with valid inequalities. Chitsaz *et al.* [16] formulate the problem as a MIP and propose a three-phase decomposition matheuristic that relies on the iterative solution of different subproblems. Brahimi and Aouam [14] propose a hybrid heuristic in which a relax-and-fix strategy is combined with a local search strategy. Qiu *et al.* [33] develop a branch-and-price heuristic by incorporating a column generation formulation based on the Dantzig-Wolfe decomposition. Zhang *et al.* [44] present an iterative MILP based heuristic that uses restricted sets of routes that are dynamically updated. Russell [38] and Solyalı and Süral [40] propose heuristics where routes are computed in advance and, in a second-stage, the predetermined routes are used to simplify the MIP models.

Although, in general, instances from such complex problems are not solved to optimality, exact approaches such as a branch-and-cut [2] and branch-and-cut-and-price algorithms [20, 33] have also been considered. Small benchmark instances can be solved to optimality, see for instance, Avella *et al.* [12].

For several related problems, where no backlog is allowed, extended formulations [2, 4, 12, 24, 37] and valid inequalities [11, 37] have been considered.

Uncertainty has also been considered for related problems. Solyalı *et al.* [39] introduce a robust optimization approach for an inventory routing problem with uncertain demands. Aghezzaf [5] considers a variant of the inventory routing optimization problem where customer demand rates and travel times are stochastic but stationary. Stochastic approaches for related inventory routing problems have been also considered [3, 7, 8, 13, 34]. Rahim *et al.* [34] propose a deterministic equivalent approximation model to the original stochastic problem. Then, using Lagrangian relaxation, the deterministic model is decomposed into an inventory allocation subproblem and a vehicle routing subproblem. Bertazzi *et al.* [13] propose a hybrid roll-out algorithm for an inventory-routing problem where an order-up-to level policy is followed for each retailer.

A classical approach for handling stochastic problems is the sample average approximation (SAA) method [28, 43]. In this method the expected value of the objective function is approximated by a sample average estimate obtained from a random sample. The resulting SAA problem is solved for different samples in order to obtain a set of candidate solutions. Then, these candidate solutions are tested on a larger sample and the best solution for that sample is chosen. Within related inventory-routing problems the SAA was used in [3, 7, 8]. In a closely related problem, Adulyasak *et al.* [3] consider two-stage and multi-stage stochastic production routing problems with demand uncertainty. A Benders decomposition approach is used to solve each sample of the SAA method to optimality. However no backlog is allowed in their model. In [7] a practical maritime inventory routing problem is solved using the SAA method. For each sample the stochastic problem was solved using a commercial solver based on a branch-and-cut algorithm with a running time limit. This solution procedure is heuristic since the instances are not solved to optimality. Consequently, the SAA method generates solutions whose objective function value vary significantly from sample to sample. In order to circumvent this lack of stability, Agra *et al.* [8] propose the use of a local branching heuristic to improve an initial solution for each sample generated within the SAA method. By generating a near local optimal solution for each sample the stability issues were mitigated.

A common approach to deal with inventory models under uncertainty is to consider the possibility of satisfying demands with delay and to allow shortfalls. This assumption aims to use the studied problems to approximate the practical ones faced by companies, and to ensure that a two-stage recourse problem has complete recourse (for each first-stage solution there is a feasible assignment of the second-stage solutions). However, as reported in [8], penalizing the backlogged demand makes the instance harder to solve by exact methods based on mixed integer formulations, since the integrality gaps of the linear solutions tend to be high. Hence, these SPIR problems tend to be even harder than the deterministic ones when no backlog is allowed.

When the SAA problems are not solved to optimality for each sample, theoretical results on convergence fail and the SAA method acts as a heuristic. Based on this observation, and on the assumption that one cannot solve efficiently these complex SPIR problems even for small size instances, a natural approach is to follow the SAA method and solve each one of the SAA problems defined for small size samples heuristically. Then, the best solution among the generated solutions is chosen. We call this the *static* approach. In contrast with this approach, in which the solution to the problem is one of the candidate solutions, we propose a new heuristic approach that explores the knowledge of the several candidate solutions in order to solve the SAA problem for a larger and more representative set of scenarios. Following this approach, a new heuristic named *Adjustable Sample Average Approximation* (ASAA) algorithm is proposed. In this algorithm, a feasible solution for each sample is generated using a heuristic scheme. Then, these solutions are explored in order to identify first-stage variables that are frequently fixed to zero or to one, and a partial fixed solution is constructed, since only some of those variables are fixed. Finally, the restricted model is solved for a larger sample keeping all the remaining variables free. The proposed adjustable heuristic approach has several advantages in relation to the classical SAA method: i) it does not require solving each sample set to optimality; ii) it does not require the use of large sample sets, and iii) it allows the final solution to be adjusted to a larger sample set since many first-stage variables are kept free. Further, the number of variables that are fixed can be easily controlled.

To the best of our knowledge, the use of the information from several solutions obtained with the SAA method to obtain a partial solution was introduced in a preliminary version of this work [10]. However, algorithms that use information of previous solutions to derive partial solutions are not new. Rochet and Taillard [35] introduce the Adaptive Memory concept to describe a set of good solutions, that are kept in a pool that is dynamically updated taking into account common features of the solutions. Based on this concept, a very general metaheuristic focused on the exploitation of strategic memory components was proposed by Glover [22]. Such a metaheuristic, known as Adaptive Memory Programming (AMP), has been applied over the years to solve several hard combinatorial optimization problems, such as vehicle routing problems [23, 41], stochastic production distribution network design [15] and supplier selection problems [42]. Another closely related concept is the Concentration Set (CS) introduced by Rosing and ReVelle [36] to solve the deterministic p-median problem heuristically. A large number of solutions are generated heuristically. Then, the best v solutions are selected to the CS and this set is used to set to one (zero) all the variables that take value one (zero) in all the v solutions. The remaining variables are kept free. Threshold values were also used in [26] for a dynamic facility location problem with capacity levels. Using a Lagrangian relaxation model, some of those levels are eliminated (the corresponding variables are fixed to zero) according to the information provided by the set of solutions.

In this paper we introduce a general approach that explores the SAA in order to handle large size instances of the production-inventory-routing problem. We discuss a mixed integer model and possible enhancements. Based on the enhanced model with valid inequalities, two main approaches are compared: a static approach and an adjustable SAA approach. In order to improve solutions, an iterated local search heuristic based on local branching is presented. The adjustable approach can be easily extended for other two-stage stochastic problems where the corresponding deterministic problems are already difficult to solve to optimality for realistic size instances.

The rest of this paper is organized as follows. The mathematical model is presented in Section 2. Model improvements, such as the valid inequalities and an extended formulation, are discussed in Section 3. The SAA method, the static heuristic approaches based on the SAA method, and the new ASAA heuristic are introduced in Section 4. Computational tests that compare variants of the static and the adjustable approaches are presented in Section 5. Final conclusions are drawn in Section 6.

2 Problem specifications and mathematical formulation

In this section we introduce the mathematical model for the SPIR problem. We assume the production plan (production periods and quantities to produce) and the routing (which clients to visit in each period) are first-stage decisions. The quantities to deliver, the amount backlogged at each time period as well as the inventory levels are adjusted to the scenario. The goal of the stochastic approach is to find the solution that minimizes the production and the routing cost plus the expected cost of both the inventory and the penalty costs for backlogged demand. Following the SAA method, the true expected cost value is replaced by the mean value of a large random sample $\Omega = \{\xi^1, \ldots, \xi^s\}$ of scenarios, obtained by the Monte Carlo method. This larger set of s scenarios is regarded as a benchmark scenario set representing the true distribution [27].

Consider a graph $G = (N, A)$ where $N = \{0, 1, \ldots, n^N\}$ represents the set of nodes and A represents the set of possible links. Set A is a subset of set $N \times N$. Node 0 denotes the producer and $N_c = \{1, \ldots, n^N\}$ is the set of clients. Set $T = \{1, \ldots, n^T\}$ is the set of periods and $K = \{1, \ldots, n^V\}$ is the set of homogeneous vehicles available.

Consider the following parameters: $d_{it}(\xi)$ is the demand of client $i \in N_c$ in period $t \in T$ in scenario $\xi \in \Omega$, I_0/I_i is the initial stock at producer/client i, $\overline{S}_0/\overline{S}_i$ is the inventory capacity at producer/client i, $\overline{P}$ is the production capacity at each time period, $\underline{Q}_{it}$ and $\overline{Q}_{it}$ are the lower and upper limits to the delivery quantity in period $t \in T$ at client i and L is the vehicle capacity. For the objective function, S is the set up cost for producing in a period, P is the production cost of a unit of product, V is the fixed vehicle usage cost, C_{ij} is the travel cost from node i to node j, $(i, j) \in A$, H_{it} is the holding cost of the product at node $i \in N$ in period $t \in T$, B_{it} is the backlog cost of the product at node $i \in N$ in period $t \in T$.

Consider the following first-stage variables: binary variables y_t are the setup variables that indicate whether there is production in period $t \in T$, variables p_t give the production level in period $t \in T$, binary variables z_{itk} indicate whether there is a visit to node $i \in N_c$ in period $t \in T$, by vehicle $k \in K$, the routing variables x_{ijtk} indicate whether vehicle $k \in K$ travels from node i to node j, $(i, j) \in A$, in period $t \in T$, v_{tk} is a binary variable that is one if vehicle $k \in K$ is used in period $t \in T$ and zero otherwise; for $(i, j) \in A$, $t \in T$, $k \in K$, f_{ijtk} is the artificial flow of a single commodity variable used to prevent cycles in the routing problem. Notice that this is not the amount transported from node i to node j in period $t \in T$ by vehicle k since that quantity depends on the scenario. Such adjustable variables could be used but would imply the use of an unnecessarily large model.

For each scenario $\xi \in \Omega$, we define the second-stage variables $q_{itk}(\xi)$ indicating the quantity delivered at node $i \in N_c$ in period $t \in T$ by vehicle k; $s_{it}(\xi)$ indicating the stock level of node $i \in N$ at the end of period $t \in T$; and $r_{it}(\xi)$ is the quantity backlogged at node $i \in N_c$ in period $t \in T$.

The SPIR problem is modeled by the following formulation.

$$
\begin{aligned}
\min \quad & \sum_{t \in T}(Sy_t + Pp_t + V\sum_{k \in K} v_{tk} + \sum_{k \in K}\sum_{(i,j) \in A} C_{ij}x_{ijtk}) \\
& \qquad + \frac{1}{\mid \Omega \mid}\sum_{\xi \in \Omega}\sum_{t \in T}\sum_{i \in N}(H_{it}s_{it}(\xi) + B_{it}r_{it}(\xi))
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
s.t. \quad & s_{0,t-1}(\xi) + p_t = \sum_{k \in K}\sum_{i \in N_c} q_{itk}(\xi) + s_{0t}(\xi), && \forall t \in T, \xi \in \Omega && (2) \\
& s_{i,t-1}(\xi) + r_{it}(\xi) + \sum_{k \in K} q_{itk}(\xi) = d_{it}(\xi) + s_{it}(\xi) + r_{i,t-1}(\xi), && && (3) \\
& && \forall i \in N_c, t \in T, \xi \in \Omega && \\
& s_{i0}(\xi) = I_i, && \forall i \in N, \xi \in \Omega && (4) \\
& s_{it}(\xi) \leq \overline{S}_i, && \forall i \in N, t \in T, \xi \in \Omega && (5) \\
& p_t \leq \overline{P}\ y_t, && \forall t \in T && (6) \\
& \underline{Q}_{it}\ z_{itk} \leq q_{itk}(\xi) \leq \overline{Q}_{it}\ z_{itk}, && \forall i \in N_c, t \in T, k \in K, \xi \in \Omega && (7) \\
& \sum_{i \in N_c} q_{itk}(\xi) \leq Lv_{tk}, && \forall t \in T, k \in K, \xi \in \Omega && (8) \\
& \sum_{j \in N} x_{ijtk} = z_{itk}, && \forall i \in N_c, t \in T, k \in K && (9) \\
& \sum_{j \in N} x_{jitk} - \sum_{j \in N} x_{ijtk} = 0, && \forall i \in N, t \in T, k \in K && (10) \\
& \sum_{j \in N} x_{0jtk} = v_{tk}, && \forall t \in T, k \in K && (11) \\
& \sum_{i \in N} f_{ijtk} - \sum_{i \in N_c} f_{jitk} = z_{jtk}, && \forall j \in N_c, t \in T, k \in K && (12) \\
& f_{ijtk} \leq n\ x_{ijtk}, && \forall i,j \in N, t \in T, k \in K && (13) \\
& y_t, v_{tk}, z_{itk}, x_{ijtk} \in \{0,1\}, && \forall i,j \in N, t \in T, k \in K && (14) \\
& p_t, f_{ijtk} \geq 0, && \forall i,j \in N, t \in T, k \in K && (15) \\
& r_{it}(\xi), s_{it}(\xi), q_{itk}(\xi) \geq 0, && \forall i \in N, t \in T, k \in K, \xi \in \Omega && (16)
\end{aligned}
$$

The objective function (1) minimizes the total cost which includes the production set up, the production, the vehicle usage, the traveling costs, and the expected holding and backlog penalty costs. Constraints (2) and (3) are the inventory conservation constraints at the producer and at the clients, respectively. Constraints (4) establish the initial inventory level at the producer and at each client. Constraints (5) impose a storage capacity at the producer and at each client. Constraints (6) impose a maximum capacity on the production at each period. Constraints (7) impose limits on the delivery quantity at each client in each period. The lower bound imposed ensures that for each scenario the delivered quantity is positive, otherwise there could be second-stage solutions with null delivery quantity. Constraints (8) establish a capacity on the quantity transported by a vehicle. Constraints (9) and (10) are the routing constraints. Constraints (11) guarantee that a vehicle leaves the producer when there are visits to clients. Constraints (12) are the flow balance constraints at clients ensuring that visits are satisfied and constraints (13) guarantee that there is flow between two nodes only if a vehicle travels between these two nodes. Constraints (14), (15) and (16) are the variable domain constraints.

The formulation (1)–(16) is denoted SPIRF.

3 Model tightening

The mixed integer model is weak in the sense that the integrality gap is usually large [2, 6]. In order to tighten the model, we discuss families of valid inequalities and the use of an extended formulation.

3.1 Extended formulation

In this subsection we present an extended formulation which results from adapting ideas from the extended formulations for related problems (see [32] for deterministic lot-sizing problems and [4, 37] for deterministic production-inventory-routing problems without backlog) to the production-inventory-routing problem with multiple scenarios.

One weakness of model SPIRF are the linking constraints (7), especially when the delivered quantities $q_{ikt}(\xi)$ can be much smaller than the upper bounds $\overline{Q}_{it}$.

By decomposing the amount delivered at each client in a given time period into the net demand of the time period destination, tighter linking constraints can be developed.

Let $nd_{it}(\xi)$ denote the net demand for client $i \in N_c$ at time period $t \in T$ when scenario $\xi \in \Omega$ occurs. Such net demand represents the minimum amount of product that must be delivered to client i at period t in order to satisfy the client demand taking into account the initial stock level, and can be computed as follows:

$$nd_{it}(\xi) = \max\{0, \sum_{\ell=1}^{t} d_{i\ell}(\xi) - I_i - \sum_{\ell=1}^{t-1} nd_{i\ell}(\xi)\}.$$

Consider the new variables $u_{it\ell}(\xi)$ indicating the fraction of the net demand of client $i \in N_c$ at time period t that is delivered in time period ℓ when scenario ξ occurs, for $t \in T, \ell \in T \cup \{n^N + 1\}$, $\xi \in \Omega$. Notice that when $\ell = n^T + 1$, variables $u_{it\ell}(\xi)$ represent an amount that is not delivered during the considered time horizon.

In order to tighten model SPIRF, the following set of constraints, that we call EF, can be added.

$$\sum_{k\in K} q_{ilk}(\xi) = \sum_{t\in T} nd_{it}(\xi)\; u_{it\ell}(\xi) \qquad \forall i\in N_c, \ell\in T, \xi\in\Omega \tag{17}$$

$$u_{it\ell}(\xi) \le \sum_{k\in K} z_{i\ell k} \qquad \forall i\in N_c, t,\ell\in T, \xi\in\Omega \tag{18}$$

$$r_{i,n^T}(\xi) = \sum_{t\in T} nd_{it}(\xi)\; u_{i,t,n^T+1}(\xi) \qquad \forall i\in N_c, \xi\in\Omega \tag{19}$$

$$\sum_{\ell\in T\cup\{n^T+1\}} u_{it\ell}(\xi) = 1 \qquad \forall i\in N_c, t\in T, \xi\in\Omega : nd_{it}(\xi) > 0 \tag{20}$$

$$u_{it\ell}(\xi) \ge 0 \qquad \forall i\in N_c, t\in T, l\in T\cup\{n^T+1\}, \xi\in\Omega \tag{21}$$

Constraints (17) relate variables $u_{it\ell}(\xi)$ to the delivery quantity variables $q_{itk}(\xi)$ and compute the quantity that is delivered to client i at time period ℓ. Constraints (18) ensure that if some fraction of net demand for client i is delivered at time period ℓ, then client i must be visited in that time period. Constraints (19) compute the demand not satisfied. Constraints (20) guarantee that demand is satisfied by production and/or is unmet during the time horizon (when $u_{i,t,n^T+1}(\xi) > 0$). Constraints (21) define the new variables as nonnegative.

The linking constraints (18) of this extended formulation are tighter than the linking constraints (7) of the original model, leading to a tighter model.

Even considering a small number of scenarios, the number of constraints and variables in (17)–(21) becomes too large, which means that it is computationally impractical to use such sets of variables and constraints with all scenarios. In Section 5.1 some strategies are proposed to overcome this problem.

3.2 Valid inequalities

From the underlying fixed-charge network flow structure of the SPIRF one can derive several families of valid inequalities [31]. Since the possible set of inequalities is large, we select several such families of inequalities based on preliminary computational tests and on the knowledge of its relevance to related problems. Some of these inequalities are adapted from deterministic lot-sizing problems [32].

The first family relates the delivered quantity $\sum_{k\in K}\sum_{i\in N_c} q_{itk}(\xi)$ with the setups. If there is no set up in time period t, then the delivered amount is bounded by the stock in the producer at the end of time period $t-1$. Otherwise, $\sum_{k\in K}\sum_{i\in N_c} q_{itk}(\xi)$ is bounded by the total vehicles capacities $n^V L$.

$$\sum_{k\in K}\sum_{i\in N_c} q_{itk}(\xi) \le n^V L\; y_t + s_{0,t-1}(\xi), \quad \forall t\in T, \xi\in\Omega \tag{22}$$

To the best of our knowledge, inequalities (22) are new, even for the deterministic case (when only one scenario is considered). These inequalities are useful to cut linear solutions with a fractional value of y_t. Their impact is greater when $n^V L$ is small, which happens, for instance, when a single small capacity vehicle is used, see Section 5.1.

The following two families of inequalities link the inventory levels (stock and backlog) with the binary variables representing either setups or the visits to clients. We notice that the structure of these families of inequalities is quite different from the ones discussed in [4] for a related problem since backlog is not considered there.

The first family is useful to cut fractional solutions when the aggregated value of the binary variables in the RHS of the inequalities is less than 1. This family of inequalities is derived from the uncapacitated lot-sizing with backlogging feasible set (see [32]) defined by inequalities (3), (4) and (7), for a client i, period t and scenario ξ:

$$r_{it}(\xi) \geq \left(\sum_{\ell=1}^{t} d_{i\ell}(\xi) - I_i \right)^+ (1 - \sum_{k \in K} \sum_{\ell=1}^{t} z_{i\ell k}), \quad \forall i \in N_c, t \in T, \xi \in \Omega \tag{23}$$

where $(x)^+ = \max\{x, 0\}$.

These inequalities force the net demand (demand that cannot be satisfied by the initial stock) at client i until period t to be satisfied with backlog in case there is no delivery until period t, that is, if $\sum_{k \in K} \sum_{\ell=1}^{t} z_{i\ell k} = 0$. Note that these inequalities are a particular case of the more general family $s_{i,h-1}(\xi) + r_{it}(\xi) \geq \sum_{\ell=h}^{t} d_{i\ell}(\xi)(1 - \sum_{k \in K} \sum_{\ell=h}^{t} z_{i\ell k})$ for $h, t \in T, h \leq t$.

The second family of inequalities is derived from the uncapacitated lot-sizing with backlogging feasible set obtained by considering the aggregated demands, stocks and backlogs.

$$\sum_{i \in N} s_{i,h-1}(\xi) + \sum_{i \in N} r_{it}(\xi) \geq \sum_{\ell=h}^{t} d_{i\ell}(\xi) \left(1 - \sum_{\ell=h}^{t} y_\ell \right), \quad \forall h, t \in T, h \leq t, \xi \in \Omega \tag{24}$$

Inequalities (24) ensure that the total demand from period h to period t must be satisfied either from stock (at the producer or at the clients) or from backlog at the clients if there is no production in that period.

For the particular case of $h = 1$, inequalities (24) can be written in the stronger format

$$\sum_{i \in N_c} r_{it}(\xi) \geq D(t, \xi) \left(1 - \sum_{\ell=1}^{t} y_\ell \right),$$

where $D(t, \xi) = \sum_{i \in N_c} \sum_{\ell=1}^{t} nd_{i\ell}(\xi)$, is the net demand until period t for scenario ξ.

Next, we introduce a family of inequalities that can be derived by Mixed Integer Rounding (MIR) [30]. Given a simple mixed integer set

$$X = \{(S, Y) \in \mathbb{R}_+ \times \mathbb{Z} \mid S + aY \geq b\},$$

where $a, b \in \mathbb{R}_+$ are arbitrary constants, the MIR inequality is given by

$$S \geq r\big(\lceil b/a \rceil - Y\big),$$

where $r = b - (\lceil b/a \rceil - 1)a$.

Constraint $\overline{P} \sum_{\ell=1}^{t} y_\ell + \sum_{i \in N_c} r_{it}(\xi) \geq D(t, \xi)$ is a relaxation of the feasible solutions set and imposes that the total production capacity plus the backlogged demand must cover the net demand of all clients until period t. Taking $S = \sum_{i \in N_c} r_{it}(\xi)$, $Y = \sum_{\ell=1}^{t} y_\ell$, $a = \overline{P}$, and $b = D(t,\xi)$, the following MIR inequality is obtained

$$\sum_{i \in N_c} r_{it}(\xi) \geq R_P(t,\xi) \left(\left\lceil \frac{D(t,\xi)}{\overline{P}} \right\rceil - \sum_{\ell=1}^{t} y_\ell \right), \quad \forall t \in T, \xi \in \Omega \tag{25}$$

where $R_P(t,\xi) = D(t,\xi) - (\lceil \frac{D(t,\xi)}{\overline{P}} \rceil - 1)\overline{P}$. Inequalities (25) state that the total setup capacity plus the backlogged demand cover the net demand of all clients until period t.

From preliminary computational experience, we observed that the number of inequalities cutting off the linear relaxation solutions is too large. Hence some strategies need to be taken in order to choose which cuts should be included. Two main strategies, Option (W) and Option (A), that differ in the way the scenarios are considered, were tested.

(W) Worst case scenario. In this option, for each family of inequalities and each set of time period and/or client indices (accordingly to the family) at most one inequality is added. The inequality added is the one that, of all the scenarios in Ω, corresponds to the scenario leading to the largest violated inequality.

(A) Aggregation of scenarios. In this option, for each family of inequalities and each set of time period and/or client indices (accordingly to the family), the set of inequalities is aggregated for all the scenarios in Ω. It adds the inequality aggregating all the scenarios leading to a violated inequality.

Option (W) tends to add too many cuts since, for a given fractional first-stage solution, the value of the second-stage solution variables, variables $s_{it}(\xi)$ and $r_{it}(\xi)$, tends to be null. Moreover, inequalities for a given scenario do not necessarily affect the solution variables of the remaining scenarios (unless they force the values of the first-stage solutions to change). Option (A) tends to add fewer cuts but these cuts are weaker than the individual inequalities for each scenario. This option resembles the scenario-group cuts approach proposed by Adulyasak *et al.* [4] where groups of scenarios are considered accordingly to the total demand, and Benders cuts are aggregated for each group. Option (A) corresponds to the case where only one group is created which, in the general framework we are proposing, is reasonable since the number of scenarios considered is small.

In Section 5 we report results derived from the tests conducted to assess the performance of these two options.

4 Solution approaches

In this section we discuss several algorithms to solve the SPIR problem. These algorithms are based on the SPIRF model which is a two-stage stochastic model.

A common approach to solve stochastic problems is the Sample Average Approximation (SAA) method [28, 43], where the true expected value of the objective function is approximated by the average value obtained for a very large sample of s scenarios. The SAA method generates M separate sample sets $\Omega_m, m \in \{1, \ldots, M\}$, each one containing $\ell \ll s$ scenarios. For each one of the scenarios set, Ω_m, the resulting SAA problem (where Ω is replaced by Ω_m in model SPIRF) is solved generating a candidate solution. Then, for each candidate solution, the first-stage solution is fixed, and the value of the objective function for a very large sample with s scenarios is computed. In the case of the two-stage model SPIRF, this value is computed by solving a pure linear programming problem on the second-stage variables.

Solving the SPIR problem to optimality, even for a very small size scenarios sample, may not be practical, given the difficulty of the problem. Therefore the M subproblems generated by the SAA method can hardly be solved to optimality. Hence, the candidate solutions will be feasible solutions obtained with heuristics.

Next, we describe two approaches to solve the SPIR problem. The first one follows the SAA method and uses heuristics to obtain the M candidate solutions. Then the best one is chosen. This will be named the static approach. The second approach uses the candidate solutions to fix part of the first-stage solution and then solves (heuristically) the resulting restricted problem for a larger sample. Since the final solution can be adjusted to the larger sample we name it the adjustable approach.

4.1 Static approach

A first heuristic consists in solving the SPIRF model by branch-and-bound with a commercial solver and imposing a running time limit. In fact, as we can observe in the computational section, this is a natural option that works well on the easiest instances where the solver can find near optimal solutions within reasonable time. For those cases, imposing a time limit does not have great impact on the quality of the solutions. Since it is well known, a large amount of running time is used to prove optimality, which is not a major relevant issue here given that the solution will be evaluated with another sample.

When the instances are harder to solve, the previous approach can perform quite badly and the quality of the solutions varies from sample to sample, leading to a large variability in the solution values. In order to circumvent this fault we used two heuristic approaches that can be taken separately or combined. One approach is to use an iterative local search method to improve the quality of the candidate solutions. The other approach is to simplify the model by pre-defining a route, making the restricted model easier to solve.

Iterative Local Search method

The quality of any feasible solution can be improved using a Local Search procedure. The local search scheme applies a local branching method based on the method proposed by Fischetti and Lodi [21] for reducing the solution space.

For a given positive integer parameter Δ_z, define the *neighborhood* $\mathcal{N}(\bar{z}, \Delta_z)$ of $\bar{z}$ as the set of feasible solutions of the SPIRF satisfying the additional local branching constraint:

$$\sum_{i\in N, t\in T, k\in K | \bar{z}_{itk}=0} z_{itk} + \sum_{i\in N, t\in T, k\in K | \bar{z}_{itk}=1} (1 - z_{itk}) \leq \Delta_z. \tag{26}$$

Hence, the neighborhood of $\bar{z}$, $\mathcal{N}(\bar{z}, \Delta_z)$, is the set of solutions that differ by a maximum of Δ_z values from the z_{itk} variables of the current solution $\bar{z}$. In fact, the linear constraint (26) limits to Δ_z the total number of binary variables z_{itk} flipping their value with respect to the solution $\bar{z}$, either from 1 to 0 or from 0 to 1.

The Local Branching heuristic amounts to adding constraint (26) to SPIRF and running the solver for a time limit of α seconds. If a better solution is found, then $\overline{z}$ is updated and the Local Branching is run again. The complete heuristic is denoted by Iterated Local Search (ILS) and is described in Algorithm 1.

Algorithm 1 Iterated Local Search.

Solve the SPIRF model, considering a single scenario, for α seconds
Save the solution $\overline{z}$
Add the remaining scenarios to the SPIRF model
repeat
    Add constraint (26) to the SPIRF model
    Solve the model for β seconds
    Update the solution $\overline{z}$
until No improvement in the objective function is observed

Using routing precedences from a defined route

The SPIRF includes the TSP as a subproblem. For each time period, the minimum cost hamiltonian cycle that includes all the visited clients must be determined. The TSP is a NP-hard problem and is often very time-consuming. Combining the TSP formulation with production and inventory makes the model SPIRF quite large. In order to simplify the model, a TSP is initially solved for all clients. The order in which clients are visited on this route is used to define a precedence relation between the clients. These precedences are used to eliminate (set to zero) those variables that violate the precedence relation. Similar heuristic approaches have been used before for deterministic problems [38, 40]. The TSP involving all the clients is solved using the *Concorde* software [18].

We name this simplification procedure as the Precedence order relation (P) technique.

4.2 Adjustable approach

In this subsection we describe the adjustable sample average approximation (ASAA) heuristic. Since this heuristic can easily be extended to other two-stage stochastic problems we introduce it as a general framework procedure.

In contrast with the SAA method, where a set of M candidate solutions are generated by solving M SAA problems and the corresponding first-stage solutions are fixed, we propose a new approach that uses the M candidate solutions to generate a partial solution. The idea is to identify within the first-stage variables those that have the same value in all (or almost all) the M generated solutions. Thus, it is likely that this value coincides with the variable value in the optimal solution. By fixing the value of those first-stage variables the problem is simplified and the solution can be completed by solving a problem with a new larger sample.

It is important that the M candidate solutions result from different solution structures, so that the variable fixing corresponds to common structures under different sample scenarios. Hence, the number of scenarios considered in each sample can be small which facilitates the generation of the M candidate solutions. Our experience (not reported in the computational tests) show that for the SPIR problem working with only one scenario leads to solutions that are quite different and few first-stage variables are fixed. Thus the number of scenarios should not be too small.

Another important ingredient is to decide which variables to fix. If a binary first-stage variable has the same value in all the candidate solutions it is natural to fix it to that value. However, we may wish to fix variables that have the same value in at least a given percentage of solutions or satisfy another criteria. Moreover, we may wish to assign different weights to each candidate solution either depending on the probability (if the solution is obtained for a scenario with higher probability its weight could be higher) or on its objective function value (a solution with a better objective function value than the others would be given a greater weight).

Hence, we define a function $f_I : \{0,1\}^M \to [0,1]$ that, for particular values $\omega_1, \ldots, \omega_M$ of a first-stage binary variable ω, assigns a value $f_I(\omega_1, \ldots, \omega_M)$ between 0 and 1. The index I represents the vector of instance parameters. Desirably, function f_I should satisfy the following condition:

Condition 1: $f_I(0^M) = 0$ and $f_I(1^M) = 1$.

Vectors 0^M and 1^M represent the null vector and the vector with all components equal to 1 of dimension M, respectively. Condition 1 ensures that if a variable ω is zero (one) in all the candidate solutions, then the value of the function will be zero (one).

Let Ξ be a subset of binary first-stage decision variables that makes the problem easier to solve when their values are fixed. Thus, this new method can be described in four steps.

First step: solve M independent problems. Generate M separate samples $\Omega_m, m \in \{1, \ldots, M\}$, of dimension ℓ_1. Then, solve the two-stage stochastic model for each sample set, Ω_m, with a time limit of α seconds (either a feasible solution is found within this time limit or it stops after the first feasible solution is found). This gives M candidate solutions, $\varphi_1, \ldots, \varphi_M$.

Second step: obtain $\widehat{M}$ partial solutions. Use the candidate solutions $\varphi_1, \ldots, \varphi_M$ to obtain partial candidate solutions by fixing the value of some first-stage variables in set Ξ and keeping all the remaining variables free.

Let φ_m^P denote the projection of the solution vector φ_m into the space of variables in Ξ. Define two threshold values, γ_1 and γ_2 and generate a new partial solution in the following way: denote by $(\varphi_m^P)_\tau$ the value of a generic first-stage decision variable τ from set Ξ in its partial solution φ_m^P, with the value of this variable in the generated partial solution, ψ^P, computed as follows

$$(\psi^P)_\tau = \begin{cases} 0, & f_I((\varphi_1^P)_\tau, \ldots, (\varphi_M^P)_\tau) < \gamma_1; \\ 1, & f_I((\varphi_1^P)_\tau, \ldots, (\varphi_M^P)_\tau) \geq \gamma_2. \end{cases}$$

Either considering $\widehat{M}$ different functions f_I or varying the parameters γ_1 and γ_2, $\widehat{M}$ different partial solutions $\psi^{P1}, \ldots, \psi^{P\widehat{M}}$ can be obtained.

Third step: complete the $\widehat{M}$ partial solutions. For each partial candidate solution $\psi^{P1}, \ldots, \psi^{P\widehat{M}}$ constructed and having some of the first-stage variables fixed, solve the two-stage stochastic model using a larger sample set with $\ell_2 \geq \ell_1$ scenarios. Use the optimal solution obtained and fix to its optimal value the first-stage variables not fixed yet. Repeating the process for each partial candidate solution yields $\widehat{M}$ candidate solutions with all the first-stage variables fixed.

Fourth step: solve $\widehat{M}$ simplified problems. With all the first-stage decision variables fixed, the two-stage stochastic model is solved for a very large sample set with $\ell_3 >> \ell_2$ scenarios. The value of the recourse variables (the second-stage variables) is computed for each scenario and, consequently, the objective function value of each solution is computed. The solution with the lowest average cost is chosen.

The second and the third steps correspond to the adjustable part of the procedure. If we let $M = \widehat{M}$ and remove the second and the third steps, the adjustable approach ASAA coincides with the SAA method.

In the computational tests we considered the following aspects:

- Only one partial solution is derived, that is, $\widehat{M} = 1$.
- The binary first-stage variables considered in set Ξ are the z_{ijk} variables.
- Function f_I is defined as follows: $f_I((\varphi_1^P)_\tau, \ldots, (\varphi_M^P)_\tau) = \sum_{m=1}^{M} w_m (\varphi_m^P)_\tau$ where the weight w_m of each solution φ_m is computed according to the corresponding objective function value. Let c^m denote the objective function value of solution φ_m (with first-stage variables fixed) computed on a larger sample of dimension $\ell^* \geq \ell_1$. Normalize these weights as follows

$$\bar{c} = \max_{k=1,\ldots,M} \{c^k\} \qquad \bar{d} = \min_{k=1,\ldots,M} \{\bar{c} - c^k | c^k \neq \bar{c}\}.$$

Then, for each solution φ_m, define its weight as

$$w_m = \frac{\bar{c} + \bar{d} - c^m}{\sum_{k=1}^{M} (\bar{c} + \bar{d} - c^k)}.$$

Proposition 1: The function f_I defined above satisfies Condition 1.

Proof: Computing the value of the function f_I for the vectors 0^M and 1^M we have

$$f_I(0, \ldots, 0) = \sum_{m=1}^{M} w_m \times 0 = 0$$

$$f_I(1, \ldots, 1) = \sum_{m=1}^{M} w_m = \sum_{m=1}^{M} \left(\frac{\bar{c} + \bar{d} - c^m}{\sum_{k=1}^{M} (\bar{c} + \bar{d} - c^k)} \right) = \frac{\sum_{m=1}^{M} (\bar{c} + \bar{d} - c^m)}{\sum_{k=1}^{M} (\bar{c} + \bar{d} - c^k)} = 1$$

- In the third step, the model for a large sample is solved by relaxing the routing variables, and all the first-stage variables are fixed, except the routing variables. Then, defining the routing variables as binary, a TSP is solved for each vehicle and each time period.

5 Computational experiments

This section reports the computational experiments carried out to illustrate the performance of the methods described to solve the SPIR problem. All tests were run using a computer with an Intel Core i7-4750HQ 2.00GHz processor and 8GB of RAM, and were conducted using the Xpress-Optimizer 28.01.04 solver with the default options. The performed experiments are based on the instances introduced in [6] which are generated as follows.

The coordinates of the n clients are randomly generated in a 100 by 100 square grid, and the producer is located in the center of the grid. For each value of n a complete graph $G = (N, A)$ is obtained and symmetric traveling costs are associated to the set of arcs. The traveling costs C_{ij} are the euclidean distance between nodes i and j in the grid.

The computational experiments used to test the improvements of both the valid inequalities and the extended formulation are based on instances with $n^N = 5, 15$ and 25 clients. For the computational experiments conducted to assess the performance of the ASAA, the number of clients used is $n^N = 10, 20, 30, 40, 50, 60, 70, 80$. In both cases, $n^T = 5, 10$ time periods are considered.

For each client and each period, the nominal demand value d_{it} is randomly generated between 40 and 80 units, and the uncertain demands vary in $[0.7d_{it}, 1.3d_{it}]$.

The initial stock I_0 at the producer is randomly generated in the interval $[0, 30]$, and the initial stock I_i at client i is randomly generated between 0 and three times the average demand of client i. The maximum inventory level $\overline{S}_i$ is 500 for all $i \in N$. The production capacity $\overline{P}$ is 80% of the average net demand. The number n^V of available homogeneous vehicles is one and two and their capacity L is set to 80% and to 40% of the average net demand, respectively. The lower $\underline{Q}_{it}$ and upper $\overline{Q}_{it}$ delivery limits are 1 and L (the vehicles capacity), respectively. The production set up cost, the unit production cost and the fixed vehicle usage cost are given by $S = 100$, $P = 1$ and $V = 50$, respectively. For all $i \in N, t \in T$, the holding cost H_{it} is 0.2 and the backlog cost B_{it} is 0.5, except for $t = n^T$ where B_{i,n^T} is 5, since this cost penalizes the demand of client i that is not satisfied during the planning horizon.

5.1 Model enhancements

In this subsection we report the computational experiments conducted to test the model enhancements, namely the extended formulation and the valid inequalities. Based on these results the final model is chosen.

Inclusion of the extended formulation

First, we evaluate the improvements observed when the extended formulation EF, described in Subsection 3.1, is used. Several strategies can be used to deal with the scenarios in the extended formulation. On the one hand, when the number of scenarios used is large the size of the formulation becomes too large. On the other hand, by considering few scenarios the model becomes weaker. To evaluate advantages and disadvantages of the use of the extended formulation, four strategies were studied:

(i) SPIRF, only SPIRF is used;

(ii) SPIRF+EF, SPIRF is used together with EF considering all the scenarios;

(iii) SPIRF+EF$_l$, SPIRF is used together with EF for the scenario with the lowest value of total net demand, $\xi^l = argmin_{\xi \in \Omega}\{\sum_{i \in N_c} \sum_{t \in T} nd_{it}(\xi)\}$;

(iv) SPIRF+EF$_g$, SPIRF is used together with EF for the scenario with the greatest value of total net demand, $\xi^g = argmax_{\xi \in \Omega}\{\sum_{i \in N_c} \sum_{t \in T} nd_{it}(\xi)\}$.

We consider instances, as described above, with $n^N = 5, 15$ and 25 clients, $n^T = 5, 10$ periods and a single vehicle. For each instance, 10 scenarios were considered and the time limit was set to 600 seconds.

Table 1 summarizes the computational experience. For each one of the four strategies and for each instance, identified by the number n^T of periods and the number n^N of clients, Table 1 displays the values of the linear programming relaxation (LR), the lower bound (LB), the upper bound (UB) and two gaps,

$$Gap_1 = \frac{BFS - LB}{BFS} \times 100 \quad \text{and} \quad Gap_2 = \frac{UB - BFS}{BFS} \times 100$$

where BFS is the value of the best feasible solution obtained among the four strategies mentioned. The values of LB and UB are obtained at the end of the running time.

		SPIRF					SPIRF+EF				
n^T	n^N	LR	LB	UB	Gap_1	Gap_2	LR	LB	UB	Gap_1	Gap_2
	5	2196	3076	3155	2.5	0.0	2275	3061	3158	3.0	0.1
5	15	5105	5841	6265	6.8	0.0	5183	5682	6420	9.3	2.5
	25	8982	9537	11395	12.9	4.1	9085	9464	20972	13.6	91.6
	5	5001	6759	6961	2.9	0.0	5026	6546	6965	6.0	0.1
10	15	13719	14884	16821	11.5	0.0	13750	14625	28887	13.1	71.7
	25	20724	21332	47110	54.7	0.0	20750	21335	48339	54.7	2.6

		SPIRF+EF$_l$					SPIRF+EF$_g$				
n^T	n^N	LR	LB	UB	Gap_1	Gap_2	LR	LB	UB	Gap_1	Gap_2
	5	2243	3068	3155	2.8	0.0	2243	3068	3155	2.8	0.0
	15	5168	5740	6265	8.4	0.0	5168	5740	6340	8.4	1.2
5	25	9062	9649	10948	11.9	0.0	9062	9649	10948	11.9	0.0
	5	5022	6729	6961	3.3	0.0	5022	6729	6961	3.3	0.0
10	15	13749	14935	17883	11.2	6.3	13749	14935	17883	11.2	5.9
	25	20730	21355	48394	54.7	2.7	20730	21360	48795	54.7	3.6

Table 1: Computational comparison of the four strategies regarding the inclusion of the extended formulation EF.

As expected, the formulation SPIRF+EF provides the best quality linear relaxation solutions, however the computational time is higher, and the lower and upper bounds (and consequently the gaps Gap_1 and Gap_2), are the worst. The differences are very small between strategies SPIRF, SPIRF+EF$_l$ and SPIRF+EF$_g$. Therefore, no clear conclusions can be drawn. Although SPIRF+EF$_l$ provides better results for small size instances, the results for $n^N = 25$, $n^T = 10$ seem to indicate that for the larger instances model SPIRF is better. Also, the gains in terms of the linear relaxation value indicate that the improvements obtained by including the complete model EF are minor when the size of the model increases.

Hence, since there is no clear benefit in the use of any one of the strategies $SPIRF + EF$, $SPIRF + EF_l$ and $SPIRF + EF_g$, mainly for the largest instances, and since the number of constraints and variables tend to increase rapidly as the size of the instances increase, in what follows the extended formulation is not included.

Inclusion of valid inequalities

Now, we assess the improvements observed when SPIRF is tightened with the inclusion of the valid inequalities described in Subsection 3.2.

Since inequalities (22) are new, an independent experiment was conducted on these inequalities for instances with $n^T = 5$ periods, $n^N = 10$ clients and $n^V = 1, 2$ vehicles. The linear gap of SPIRF (Gap_L) and the linear gap of SPIRF tightened with inequalities (22) (Gap_T) were computed for L, the vehicles capacity, which was set to 20%, 40%, 60% and 80% of the average net demand. The average gap reduction obtained by the inclusion of the inequalities, computed by $\frac{Gap_L - Gap_T}{Gap_T} \times 100$, was $27.5, 20.6, 10.6$ and 11.1, respectively, for one vehicle, and $26.2, 17.8, 1.4$ and 0, respectively, for two vehicles. Thus, the impact of these inequalities is greater for small values of L and for one vehicle.

The valid inequalities (22)–(25) are dynamically added to the SPIRF and three strategies are compared:

(i) SPIRF, only SPIRF is used, no inequalities are included.

(ii) SPIRF-W, corresponds to Option (W) for the inclusion of valid inequalities discussed in Section 3.2.

(iii) SPIRF-A, this strategy corresponds to Option (A) applied to inequalities (22), (23) and (24) and Option (W) applied to inequalities (25). Notice that Option (A), aggregation of the inequalities for all the scenarios, applied to inequalities (25) leads to very weak inequalities.

Twelve sets of instances were used, six considering one vehicle and six considering two vehicles. For each set, five instances were generated considering different samples of demand values. A time limit of 600 seconds was imposed. The obtained results are displayed in Table 2, Figure 1 and Figure 2.

Table 2 displays the results for the three strategies: SPIRF, SPIRF-A, and SPIRF-W. Each line has the results for an instance set identified by its number n^V of vehicles, its number n^T of periods, and its number n^N of clients. Columns identified with $\#vi$ display the average number of valid inequalities added to the SPIRF. Columns identified with $\#nodes$ display the average number of nodes solved within the branch-and-bound algorithm. Columns named Gap display the average gap, computed according to $\frac{\overline{UB} - \overline{LB}}{\overline{LB}} \times 100$, where $\overline{UB}$ and $\overline{LB}$ denote the average upper bound value and the average lower bound value, respectively.

Table 2 shows that the average number of inequalities added to the SPIRF in strategy SPIRF-A is lower than the average number of inequalities added to the SPIRF in strategy

n^V	n^T	n^N	SPIRF		SPIRF-A			SPIRF-W		
			#nodes	*Gap*	*#vi*	*#nodes*	*Gap*	*#vi*	*#nodes*	*Gap*
1	5	5	2482	0.0	18	2464	0.0	50	2812	0.0
		15	44696	2.4	65	19118	1.5	82	38999	1.5
		25	6772	5.9	108	4039	4.1	185	5885	4.5
	10	5	35385	0.0	28	30656	0.0	125	30909	0.0
		15	9185	2.1	90	4891	1.9	215	7971	2.2
		25	89	11.4	210	2213	7.7	299	591	8.5
2	5	5	119675	4.9	23	82384	3.7	55	129334	4.7
		15	2945	34.8	76	7709	26.6	105	7785	28.2
		25	925	40.8	139	1326	31.3	174	724	35.7
	10	5	10730	7.0	40	12201	6.4	132	17016	6.6
		15	44	121.0	110	1792	54.2	215	2147	97.8
		25	74	40.3	233	4079	26.6	287	2695	28.9

Table 2: Computational results for the three strategies: SPIRF, SPIRF-A and SPIRF-W.

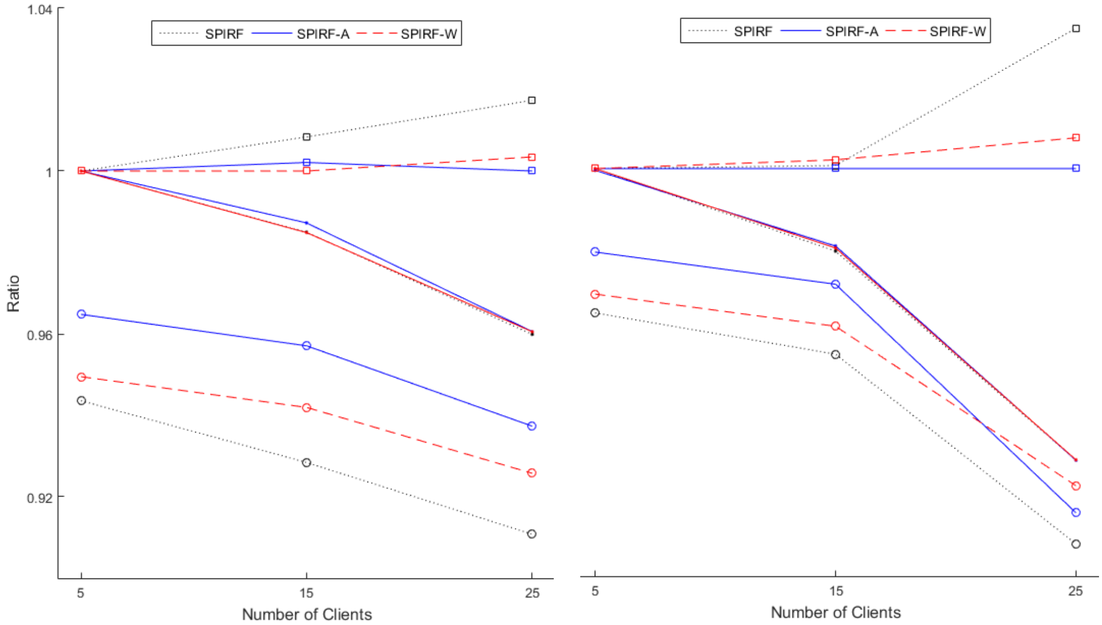
SPIRF-W. Also the number of nodes solved is lower. Columns *Gap* show that the average gap is reduced by the inclusion of valid inequalities and smaller average gaps are always obtained with strategy SPIRF-A. It is also possible to observe that instances with more than one vehicle are more difficult to solve, even for small size instances.

Figure 1 and Figure 2 display the relation between the average linear relaxation, average lower bound, and average upper bound values for the three strategies, for each instance set. Since these values can vary considerably from instance to instance, for better comparison each value was divided by the best (lower) upper bound of the corresponding instance set.

Figure 1 depicts results for the single-vehicle case, while Figure 2 depicts results for the case where two vehicles are available. In both figures, the values obtained by using approaches SPIRF-A and SPIRF-W are associated with the solid and dashed lines, respectively. The dotted lines are associated with SPIRF, the case where no inequalities are added. The circle points (in the first/down set of three lines) represent the ratio between the average linear relaxation and the best upper bound value. Solid points (in the second/middle set of three lines) represent the ratio between the average lower bound and the best upper bound value. Square points (in the third/upper set of three lines) represent the ratio between the average upper bound and the best upper bound value.

Figure 1 and Figure 2 help to explain the gaps computed in Table 2, since the largest average lower bound and smallest average upper bound values are obtained with strategy SPIRF-A, except in one case. Note that, for the three approaches, the average lower bound values are very close.

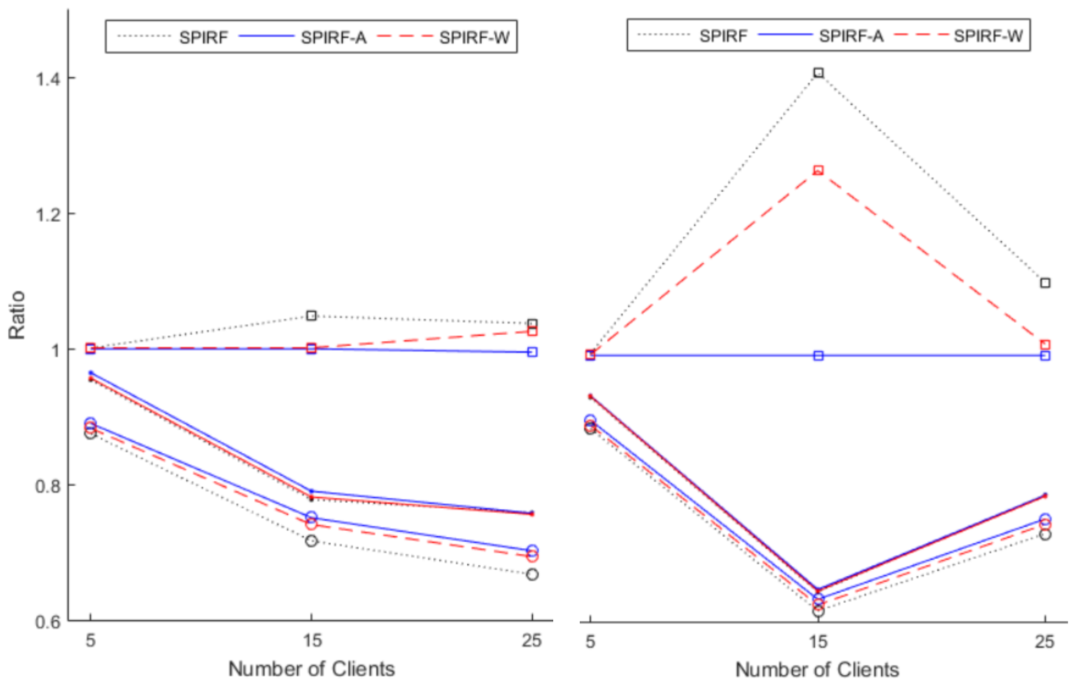
Since the best results are obtained with strategy SPIRF-A, in what follows, the SPIRF will be used together with the inclusion of inequalities following strategy SPIRF-A.



(a) Results for instances with 5 time periods

(b) Results for instances with 10 time periods

Figure 1: Comparison of average linear relaxation, average lower bound, and average upper bound values for instances using 1 vehicle.



(a) Results for instances with 5 time periods

(b) Results for instances with 10 time periods

Figure 2: Comparison of average linear relaxation, average lower bound, and average upper bound values for instances using 2 vehicles.

5.2 Computational comparison of the static and adjustable strategies

Extensive computational experiments to assess the performance of the proposed Adjustable Sample Average Approximation method are presented. Three variants of the ASAA, that differ from each other in the strategy used to obtain the initial candidate solutions, are compared. These three variants of the ASAA method are:

(i) $ASAA_{ILS}$, the ILS procedure is used;

(ii) $ASAA_{ILS+P}$, the ILS procedure is used and precedence relations in the routing are considered;

(iii) $ASAA_R$, the routing variables x_{ijtk} are relaxed.

These three variants were tested against the following four static approaches:

(i) *SAA*, the SAA method;

(ii) SAA_{ILS}, the SAA with the ILS procedure;

(iii) SAA_{ILS+P}, the SAA with the ILS procedure in which precedence relations in the routing are considered;

(iv) *EVP*, the Expected Value Problem (EVP), which corresponds to solving the deterministic problem with all the uncertainty parameters replaced by their expected values.

Whenever the ILS procedure is used, the initial scenario considered is the one that corresponds to the EVP. However, since we are dealing with difficult instances, usually the corresponding EVP problem cannot be solved to optimality within a reasonable amount of running time. Thus a two-stage heuristic procedure to obtain a feasible solution is followed. First, the EVP with the routing variables x_{ijtk} relaxed is solved with a time limit of 300 seconds. Then, all the visit variables z_{itk} are fixed and n^T pure TSP problems are solved by using the software *Concorde*. Even when the number of visit variables that are allowed to change their value in the local branching constraint (26) is low, the time used to solve each subproblem is large, since all the routing variables remain free. Hence, in order to solve such subproblems faster, we used a local branching constraint similar to (26) for the routing variables x_{ijtk}, in which the number of variables that are allowed to flip their value is three times the value used in the local branching constraint associated to the visit variables.

To evaluate the performance of these seven strategies to solve the SPIRF, 32 instances, randomly generated as described before, are used. For each instance and each strategy, except for strategy named *EVP* (corresponding to solve the EVP), the number of candidate solutions used is $M = 10$ and each candidate solution is obtained by using a sample of $\ell_1 = 10$ scenarios and a time limit of 300 seconds. However, when no integer solution is found within this time limit, the procedure only stops when the first integer solution is obtained. For strategy *EVP*, only one candidate solution is obtained using a single scenario, with a time limit of 3600 seconds (one hour). For all strategies, the dimension of the final sample used is $\ell_3 = 1000$.

For the ASAA variants, a sample of $\ell^* = 25$ scenarios is used to define the weight of each partial candidate solution in the second step and several strategies for fixing variables z_{itk} are used, corresponding to different choices of the parameters γ_1 and γ_2. The threshold values 0, 0.05, 0.15 and 0.25 were tested for γ_1, while γ_2 is set to $1 - \gamma_1$. In the third step, when solving the simplified model, a sample of $\ell_2 = 50$ scenarios is used.

The obtained results are displayed in Table 3 and Table 4. Each strategy is identified in the first line of the table and each instance is identified by its number n^V of vehicles, its number n^T of periods and its number n^N of clients in the first three columns.

In Table 3, the computational time (in seconds) required by each approach to obtain the final solution is displayed in columns named *Time*. For the adjustable approaches, the computational time values displayed correspond to the value obtained for γ_1, the one which required more computational time. In columns *VSS* an approximation of the Value of the Stochastic Solution is displayed for each approach considered. Notice that, as explained before, it is not possible to obtain the exact solution of the Expected Value Problem, thus all the values are computed according to the heuristic solution obtained for the *EVP*. For the adjustable approaches, the approximation for the value of the optimal solution is computed based on the best solution obtained for the different threshold values of γ_1.

n^V	n^T	n^N	EVP	SAA		SAA_{ILS}		$ASAA_{ILS}$		SAA_{ILS+P}		$ASAA_{ILS+P}$		$ASAA_R$	
			$Time$	VSS	$Time$	VSS	$Time$	VSS	$Time$	VSS	$Time$	VSS	$Time$	VSS	$Time$
		10	11	2	525	2	183	2	138	2	173	2	138	-210	86
		20	3614	15	3142	20	1213	20	1080	19	963	22	839	-120	1824
		30	3630	-208	3300	5	1599	28	1311	-1	1600	32	1319	-308	3020
	5	40	3654	-412	3544	-169	1908	33	1392	28	1879	61	1340	-212	891
		50	3683	-454	3818	60	2430	102	1628	103	2438	103	1628	-71	1836
		60	3716	-19932	4147	195	2722	233	1605	229	2737	225	1611	300	3019
		70	3756	-45757	4550	644	3017	989	1519	785	3209	1034	1698	1110	3080
		80	3395	-48852	5035	1871	3328	2037	2512	2168	3529	2440	2019	1586	3105
1		10	3610	-18	3095	16	899	16	822	11	1586	17	1379	-64	3013
		20	3630	-332	3300	63	1611	160	1334	50	1709	105	1401	-25	3021
		30	3666	-15013	3638	1010	1924	1089	1334	1023	1979	1134	1468	794	3163
	10	40	3712	-84355	4070	498	1919	509	915	628	2427	786	1323	109	3043
		50	3768	-97813	4631	3276	3103	4222	1635	3454	3072	4220	1500	4740	3138
		60	3842	-117460	5307	4897	4610	6084	2457	4986	4320	6083	2024	4536	3166
		70	3935	-117187	6138	6047	5801	7395	3844	6676	5269	7397	2046	5769	3187
		80	3365	-161839	7042	3406	5532	6517	3921	6601	5439	6946	2569	3361	3214
		10	424	20	3051	20	481	20	437	22	456	22	407	-52	3005
		20	3615	-440	3152	26	2552	33	2422	65	2557	87	2407	-70	3007
		30	3632	-294	3303	70	3969	131	3687	228	3764	229	3462	7	3012
	5	40	3657	-25691	3505	-218	3972	112	3580	355	3650	362	3164	-197	3024
		50	3687	-32493	3765	67	4223	159	3490	649	3332	700	2487	658	3028
		60	3721	-40248	4077	532	5212	740	4144	692	3919	864	2722	532	3043
		70	3763	-46376	4528	1524	5543	1527	4124	1942	5238	2067	3668	1142	3062
		80	4021	-45704	4996	1313	6182	1595	4071	3250	6182	3303	4887	2075	3102
2		10	3611	-131	3104	13	3104	26	3013	14	2807	36	2722	-459	3023
		20	3634	-34586	3291	-231	4008	91	3692	77	3934	230	3631	-120	3032
		30	3671	-53783	3583	1139	3936	1286	3281	1013	2353	1862	1714	1539	3029
	10	40	3718	-79719	3980	1329	3842	1393	2721	1501	4130	1546	3059	160	3043
		50	3777	-95142	4501	1914	3723	2401	2198	2165	5338	2782	3650	934	3132
		60	3848	-124200	7350	-1757	6923	589	4649	3047	5626	3420	3111	1265	3145
		70	3926	-126009	6145	649	7483	6526	4346	1129	7290	6647	3894	5385	3168
		80	4753	-6047	9681	145609	8806	157775	4421	156951	6644	157006	2985	158172	3206

Table 3: Computational time and approximation of the Value of the Stochastic Solution for all the approaches.

The computational times presented in Table 3 for all the approaches are very close and, in general, the lowest ones correspond to the approaches in which precedence relations are used. Furthermore, the additional computational time required to improve the solution in the adjustable approaches is lower than the time required to evaluate all the 10 candidate solutions in the final step of the corresponding static approaches.

The cost of the solutions obtained by each one of the strategies is reported in Table 4.

In each set of four columns associated with each variant of the ASAA, the results for the different threshold values of γ_1 are reported.

The results displayed in Table 4 show that the cost of the solutions obtained by SAA tend to be excessively high for medium and large size instances. However, for the smallest instances, the cost of the solutions obtained by strategy $ASAA_R$, where the structure of the problem changes due to the relaxation of the routing variables, is even higher than the one obtained by SAA.

The solutions obtained by the static strategies that use the ILS procedure, strategies SAA_{ILS} and SAA_{ILS+P}, are in general better than the ones obtained by solving the *EVP*. Furthermore, such solutions can be improved by using the corresponding adjustable strategies, $ASAA_{ILS}$ and $ASAA_{ILS+P}$, mainly when the size of the instances increases. This means that the second and third steps in the adjustable approach improve the quality of the solutions. Notice that these two adjustable approaches iterate from the solutions obtained by the corresponding static version.

The solutions obtained by SAA_{ILS+P} have, in general, a lower cost than the ones obtained by SAA_{ILS} and, in some cases, there is a big difference between the cost of these solutions. The same observation applies when comparing the strategies SAA_{ILS+P} and SAA_{ILS}. This means that the use of precedence relations can be very useful for such instances, mainly for the medium and large size instances, since fixing the routing variables *a priori* make the instances easier to solve.

In strategy $ASAA_R$, where the routing variables are relaxed, integer solutions are found quickly. For small size instances, the solutions obtained by this variant are the ones with the worst cost. However, when the size of the instances increases, the computational time required to obtain solutions with the relaxation strategy is kept small and much better solutions than the ones obtained by SAA and *EVP* are obtained.

The computational results obtained for the adjustable approaches suggest that different good solutions can be obtained from different threshold values, and in some cases, different threshold values may result in the same solution. In fact, there is no specific threshold value that always leads to the best solution. For the large size instances, better results are obtained by considering threshold values different from zero, which corresponds to fixing more variables than the ones that take the same value in all the candidate solutions. However, the threshold value γ_1 used should not be close to 0.5, since the quality of the solution can be lost when many variables are fixed *a priori.*

From a general point of view, the computational results presented in Table 3 and Table 4 allow us to conclude that in almost all the cases, the solutions obtained by the static approaches can be improved by the second and third steps of the adjustable approaches using a short running time. Hence, the proposed adjustable strategies prove to be effective and efficient, mainly for medium and large instances, since good solutions can be obtained in a short time. Moreover, among the three adjustable strategies tested, strategy $ASAA_{ILS+P}$ is the one that, in general, leads to the best results, solution and time.

n^V	n^T	n^N	EVP	SAA	SAA_{ILS}	$ASAA_{ILS}$				SAA_{ILS+P}	$ASAA_{ILS+P}$				$ASAA_R$			
						0	0.05	0.15	0.25		0	0.05	0.15	0.25	0	0.05	0.15	0.25
		10	5151	5149	5149	5149	5149	5149	5149	5149	5149	5149	5149	5149	5361	5361	5361	5361
		20	9069	9054	9049	9049	9049	9049	9049	9050	9047	9047	9051	9051	9189	9189	9295	9295
		30	12336	12544	12331	12331	12308	12308	12308	12337	12316	12320	12304	12304	12668	12644	12679	12679
	5	40	18468	18880	18637	18449	18454	18435	18611	18440	18416	18416	18407	18407	18680	18741	18701	18741
		50	22495	22949	22435	22393	22403	22428	22442	22393	22393	22393	22397	22397	22566	22616	22822	22786
		60	27308	47240	27113	27080	27075	27077	27095	27079	27140	27120	27083	27091	27035	27035	27080	27008
		70	31701	77458	31057	30732	30721	30712	30712	30916	30690	30683	30683	30667	31073	30954	30887	30591
		80	35152	84004	33281	33151	33115	33173	33115	32984	32718	32718	32712	32749	33566	33589	33606	33640
1		10	12936	12954	12920	12920	12920	12920	12920	12925	12928	12919	12919	12928	13000	13030	13027	13335
		20	24010	24342	23947	23850	23850	23929	24008	23960	23905	23905	23927	23947	24659	24084	24035	24097
		30	35527	50540	34517	34438	34478	34508	34524	34504	34393	34399	34441	34504	35415	35312	34805	34733
	10	40	47047	131402	46549	46538	46538	46684	46684	46419	46261	46261	64321	46600	46972	46938	47047	47524
		50	59861	157674	56585	55694	55639	55664	55668	56407	55641	55649	55649	55649	55791	55791	55627	55121
		60	73463	190923	68566	67423	67400	67420	67379	68477	67458	67394	67387	67380	68931	68948	68928	68927
		70	81309	198496	75262	74030	73972	73914	73914	74633	73929	73942	73912	73912	75548	75571	75540	75608
		80	91870	253709	88464	85353	85355	85392	85389	85269	84985	84924	84958	84958	88509	88509	88561	88573
		10	5624	5604	5604	5604	5604	5604	5610	5602	5602	5602	5602	5604	5690	5676	5693	5716
		20	9597	10037	9571	9564	9564	9564	9581	9532	9575	9571	9510	9515	9799	9810	9769	9667
		30	12916	13210	12846	12785	12785	12812	12837	12688	12688	12688	12688	12687	14178	14432	12909	13381
	5	40	19114	44805	19332	19002	19110	19229	19268	18759	18752	18752	18752	18752	20607	20272	19708	19311
		50	23456	55949	23389	23382	23297	23333	23350	22807	22785	22756	22795	22798	22972	22972	22798	23511
		60	28038	68286	27506	27298	27350	27381	27381	27346	27174	27174	27205	27245	27968	28006	27506	27676
		70	33052	79428	31528	31556	31525	31614	31614	31110	30985	30985	30997	30993	31910	31946	32112	31916
		80	37050	82754	35737	35459	35455	35500	35500	33800	33747	33764	33790	33790	36727	36727	34976	34975
2		10	13970	14101	13957	13944	13944	13956	13958	13956	13934	13934	13934	13970	18487	14582	14651	14429
		20	25533	60119	25764	25442	25442	25507	25868	25456	25303	25401	25418	25517	26209	26388	25653	25894
		30	37262	91045	36123	35976	36000	36070	36070	36249	35400	35400	35400	35400	36611	36809	35723	38598
	10	40	49025	128744	47696	47632	47632	47707	47707	47524	47479	47479	47504	47572	48865	49005	49020	51836
		50	60677	155819	58763	58276	58292	58381	58381	58512	57895	57977	58228	58231	59743	59884	60590	61455
		60	71961	196161	73718	71376	71372	71484	71484	68914	68541	68544	68607	68617	70696	71696	71040	71689
		70	86622	212631	85973	80096	80097	80109	80123	85493	80015	79975	79991	80141	81237	81423	82165	82374
		80	250219	256266	104610	92444	92444	92462	92465	93268	93213	93213	93213	93213	92050	92047	92386	93086

Table 4: Computational results for comparison of several strategies for solving a two-stage stochastic model.

Results not reported here show that each subproblem associated to the determination of each candidate solution is highly influenced by the variation of the demand values. Hence, in order to validate our conclusions, further tests were conducted for the large size instances in which the demand samples are derived from a normal distribution with parameters d_{it} (mean) and $0.15d_{it}$ (standard deviation), and for a gamma distribution with parameters $(1/0.15)^2$ (shape) and $1/(0.15^2 d_{it})$ (rate). For these results we compare the EVP, the best static approach SAA_{ILS+P}, and the best adjustable approach $ASAA_{ILS+P}$ using the threshold value 0.15. The results are reported in Tables 5 and 6, respectively.

			EVP		SAA_{ILS+P}			$ASAA_{ILS+P}$		
n^V	n^T	n^N	Time	Cost	Time	Cost	VSS	Time	Cost	VSS
1	5	60	3710	27285	2361	26802	483	1531	26590	695
		70	3748	31667	2874	30442	1225	1740	30204	1463
		80	3789	35110	3207	32962	2148	1745	32729	2381
	10	60	4014	84826	6718	67447	17379	3456	67170	17656
		70	10191	81235	7215	73633	7602	2809	73290	7945
		80	3973	96860	6250	85318	11542	3316	84981	11879
2	5	60	3708	28017	2534	27388	629	1720	27237	780
		70	3745	33008	3875	31139	1869	2765	31014	1994
		80	3805	37015	3999	33866	3149	2409	33704	3311
	10	60	4021	71910	6349	68873	3037	3031	68527	3383
		70	10291	86527	7821	75803	10724	3415	75558	10969
		80	3985	250198	5755	93426	156772	2725	92152	158046

Table 5: Computational results obtained for the case where the demands follow the normal distribution.

			EVP		SAA_{ILS+P}			$ASAA_{ILS+P}$		
n^V	n^T	n^N	Time	Cost	Time	Cost	VSS	Time	Cost	VSS
1	5	60	3722	26495	2982	25958	537	2056	25721	774
		70	3759	30734	3935	29484	1250	2713	29246	1488
		80	3802	34103	4657	31891	2212	3091	31625	2478
	10	60	3835	82949	4101	65265	17684	2271	64960	17989
		70	9933	78959	4527	71249	7710	2185	70858	8101
		80	3955	91075	6303	82412	8663	3513	82045	9030
2	5	60	3707	27202	3247	26570	632	2441	26414	788
		70	3747	32083	4333	30193	1890	3207	30023	2060
		80	3791	35962	4550	32809	3153	3072	32587	3375
	10	60	3829	69788	4772	66794	2994	2990	66350	3438
		70	10034	84272	6013	73420	10852	3663	73172	11100
		80	3990	246986	6014	90394	156592	2944	89429	157557

Table 6: Computational results obtained for the case where the demands follow the gamma distribution.

Both tables show that the static and the adjustable approaches lead to gains when compared to the feasible solution obtained by solving the EVP. Although the running times are similar (which is expected since the first step in both approaches is the same and corresponds to computing the set of candidate solutions, which is the most time consuming step), the adjustable approach is always slightly faster. In relation to the quality of the feasible solution, the adjustable approach is always better than the static one.

6 Conclusion

A stochastic production-inventory-routing problem with recourse is considered. A sample average approximation method is applied where the stochastic problem is solved for several samples and the best solution among the solutions obtained is selected. Since the stochastic problem is very complex and problem instances can hardly be solved to optimality within reasonable amount of running time, two different approaches are proposed: a static approach that consists of using heuristics to generate a solution for the stochastic problem defined for each sample; and a new adjustable heuristic procedure where the generated solutions are used to fix part of the first-stage solution (corresponding to those variables which have common values in the most of the candidate solutions). The partial solution is completed by solving a new restricted problem defined for a larger sample.

Several algorithms were tested for each approach. Computational results have shown that the static procedure performs well on small size instances using as heuristic the (time) truncated branch-and-bound algorithm. For medium size instances the static procedure performs well if an iterated local search heuristic is used to improve the initial candidate solutions. However, such solutions can be improved by using the adjustable approach with a short computational time.

Future research would be to extend the adjustable approach introduced here to other complex two-stage stochastic problems, such as network design problems.

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