APPROXIMATION IN GENERALIZED MORREY SPACES*

ALEXANDRE ALMEIDA AND STEFAN SAMKO

Abstract. In this paper we study the approximation of functions from generalized Morrey spaces by nice functions. We introduce a new subspace whose elements can be approximated by infinitely differentiable compactly supported functions. This provides, in particular, an explicit description of the closure of the set of such functions in generalized Morrey spaces.

1. Introduction

Morrey spaces play an important role in applications to regularity properties of solutions to PDE including heat equations and Navier-Stokes equations, see [31, 32] and references therein for further details. The classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$, $1 \leq p < \infty$, $0 \leq \lambda \leq n$, consist of all locally p-integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{r^\lambda}\int_{B(x,r)} |f(y)|^p\, dy\right)^{1/p} < \infty. \tag{1.1}$$

Straightforward calculations show that

$$\|f(t\cdot)\|_{L^{p,\lambda}(\mathbb{R}^n)} = t^{\frac{\lambda-n}{p}}\, \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}\,, \qquad t>0,$$

which implies a modification of the scaling factor in comparison with L^p-spaces. This property reveals the homogeneous nature of the spaces $L^{p,\lambda}(\mathbb{R}^n)$ and it is very useful in the study of partial differential equations.

The theory of Morrey spaces goes back to Morrey [16] who considered related integral inequalities in the study of solutions to nonlinear elliptic equations. In the form of Banach spaces of functions, called thereafter Morrey spaces, the ideas of Morrey [16] were further developed by Campanato [7] and Peetre [19]. We refer to the books [1, 11, 21, 30, 31] and the overview [22] for additional references and basic properties of these spaces, including some generalizations. We also refer to [2], [24], [32] for a discussion of harmonic analysis in Morrey spaces.

Although Morrey spaces may describe local properties of functions better than Lebesgue spaces, they miss some important properties like separability and approximation by nice functions. It is known that there are Morrey functions that cannot be approximated even by continuous functions (see [37] for examples and further details). In [37] Zorko has observed that the set of Morrey functions for which the translation is continuous in

Date: July 29, 2018.

2010 *Mathematics Subject Classification.* 46E30, 42B35, 42B20.

Key words and phrases. Generalized Morrey spaces, vanishing properties, approximation, convolution.

* Accepted manuscript in GMJ [A. Almeida, S. Samko: Approximation in generalized Morrey spaces, Georgian Math. J. (2018), 25(2), 155–168]. The final publication is available at https://doi.org/10.1515/gmj-2018-0023.

The first author was supported by the Portuguese Foundation for Science and Technology (FCT – Fundação para a Ciência e a Tecnologia), through CIDMA – Center for Research and Development in Mathematics and Applications, within project UID/MAT/04106/2013.

Morrey norm plays an important role in approximation. These facts were sketched in [37, Proposition 3] (see also [8] and [14]).

On bounded domains, the so-called *Zorko subspace* can be characterized by an integral mean vanishing property at the origin (cf. Remark 5.8) and its elements can be approximated by compactly supported bounded functions (cf. [8, Lemma 1.1]). It turns out that these features are no longer true for unbounded domains, and a description of this set of functions in easily verified terms seems to be a difficult task in this setting.

Using vanishing type conditions, in [3] the authors have introduced a new (closed) subspace of $L^{p,\lambda}(\mathbb{R}^n)$, strictly smaller than Zorko class, and have shown that all elements in this new class, denoted by $V^{(*)}_{0,\infty}L^{p,\lambda}(\mathbb{R}^n)$, can be approximated by C_0^∞-functions in Morrey norm. In particular, it was obtained in [3] an explicit description of the closure of $C_0^\infty(\mathbb{R}^n)$ in $L^{p,\lambda}(\mathbb{R}^n)$. This closure plays an important role in harmonic analysis on Morrey spaces, including Calderón-Zygmund theory, since its dual provides a predual space (cf. [1, 2, 25, 32]). The description of such closure was given in similar, but different terms, in paper [36]. In [12], [13] the authors used the closure of $C_0^\infty(\mathbb{R}^n)$ (and the closure of the set of compactly supported bounded functions) in Morrey spaces in the study of complex interpolation, but did not provide a characterization for such closures.

The main goal of this paper is to extend the approximation scheme developed in [3] for classical Morrey spaces to the case of generalized Morrey spaces. The latter are defined by replacing the power r^λ in (1.1) by a more general positive function $\varphi(r)$ (cf. (2.1), (2.2)), mainly satisfying monotonicity type conditions. Such spaces proved to be useful in the study of critical Sobolev type embeddings (cf. [27], [35]).

Using appropriate vanishing properties we identify a closed subspace of $L^{p,\varphi}(\mathbb{R}^n)$ whose elements may be approximated by C_0^∞-functions in the generalized Morrey norm. We also show how the various Morrey subspaces, including a generalized version of Zorko class, are related to each other by proving embeddings and presenting examples showing its strictness. Moreover, we prove that such vanishing properties are preserved by convolution operators with integrable kernels.

The paper is organized as follows. After some notation and preliminaries on generalized Morrey spaces, in Section 3 we introduce a generalization of the Zorko space and new generalized vanishing Morrey subspaces. The invariance of such subspaces with respect to convolution operators with integrable kernels is studied in Section 4. One of the main results in this section asserts that the convolution of Morrey functions with some special good kernels always produce bounded functions. The relation between all the subspaces is discussed in Section 5, including examples showing the difference between them. The main results on approximation are given in Section 6. In this section we study the approximation by nice functions in various generalized Morrey subspaces. The approximation of Morrey functions having all the vanishing properties by C_0^∞-functions in generalized Morrey norm is of special interest. In particular, the set of such Morrey functions provides an explicit description of the closure of $C_0^\infty(\mathbb{R}^n)$ in the generalized Morrey spaces $L^{p,\varphi}(\mathbb{R}^n)$.

2. Preliminaries on generalized Morrey spaces

We use the following notation: $B(x,r)$ is the open ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ and radius $r > 0$. If $E \subseteq \mathbb{R}^n$ is a measurable set, then $|E|$ denotes its (Lebesgue) measure and χ_E denotes its characteristic function. The measure of the unit ball in $\mathbb{R}^n$ is denoted by ω_n and S^{n-1} stands for the unit sphere. We use the notation $X \hookrightarrow Y$ for continuous embeddings from the normed space X into the normed space Y. We use c as a generic positive constant, i.e., a constant whose value may change with each appearance. The expression $f \lesssim g$ means that $f \le c\, g$ for some independent constant c, and $f \approx g$ means $f \lesssim g \lesssim f$.

A function $\varphi : (0,\infty) \to (0,\infty)$ is said to be *almost increasing* (resp., *almost decreasing*) if $\varphi(s) \lesssim \varphi(t)$ (resp., $\varphi(s) \gtrsim \varphi(t)$) for $s \le t$.

As usual the class $C_0^\infty(\mathbb{R}^n)$ consists of all complex-valued infinitely differentiable functions on $\mathbb{R}^n$ with compact support, and $L^p(\mathbb{R}^n)$ stands for the classical Lebesgue space normed by

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx\right)^{1/p}, \qquad 1 \le p < \infty.$$

In the sequel we use the following notation for locally integrable functions f on $\mathbb{R}^n$:

$$\mathfrak{M}_{p,\varphi}(f;x,r) := \frac{1}{\varphi(r)} \int_{B(x,r)} |f(y)|^p \, dy\,, \qquad x \in \mathbb{R}^n, \quad r > 0. \tag{2.1}$$

Definition 2.1. *Let $1 \le p < \infty$ and let $\varphi : (0,\infty) \to (0,\infty)$ be a measurable function. The generalized Morrey space $L^{p,\varphi}(\mathbb{R}^n)$ consists of all locally p-integrable functions f on $\mathbb{R}^n$ with finite norm*

$$\|f\|_{p,\varphi} := \sup_{x \in \mathbb{R}^n,\, r>0} \mathfrak{M}_{p,\varphi}(f;x,r)^{1/p}. \tag{2.2}$$

Such a spaces already appeared in [15] and [17]. Nowadays one can find many generalizations of Morrey spaces in the literature. We refer to survey paper [22] and [28] for further references and historical remarks. The interest in studying generalized Morrey spaces comes not only from theoretical reasons, but also from their important role in applications (cf. [29]), including the study of optimal Sobolev type embeddings for critical exponents (see, for instance, [27] and [35]).

If $\varphi(r) = r^\lambda$, $0 \le \lambda \le n$, then $L^{p,\varphi}(\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n)$ are the well-known classical Morrey spaces, which in turn coincide with $L^p(\mathbb{R}^n)$ when $\lambda = 0$.

It is not hard to see that

$$L^{p,\varphi}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n) \quad \text{if} \quad \sup_{r>0} \frac{\varphi(r)}{r^n} < \infty$$

and

$$L^\infty(\mathbb{R}^n) \hookrightarrow L^{p,\varphi}(\mathbb{R}^n) \quad \text{if} \quad \inf_{r>0} \frac{\varphi(r)}{r^n} > 0.$$

Consequently,

$$L^{p,\varphi}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n) \quad \text{when} \quad \varphi(r) \approx r^n.$$

In general we have to require some assumptions on the function parameter φ in order to ensure good properties for the spaces $L^{p,\varphi}(\mathbb{R}^n)$. Let us consider the following class.

Definition 2.2. *The class Φ consists of all measurable functions $\varphi : (0,\infty) \to (0,\infty)$ such that*

(1) *φ is almost increasing;*
(2) *$\varphi(t)/t^n$ is almost decreasing.*
(3) *$\inf_{t>\delta} \varphi(t) > 0$ for every $\delta > 0$.*

The conditions above defining the class Φ are widely used in papers on generalized Morrey spaces. As observed in [18], if $\varphi \in \Phi$ then the space $L^{p,\varphi}(\mathbb{R}^n)$ is non-trivial. For example, characteristic functions on balls belong to $L^{p,\varphi}(\mathbb{R}^n)$ under such assumptions on φ. Moreover, for any $x_0 \in \mathbb{R}^n$ and $r_0 > 0$ there holds

$$\|\chi_{B(x_0,r_0)}\|_{p,\varphi}^p \approx \frac{r_0^n}{\varphi(r_0)}, \qquad x_0 \in \mathbb{R}^n, \quad r_0 > 0 \tag{2.3}$$

(cf. [9, Proposition A]). It is clear that if $\varphi \lesssim \psi$ then $L^{p,\varphi}(\mathbb{R}^n) \hookrightarrow L^{p,\psi}(\mathbb{R}^n)$. Moreover, using (2.3) we can see that, for $\varphi, \psi \in \Phi$, the condition $\varphi \lesssim \psi$ is also necessary for the above embedding.

There are Morrey funtions on $\mathbb{R}^n$ which are not in $L^p(\mathbb{R}^n)$. Indeed, if $\varphi \in \Phi$ and, for some $\varepsilon > 0$, the function $\varphi(t)/t^\varepsilon$ is almost increasing, then it is known that

$$\left(\frac{\varphi(|x|)}{|x|^n}\right)^{1/p} \in L^{p,\varphi}(\mathbb{R}^n) \qquad \text{but} \qquad \left(\frac{\varphi(|x|)}{|x|^n}\right)^{1/p} \notin L^p(\mathbb{R}^n), \tag{2.4}$$

see, for example, [4, Lemma 2.1] and [10, Lemma 2.4].

3. New and known Morrey subspaces

It is known that approximations to the identity do not behave well in Morrey spaces, since these spaces may contain functions with singularities like (2.4). The lack of such property has motivated the consideration of appropriate Morrey subspaces mainly in the case of power functions $\varphi(r) = r^\lambda$. For example, it was introduced in [37] the subset $\mathbb{L}^{p,\lambda}(\mathbb{R}^n)$ consisting of all Morrey functions for which the translation is continuous in Morrey norm. We introduce a generalized Zorko subspace as follows:

Definition 3.1. *For $1 \le p < \infty$ and $\varphi \in \Phi$, we consider*

$$\mathbb{L}^{p,\varphi}(\mathbb{R}^n) := \left\{ f \in L^{p,\varphi}(\mathbb{R}^n) : \quad \|\tau_\xi f - f\|_{p,\varphi} \to 0 \quad \textit{as} \quad \xi \to 0 \right\}, \tag{3.1}$$

where $\tau_\xi f := f(\cdot - \xi)$, $\xi \in \mathbb{R}^n$.

We also consider some Morrey classes by using vanishing type properties related to the behavior of (2.1) at the origin and at infinity.

Definition 3.2. *Let $1 \le p < \infty$ and $\varphi \in \Phi$. The class $V_0L^{p,\varphi}(\mathbb{R}^n)$ consists of all those functions $f \in L^{p,\varphi}(\mathbb{R}^n)$ such that*

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{M}_{p,\varphi}(f; x, r) = 0. \tag{V_0}$$

Similarly, $V_\infty L^{p,\varphi}(\mathbb{R}^n)$ is the set of all $f \in L^{p,\varphi}(\mathbb{R}^n)$ such that

$$\lim_{r\to \infty} \sup_{x\in\mathbb{R}^n} \mathfrak{M}_{p,\varphi}(f; x, r) = 0. \tag{V_∞}$$

We shall call $V_0L^{p,\varphi}(\mathbb{R}^n)$ and $V_\infty L^{p,\varphi}(\mathbb{R}^n)$ *vanishing Morrey spaces*, at the origin and at infinity, respectively. As usual, in the classical case of $\varphi(r) = r^\lambda$ we write $V_0L^{p,\lambda}(\mathbb{R}^n)$ and $V_\infty L^{p,\lambda}(\mathbb{R}^n)$.

While the space $V_0L^{p,\lambda}(\mathbb{R}^n)$ was introduced by Chiarenza and Franciosi [8] (on bounded domains) in the study of elliptic equations (see also Vitanza [33, 34] for regularity results for elliptic equations with coefficients in such subspace), the vanishing space $V_\infty L^{p,\lambda}(\mathbb{R}^n)$ was recently introduced in [3] and in [36].

Remark 3.3. The generalized vanishing space $V_0L^{p,\varphi}(\mathbb{R}^n)$ has been already considered in other papers (see, for instance, [6], [20], [23], [26] for harmonic analysis in such spaces). As regards its generalized counterpart at infinity, $V_\infty L^{p,\varphi}(\mathbb{R}^n)$, up to author's knowledge it is being consider for the first time in the present paper.

The space $V_0L^{p,\varphi}(\mathbb{R}^n)$ is non-trivial if $\varphi \in \Phi$ and

$$\lim_{r\to 0} \frac{r^n}{\varphi(r)} = 0 \tag{3.2}$$

since then it contains bounded functions with compact support. Note also that bounded Morrey functions always satisfy the vanishing property (V_0) when (3.2) holds. We state this property as a separate lemma.

Lemma 3.4. *Let $1 \le p < \infty$ and let $\varphi \in \Phi$. If, in addition, φ satisfies* (3.2)*, then*

$$L^{p,\varphi}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \subset V_0L^{p,\varphi}(\mathbb{R}^n).$$

As regards to $V_\infty L^{p,\varphi}(\mathbb{R}^n)$, it is not hard to see that it is non-trivial if $\frac{\varphi(r)}{r^\varepsilon}$ is almost increasing for some $\varepsilon > 0$ and $\inf_{r>\delta} \varphi(r) > 0$ for all $\delta > 0$. In fact, under such assumptions on φ, compactly supported bounded functions belong to this vanishing space. We also observe the obvious fact that

$$V_0L^{p,\varphi}(\mathbb{R}^n) = L^p(\mathbb{R}^n) \qquad \text{when} \qquad \varphi(r) \approx 1.$$

Following the classical case studied in [3], we also introduce a new subspace by using another vanishing property related to truncations in large balls. Below we use the notation

$$\mathcal{A}_{N,p}(f) := \sup_{x\in\mathbb{R}^n} \int_{B(x,1)} |f(y)|^p\, \chi_N(y)\, dy\,, \qquad \text{where } \chi_N := \chi_{\mathbb{R}^n\setminus B(0,N)}\,, \quad N \in \mathbb{N}.$$

Definition 3.5. *For $1 \le p < \infty$ and $\varphi \in \Phi$, we define $V^{(*)}L^{p,\varphi}(\mathbb{R}^n)$ as the set of all functions $f \in L^{p,\varphi}(\mathbb{R}^n)$ having the vanishing property*

$$\lim_{N\to\infty} \mathcal{A}_{N,p}(f) = 0. \tag{V^*}$$

Remark 3.6. As shown in [3, Lemma 3.4], a Morrey function f satisfies property (V^*) if and only if

$$\lim_{N\to\infty} \sup_{x\in\mathbb{R}^n} \int_{B(x,r)} |f(y)|^p\, \chi_N(y)\, dy = 0 \tag{3.3}$$

uniformly in $r \in]0, R_0]$, for any fixed $R_0 > 0$. Note also that the generalized parameter φ does not interfere in the vanishing property (V^*) itself.

By the Lebesgue dominated convergence theorem, we can see that every L^p-function has property (V^*) and hence $V^{(*)}L^{p,\varphi}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ when $\varphi(r) \approx 1$. Nevertheless, there are Morrey functions which fail to have this vanishing property. An example of such a function is given in Example 5.4 below.

The subspaces $V_0L^{p,\varphi}(\mathbb{R}^n)$ and $V^{(*)}L^{p,\varphi}(\mathbb{R}^n)$ make difference only when we are dealing with Morrey spaces different from Lebesgue spaces. Indeed, it is easy to check that all L^p-functions have the vanishing properties (V_0) and (V^*). On the other hand, the translation operator is continuous in L^p-norm. Therefore, for any $p \in [1, \infty)$ we have the coincidence

$$\mathbb{L}^{p,\varphi}(\mathbb{R}^n) = V_0L^{p,\varphi}(\mathbb{R}^n) = V^{(*)}L^{p,\varphi}(\mathbb{R}^n) = L^p(\mathbb{R}^n) \qquad \text{if} \quad \varphi(r) \approx 1.$$

We end this section by defining a smaller Morrey class which will play an important role later on.

Definition 3.7. *Let $1 \le p < \infty$ and $\varphi \in \Phi$. We set*

$$V^{(*)}_{0,\infty} L^{p,\varphi}(\mathbb{R}^n) := V_0 L^{p,\varphi}(\mathbb{R}^n) \cap V_\infty L^{p,\varphi}(\mathbb{R}^n) \cap V^{(*)} L^{p,\varphi}(\mathbb{R}^n).$$

Remark 3.8. It can be shown that all the vanishing subsets $\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$, $V_0 L^{p,\varphi}(\mathbb{R}^n)$, $V_\infty L^{p,\varphi}(\mathbb{R}^n)$ and $V^{(*)} L^{p,\varphi}(\mathbb{R}^n)$ (and consequently, $V^{(*)}_{0,\infty} L^{p,\varphi}(\mathbb{R}^n)$) are closed in $L^{p,\varphi}(\mathbb{R}^n)$ (see [3, Lemma 3.7] for a proof in the classical case).

4. Convolution in Morrey spaces

The classical Young's inequality for convolutions with kernels in $L^1(\mathbb{R}^n)$, known for L^p-spaces, holds also for Morrey spaces. In a more general definition of Morrey spaces, Young's inequality was recently studied in [5]. Here we are interested in the preservation of the vanishing properties (V_0), (V_∞) and (V^*) by convolution operators.

The following two lemmas can be proved using Minskowski's inequality and a simple change of variables. Further details can be found in the proof of [3, Theorem 3.8].

Lemma 4.1. *Let $1 \le p < \infty$ and $\varphi \in \Phi$. If $\mathcal{K} \in L^1(\mathbb{R}^n)$ and f is locally p-integrable on $\mathbb{R}^n$, then*

$$\mathfrak{M}_{p,\varphi}\big(\mathcal{K} * f; x, r\big) \le \|\mathcal{K}\|_1^p \sup_{z \in \mathbb{R}^n} \mathfrak{M}_{p,\varphi}(f; z, r) \tag{4.1}$$

for every $x \in \mathbb{R}^n$ and $r > 0$. Consequently,

$$\|\mathcal{K} * f\|_{p,\varphi} \le \|\mathcal{K}\|_1 \, \|f\|_{p,\varphi}.$$

In the next lemma we use the interpretation

$$\mathcal{A}_{a,p}(f) := \sup_{x \in \mathbb{R}^n} \int_{B(x,1)} |f(y)|^p \, \chi_a(y) \, dy \,, \qquad a \in \mathbb{R},$$

where

$$\chi_a := \chi_{\mathbb{R}^n \setminus B(0,a)} \text{ if } a > 0 \quad \text{and} \quad \chi_a \equiv 1 \text{ if } a \le 0.$$

Lemma 4.2. *Let $1 \le p < \infty$. If $\mathcal{K} \in L^1(\mathbb{R}^n)$ and f is locally integrable on $\mathbb{R}^n$, then*

$$\big[\mathcal{A}_{N,p}(\mathcal{K} * f)\big]^{1/p} \le \int_{\mathbb{R}^n} |\mathcal{K}(z)| \, \big[\mathcal{A}_{N-|z|,p}(f)\big]^{1/p} \, dz, \tag{4.2}$$

for any $N \in \mathbb{N}$.

From (4.1) we easily see that both vanishing properties, at the origin and at infinity, are preserved by convolutions with integrable kernels. Moreover, the same holds for the vanishing property (V^*). Indeed, by (4.2) and the Lebesgue dominated convergence theorem we conclude that

$$\mathcal{A}_{N,p}(\mathcal{K} * f) \to 0 \quad \text{as} \quad N \to \infty.$$

Corollary 4.3. *Let $1 \le p < \infty$ and $\varphi \in \Phi$. Then the Morrey subspaces $V_0 L^{p,\varphi}(\mathbb{R}^n)$, $V_\infty L^{p,\varphi}(\mathbb{R}^n)$ and $V^{(*)} L^{p,\varphi}(\mathbb{R}^n)$ are all invariant with respect to convolution operators with a kernel $\mathcal{K} \in L^1(\mathbb{R}^n)$:*

$$f \in V_0 L^{p,\varphi}(\mathbb{R}^n) \implies \mathcal{K} * f \in V_0 L^{p,\varphi}(\mathbb{R}^n)\,, \qquad f \in V_\infty L^{p,\varphi}(\mathbb{R}^n) \implies \mathcal{K} * f \in V_\infty L^{p,\varphi}(\mathbb{R}^n)$$

$$\textit{and} \qquad f \in V^{(*)} L^{p,\varphi}(\mathbb{R}^n) \implies \mathcal{K} * f \in V^{(*)} L^{p,\varphi}(\mathbb{R}^n).$$

We already know (by Lemma 4.1) that the convolution operator with an integrable kernel is bounded on $L^{p,\varphi}(\mathbb{R}^n)$. The next result shows that the outcome convolution of a Morrey function is bounded when the kernel has additional good properties.

Theorem 4.4. *Let $1 \le p < \infty$ and $\varphi \in \Phi$. Let $\mathcal{K}(x) = \widetilde{k}(|x|)$ be an integrable kernel in $\mathbb{R}^n$ with $\widetilde{k}$ a non-negative C^1 function on $[0,\infty)$. Suppose that*

$$\widetilde{k}(r)\, r^{\frac{n}{p'}}\, \varphi(r)^{\frac{1}{p}} \to 0 \quad \text{as} \quad r \to \infty \tag{4.3}$$

and

$$C(\mathcal{K},\varphi,p) := \int_0^\infty |\widetilde{k}'(r)|\, r^{\frac{n}{p'}}\, \varphi(r)^{\frac{1}{p}} dr < \infty. \tag{4.4}$$

Then the convolution operator with kernel $\mathcal{K}$ is bounded from $L^{p,\varphi}(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ and

$$\|\mathcal{K} * f\|_\infty \le \omega_n^{\frac{1}{p'}}\, C(\mathcal{K},\varphi,p)\, \|f\|_{p,\varphi}$$

for every $f \in L^{p,\varphi}(\mathbb{R}^n)$.

Proof. Passing to polar coordinates, we get

$$\int_{\mathbb{R}^n} \mathcal{K}(y)\, |f(x-y)|\, dy = \int_0^\infty \int_{S^{n-1}} \widetilde{k}(r)\, |f(x-r\sigma)|\, r^{n-1}\, d\sigma\, dr = \int_0^\infty \widetilde{k}(r)\, dF(r) \tag{4.5}$$

where

$$F(r) = \int_0^r G(s)\, ds \quad \text{with} \quad G(s) = \int_{S^{n-1}} |f(x-s\sigma)|\, s^{n-1}\, d\sigma.$$

Integrating by parts we can write the right-hand side of (4.5) as

$$-\int_0^\infty \widetilde{k}'(r)\, F(r)\, dr = -\int_0^\infty \widetilde{k}'(r) \int_{B(0,r)} |f(x-y)|\, dy\, dr = -\int_0^\infty \widetilde{k}'(r) \int_{B(x,r)} |f(y)|\, dy\, dr,$$

where we noted (by Hölder's inequality) that

$$F(r) = \int_{B(x,r)} |f(y)|\, dy \le (\omega_n r^n)^{\frac{1}{p'}}\, \varphi(r)^{\frac{1}{p}}\, \|f\|_{p,\varphi}$$

and used (4.3) in particular. Therefore, with the help of Hölder's inequality and condition (4.4), the convolution may be estimated as follows at any $x \in \mathbb{R}^n$:

$$|\mathcal{K} * f(x)| \le \int_0^\infty |\widetilde{k}'(r)| \int_{B(x,r)} |f(y)|\, dy\, dr \le \omega_n^{\frac{1}{p'}}\, C(\mathcal{K},\varphi,p)\, \|f\|_{p,\varphi}$$

The claim is proved. □

Remark 4.5. An example of a radial kernel satisfying the assumptions of Theorem 4.4 is the heat kernel

$$\mathcal{K}_t(x) = c_n\, t^{-\frac{n}{2}}\, e^{-\frac{|x|^2}{4t}}, \qquad t > 0, \tag{4.6}$$

where $c_n > 0$ is a certain normalization constant depending on n. This special case was considered in [14] in the case of classical Morrey spaces.

We end this section with an observation on the approximation of Morrey functions with Zorko property by standard mollifiers. This will be useful later on.

Consider the usual dilations $\mathcal{K}_t(x) = t^{-n}\mathcal{K}(x/t)$, $t > 0$, where $\mathcal{K}$ is an integrable function on $\mathbb{R}^n$ with $\|\mathcal{K}\|_1 = 1$. By straightforward calculations and Minkowski's integral inequality we get

$$\|f * \mathcal{K}_t - f\|_{p,\varphi} \le \int_{\mathbb{R}^n} \|\tau_{tz} f - f\|_{p,\varphi}\, |\mathcal{K}(z)|\, dz.$$

If $f \in \mathbb{L}^{p,\varphi}(\mathbb{R}^n)$ we have $\|\tau_{tz}f - f\|_{p,\varphi} \to 0$ as $t \to 0$. Since $\|\tau_{tz}f - f\|_{p,\varphi} \leq 2\|f\|_{p,\varphi}$ for any $t > 0$ and $z \in \mathbb{R}^n$, we see that

$$\|f * \mathcal{K}_t - f\|_{p,\varphi} \to 0 \quad \text{as} \quad t \to 0 \tag{4.7}$$

by the Lebesgue theorem.

5. Strict embeddings between Morrey subspaces

5.1. **Examples and embeddings between vanishing Morrey spaces.** We consider the families of functions given by

$$g_\alpha(x) := \left(\frac{\varphi(|x|)}{|x|^{n-\alpha}}\right)^{1/p} \chi_{B(0,1)}(x) \qquad \text{and} \qquad h_\beta(x) := \left(\frac{\varphi(|x|)}{|x|^{n+\beta}}\right)^{1/p} \chi_{\mathbb{R}^n \setminus B(0,1)}(x)$$

for $\alpha, \beta \geq 0$. In the limiting cases $\alpha = 0 = \beta$ we already know that that $g_\alpha, h_\beta \in L^{p,\varphi}(\mathbb{R}^n)$ when $\varphi \in \Phi$ and $\frac{\varphi(t)}{t^\varepsilon}$ is almost increasing for some $\varepsilon > 0$ (cf. [4, Lemma 2.1]). For the non-limiting cases we have the following lemma:

Lemma 5.1. *Let $\varphi \in \Phi$ and $1 \leq p < \infty$.*
(i) If $\alpha > 0$ and $\varphi(t)/t^{n-\delta}$ is almost decreasing for some $\delta \in (0, \alpha]$, then $g_\alpha \in V_0 L^{p,\varphi}(\mathbb{R}^n)$.
(ii) If $\beta > 0$ and $\varphi(t)/t^\varepsilon$ is almost increasing for some $\varepsilon > \beta$, then $h_\beta \in V_\infty L^{p,\varphi}(\mathbb{R}^n)$.

Proof. Step 1: We prove (i). Let $x \in \mathbb{R}^n$ and $r > 0$.
If $|x| < 2r$ we have

$$\mathfrak{M}_{p,\varphi}(g_\alpha; x, r) \leq \frac{1}{\varphi(r)} \int_{B(0,\min\{1,3r\})} \frac{\varphi(|y|)}{|y|^{n-\alpha}}\, dy$$

since $B(x,r) \subset B(0,3r)$. Passing to polar coordinates in the last integral and using the fact that φ is almost increasing and $\varphi(t)/t^n$ is almost decreasing, we get

$$\mathfrak{M}_{p,\varphi}(g_\alpha; x, r) \lesssim \min\{1, r\}^\alpha, \tag{5.1}$$

with the implicit constant independent of x and r.
In the case $|x| \geq 2r$ we have $|y| > |x - y|$. Hence

$$\mathfrak{M}_{p,\varphi}(g_\alpha; x, r) \leq \frac{1}{\varphi(r)} \int_{B(x,r)\cap B(x,1)} \frac{\varphi(|y|)}{|y|^{n-\alpha}} \chi_{B(0,1)}(y)\, dy \leq \frac{1}{\varphi(r)} \int_{B(x,\min\{1,r\})} \frac{\varphi(|y|)}{|y|^{n-\delta}}\, dy$$

where we used that $\delta \leq \alpha$. Since $\varphi(t)/t^{n-\delta}$ is almost decreasing the last integral can be estimated from above by

$$\int_{B(x,\min\{1,r\})} \frac{\varphi(|x-y|)}{|x-y|^{n-\delta}}\, dy = \int_{B(0,\min\{1,r\})} \frac{\varphi(|z|)}{|z|^{n-\delta}}\, dz.$$

Passing again to polar coordinates and observing that φ is almost increasing, we obtain the final estimate

$$\mathfrak{M}_{p,\varphi}(g_\alpha; x, r) \lesssim \min\{1, r\}^\delta. \tag{5.2}$$

By (5.1) and (5.2) we see that $g_\alpha \in V_0 L^{p,\varphi}(\mathbb{R}^n)$.
Step 2: We prove (ii). If $|x| < 2r$ we use again that $B(x,r) \subset B(0,3r)$ and obtain

$$\mathfrak{M}_{p,\varphi}(h_\beta; x, r) \leq \frac{1}{\varphi(r)} \int_{1\leq|y|\leq 3r} \frac{\varphi(|y|)}{|y|^{n+\beta}}\, dy$$

when $3r \geq 1$ (otherwise the integral vanishes taking into account the definition of h_β). Writing the last integral in polar coordinates and noting that $\varphi(t)/t^\varepsilon$ is almost increasing and that $\varphi(t)/t^n$ is almost decreasing, we estimate the right-hand side as

$$\frac{1}{\varphi(r)} \int_{1\leq |y| \leq 3r} \frac{\varphi(|y|)}{|y|^{n+\beta}}\, dy \lesssim \frac{\varphi(3r)}{\varphi(r)\, r^\varepsilon} \int_1^{3r} \rho^{\varepsilon-\beta-1} d\rho \lesssim r^{-\beta}$$

Hence, in the case $|x| < 2r$, there holds the estimate

$$\mathfrak{M}_{p,\varphi}(h_\beta; x, r) \lesssim \max\{1, r\}^{-\beta}, \tag{5.3}$$

with the implicit constant not depending on x and r. We can check that estimate (5.3) also holds in the remaining case $|x| > 2r$. This can be handled following the corresponding case in Step 1, but now using that $\varphi(t)/t^n$ is almost decreasing and $\varphi(t)/t^\varepsilon$ is almost increasing (with $\beta < \varepsilon$). □

The previous lemma and similar calculations used in its proof prompt us to exhibit the following example which summarizes the main conclusions on the preciseness of embeddings.

Example 5.2. Let $1 \leq p < \infty$ and $\varphi \in \Phi$. Define

$$f_{\alpha,\beta}(x) := g_\alpha(x) + h_\beta(x) = \begin{cases} \left(\frac{\varphi(|x|)}{|x|^{n-\alpha}}\right)^{1/p}, & |x| < 1, \\ \left(\frac{\varphi(|x|)}{|x|^{n+\beta}}\right)^{1/p}, & |x| \geq 1. \end{cases}$$

If $\alpha, \beta > 0$ and there exist $\delta \in (0, \alpha]$ and $\varepsilon > \beta$ such that $\varphi(r)/r^{n-\delta}$ is almost decreasing and $\varphi(r)/r^\varepsilon$ is almost increasing, then

$$f_{\alpha,\beta} \in V_0 L^{p,\varphi}(\mathbb{R}^n) \cap V_\infty L^{p,\varphi}(\mathbb{R}^n).$$

Moreover,

$$f_{\alpha,0} \in V_0 L^{p,\varphi}(\mathbb{R}^n) \quad \text{but} \quad f_{\alpha,0} \notin V_\infty L^{p,\varphi}(\mathbb{R}^n)$$

and

$$f_{0,\beta} \in V_\infty L^{p,\varphi}(\mathbb{R}^n) \quad \text{but} \quad f_{0,\beta} \notin V_0 L^{p,\varphi}(\mathbb{R}^n).$$

In the limiting case $\alpha = 0 = \beta$ there holds

$$f_{\alpha,\beta} \in L^{p,\varphi}(\mathbb{R}^n) \quad \text{but} \quad f_{\alpha,\beta} \notin V_0 L^{p,\varphi}(\mathbb{R}^n) \cup V_\infty L^{p,\varphi}(\mathbb{R}^n),$$

where the failure of the vanishing properties (V_0) and (V_∞) can be seen from the estimate

$$\mathfrak{M}_{p,\varphi}(f_{0,0}; 0, r) = \frac{1}{\varphi(r)} \int_{B(0,r)} \frac{\varphi(|y|)}{|y|^n}\, dy \gtrsim 1$$

(with the implicit constant independent of $r > 0$), which follows by the monotonicity of $\varphi(r)/r^n$.

Theorem 5.3. *Let $1 \leq p < \infty$ and $\varphi \in \Phi$. If, in addition, φ satisfies* (3.2) *and*

$$\lim_{r\to\infty} \frac{\log^p r}{\varphi(r)} = 0, \tag{5.4}$$

then there are functions in $V_0 L^{p,\varphi}(\mathbb{R}^n) \cap V_\infty L^{p,\varphi}(\mathbb{R}^n)$ which do not have property (V^*).

Proof. We take the same example given in [3, Example 3.6]. Let f_0 be the function defined by

$$f_0(x) := \sum_{k=2}^{\infty} \chi_{B_k}(x)\,,$$

where $B_k = B(2^k \mathrm{e}_1, 1)$, $k \in \mathbb{N}$, with $\mathrm{e}_1 = (1, 0, \ldots, 0)$. The calculations involving the vanishing properties are quite similar to those presented in the proof of Theorem 4.1 in [3]. By this reason we shall skip many details here. Nevertheless, for reader's convenience, we present some steps to show where the assumptions on φ are used.

Since $\mathcal{A}_{N,p}(f_0) \geq |B(0,1)|$ for any $N \in \mathbb{N}$, then f_0 fails to have the vanishing property (V^*). In order to see that f_0 belongs to the other vanishing spaces, we need to count the number of balls B_k intersecting $B(x,r)$ (for any fixed $x \in \mathbb{R}^n$ and $r > 0$). If $r \leq 1$ there exists at most one such a ball. Hence

$$\mathfrak{M}_{p,\varphi}(f_0; x, r) \lesssim \frac{r^n}{\varphi(r)} \quad \text{for} \quad r \leq 1, \tag{5.5}$$

with the implicit constant independent of x and r. From (5.5) and (3.2) we conclude that f_0 satisfies property (V_0). The case of large values of r is technically more complicated. Following [3] we get

$$\mathfrak{M}_{p,\varphi}(f_0; x, r) \leq \frac{\log^p(4r)}{\varphi(r)} \quad \text{for} \quad r > 1. \tag{5.6}$$

Therefore, by (5.6) and (5.4), we see that f_0 also satisfies property (V_∞). The proof is complete. $\square$

Let us explicit the example contained in the proof above.

Example 5.4. Let f_0 be the function given by

$$f_0(x) := \sum_{k=2}^{\infty} \chi_{B_k}(x) \qquad (\text{with } B_k = B(2^k \mathrm{e}_1, 1)).$$

If $1 \leq p < \infty$ and $\varphi \in \Phi$ satisfies the conditions (3.2) and (5.4), then we have

$$f_0 \in V_0 L^{p,\varphi}(\mathbb{R}^n) \cap V_\infty L^{p,\varphi}(\mathbb{R}^n) \quad \text{but} \quad f_0 \notin V^{(*)} L^{p,\varphi}(\mathbb{R}^n).$$

Corollary 5.5. *Under the same assumptions of Theorem 5.3, there holds*

$$V_{0,\infty}^{(*)} L^{p,\varphi}(\mathbb{R}^n) \ \subsetneqq\ V_0 L^{p,\varphi}(\mathbb{R}^n) \cap V_\infty L^{p,\varphi}(\mathbb{R}^n) \ \subsetneqq\ V_0 L^{p,\varphi}(\mathbb{R}^n) \ \subsetneqq\ L^{p,\varphi}(\mathbb{R}^n).$$

5.2. **Embeddings involving the generalized Zorko space.** We have seen in the previous section that the new subspace $V_{0,\infty}^{(*)} L^{p,\varphi}(\mathbb{R}^n)$ is strictly smaller than the intersection $V_0 L^{p,\varphi}(\mathbb{R}^n) \cap V_\infty L^{p,\varphi}(\mathbb{R}^n)$. We will see that $V_{0,\infty}^{(*)} L^{p,\varphi}(\mathbb{R}^n)$ is also strictly smaller than the generalized Zorko subspace $\mathbb{L}^{p,\varphi}(\mathbb{R}^n)$ (recall Definition 3.1).

First we observe the connection of the Zorko space with the vanishing space $V_0 L^{p,\varphi}(\mathbb{R}^n)$. The next result is known in the classical case of $\varphi(r) = r^\lambda$, $0 \leq \lambda \leq n$, see [14, Corollary 3.3].

Theorem 5.6. *Let $1 \leq p < \infty$ and $\varphi \in \Phi$. If φ also satisfies* (3.2)*, then*

$$\mathbb{L}^{p,\varphi}(\mathbb{R}^n) \ \subset\ V_0 L^{p,\varphi}(\mathbb{R}^n). \tag{5.7}$$

Proof. Let $f \in \mathbb{L}^{p,\varphi}(\mathbb{R}^n)$ and consider the heat kernel $\mathcal{K}_t(x)$ given by (4.6). Recalling the discussion in the very end of Section 4, we have

$$\|f * \mathcal{K}_t - f\|_{p,\varphi} \to 0 \quad \text{as} \quad t \to 0.$$

On the other hand, by Lemma 4.1 and Theorem 4.4 (and also Remark 4.5), we have, for any $t > 0$,

$$f * \mathcal{K}_t \in L^{p,\varphi}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n).$$

Therefore, $f * \mathcal{K}_t \in V_0 L^{p,\varphi}(\mathbb{R}^n)$ in view of Lemma 3.4. Since $V_0 L^{p,\varphi}(\mathbb{R}^n)$ is closed we conclude that $f \in V_0 L^{p,\varphi}(\mathbb{R}^n)$. □

The strictness of the embedding (5.7) was proved in [14, Example 3.4] in the case of classical Morrey spaces. Below we use the same example from [14] to show that there are functions satisfying both vanishing properties at the origin and at infinity but not having Zorko property, when, in addition to the conditions in Theorem 5.6, φ also satisfies (5.4). It shows, in particular, that embedding (5.7) is strict under appropriate assumptions.

Example 5.7. For simplicity consider the case $n = 1$. Define

$$f_1(x) := \sum_{k=1}^{\infty} \eta_k(x - 2^k) \qquad \text{where} \qquad \eta_k(y) = \sin(2k\pi y)\, \chi_{(0,1)}(y), \quad k \in \mathbb{N}.$$

It is clear that f_1 is bounded and $\|f_1\|_\infty = 1$. Thus, for every x and r, the modular $\mathfrak{M}_{p,\varphi}(f_1; x, r)$ has the same bound as in (5.5). To control the behavior of the modular for large values of r we need to count the number of intervals $(2^k, 1 + 2^k)$ contained in $(x - r, x + r)$. Using similar arguments to those used in the proof of Theorem 5.3, we can arrive at an estimate similar to (5.6). Taking into account the assumptions on φ, we conclude that $f_1 \in V_0 L^{p,\varphi}(\mathbb{R}^n) \cap V_\infty L^{p,\varphi}(\mathbb{R}^n)$.

In order to show that $f_1 \notin \mathbb{L}^{p,\varphi}(\mathbb{R}^n)$ first we calculate $\tau_{\frac{1}{2k}} \eta_k - \eta_k$ in the intervals $(0, 1 + \frac{1}{2k})$, $k \in \mathbb{N}$. Following [14, p.137], after some calculations we get

$$\int_{B\left(2^k + \frac{1}{2}, 1\right)} \left|\tau_{\frac{1}{2k}} f_1(y) - f_1(y)\right|^p dy \ge 4^{-p}.$$

Therefore,

$$\left\|\tau_{\frac{1}{2k}} f_1 - f_1\right\|_{p,\varphi} \ge c_\varphi > 0$$

for some constant c_φ independent of k. Hence we have

$$f_1 \in V_0 L^{p,\varphi}(\mathbb{R}^n) \cap V_\infty L^{p,\varphi}(\mathbb{R}^n) \quad \text{but} \quad f_1 \notin \mathbb{L}^{p,\varphi}(\mathbb{R}^n).$$

In particular, by Theorem 5.6 we have the strict embedding

$$\mathbb{L}^{p,\varphi}(\mathbb{R}^n) \subsetneqq V_0 L^{p,\varphi}(\mathbb{R}^n).$$

Remark 5.8. It is known that the vanishing Morrey space at the origin and the Zorko space coincide on bounded domains. At least in the standard case of power functions $\varphi(t) = t^\lambda$, this follows from [8, Lemma 1.2] and [14, Corollary 3.3]. In a sense the bounded domains setting is much better to deal with since then Morrey functions are always in L^p.

As mentioned above, we want to show that our space $V_{0,\infty}^{(*)} L^{p,\varphi}(\mathbb{R}^n)$ is even strictly smaller than the Zorko space. To this end we need an approximation argument which already touches the main topic of this paper. Note that the approximation by nice functions in Morrey subspaces will be discussed in detail in Section 6 below. By convenience, the proof of the next lemma is postponed to Section 6 (see Step 2 in the proof of Theorem 6.3).

Lemma 5.9. *Let $1 \le p < \infty$ and $\varphi \in \Phi$. Then every function from $V_{0,\infty}^{(*)}L^{p,\varphi}(\mathbb{R}^n)$ can be approximated in Morrey norm by compactly supported functions.*

Theorem 5.10. *For any $1 \le p < \infty$ and $\varphi \in \Phi$ we have $V_{0,\infty}^{(*)}L^{p,\varphi}(\mathbb{R}^n) \subset \mathbb{L}^{p,\varphi}(\mathbb{R}^n)$.*

Proof. Let $f \in V_{0,\infty}^{(*)}L^{p,\varphi}(\mathbb{R}^n)$. For any $\varepsilon > 0$ there exists $g \in L^{p,\varphi}(\mathbb{R}^n)$ with compact support such that

$$\|f - g\|_{p,\varphi} < \varepsilon/4$$

(cf. Lemma 5.9). Since for any $\xi \in \mathbb{R}^n$ we have

$$\|\tau_\xi f - f\|_{p,\varphi} \le 2\,\|f - g\|_{p,\varphi} + \|\tau_\xi g - g\|_{p,\varphi}$$

it suffices to show that the second norm is less that $\varepsilon/2$ for small values of $|\xi|$. By the vanishing properties (V_0) and (V_∞), one can find $r_0, r_1 > 0$, with $r_0 < r_1$, such that

$$S_1 := \sup_{x\in\mathbb{R}^n, 0<r<r_0} \mathfrak{M}_{p,\varphi}(\tau_\xi g - g; x, r) \le 2^p \sup_{x\in\mathbb{R}^n, 0<r<r_0} \mathfrak{M}_{p,\varphi}(g; x, r) < (\varepsilon/2)^p$$

and

$$S_2 := \sup_{x\in\mathbb{R}^n, r>r_1} \mathfrak{M}_{p,\varphi}(\tau_\xi g - g; x, r) \le 2^p \sup_{x\in\mathbb{R}^n, r>r_1} \mathfrak{M}_{p,\varphi}(g; x, r) < (\varepsilon/2)^p.$$

For such fixed r_0 and r_1, we estimate

$$\|\tau_\xi g - g\|_{p,\varphi}^p \le \max\{S_1, S_2, S_3\}$$

with

$$S_3 := \sup_{x\in\mathbb{R}^n, r_0\le r\le r_1} \mathfrak{M}_{p,\varphi}(\tau_\xi g - g; x, r).$$

Since $\inf_{r\ge r_0} \varphi(r) > 0$, we have

$$S_3 \lesssim \sup_{x\in\mathbb{R}^n} \int_{B(x,r_1)} |g(y-\xi) - g(y)|^p\, dy \le \max\left\{ \sup_{|x|<M} (\cdots),\ \sup_{|x|>M} (\cdots) \right\}.$$

where $M > 0$ is chosen below. Since g has compact support there exists $K > 0$ such that $g(u) = 0$ if $|u| > K$. In the case $|x| > M$ we have

$$\int_{|y-x|<r_1} |g(y-\xi) - g(y)|^p\, dy = \int_{|z|<r_1} |g(z+x-\xi) - g(z+x)|^p\, dz.$$

Hence, if we choose $M > r_1 + K + 1$ then $g(z+x) = 0$ and $g(z+x-\xi) = 0$ for small values of $|\xi|$, say $|\xi| < 1$, since

$$|z+x| > M - r_1 > K \quad \text{and} \quad |z+x-\xi| \ge |z+x| - |\xi| > K.$$

Let then $M > r_1 + K + 1$ be fixed and let us now estimate the integral when $|x| < M$. In this case we are just taking the L^p-norm on a ball centered at the origin with fixed radius, precisely $B(0, r_1 + M + 1)$, again for $|\xi| < 1$. Therefore we also obtain

$$S_3 < (\varepsilon/2)^p$$

by the continuity of the L^p-norm with respect to translations. □

We end this section with the following chain of strict embeddings:

Corollary 5.11. *Let $1 \le p < \infty$ and $\varphi \in \Phi$ satisfying* (3.2) *and* (5.4). *Then*

$$V_{0,\infty}^{(*)}L^{p,\varphi}(\mathbb{R}^n) \subsetneqq \mathbb{L}^{p,\varphi}(\mathbb{R}^n) \subsetneqq V_0L^{p,\varphi}(\mathbb{R}^n) \subsetneqq L^{p,\varphi}(\mathbb{R}^n).$$

6. Approximation in subspaces of generalized Morrey spaces

As observed by Zorko [37] in the classical case $\varphi(r) = r^\lambda$, $\lambda \in (0,n)$, there are Morrey functions that cannot be approximated even by continuous functions. It is the case of functions with the form (2.4). This fact has motivated the introduction of the subspace $\mathbb{L}^{p,\lambda}(\mathbb{R}^n)$. It turns out that even in this subspace we can not approximate (in Morrey norm) by infinitely differentiable compactly supported functions.

Recalling the details given in the end of Section 4, one knows that if $f \in \mathbb{L}^{p,\varphi}(\mathbb{R}^n)$ and $\mathcal{K} \in L^1(\mathbb{R}^n)$, with $\|\mathcal{K}\|_1 = 1$, then $\|f * \mathcal{K}_t - f\|_{p,\varphi} \to 0$ as $t \to 0$ (cf. (4.7)). If, in addition, the kernel $\mathcal{K}$ is smooth, say $\mathcal{K}$ is a Schwartz function, then the mollifiers $f * \mathcal{K}_t \in \mathbb{L}^{p,\varphi}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ for any $t > 0$ (note that the convolution is invariant with respect to translations). Consequently, we derive the following result:

Theorem 6.1. *Let $\varphi \in \Phi$ and $1 \le p < \infty$. Then every Morrey function with Zorko property can be approximated in Morrey norm by C^∞-functions. Moreover, we have*

$$\overline{\mathbb{L}^{p,\lambda}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)} = \mathbb{L}^{p,\varphi}(\mathbb{R}^n).$$

Now we discuss the approximation of Morrey functions having both vanishing properties at the origin and at infinity.

Theorem 6.2. *Let $\varphi \in \Phi$ and $1 \le p < \infty$. If $f \in V_0L^{p,\varphi}(\mathbb{R}^n) \cap V_\infty L^{p,\varphi}(\mathbb{R}^n)$ is uniformly continuous, then f can be approximated in Morrey norm by functions from $V_0L^{p,\varphi}(\mathbb{R}^n) \cap V_\infty L^{p,\varphi}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$.*

Proof. Take a Schwartz kernel $\mathcal{K}$ with $\|\mathcal{K}\|_1 = 1$. Then, by Corollary 4.3, we have $f * \mathcal{K}_t \in V_0L^{p,\varphi}(\mathbb{R}^n) \cap V_\infty L^{p,\varphi}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ for any $t > 0$.
It remains to show that $f * \mathcal{K}_t \to f$ in $L^{p,\varphi}(\mathbb{R}^n)$ as $t \to 0$. Let $\varepsilon > 0$. For any $x \in \mathbb{R}^n$, $r > 0$ and $t > 0$, we have

$$\left(\int_{B(x,r)} |(f * \mathcal{K}_t)(y) - f(y)|^p \, dy\right)^{1/p} \le \int_{\mathbb{R}^n} |\mathcal{K}_t(z)| \left(\int_{B(x,r)} |f(y-z) - f(y)|^p \, dy\right)^{1/p} dz.$$

Since $f \in V_0L^{p,\varphi}(\mathbb{R}^n) \cap V_\infty L^{p,\varphi}(\mathbb{R}^n)$, there are $r_0, r_1 > 0$ such that

$$\frac{1}{\varphi(r)} \int_{B(x,r)} |f(y-z) - f(y)|^p \, dy < \varepsilon$$

for every $r < r_0$ or $r > r_1$ (and all $x, z \in \mathbb{R}^n$). We also have

$$\sup_{r>0} \frac{1}{\varphi(r)} \int_{B(x,r)} |(f * \phi_t)(y) - f(y)|^p \, dy \le \frac{1}{\inf_{r \ge r_0} \varphi(r)} \max\{\varepsilon, S_{r_0,r_1}(x,t)^p\}, \tag{6.1}$$

where

$$S_{r_0,r_1}(x,t) := \int_{\mathbb{R}^n} |\mathcal{K}_t(z)| \left(\int_{B(x,r_1)} |f(y-z) - f(y)|^p \, dy\right)^{1/p} dz.$$

By the uniform continuity of f one can find $\delta > 0$ such that $|f(y-z) - f(y)| < \varepsilon$ for any y and z with $|z| < \delta$. For such fixed δ we split the outer integral above into

$$S_{r_0,r_1}(x,t) = \int_{|z|<\delta} (\cdots)\, dz + \int_{|z|\ge\delta} (\cdots)\, dz. \tag{6.2}$$

For the first integral we use the uniform continuity of f and get

$$\int_{|z|<\delta} (\cdots)\, dz \le \varepsilon \, |B(x,r_1)|^{1/p} \int_{|z|<\delta} |\mathcal{K}_t(z)| \, dz \le |B(0,1)| \, r_1^{n/p} \, \varepsilon \tag{6.3}$$

for every $x \in \mathbb{R}^n$ and $t > 0$. In the second integral in (6.2), we use the fact that $f \in L^{p,\varphi}(\mathbb{R}^n)$ and

$$\int_{|z|\geq\delta} |\mathcal{K}_t(z)|\, dz \lesssim \int_{|z|\geq\delta} \frac{t}{|z|^{n+1}}\, dz \lesssim t$$

(where the implicit constant is independent of t) to derive the inequality

$$\int_{|z|\geq\delta} (\cdots)\, dz \lesssim \sup_{r\in[r_0,r_1]} \varphi(r)^{1/p}\, t\, \|f\|_{p,\varphi} \tag{6.4}$$

with the implicit constant depending only on $\mathcal{K}$, n and δ. Taking into account (6.3) and (6.4) in (6.2), from (6.1) we obtain

$$\sup_{x\in\mathbb{R}^n, r>0} \mathfrak{M}_{p,\varphi}(f * \mathcal{K}_t - f; x, r) \lesssim \varepsilon$$

for sufficiently small $t > 0$. This implies $\|f * \mathcal{K}_t - f\|_{p,\varphi} \to 0$ as $t \to 0$, and hence the proof is complete. □

Finally we discuss the approximation of Morrey functions having all the vanishing properties. In particular, property (V^*) allow us to approximate by compactly supported functions.

Theorem 6.3. *Let $\varphi \in \Phi$ and $1 \leq p < \infty$. Then every function in $V_{0,\infty}^{(*)} L^{p,\varphi}(\mathbb{R}^n)$ can be approximated in Morrey norm by C_0^∞-functions.*

Proof. We split the proof into two steps.

Step 1: The claim holds true for functions $f \in V_{0,\infty}^{(*)} L^{p,\varphi}(\mathbb{R}^n)$ with compact support. In fact, if we take a C_0^∞ kernel $\mathcal{K}$, then the mollifiers $f * \mathcal{K}_t$ have compact support and belong to $V_{0,\infty}^{(*)} L^{p,\varphi}(\mathbb{R}^n)$ (cf. Corollary 4.3). Moreover, they approximate f in Morrey norm (recall again the discussion in the end of Section 4 leading to (4.7)).

Step 2: We show now that functions from $V_{0,\infty}^{(*)} L^{p,\varphi}(\mathbb{R}^n)$ can be approximated by compactly supported functions in Morrey norm. Let $f \in V_{0,\infty}^{(*)} L^{p,\varphi}(\mathbb{R}^n)$. As before let $\chi_k := \chi_{\mathbb{R}^n\setminus B(0,k)}$, $k \in \mathbb{N}$. For each k, set

$$f_k = f \quad \text{on the ball } B(0,k) \qquad \text{and} \quad f_k = 0 \ \text{ otherwise.}$$

Let $\varepsilon > 0$ be arbitrary. Again by the vanishing properties (V_0) and (V_∞) there exist $r_0, r_1 > 0$ such that

$$\sup_{x\in\mathbb{R}^n} \mathfrak{M}_{p,\varphi}\big(f - f_k; x, r\big) = \sup_{x\in\mathbb{R}^n} \mathfrak{M}_{p,\varphi}\big(f\chi_k; x, r\big) < \varepsilon$$

for all $k \in \mathbb{N}$ and every $r < r_0$ or $r > r_1$. Hence

$$\|f - f_k\|_{p,\varphi}^p < \max\{\varepsilon, S_{r_0,r_1}(k)\}$$

where

$$S_{r_0,r_1}(k) := \sup_{x\in\mathbb{R}^n, r\in[r_0,r_1]} \mathfrak{M}_{p,\varphi}\big(f\chi_k; x, r\big).$$

Now, by the vanishing property (V^*) and Remark 3.6, we get

$$S_{r_0,r_1}(k) \leq \sup_{x\in\mathbb{R}^n} \frac{1}{\inf_{r\geq r_0} \varphi(r)} \int_{B(x,r_1)} |f(y)|^p\, \chi_k(y)\, dy < \varepsilon$$

for all k large enough. Therefore,

$$\|f - f_k\|_{p,\varphi} \to 0 \quad \text{as} \quad k \to \infty,$$

which completes the proof. □

Since $V_{0,\infty}^{(*)}L^{p,\varphi}(\mathbb{R}^n)$ is closed (cf. Remark 3.8), from Theorem 6.3 we are ready to give an explicit description of the closure of $C_0^\infty(\mathbb{R}^n)$ in the generalized Morrey space.

Corollary 6.4. *Let* $1 \leq p < \infty$ *and let* $\varphi \in \Phi$ *satisfy* (3.2) *and* (5.4)*. Then the class* $C_0^\infty(\mathbb{R}^n)$ *is dense in* $V_{0,\infty}^{(*)}L^{p,\varphi}(\mathbb{R}^n)$*. Moreover,* $V_{0,\infty}^{(*)}L^{p,\varphi}(\mathbb{R}^n)$ *coincides with the closure of* $C_0^\infty(\mathbb{R}^n)$ *in* $L^{p,\varphi}(\mathbb{R}^n)$*.*

Acknowledgements

We would like to thank W. Sickel for calling our attention to the paper [36]. We also thank the referee for the constructive comments on the manuscript.

References

[1] D.R. Adams: *Morrey Spaces*, Lecture Notes in Applied and Numerical Harmonic Analysis, Birkhäuser, 2015.
[2] D.R. Adams, J. Xiao: *Morrey spaces in harmonic analysis*, Ark. Mat. **50** (2012), 201–230.
[3] A. Almeida, S. Samko: *Approximation in Morrey spaces*, J. Funct. Anal. **272** (2017), 2392–2411.
[4] A. Almeida, S. Samko: *Embeddings of local generalized Morrey spaces between weighted Lebesgue spaces*, Nonlinear Anal. **164** (2017), 67–76.
[5] V.I. Burenkov, T.V. Tararykova: *An analog of Young's inequality for convolutions of functions for general Morrey-type spaces*, Proc. Steklov Inst. Math. **293** (2016), 107–126 [translation from Tr. Mat. inst. Steklova 293 (2016), 113–132].
[6] E. Burtseva, N. Samko: *Weighted Adams type theorem for the Riesz fractional integral in generalized Morrey spaces*, Fract. Calc. Appl. Anal. **19** (2016), 954–972.
[7] S. Campanato: *Proprietà di una famiglia di spazi funzioni*, Ann. Scuola Norm. Sup. Pisa **18** (1964), 137–160.
[8] F. Chiarenza, M. Franciosi: *A generalization of a theorem by C. Miranda*, Ann. Mat. Pura Appl., IV. Ser. 161, (1992) 285–297.
[9] Eridani, H. Sawano: *Fractional integral operators in generalized Morrey spaces defined on metric measure spaces*, Proc. A. Razmadze Math. Inst. **158** (2012), 13–24.
[10] Eridani, G. Gunawan, E. Nakai, H. Sawano: *Characterizations of the generalized fractional integral operators on Morrey spaces*, Math. Ineq. Appl. **17** (2014), 761–777.
[11] M. Giaquinta: *Multiple integrals in the calculus of variations and non-linear elliptic systems*, Princeton Univ. Press, Princeton, 1983.
[12] D. Hakim, S. Nakamura, Y. Sawano: *Complex interpolation of smoothness Morrey subspaces*, Constr. Approx. **46** (2017), 489–563.
[13] D. Hakim, Y. Sawano: *Calderón's first and second complex interpolations of closed subspaces of Morrey spaces*, J. Fourier Anal. Appl. **23** (2017), 1195–1226.
[14] T. Kato: *Strong solutions of the Navier-Stokes equation in Morrey spaces*, Bol. Soc. Bras. Mat. **22** (1992), 127–155.
[15] T. Mizuhara: *Boundedness of some classical operators on generalized Morrey spaces*, ICM-90 Satellite Conf. Proc., S. Igari (Ed.) (1991), pp. 183-189.
[16] C.B. Morrey: *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), 126–166.
[17] E. Nakai: *Hardy-Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces*, Math. Nachr. **166** (1994), 95–103.
[18] E. Nakai: *A characterization of pointwise multipliers on the Morrey spaces*, Sci. Math. **3** (2000), 445–454.
[19] J. Peetre: *On the theory of $\mathcal{L}_{p,\lambda}$ spaces*, J. Funct. Anal. **4** (1969), 71–87.
[20] L.E. Persson, M.A. Ragusa, N. Samko, P. Wall: *Commutators of Hardy operators in vanishing Morrey spaces*, AIP Conf. Proc. 1493 (2012), 859–866.
[21] L. Pick, A. Kufner, O. John, S. Fučík: *Function spaces*, Vol. 1, 2nd. ed., De Gruyter Series in Nonlinear Analysis and Applications 14, Berlin, 2013.

[22] H. Rafeiro, N. Samko, S. Samko: *Morrey-Campanato spaces: an overview*, In: Karlovich, Yi., Rodino, L., Silbermann, B., Spitkovsky, IM. (eds.) *Operator Theory, Pseudo-Differential Equations and Mathematical Physics, Advances and Applications*, vol. 228, pp. 293–323, Springer, Basel, 2013.
[23] M.A. Ragusa: *Commutators of fractional integral operators on vanishing Morrey spaces*, J. Global Optim. **40** (2008) 361–368.
[24] M. Rosenthal, H. Triebel: *Calderón-Zygmund operators in Morrey spaces*, Rev. Mat. Complut. **27** (2014), 1–11.
[25] M. Rosenthal, H. Triebel: *Morrey spaces, their duals and preduals*, Rev. Mat. Complut. **28** (2015), 1–30.
[26] N. Samko: *Maximal, potential and singular operators in vanishing generalized Morrey spaces*, J. Glob. Optim. **57** (2013), 1385–1399.
[27] H. Sawano, H. Wadade: *On the Gagliardo-Nirenberg type inequality in the critical Sobolev-Morrey space*, J. Fourier Anal. Appl. **19** (2013), 20–47.
[28] Y. Sawano, H. Gunawan, V. Guliyev, H. Tanaka: *Morrey spaces and related function spaces*, J. Funct. Spaces, Vol. 2014, Article ID867192.
[29] L.G. Softova: *Morrey-type regularity of solution to parabolic problems with discontinuous data*, Manuscr. Math. **136** (2011), 365–382.
[30] M.E. Taylor: *Tools for PDE: Pseudodifferential Operators, Paradifferential Operators and Layer Potentials*, Math. Surveys and Monogr., Vol. 81, AMS, Providence, 2000.
[31] H. Triebel: *Local function spaces, Heat and Navier-Stokes equations*, EMS Tracts in Mathematics 20, 2013.
[32] H. Triebel: *Hybrid function spaces, Heat and Navier-Stokes equations*, EMS Tracts in Mathematics 24, 2015.
[33] C. Vitanza: *Functions with vanishing Morrey norm and elliptic partial differential equations*. In: Proceedings of Methods of Real Analysis and Partial Differential Equations, Capri, pp. 147–150, Springer, 1990.
[34] C. Vitanza: *Regularity results for a class of elliptic equations with coefficients in Morrey spaces.* Ric. Mat. **42**, No.2 (1993), 265–281 .
[35] H. Wadade: *Optimal embeddings of critical Sobolev-Lorentz-Zygmund spaces*, Stud. Math. **223**, no.1 (2014), 77–95.
[36] W. Yuan, W. Sickel, D. Yang: *Calderón's first and second complex interpolations of closed subspaces of Morrey spaces*, Sci. Math. China **58** (2015), 1835–1908.
[37] C. Zorko: *Morrey spaces*, Proc. Amer. Math. Soc. **98** (1986), 586–592.

Department of Mathematics, Center for R&D in Mathematics and Applications, University of Aveiro, 3810-193 Aveiro, Portugal
Email address: `jaralmeida@ua.pt`

University of Algarve, Department of Mathematics, Campus de Gambelas, 8005-139 Faro, Portugal
Email address: `ssamko@ualg.pt`