MIFPA-10-17
DAMTP-2010-43

Correlation Functions and Hidden Conformal Symmetry of Kerr Black Holes

Melanie Becker[†], Sera Cremonini[†,§], Waldemar Schulgin[†*]

[†] *George and Cynthia Mitchell Institute for Fundamental Physics and Astronomy, Texas A&M University, College Station, TX 77843–4242, USA*

and

[§] *Centre for Theoretical Cosmology, DAMTP, CMS, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WA, UK*

Abstract

Extremal scalar three-point correlators in the near-NHEK geometry of Kerr black holes have recently been shown to agree with the result expected from a holographically dual non-chiral two-dimensional conformal field theory. In this paper we extend this calculation to extremal three-point functions of scalars in a general Kerr black hole which need not obey the extremality condition $M = \sqrt{J}$. It was recently argued that for low frequency scalars in the Kerr geometry there is a dual conformal field theory description which determines the interactions in this regime. Our results support this conjecture. Furthermore, we formulate a recipe for calculating finite-temperature retarded three-point correlation functions which is applicable to a large class of (even non-extremal) correlators, and discuss the vanishing of the extremal couplings.

[*]Electronic address: mbecker,sera,schulgin@physics.tamu.edu

Contents

1. INTRODUCTION

In the original formulation of the Kerr/CFT correspondence [1], quantum gravity in the NHEK (near-horizon extreme Kerr) geometry was conjectured to have a dual description in terms of a *chiral*, left-moving two-dimensional CFT with central charge $c_L = 12J$, where J denotes the black hole angular momentum. This conjecture was motivated by the fact that the asymptotic symmetry group of the NHEK geometry was shown to be one copy of the conformal group. Support for the proposal emerged from the agreement between the macroscopic Bekenstein-Hawking entropy and the microscopic entropy of the dual CFT, whose computation relied crucially on knowledge of the central charge $c_L = 12J$ of the Virasoro algebra [1]. The presence of only a chiral half of the CFT can be explained by the fact that at extremality the black hole horizon rotates at the speed of light. Since both edges of the forward light-cone coincide as the horizon is approached, all physical excitations are forced to rotate chirally with the black hole.

A generalization of the original Kerr/CFT conjecture to the near-NHEK geometry (near-extremal, near-horizon Kerr) emerged shortly thereafter [2]. The near-extremal case allows for some energy above extremality – the forward light cones no longer coincide, and right-moving excitations are now possible, in addition to left-movers. The theory has been conjectured [2] to be dual to a 2D *non-chiral* CFT with $c_L = c_R = 12J$. Even though finding appropriate boundary conditions allowing for both left-movers and right-movers has proven elusive, several studies offer support for the conjecture [2–4]. In addition to the matching of microscopic and macroscopic entropies, it is also supported by the agreement between finite-temperature two-point correlation functions – on the CFT side – and black hole scattering amplitudes – on the gravity side – for frequencies close to the superradiant bound (see [2, 5, 6]).

At the level of correlation functions, a more systematic check was provided only recently by the calculation of extremal three-point functions of scalars in the near-NHEK geometry [7], arising from interaction terms of the form $\sim \lambda\, \Phi_{h_1} \Phi_{h_2} \Phi_{h_3}$. On the CFT side, these are finite-temperature correlators of the form $\langle \mathcal{O}_{h_1} \mathcal{O}_{h_2} \mathcal{O}_{h_3} \rangle$. Such correlators are "extremal" when the conformal weights h_i of the operators $\mathcal{O}_{h_i}$ dual to the scalar fields obey $h_3 = h_1 + h_2$. Such a restriction dramatically simplifies the form of the CFT three-point function, reducing it to a product of two two-point functions, while still providing a strong test for the conjecture.

In an interesting new development [8], the wave-equation for a *low-frequency* scalar in the background of a general Kerr black hole has been shown to exhibit an $SL(2,R)_L \times SL(2,R)_R$ symmetry. In addition to the requirement of low frequencies, *i.e.* $\omega M \ll 1$, the conformal symmetry arises from the near-region of Kerr, specified by $r \ll 1/\omega$. Since these conditions don't place any restrictions on the temperature (for very small frequencies, r becomes arbitrarily large), the analysis of [8] provides additional evidence for the validity of the Kerr/CFT conjecture for general values of mass and angular momentum. In particular, it justifies using the Cardy formula to compute the microscopic entropy, since it allows one to take the temperatures to be large compared to the central charges. Scattering computations of low-frequency scalars in the near-region of Kerr were again shown to reproduce two-point correlation functions on the CFT side [8, 9], in close analogy with the near-NHEK scattering calculations. Finally, we emphasize that thus far the $SL(2,R)_L \times SL(2,R)_R$ is only understood as a symmetry of the equation of motion, and *not* an isometry of the metric – hence the name "hidden" conformal symmetry. While this conformal symmetry acts locally on the space of solutions, it is obstructed globally by periodic identification of the azimuthal angle ϕ. For recent extensions of [8] see [9–14].

In this note we would like to extend our previous near-NHEK computation of scalar three-point correlation functions to the low-frequency case analyzed in [8], in the background of a general Kerr black hole. Although we restrict our attention to the particular case of extremal correlators, our computation can also be applied to non-extremal conformal weights, as we explain in Section 5. The calculation proceeds along the lines of [7]. On the gravity side, we will see that the dominant contribution to the three-point function comes from a term which contains a divergent factor $\propto \frac{1}{h_3-h_1-h_2}$. In particular, the gravity result reproduces the expected CFT correlators,

$$\lambda \int \Phi_{h_1}\Phi_{h_2}\Phi_{h_1+h_2} \sim \lambda \, \frac{1}{h_3-h_1-h_2} \, \langle \mathcal{O}_{h_1}\mathcal{O}_{h_2}\mathcal{O}_{h_1+h_2}\rangle \,, \tag{1.1}$$

and tells us that the coupling λ of the cubic interaction should vanish for extremal correlators,

$$\lambda_{\,\text{extremal}} \propto h_3 - h_1 - h_2 \,, \tag{1.2}$$

in direct analogy with standard AdS/CFT studies of extremal correlators [15–17]. The vanishing of the extremal coupling is expected – and dictated – from the structure of conformal anomalies (see e.g. [17, 18]) in theories that admit a Coulomb branch, as we explain in Section 4. This check provides one more piece of evidence for the existence of a dual, non-chiral CFT description of general Kerr black holes. We emphasize that our three-point function calculation can be directly generalized to a variety of backgrounds, as we explain in Section 5, and is therefore in some sense "universal." Because of its relative simplicity, in Section 5 we will outline its main features, so that it can be easily adopted in related contexts.

This paper is organized as follows. In Section 2 we review some highlights of the hidden conformal symmetry found in [8]. Section 3 is devoted to the calculation of the extremal three-point correlation function, for low-frequency scalars in a general four-dimensional Kerr black hole. In Section 4 we explain why the coupling of extremal correlators is expected to vanish, and its relation to the conformal anomaly. Section 5 offers a summary of the main ingredients of our three-point function calculation (touching on [7] as well as this note), including comments on its applicability to non-extremal correlators. We conclude with a discussion of our results, open problems and work in progress.

2. HIDDEN CONFORMAL SYMMETRY

It has been recently observed in [8] that *at low frequencies* the wave equation for a scalar field incident on a Kerr black hole exhibits two-dimensional conformal symmetry, for *generic* non-extreme values of the mass, $M \neq \sqrt{J}$. In particular, [8] has been able to recast the scalar wave equation (for sufficiently low frequencies, $\omega M \ll 1$ and in the region where $r \ll \frac{1}{\omega}$) in terms of the generators of an $SL(2,R)_L \times SL(2,R)_R$ symmetry. This symmetry, however, is not an isometry of the geometry (unlike the NHEK case, where there was an $SL(2,R) \times U(1)$ isometry), but just a statement about what certain scalar modes see while scattering off the Kerr black hole. Thus, this is an instance where the solution space has the requisite conformal symmetry, but the space on which the field propagates does not. Let's outline how this works, following the discussion of [8].

Consider a massless scalar field incident on a Kerr black hole,

$$\Phi(t,r,\theta,\phi) = e^{-i\omega t + im\phi}\phi(r,\theta). \tag{2.1}$$

It is well-known that the wave equation $\Box\Phi = 0$ in the Kerr background separates. Thus, for a field of the form

$$\Phi(t,r,\theta,\phi) = e^{-i\omega t + im\phi}R(r)S(\theta), \tag{2.2}$$

the wave equation for the *radial* wavefunction is given by

$$\left[\partial_r \Delta \partial_r + \frac{(2Mr_+\omega - am)^2}{(r-r_+)(r_+-r_-)} - \frac{(2Mr_-\omega - am)^2}{(r-r_-)(r_+-r_-)} + \Big(r^2 + 2M(r+2M)\Big)\omega^2\right] R(r) = K_l R(r)\,, \tag{2.3}$$

with

$$\Delta = r^2 + a^2 - 2Mr\,, \quad a = \frac{J}{M}\,, \quad r_\pm = M \pm \sqrt{M^2 - a^2}\,. \tag{2.4}$$

The radial equation can be greatly simplified by dropping the last term, which is $\mathcal{O}(\omega^2)$. More precisely, in the limit where the wavelength of the scalar excitation is much larger than the curvature radius,

$$\omega \ll \frac{1}{M}\,, \tag{2.5}$$

and in the "near region" $r \ll 1/\omega$, the angular wave equation reduces to the standard laplacian on a two-sphere S^2, the separation constant becomes $K_l = l(l+1)$ and the radial equation (2.3) simplifies to

$$\left[\partial_r \Delta \partial_r + \frac{(2Mr_+\omega - am)^2}{(r-r_+)(r_+-r_-)} - \frac{(2Mr_-\omega - am)^2}{(r-r_-)(r_+-r_-)}\right] R(r) = l(l+1)R(r)\,. \tag{2.6}$$

Note that the near region $r \ll 1/\omega$ is *not* the near-horizon region of the Kerr black hole. In fact, when ω is small enough, r can become arbitrarily large, in contrast to the NHEK geometry. The crucial observation of [8] is that the near-region radial wave equation (2.6) can be recast in terms of appropriately identified $SL(2,R)_L \times SL(2,R)_R$ generators:

$$H^2\Phi = \bar{H}^2\Phi = l(l+1)\Phi\,, \tag{2.7}$$

where $H^2 = \frac{1}{2}(H_1H_{-1} + H_{-1}H_1) - H_0^2$, and an analogous expression for $\bar{H}^2$, denote the $SL(2,R)$ Casimirs. We refer the reader to [8] for the explicit form of the generators.

The solution to the radial wave equation for low frequency scalars $\omega M \ll 1$ and incoming boundary conditions at the horizon has been known for a long time[1]. In the notation of [8] it takes the form

$$R(r) = \Big(\frac{r-r_+}{r-r_-}\Big)^{\frac{-2iMr_+(\omega-m\Omega)}{r_+-r_-}} (r-r_-)^{-l-1}\, F\Big(\alpha,\beta;\gamma;\frac{r-r_+}{r-r_-}\Big)\,, \tag{2.8}$$

with

$$\alpha \equiv 1+l-\frac{4iM}{r_+-r_-}(M\omega - r_+ m\Omega)\,,\quad \beta\equiv 1+l-2iM\omega\,,\quad \gamma\equiv 1-\frac{4iMr_+}{r_+-r_-}(\omega-m\Omega)\,. \tag{2.9}$$

In the asymptotic regime $r \gg M$ of the near region $r \ll \frac{1}{\omega}$ it reduces to the simple form

$$R(r) \sim Ar^l + Br^{-l-1}. \tag{2.10}$$

The coefficients are given by

$$A = \frac{\Gamma\left(1-i\frac{4Mr_+}{r_+-r_-}(\omega-m\Omega)\right)\Gamma(1+2l)}{\Gamma(1+l-2iM\omega)\Gamma\left(1+l-\frac{i4M^2}{r_+-r^-}\omega+\frac{i4Mr_+\Omega}{r_+-r_-}m\right)}\,, \tag{2.11}$$

where the horizon angular velocity is $\Omega = \frac{a}{r_+^2} \sim \frac{a}{2Mr_+}$, and

$$B = \frac{\Gamma(-2l-1)\Gamma\left(1-i\frac{4Mr_+}{r_+-r_-}(\omega-m\Omega)\right)}{\Gamma(-l-2iM\omega)\Gamma\left(-l-\frac{i4M^2}{r_+-r_-}\omega+\frac{i4Mr_+\Omega}{r_+-r_-}m\right)}\,. \tag{2.12}$$

As we will see in the next section, this asymptotic expansion will play a key role in the computation of extremal three-point functions.

3. EXTREMAL THREE-POINT CORRELATORS

Extremal three-point functions of scalars in the near-NHEK geometry were shown to agree with the corresponding finite temperature conformal field theory correlators in [7]. Here we would like to perform an analogous computation of extremal three-point correlators, but in the general Kerr

[1] For the computation of the absorption cross-section see. e.g. [19, 20].

black hole geometry, under the assumption that the scalar fields have low frequency, $\omega \ll 1/M$. The calculation can be done in a manner analogous to [7], so we shall be brief. For a more detailed treatment of the calculation we refer the reader to [7].

On the conformal field theory side, we are interested in computing finite temperature three-point correlators of the form $\langle \mathcal{O}_{h_1}\mathcal{O}_{h_2}\mathcal{O}_{h_3}\rangle$, where the h_i denote the conformal dimensions of the operators. Since the conjectured dual CFT is non-chiral, we will be dealing with two sectors – right (left) movers at temperature T_R (T_L). On the gravity side, we will be looking at three-point functions of scalar fields Φ_h dual to the operators we are interested in. As shown in [7], the finite temperature CFT three-point correlator[2] can be obtained, on the gravity side, by the bulk integral over three bulk-to-boundary propagators,

$$\langle \mathcal{O}_{h_1}(\vec{x}_1)\mathcal{O}_{h_2}(\vec{x}_2)\mathcal{O}_{h_3}(\vec{x}_3)\rangle \sim \int_{r_+}^{r_{\text{eff}}} d\vec{x}\, dr\, \sqrt{-g} K_{h_1}(r,\vec{x};\vec{x}_1) K_{h_2}(r,\vec{x};\vec{x}_2) K_{h_3}(r,\vec{x};\vec{x}_3)\,. \tag{3.1}$$

The radial integral runs from the black hole horizon r_+ to the location r_{eff} of the "effective boundary" where the dual CFT lives. For the case of [8], the boundary is specified by $M \ll r \ll \frac{1}{\omega}$. Note that by making ω arbitrarily small, one can push the boundary to infinity. We will discuss the prescription (3.1) – and its applicability to the finite temperature case – in the next section. For now we just mention that the bulk-to-boundary propagator is constructed so that it obeys incoming wave boundary conditions at the black hole horizon.

Computing the three-point function using the exact solution to the wave equation, i.e. the full hypergeometric function $F(\alpha, \beta; \gamma; \frac{r-r_-}{r-r_+})$ is challenging, and moreover is not needed for extremal correlators. In fact, the wavefunction can be approximated in various regions, and can then be used in a straightforward way for the computation of n-point correlation functions. In [7] we approximated the wavefunction near the horizon, near the boundary, and in the intermediate region, by making appropriate expansions in $\tau_H \sim r_+ - r_-$. After computing the resulting three-point integral we took the conformal weights to be extremal, $h_3 = h_1 + h_2$, which simplified the result significantly. However, [7] does contain an (approximated) expression for the three-point function which is valid for generic, non-extremal conformal weights, under the assumption that τ_H is small.

Here we would like to adopt an alternative derivation of the three-point correlator, which is more concise than that given in [7], and is particularly appropriate to the special case of extremal correlators. We should note, however, that this derivation fails for non-extremal correlators – for generic conformal weights the reader should follow the procedure of [7], which we spell out in the next section.

We start by expanding the radial wavefunction (2.8) (which coincides with the momentum-space propagator [7]) around the boundary, which is specified by $r \gg M$ as long as the condition $r \ll \frac{1}{\omega}$ is satisfied. By taking $r \gg r_+ - r_- = 2\sqrt{M^2 - a^2} \sim 2M$, the hypergeometric function can be

[2] As we will explain in Section 5, this is a *retarded* correlator.

approximated by the infinite series

$$\lim_{r\gg M} {}_2F_1\left(\alpha,\beta;\gamma;\frac{r-r_-}{r-r_+}\right) = \left(\frac{r}{r_--r_+}\right)^{\alpha+\beta-\gamma}\left[\frac{\Gamma(\alpha+\beta-\gamma)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}+\mathcal{O}\left(\frac{r_+-r_-}{r}\right)\right] + \left[\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}+\mathcal{O}\left(\frac{r_+-r_-}{r}\right)\right], \tag{3.2}$$

allowing one to rewrite the wavefunction as

$$R(r) = A\,r^l\,(r_--r_+)^{-2l-1}+\mathcal{O}\Big((r_--r_+)^{-2l}r^{l-1}\Big) + B\,r^{-1-l}+\mathcal{O}\Big((r_--r_+)r^{-2-l}\Big). \tag{3.3}$$

The coefficients A, B are precisely the ones given in (2.11) and (2.12). At any value of r along the integration region from the outer horizon to the effective boundary the series is convergent. Note that we are *not* truncating the series in (3.3), although we are using a suggestive notation for the subleading terms $\mathcal{O}(\ldots)$, for reasons which will become clear.

Typically if one is interested only in the boundary behavior of the wavefunction – as in two-point correlation function calculations – the expansion (3.3) is truncated, and one keeps only the leading normalizable and non-normalizable modes, as in (2.10). However, we are interested in using (3.3) to compute the three-point function integral, which requires integrating over the entire bulk, and we should therefore keep as many terms in the expansion as possible. Eventually, when the radial coordinate becomes close to the horizon, $r\sim r_+$, this approximation to the wavefunction will break down[3], and we will be forced to approximate $R(r)$ in a different manner. It is straightforward to approximate the full solution near the horizon r_+, as well as when $r\sim r_+-r_-$. For explicit details of the approximations in the near-NHEK region of Kerr we refer the reader to [7]. It turns out that for the types of correlators we are interested in, the contribution to the bulk integral comes *from the boundary* and will contain a divergent term, which dominates all remaining (finite) terms, giving an exact expression for the three-point function integral. Thus, because of this divergence we don't need the (finite) terms arising from the regions $r\gtrsim r_+$.

The solution of the radial wave equation (3.3) matches the *momentum-space* bulk-to-boundary propagator [7]. It is more convenient to perform our computation in momentum space. Using the relation between the momentum and coordinate space propagator

$$K(r,t',\phi';t,\phi) = \int dm\int d\omega\;\tilde{K}(r,m,\omega)\;e^{-im(\phi-\phi')}e^{i\omega(t-t')}, \tag{3.4}$$

and the expression for the three-point correlator in coordinate space

$$\langle O(t_1,\phi_1)\,O(t_2,\phi_2)\,O(t_3,\phi_3)\rangle \sim \int d\phi' dt' dr\,\sqrt{-g}K_1(r,t',\phi';t_1,\phi_1)K_2(r,t',\phi';t_2,\phi_2)K_3(r,t',\phi';t_3,\phi_3), \tag{3.5}$$

[3] Note that in the wavefunction expansion we implicitly assumed that $\frac{r-r_+}{r_--r_+}\approx\frac{r}{r_--r_+}$.

it is a simple task to a obtain a corresponding expression for the correlator in momentum space:

$$\begin{aligned} V_3^{m.s.} &= \langle O(m_1,\omega_1)\, O(m_2,\omega_2)\, O(m_3,\omega_3)\rangle \\ &= \delta(m_1+m_2+m_3)\,\delta(\omega_1+\omega_2+\omega_3)\int_{r_+}^{r_{\rm eff}} dr\,\sqrt{-g}\;\tilde K_1\tilde K_2\tilde K_3\,. \end{aligned} \tag{3.6}$$

Taking into account the normalization appropriately [7] and using the relation between the angular momentum and the conformal weight, which can be read off from (2.7)

$$h = l\,, \tag{3.7}$$

we find the final expression for the momentum-space bulk-to-boundary propagator:

$$\begin{aligned} \tilde K(r,m,\omega) &= r^h + \mathcal{O}\Big((r_- - r_+)r^{h-1}\Big) \\ &+ \frac{B}{A}\, r^{-h-1}\,(r_- - r_+)^{2h+1} + \mathcal{O}\Big((r_- - r_+)^{2h+2} r^{-2-h}\Big)\,. \end{aligned} \tag{3.8}$$

Inserting (3.8) into the three-point integral (3.6) and performing the integration term by term will give us one divergent term and infinitely many finite terms (part of which can be identified as contact terms). For the full details of the calculation we again refer the reader to [7]. Here we note that the terms one gets are of the form:

$$\begin{aligned} V_3^{m.s.} &= \int_{r_+}^{r_{\rm eff}} \sqrt{-g}\,\tilde K_{h_1}\tilde K_{h_2}\tilde K_{h_3} = \Bigg[\text{contact terms} + \frac{B_1B_2B_3}{A_1A_2A_3}\frac{r^{-h_1-h_2-h_3+1}}{1-h_1-h_2-h_3}(r_- - r_+)^{2h_1+2h_2+2h_3+3} \\ &+\frac{B_1B_2}{A_1A_2}\,\frac{r^{h_3-h_1-h_2}}{h_3-h_1-h_2}\,(r_- - r_+)^{2h_1+2h_2+2} + \frac{B_2B_3}{A_2A_3}\,\frac{r^{h_1-h_2-h_3}}{h_1-h_2-h_3}\,(r_- - r_+)^{2h_2+2h_3+2} \\ &+\frac{B_1B_3}{A_1A_3}\,\frac{r^{h_2-h_1-h_3}}{h_2-h_1-h_3}\,(r_- - r_+)^{2h_1+2h_3+2} + \ldots\Bigg]_{r_+}^{r_{\rm eff}}\,. \end{aligned} \tag{3.9}$$

We should mention that the metric determinant in the final bulk integral is $\sqrt{-g}\sim r$. Also, since we are working in momentum space the contact terms do not have the standard form $\delta(\vec{x}_i - \vec{x}_j)$. We recognize them as coming from terms that only depend on a single ratio B/A, or from terms with no dependence on A, B.[4]

In the extremal limit $h_3 = h_1 + h_2$ the final result for the tree-point function is

$$V_3^{ms} \sim (T_L T_R)^{2h_1+2h_2+2}\left(\frac{1}{h_3-h_2-h_1}\frac{B_1B_2}{A_1A_2} + \text{finite}\right)\delta(m_1+m_2+m_3)\,\delta(\omega_1+\omega_2+\omega_3)\,, \tag{3.10}$$

where we used the relation (assuming $r_+ \gg r_+ - r_-$)

$$T_L T_R = \frac{1}{(4\pi)^2}\frac{(r_+ + r_-)(r_+ - r_-)}{r_- r_+} \approx \frac{1}{8\pi^2}\frac{r_+ - r_-}{r_-}\,. \tag{3.11}$$

The coefficients $A(\omega, m)$ and $B(\omega, m)$ are given by (2.11) and (2.12). Here $A_1 \equiv A(\omega_1, m_1), A_2 \equiv$

[4] For example, since the B_1/A_1 term will only contain dependence on (m_1, ω_1), its Fourier transform over $(m_2, m_3, \omega_2, \omega_3)$ is trivial, and gives $\sim \delta(t_2 - t_3)\delta(\phi_2 - \phi_3)$.

$A(\omega_2, m_2)$ and similarly for the B's. As in [7] this correlator was computed by analytic continuation from the non-extremal case and it clearly diverges in the extremal limit $h_3 = h_1 + h_2$. As shown in [7], the three-point correlator (3.10) exactly agrees with the result expected for an extremal correlator in a finite temperature non-chiral CFT:

$$\begin{aligned}\langle \mathcal{O}(\omega_{L_1}, \omega_{R_1}) \mathcal{O}(\omega_{L_2}, \omega_{R_2}) \mathcal{O}(\omega_{L_3}, \omega_{R_3}) \rangle \ &\sim \ \delta(\omega_{L_1} + \omega_{L_2} + \omega_{L_3})\, \delta(\omega_{R_1} + \omega_{R_2} + \omega_{R_3}) \\ &\times \ \langle \mathcal{O}(\omega_{L_1}, \omega_{R_1}) \mathcal{O}(0,0) \rangle \ \langle \mathcal{O}(\omega_{L_2}, \omega_{R_2}) \mathcal{O}(0,0) \rangle \,,\end{aligned}$$

once we recall that each two-point function is simply given by $G_R(\omega_L, \omega_R) = \frac{B(\omega_L, \omega_R)}{A(\omega_L, \omega_R)}$, and identify the left- and right-moving frequencies appropriately:

$$\omega_L = \frac{2M^3}{J}\,\omega\,, \qquad \omega_R = \frac{2M^3}{J}\,\omega - m\,. \tag{3.12}$$

The above divergence is a reflection of the vanishing of the coupling of the cubic interaction $\lambda\Phi^3$ in the extremal case,

$$\lambda \propto h_3 - h_2 - h_1, \tag{3.13}$$

This ensures that the full three-point function stays finite, as we already briefly discussed. Such a behavior of the cubic coupling for extremal correlators can be understood in the context of holographic renormalization, as we explain in the next section.

4. EXTREMAL COUPLINGS AND CONFORMAL ANOMALY

As we saw in Section 3, the leading contribution to the three-point correlator in the extremal case $h_3 = h_1 + h_2$ is a term of the form

$$\lambda\,\frac{1}{h_3 - h_1 - h_2}\,\frac{B_1 B_2}{A_1 A_2}\,, \tag{4.1}$$

which, as we recall from [7], results from a log term in disguise. The fact that the bulk integral diverges when the conformal weights are extremal indicates that the overall coupling λ of the interaction should vanish, as was already noted in [7].

The same structure – the vanishing of extremal couplings – was found in the original AdS/CFT three-point calculations (see e.g. [15] and [16]). In that context, the couplings could be obtained by direct supergravity reduction, and were explicitly shown to vanish. However, it was later understood that – for theories that admit a Coulomb branch – the vanishing of the couplings of extremal correlators is dictated by the structure of the conformal anomalies[5]. Here we would like to briefly outline why this is so, borrowing mainly from [17].

An important feature of the gauge/gravity correspondence is the so-called UV/IR connection, *i.e.* the standard field theoretic UV divergences are related to gravitational IR divergences due to the infinite volume of AdS. On the gravity side, such IR divergences – being long distance effects – clearly come from the boundary of the spacetime, and the procedure developed to deal with them

[5] We are very grateful to K. Skenderis for pointing this out to us.

is known as holographic renormalization [21, 22]. At the core of the holographic renormalization scheme is the addition of appropriate counterterms, constructed to render the action and correlation functions finite. In particular, the computation of correlation functions can be reformulated in terms of *renormalized* 1-point functions in the presence of sources. Specifically, for a CFT operator $O(x)$ of dimension h, dual to a bulk scalar field $\Phi_h(x,r)$ in AdS_{d+1}, one has:

$$\begin{aligned} \langle O(x) \rangle &= \frac{1}{\sqrt{g_0(x)}} \frac{\delta S_{ren}}{\delta \phi_0(x)} \sim \phi_{2h-d}(x)\,, \\ \langle O(x_1) \dots O(x_n) \rangle &\sim \frac{\delta \phi_{2h-d}(x_1)}{\delta \phi_0(x_2) \dots \delta \phi_0(x_n)} |_{\phi_0 = 0}\,. \end{aligned} \tag{4.2}$$

Here S_{ren} denotes the renormalized on-shell action, which includes all the appropriate counterterms, while $\phi_{2h-d}(x)$ is a term appearing in the asymptotic expansion[6] of the scalar field Φ near the boundary:

$$\Phi_h(x,r) = r^{d-h} \left(\phi_0 + r^2 \phi_2 + \dots \right) + r^h \left(\phi_{2h-d} + \log r^2 \ \psi_{2h-d} \right) + \dots\,. \tag{4.3}$$

The coefficient ψ_{2h-d} of the log term as well as the coefficients ϕ_n in the asymptotic expansion with $n < 2h - d$ are uniquely determined from the scalar field equation in terms of the source ϕ_0. On the other hand, ϕ_{2h-d} is left undetermined by the field equation, but is specified by the vacuum expectation value $\langle O \rangle$ of the dual operator, as shown by (4.2).

Moreover, the coefficient of the log term, ψ_{2h-d}, is directly related to conformal anomalies (see e.g. [18] and [21]). A crucial point for understanding the vanishing of the extremal couplings is that ψ_{2h-d} vanishes when the source is set to zero. Thus, when $\phi_0 = 0$ the expansion of the scalar field should not contain any logarithmic terms. But in the studies of extremal correlators in the AdS/CFT context (see e.g. [17]), it was shown that such couplings arise quite generically precisely from terms in the scalar field expansion which are logarithmic.

To sketch how this works, let's consider a particular example from the Coulomb branch of $\mathcal{N} = 4$ SYM, where chiral primaries get a VEV. Consider the equation of motion for the scalar field Φ_4 dual to a dimension 4 operator,

$$\Box \Phi_4 = \lambda \Phi_2^2 + \dots\,. \tag{4.4}$$

From the form of this equation, it is clear that λ is the coupling of an interaction term $\sim \lambda\, \Phi_4 \Phi_2 \Phi_2$. Moreover, since this is precisely an extremal correlator ($h_1 = h_2 = 2$, $h_3 = 4$ satisfy $h_3 = h_1 + h_2$), λ is an extremal coupling. Plugging into the equation above the near-boundary expansion for the field Φ_2 (dual to a dimension 2 operator)

$$\Phi_2 = \langle O \rangle \ r^2 + \dots \tag{4.5}$$

and, neglecting terms that are irrelevant for this analysis, one can solve (4.4) and obtain:

$$\Phi_4 = r^4 \log r^2 \lambda \langle O \rangle^2 + \dots\,. \tag{4.6}$$

[6] The asymptotic expansion is performed using Graham-Fefferman coordinates $ds^2 = \frac{1}{r^2}(dr^2 + g_{ij}(x,r)\, dx^i dx^j)$.

Since log terms must be absent when the sources are set to zero, the extremal coupling λ must be forced to vanish. This argument – which relies crucially on the fact that log terms are proportional to sources and thus evaluate to zero on the Coulomb branch – can be extended to all extremal couplings.

In summary, if a CFT has a Coulomb branch, then in any holographic realization the extremal couplings should vanish. It is plausible that a similar argument can be made for the extremal correlators that were computed herein. It would be interesting to understand this in detail, especially given that not much is known about detailed properties of the CFT dual to Kerr.

5. THREE-POINT CORRELATORS RECIPE

The development of a prescription for computing generic real-time, finite-temperature n-point correlation functions in the gauge/gravity duality has proven very challenging. A rather simple Minkowski recipe for two-point functions has been formulated (first by [23] and subsequently by [24]), and has become widely accepted. However, much less explored is the case of three-point (and higher n-point) functions (see [25] for the main work, and more recently [26]). Thus, our three-point correlation function calculation (in [7] for a superradiant mode in the near-NHEK geometry of Kerr, and here for the low-frequency scalars in Kerr) serves a dual purpose, providing:

- a non-trivial check of the conjectured Kerr/CFT correspondence in the presence of both right- and left-moving excitations
- an explicit computation of finite temperature three-point functions for the rather non-trivial background of a rotating black hole, relying on a relatively simple adaptation of the Euclidean AdS/CFT prescription.

Since our computation [7] can be applied to a much larger class of backgrounds than just the four-dimensional Kerr black hole, it is worth emphasizing the main ingredients:

1. On the gravity side of the correspondence, we adopted the Euclidean three-point function AdS/CFT prescription, encoded schematically in Witten's diagrams. We reproduced (in [7] as well as in this paper) the finite-temperature CFT three-point correlator by computing the bulk integral of three bulk-to-boundary propagators:

$$\langle \mathcal{O}_{h_1}(x)\mathcal{O}_{h_2}(y)\mathcal{O}_{h_3}(z)\rangle \sim \int_{r_{horizon}}^{r_{boundary}} dr dx' K_{h_1}(r,x';x)K_{h_2}(r,x';y)K_{h_3}(r,x';z)\,. \tag{5.1}$$

   The bulk-to-boundary propagator is constructed with incoming boundary conditions at the horizon [7]. Thus, what we are computing is the *retarded* real-time three-point function[7], which can be obtained by analytic continuation in frequency space from the imaginary-time, finite-temperature correlator [27].

2. We restricted our attention to Matsubara frequencies, for which

$$G_R(\omega) = G_E(i\omega_E)\,, \tag{5.2}$$

[7] See [26] for another recent implementation.

i.e. the retarded Green's function and the Euclidean two-point function coincide, provided the frequencies are analytically continued. Working at the Matsubara frequencies ensures that the Euclidean prescription (5.1), upon analytically continuing the frequencies, yields the correct retarded real-time three-point function.

3. Computing the three-point function (5.1) using the exact solution to the wave equation (e.g. in our case the full hypergeometric function $F(a, b;, c; z)$) is challenging, and moreover is not needed. The wavefunction can be approximated (see [7] for a detailed analysis) by appropriate expansions near the black hole horizon (Region I), near the boundary (Region III), and in the intermediate region (Region II). The bulk integral can then be estimated by patching together the contributions from the various regions. This procedure works for non-extremal correlators as well as extremal. The typical scalar field expansion near the boundary (focusing on the radial wavefunction) is of the form

$$\Phi(r) \sim A r^{-\Delta_-} + B r^{-\Delta_+} \,. \tag{5.3}$$

The coefficients A, B play a crucial role, for example, in the determination of the retarded two-point function:

$$G_R \sim \frac{B}{A} \,, \tag{5.4}$$

and turn out to also play a key role in the three-point function computation [7].

4. We focused on extremal correlators $\langle \mathcal{O}_{h_1} \mathcal{O}_{h_2} \mathcal{O}_{h_3} \rangle$, for which the sum of any two of the conformal weights equals the third, e.g. $h_3 = h_1 + h_2$. In the case of extremal conformal weights several simplifications arise:

   - On the CFT side, the three-point correlator reduces to the product of two two-point correlators. Working now explicitly in momentum space, and denoting by (ω_L, ω_R) the left and right-moving frequencies, we have schematically:

$$\begin{aligned} \langle \mathcal{O}(\omega_{L_1}, \omega_{R_1}) \mathcal{O}(\omega_{L_2}, \omega_{R_2}) \mathcal{O}(\omega_{L_3}, \omega_{R_3}) \rangle \; &\sim \; \delta(\omega_{L_1} + \omega_{L_2} + \omega_{L_3})\, \delta(\omega_{R_1} + \omega_{R_2} + \omega_{R_3}) \\ &\times \; \langle \mathcal{O}(\omega_{L_1}, \omega_{R_1}) \mathcal{O}(0,0) \rangle \; \langle \mathcal{O}(\omega_{L_2}, \omega_{R_2}) \mathcal{O}(0,0) \rangle \,. \end{aligned}$$

   - On the gravity side, when $h_3 = h_1 + h_2$ the crucial contribution to the bulk integral (5.1) comes from the boundary region. Thus, it results from the wavefunction approximation (5.3), and takes the form

$$\sim \frac{1}{h_3 - h_1 - h_2} \, \frac{B(\omega_{L_1}, \omega_{R_1})}{A(\omega_{L_1}, \omega_{R_1})} \frac{B(\omega_{L_2}, \omega_{R_2})}{A(\omega_{L_2}, \omega_{R_2})} \,. \tag{5.5}$$

     This quantity is divergent since $h_3 = h_1 + h_2$. Apart from contact terms, all other contributions to the bulk integral (coming from Regions I and II) are *finite*, and can therefore be neglected compared to (5.5).

   - Using the well-known relation $G_R \sim \frac{B}{A}$ between the asymptotic expansion of the scalar field and the retarded Green's function, we conclude that the three-point function becomes

$$\sim \frac{1}{h_3 - h_1 - h_2} \langle \mathcal{O}(\omega_{L_1}, \omega_{R_1}) \mathcal{O}(0,0) \rangle \, \langle \mathcal{O}(\omega_{L_2}, \omega_{R_2}) \mathcal{O}(0,0) \rangle \,, \tag{5.6}$$

which is precisely of the expected form (5.5), apart from the overall divergent prefactor.

- The divergence when $h_3 = h_1 + h_2$ indicates that the coupling λ of the interaction term $\lambda \int \Phi_{h_1}(x) \Phi_{h_2}(y) \Phi_{h_1+h_2}(z)$ for extremal correlators should contain a vanishing prefactor,

$$\lambda \propto (h_3 - h_1 - h_2) \,, \tag{5.7}$$

ensuring that the entire correlator is finite.

5. The strategy of approximating the entire bulk integral in the various regions I, II and III is valid for extremal as well as generic conformal weights. The advantage of the extremal computation is that the result in that case is exact, since one finds a divergent term that dominates over all other contributions. In the case of generic, non-extremal weights, on the other hand, the three-point function result we presented in [7] is only an approximation of the full three-point function. Nonetheless, it should be a good enough approximation to capture the essential features of the CFT.

6. Our approximation scheme and three-point function results should apply quite generically to a large class of black hole backgrounds. The main ingredients which we needed were the knowledge of the Matsubara frequencies, and a boundary expansion the wavefunction of the form (5.3), which is quite generic in backgrounds that are asymptotically AdS, or at least contain an AdS factor somewhere, which plays a crucial role.

6. CONCLUSIONS

In this paper we have shown that the prescription developed in [7] for calculating finite-temperature extremal three-point correlation functions can be applied not only to the near-NHEK geometry, but also to the more recent situation [8] describing low-frequency scalars in a general Kerr black hole background. Our results provide further support for the conjectured holographic description of Kerr black holes in terms of a dual non-chiral conformal field theory. The crucial ingredient that allowed us to straightforwardly apply the techniques of [7] to the present situation is the presence of an asymptotic expansion for the wavefunction of the form

$$R \sim A r^{-\Delta_-} + B r^{-\Delta_+} \,, \tag{6.1}$$

with A and B denoting, respectively, the coefficients of the leading non-normalizable and normalizable modes. For the case of extremal correlators this boundary behavior was enough to compute the three-point function. An asymptotic expansion of the form (6.1) arises generically from the near-boundary behavior of scalars in the background of asymptotically AdS black holes, as well as the near-horizon region of rotating Kerr black holes. In the case considered here, it is a consequence of the $SL(2,R)_L \times SL(2,R)_R$ symmetry of the wave equation for a low-frequency scalar in the background of a general four-dimensional Kerr black hole – not just in its near-horizon region.

We would like to claim that our prescription for calculating extremal three-point functions, which we outlined in Section 5, applies universally to all situations in which an asymptotic expansion of the field of the form (6.1) can be found, and will yield an expression of the form:

$$\lambda \, \frac{1}{h_3 - h_1 - h_2} \, \frac{B_1 \, B_2}{A_1 \, A_2} \, .$$

In such situations agreement with a dual conformal field theory is guaranteed by our prescription. This is somewhat analogous to the universality of the retarded Green's function expression, which is also determined in a simple way in terms of B and A:

$$G_R = B/A \, .$$

While the fine details of the comparison will change for different backgrounds, the main features will go through. For example, we expect our method to work for the BTZ black hole [28], for which the retarded Green's function was computed in [23] and [29], as well as for the warped AdS_3 black hole solutions constructed in [30] for which the real time two-point function was computed in [31]. It should also work for generic extremal black holes, for which the near horizon geometry was shown to include an AdS_2 component in [32], as well as for higher-dimensional generalizations [33]. In all of these cases, three-point functions should be easily computable applying the prescription presented herein. While our calculation of extremal correlators is exact, our method can also be used to obtain an approximate expression for the three-point function for generic conformal weights. We anticipate that such an approximation should be able to capture the essential features of the corresponding dual conformal field theory correlators.

A calculation of the renormalized three-point correlation function using holographic renormalization is an interesting question to which we hope to return to in the near future [34]. Finally, we should emphasize that very little is known about specific properties of the CFT dual to Kerr. As we have seen, there is a remarkable similarity between our results and those found in standard AdS/CFT studies of extremal correlators. It would be interesting to ask whether one can extract any new information on the structure of the CFT dual to Kerr, from the lessons learned from the string-theoretic AdS/CFT results. For the case of Kerr black holes, and in particular the setup studied in this paper, any potential overlap would clearly be with the work on AdS_3 string theory compactifications.

Acknowledgments

We would like to thank K. Becker, A. Castro, G. Compere, D. Freedman, K. Skenderis, M. Taylor, D.T. Son, A. Strominger and C. Warnick for useful discussions. MB and WS would like to thank DAMPT (Cambridge) for the kind hospitality during the final stages of this project. Long walks through the beautiful Grantchester meadows provided additional inspiration. The work of MB and WS was supported by NSF under PHY-0505757 and the University of Texas A&M. SC has been supported by the Cambridge-Mitchell Collaboration in Theoretical Cosmology and the

Mitchell Family Foundation.

---

[1] M. Guica, T. Hartman, W. Song and A. Strominger, "The Kerr/CFT Correspondence," Phys. Rev. D **80**, 124008 (2009) [arXiv:0809.4266 [hep-th]].

[2] I. Bredberg, T. Hartman, W. Song and A. Strominger, "Black Hole Superradiance From Kerr/CFT," JHEP **1004**, 019 (2010) [arXiv:0907.3477 [hep-th]].

[3] A. Castro and F. Larsen, "Near Extremal Kerr Entropy from AdS_2 Quantum Gravity," JHEP **0912**, 037 (2009) [arXiv:0908.1121 [hep-th]].

[4] Y. Matsuo, T. Tsukioka and C. M. Yoo, "Another Realization of Kerr/CFT Correspondence," Nucl. Phys. B **825**, 231 (2010) [arXiv:0907.0303 [hep-th]].

[5] M. Cvetic and F. Larsen, "Greybody Factors and Charges in Kerr/CFT," JHEP **0909**, 088 (2009) [arXiv:0908.1136 [hep-th]].

[6] B. Chen and C. S. Chu, "Real-time correlators in Kerr/CFT correspondence," JHEP **1005**, 004 (2010) [arXiv:1001.3208 [hep-th]].

[7] M. Becker, S. Cremonini and W. Schulgin, "Extremal Three-point Correlators in Kerr/CFT," arXiv:1004.1174 [hep-th].

[8] A. Castro, A. Maloney and A. Strominger, "Hidden Conformal Symmetry of the Kerr Black Hole," arXiv:1004.0996 [hep-th].

[9] B. Chen and J. Long, "Real-time Correlators and Hidden Conformal Symmetry in Kerr/CFT Correspondence," arXiv:1004.5039

[10] J. Rasmussen, "On the CFT duals for near-extremal black holes," arXiv:1005.2255 [hep-th].

[11] D. Chen, P. Wang and H. Wu, "Hidden conformal symmetry of rotating charged black holes," arXiv:1005.1404 [gr-qc].

[12] Y. Q. Wang and Y. X. Liu, "Hidden Conformal Symmetry of the Kerr-Newman Black Hole," arXiv:1004.4661 [hep-th].

[13] C. M. Chen and J. R. Sun, "Hidden Conformal Symmetry of the Reissner-Nordstrøm Black Holes," arXiv:1004.3963 [hep-th].

[14] C. Krishnan, "Hidden Conformal Symmetries of Five-Dimensional Black Holes," arXiv:1004.3537 [hep-th].

[15] S. Lee, S. Minwalla, M. Rangamani and N. Seiberg, "Three-point functions of chiral operators in D = 4, N = 4 SYM at large N," Adv. Theor. Math. Phys. **2**, 697 (1998) [arXiv:hep-th/9806074].

[16] E. D'Hoker, D. Z. Freedman, S. D. Mathur, A. Matusis and L. Rastelli, "Extremal correlators in the AdS/CFT correspondence," arXiv:hep-th/9908160.

[17] K. Skenderis and M. Taylor, "Kaluza-Klein holography," JHEP **0605**, 057 (2006) [arXiv:hep-th/0603016].

[18] A. Petkou and K. Skenderis, "A non-renormalization theorem for conformal anomalies," Nucl. Phys. B **561**, 100 (1999) [arXiv:hep-th/9906030].

[19] J. M. Maldacena and A. Strominger, "Universal low-energy dynamics for rotating black holes," Phys. Rev. D **56**, 4975 (1997) [arXiv:hep-th/9702015].

[20] M. Cvetic and F. Larsen, "Greybody factors for rotating black holes in four dimensions," Nucl. Phys. B **506**, 107 (1997) [arXiv:hep-th/9706071].

[21] S. de Haro, S. N. Solodukhin and K. Skenderis, "Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence," Commun. Math. Phys. **217**, 595 (2001) [arXiv:hep-th/0002230].

[22] M. Bianchi, D. Z. Freedman and K. Skenderis, "Holographic Renormalization," Nucl. Phys. B **631**, 159 (2002) [arXiv:hep-th/0112119].

[23] D. T. Son and A. O. Starinets, "Minkowski-space correlators in AdS/CFT correspondence: Recipe and applications," JHEP **0209**, 042 (2002) [arXiv:hep-th/0205051].

[24] C. P. Herzog and D. T. Son, "Schwinger-Keldysh propagators from AdS/CFT correspondence," JHEP **0303**, 046 (2003) [arXiv:hep-th/0212072].

[25] K. Skenderis and B. C. van Rees, "Real-time gauge/gravity duality: Prescription, Renormalization and Examples," JHEP **0905**, 085 (2009) [arXiv:0812.2909 [hep-th]].

[26] E. Barnes, D. Vaman, C. Wu and P. Arnold, "Real-time finite-temperature correlators from AdS/CFT," arXiv:1004.1179 [hep-th].

[27] R. Kobes, "A Correspondence Between Imaginary Time and Real Time Finite Temperature Field Theory," Phys. Rev. D **42**, 562 (1990).

[28] M. Banados, C. Teitelboim and J. Zanelli, "The Black hole in three-dimensional space-time," Phys. Rev. Lett. **69**, 1849 (1992) [arXiv:hep-th/9204099].

[29] N. Iqbal and H. Liu, "Real-time response in AdS/CFT with application to spinors," Fortsch. Phys. **57**, 367 (2009) [arXiv:0903.2596 [hep-th]].

[30] D. Anninos, W. Li, M. Padi, W. Song and A. Strominger, "Warped AdS_3 Black Holes," JHEP **0903**, 130 (2009) [arXiv:0807.3040 [hep-th]].

[31] B. Chen, B. Ning and Z. b. Xu, "Real-time correlators in warped AdS/CFT correspondence," JHEP **1002**, 031 (2010) [arXiv:0911.0167 [hep-th]].

[32] H. K. Kunduri, J. Lucietti and H. S. Reall, "Near-horizon symmetries of extremal black holes," Class. Quant. Grav. **24**, 4169 (2007) [arXiv:0705.4214 [hep-th]].

[33] H. Lu, J. Mei, C. N. Pope, "Kerr/CFT Correspondence in Diverse Dimensions," JHEP **0904**, 054 (2009). [arXiv:0811.2225 [hep-th]].

[34] M. Becker, S. Cremonini and W. Schulgin, work in progress.