Annals of Studies in Science and Humanities
Volume 1 Number 1 (2015) : 10–16

http://journal.carsu.edu.ph/
Online ISSN 02408-3631

Counting Restricted One-to-One Functions Under a More General Condition

Mary Joy R. Latayada[1,*], *Roberto R. Corcino*[2]

[1] Department of Mathematics, Caraga State University, Butuan City, Philippines
[2] Department of Mathematics, Cebu Normal University, Cebu City, Philippines

Received: September 15, 2014 Accepted: Nov 25, 2014

ABSTRACT

In this paper, we obtain some explicit and recurrence formulas in counting the number restricted one-to-one functions $f|_S : N_m \longrightarrow N$, $S \subseteq N_m$ under certain conditions.

Key words: counting functions, recurrence relation, restricted one-to-one function, stirling numbers

1 Introduction

The study of counting functions was first considered by Cantor (Zehna and Johnson, 1972) when he attempted to give meaning to power of cardinal numbers. Cantor obtained that the number of possible functions from an m-set to an n-set is equal to n^m in which $(n)_m = n(n-1)(n-2)\ldots(n-m+1)$ of these are one-to-one functions. We can further obtained that, by making use of the classical Stirling numbers of the second kind $S(n,k)$, the number of onto functions is $n!S(m,n)$, see Chen and Kho (1992). Corcino et al. (2005) obtained that the number of restricted functions $f|_S : N_m \longrightarrow N_n$ for all $S \subseteq N_m$ where $N_m = \{1, 2, \ldots, m\}$ is equal to $(n+1)^m$. These restricted functions are the same as the restricted functions $f|_S : N_m \longrightarrow N$ such that $f(a) \leq n$ $\forall a \in S$.

Recently, Corcino et al. (2005) established some formulas in counting restricted functions $f|_S : N_m \longrightarrow N$, $S \subseteq N_m$ under each of the following conditions:

(i) $f(a) \leq g(a)$, $\forall a \in S$ where g is any nonnegative real-valued continuous

* Corresponding Author
Email:mj_regidor@yahoo.com

function.

(ii) $g_1(a) \leq f(a) \leq g_2(a)$, $\forall a \in S$ where g_1 and g_2 are any two nonnegative real-valued continuous functions.

Under condition (i), with $\hat{\psi}_{i,m} = \bigcup_{S_i \subseteq N_m} \{f|_{S_i}\}$, the following formulas were obtained.

$$|\hat{\psi}_{i,m}| = \sum_{1 \leq j_1 < j_2 < \ldots < j_i \leq m} \prod_{l=1}^{i} \lceil g(j_l) \rceil \tag{1}$$

$$|\hat{\psi}_{i,m+1}| = |\hat{\psi}_{i,m}| + \lceil g(m+1) \rceil |\hat{\psi}_{i-1,m}| \tag{2}$$

$$|\hat{\psi}_m| = \prod_{i=1}^{m} (1 + \lceil g(i) \rceil), \qquad \hat{\psi}_m = \bigcup_{i=0}^{m} \hat{\psi}_{i,m}. \tag{3}$$

While under condition (ii), with $\tilde{\psi}_{i,m} = \bigcup_{S_i \subseteq N_m} \{f|_{S_i}\}$, the following formulas were obtained,

$$|\tilde{\psi}_{i,m}| = \sum_{1 \leq j_1 < j_2 < \ldots < j_i \leq m} \prod_{l=1}^{i} \tilde{g}(j_l),$$

$$|\tilde{\psi}_m| = \prod_{i=1}^{m} (1 + \tilde{g}(i)).$$

where $\tilde{g}(j_l) = \lceil g_2(j_l) \rceil - \lfloor g_1(j_l) \rfloor + 1$. Note that $|\hat{\psi}_{i,m}|$ and $|\tilde{\psi}_{i,m}|$ count the total number of restricted functions under conditions (i) and (ii), respectively.

In this paper, we count the number of restricted one-to-one functions under conditions (i) and (ii), however, the functions g and g_1 must be increasing while g_2 must be decreasing.

2 One-to-One Functions Under (i)

Let $\hat{Y}_{i,m} = \bigcup_{S_i \subseteq N_m} \{f|_{S_i}\}$ where f is a one-to-one function from N_m to N such that $m \leq n$ and $f(a) \leq g(a)$, $\forall a \in N_m$ for some nonnegative real-valued continuous function g which is increasing on the closed interval $[1, m]$. If $S_i = \{j_1, j_2 \ldots j_i\}$ where $j_1 < j_2 < \ldots < j_i$, then

$$|\{f|_{S_i}\}| = \prod (\lceil g(j_l) \rceil - l + 1$$

where $\lceil g(j_l) \rceil$ is the greatest integer that is less than or equal to a real number $g(j_l)$. Thus, we have

$$|\hat{Y}_{i,m}| = \sum_{1 \leq j_1 < j_2 < \ldots < j_i \leq m} \prod_{l=1}^{i} (\lceil g(j_l) - l + 1)$$

which is parallel to formula (1). To state this result formally, we have the following proposition.

Proposition 1. *Let f be a one-to-one function from N_m to N such that $f(a) \leq g(a)$ $\forall a \in N_m$ for some nonnegative real-valued continuous function g which is increasing or constant on the closed interval $[1, m]$. Then the number $|\hat{Y}_i, m|$ of restricted one-to-one functions $f|_{S_i}$ over all $S_i \subseteq N_m$ such that $|S_i| = i$ is given by*

$$|\hat{Y}_{i,m}| = \sum_{1 \leq j_1 < j_2 < \ldots < j_i \leq m} \prod_{l=1}^{i} (\lceil g(j_l) \rceil - l + 1)$$

where $|\hat{Y}_0, m| = 1$, $|\hat{Y}_i, m| = 0$ when $i > m$.

Remarks. (1) When $g(a) = n$, $\forall a \in N_m$, we have

$$|\hat{Y}_{i,m}| = \sum_{1 \leq j_1 < j_2 < \ldots < j_i \leq m} \prod_{l=1}^{i} (\lceil n - l + 1).$$

Since the number of terms of the sum is $\binom{m}{i}$,

$$|\hat{Y}_{i,m}| = \binom{m}{i} (n)_i, \quad (n)_i = n(n-1)(n-2) \ldots (n-i+1).$$

This is the number of restricted one-to-one functions $f|_{S_i}$ from N_m to N_n such that $|S_i| = i$.

(2) When $g(a) = a$, $\forall a \in N_m$, we have

$$|\hat{Y}_{i,m}| = \sum_{1 \leq j_1 < j_2 < \ldots < j_i \leq m} \prod_{l=1}^{i} (\lceil j_l - l + 1).$$

Parallel to (2), we have the following recurrence relation which is useful in computing the first values of $|\hat{Y}_{i,m}|$.

Proposition 2. *The number $|\hat{Y}_i, m|$ satisfies the following recurrence relation*

$$|\hat{Y}_{i,m+1}| = |\hat{Y}_{i,m}| + (\lceil g(m+1) \rceil - i + 1)|\hat{Y}_{i-1,m}|$$

with initial condition $|\hat{Y}_{0,0}| = 1$, $|\hat{Y}_{i,m}| = 0$ for $i > m$ or $i < 0$.

Proof. We know that $|\hat{Y}_{i,m+1}|$ counts the number of restricted one-to-one functions $f|_{S_i}$ over all $S_i \subseteq N_{m+1}$. This can be done by considering the following cases:

1. Case 1. $m + 1 \notin S_i$.
   Under this case, the number of such restricted one-to-one functions $f|_{S_i}$ over all $S_i \subseteq N_{m+1}$ is $|\hat{Y}_{i,m}|$.

2. Case 2. $m+1 \in S_i$.
   In this case, we consider the following sequence of events:

   (a) counting restricted functions $f|_{S_i}$ over all $S_{i-1} \subseteq N_m$ which is equal to $|\hat{Y}_{i,m}|$.

   (b) insert $m+1$ to every S_{i-1} and map $m+1$ to any of the natural number $1, 2, \ldots, \lceil g(m+1) \rceil - i + 1$. Since $m+1$ is the ith element of $S_i = S_{i-1} \cup \{m+1\}$, by Multiplication Principle (MP), the number of such restricted functions $f|_{S_i}$ with $S_i = S_{i-1} \cup \{m+1\}$ is equal to $(\lceil g(m+1) \rceil - i + 1)|\hat{Y}_{i-1,m}|$.

Thus, by Addition Principle (AP), we prove the proposition. □

If we let $\hat{Y}_m = \bigcup_{i=1}^{m} |\hat{Y}_{i,m}|$ then $|\hat{Y}_m| = \sum_{i=1}^{m} |\hat{Y}_{i,m}|$. Using Proposition 2.2, we have

$$
\begin{aligned}
|\hat{Y}_m| &= \sum_{i=0}^{m} \left\{ |\hat{Y}_{i,m-1}| + (\lceil g(m) \rceil - i + 1)|\hat{Y}_{i-1,m-1}| \right\} \\
&= \sum_{i=0}^{m-1} |\hat{Y}_{i,m-1}| + (\lceil g(m) \rceil \sum_{i=0}^{m-1} |\hat{Y}_{i,m-1}| - \sum_{i=0}^{m} i|\hat{Y}_{i,m-1}| \\
&= |\hat{Y}_{m-1}|(1 + \lceil g(m) \rceil) - \sum_{i=0}^{m} i|\hat{Y}_{i,m-1}|
\end{aligned}
$$

When $m = 1$,

$$
|\hat{Y}_1| = |\hat{Y}_0|(1 + \lceil g(1) \rceil) - \sum_{i=0}^{0} i|\hat{Y}_{i,0}| = (1 + \lceil g(1) \rceil) - \sum_{i=0}^{0} i|\hat{Y}_{i,0}|.
$$

When $m = 2$,

$$
\begin{aligned}
|\hat{Y}_2| &= |\hat{Y}_1|(1 + \lceil g(2) \rceil) - \sum_{i=0}^{1} i|\hat{Y}_{i,1}| \\
&= (1 + \lceil g(1) \rceil)(1 + \lceil g(2) \rceil) - (1 + \lceil g(1) \rceil) \sum_{i=0}^{0} i|\hat{Y}_{i,0}| - \sum_{i=0}^{1} i|\hat{Y}_{i,1} \\
&= \prod_{i=1}^{2}(1 + \lceil g(i) \rceil) - \sum_{i=0}^{1} \left\{ \prod_{l=j+1}^{1} (1 + \lceil g(l+1) \rceil) \right\} \left\{ \sum_{i=0}^{j} i|\hat{Y}_{i,j} \right\}
\end{aligned}
$$

When $m = 3$,

$$\begin{aligned}
|\hat{Y}_3| &= |\hat{Y}_2|(1+\lceil g(3)\rceil) - \sum_{i=0}^{2} i|\hat{Y}_{i,2}| \\
&= (1+\lceil g(1)\rceil)(1+\lceil g(2)\rceil)(1+\lceil g(3)\rceil) - (1+\lceil g(2)\rceil)(1+\lceil g(3)\rceil)\sum_{i=0}^{0} i|\hat{Y}_{i,0}| \\
&\quad -(1+\lceil g(1)\rceil)\sum_{i=0}^{1} i|\hat{Y}_{i,1}\sum_{i=0}^{2} i|\hat{Y}_{i,2}| \\
&= \prod_{i=1}^{3}(1+\lceil g(i)\rceil) - \sum_{i=0}^{2}\left\{\prod_{l=j+1}^{2}(1+\lceil g(l+1)\rceil)\right\}\left\{\sum_{i=0}^{j} i|\hat{Y}_{i,j}\right\} \\
&\vdots \\
|\hat{Y}_m| &= |\hat{Y}_{m-1}|(1+\lceil g(m)\rceil) - \sum_{i=0}^{m-1} i|\hat{Y}_{i,m-1}| \\
&= \prod_{i=1}^{m}(1+\lceil g(i)\rceil) - \sum_{i=0}^{m-1}\left\{\prod_{l=j+1}^{m-1}(1+\lceil g(l+1)\rceil)\right\}\left\{\sum_{i=0}^{j} i|\hat{Y}_{i,j}\right\}
\end{aligned}$$

where $\prod_{l=m}^{m-1}(1+\lceil g(l+1)\rceil) = 1$. Note that $\hat{Y}_m$ is the set of all restricted one-to-one functions $f|_S$. Thus, $|\hat{Y}_m|$ is the total number of restricted one-to-one functions $f|_S$.

Proposition 3. *The total number $|\hat{Y}_m|$ of one-to-one restricted functions f from N_m to N such that $f(a) \leq g(a)$, $\forall a \in N_m$ where g is any nonnegative real-valued continuous function which is increasing on $[1, m]$ is given by*

$$|\hat{Y}_m| = \prod_{i=1}^{m}(1+\lceil g(i)\rceil) - \sum_{i=0}^{m-1}\sum_{i=0}^{j} i|\hat{Y}_{i,j}|\prod_{l=j+1}^{m-1}(1+\lceil g(l+1)\rceil).$$

As a direct consequence of this proposition, we have the following corollary.

Corollary 1. *The total number $|\hat{Y}_m^c|$ of restricted functions f from N_m to N which are not one-to-one such that $f(a) \leq g(a)$, $\forall a \in N_m$ where g is any nonnegative real-valued continuous function which is increasing on $[1, m]$ is given by*

$$|\hat{Y}_m^c| = \sum_{i=0}^{m-1}\sum_{i=0}^{j} i|\hat{Y}_{i,j}\prod_{l=j+1}^{m-1}(1+\lceil g(l+1)\rceil).$$

Proof. Note that from (3) the total number $|\hat{\psi}_m|$ of restricted functions is given by

$$|\hat{\psi}_m| = \prod_{i=1}^{m}(1 + \lceil g(i) \rceil).$$

Hence, the total number $|\hat{Y}_m^c|$ of restricted functions which are not one-to-one is

$$|\hat{Y}_m^c| = \prod_{i=0}^{m}(1 + \lceil g(i) \rceil) - |\hat{Y}_m|.$$

Applying Proposition 3, we prove the corollary. □

3 One-to-One Functions Under (ii)

In this section, we count the number of restricted one-to-one functions $f|_S : N_m \longrightarrow N$, $S \subseteq N_m$ under condition (ii). The following proposition gives an explicit formula for this number.

Proposition 4. *Let f be a one-to-one function from N_m to N such that $g_1(a) \leq f(a) \leq g_2(a), \forall a \in N_m$ where g_1 and g_2 are decreasing and, respectively, increasing nonnegative real-valued continuous functions on $[0, m]$. Then the number $|\tilde{Y}_{i,m}|$ of restricted one-to-one functions $f|_{S_i}$ over all $S_i \subseteq N_m$ such that $|S_i| = i$ is given by*

$$|\tilde{Y}_{i,m}| = \sum_{1 \leq j_1 < j_2 < \ldots < j_i \leq m} \prod_{l=1}^{i}(\tilde{g}(j_l) - l + 1)$$

where $\tilde{g}(j_l) = \lceil g_2(j_l) \rceil - \lfloor g_1(j_l) \rfloor + 1$.

Proof. It can easily be seen that if $S_i = \{j_1, j_2, \ldots, j_i\}$ where $j_1 < j_2 < \ldots < j_i$, then to form a one-to-one function, j_1 can be mapped to any of the $\lceil g_2(j_1) \rceil - \lfloor g_1(j_1) \rfloor + 1$ natural numbers and j_2 can be mapped to any of the $\lceil g_2(j_2) \rceil - \lfloor g_1(j_2) \rfloor$ natural numbers, and so on until j_i can be mapped to any of the $\lceil g_2(j_i) \rceil - \lfloor g_1(j_i) \rfloor - i + 1$ natural numbers where $\lfloor g_1(j_i) \rfloor$ is the least integer greater than or equal to $g_1(j_i)$. Hence, the number of possible one-to-one functions that can be formed from S_i to N such that $g_1(a) \leq f(a) \leq g_2(a), \forall a \in N_m$ is

$$\prod_{l=1}^{i}(\lceil g_2(j_l) \rceil - \lfloor g_1(j_l) \rfloor - l + 1)$$

Thus, with $\tilde{Y}_{i,m} = \bigcup_{S_i \subseteq N_m} \{f|S_i\}$, we have

$$|\tilde{Y}_{i,m}| = \sum_{1 \leq j_1 < j_2 < \ldots < j_i \leq m} \prod_{l=1}^{i}(\lceil g_2(j_l) \rceil - \lfloor g_1(j_l) \rfloor - l + 1).$$

□

Now, using the same argument as in the proof of Proposition 2.2, we can easily derive the following recurrence relation for $\tilde{Y}_{i,m}$.

Proposition 5. *The number* $|\tilde{Y}_{i,m}|$ *satisfies the following recurrence relation*

$$|\tilde{Y}_{i,m+1}| = |\tilde{Y}_{i,m}| + (\tilde{g}(m+1) - i + 1)|\tilde{Y}_{i-1,m}|$$

with initial condition $|\tilde{Y}_{0,0}| = 1$, $|\tilde{Y}_{i,m}| = 0$ *for* $i > m$ *or* $i < 0$.

By making use of Proposition 5 and following the same argument in the proof of Proposition 3, we obtain

$$|\tilde{Y}_m| = \prod_{i=1}^{m}(1 + \tilde{g}(i)) - \sum_{i=0}^{m-1}\sum_{i=0}^{j} i|\tilde{Y}_{i,j}| \prod_{l=j+1}^{m-1}(1 + \tilde{g}(l+1))$$

where $|\tilde{Y}_m|$ is the total number of restricted one-to-one functions from N_m to N such that $g_1(a) \leq f(a) \leq g_2(a), \forall a \in N_m$. Thus, the total number of restricted functions which are not one-to-one is given by

$$|\tilde{Y}_m^c| = \sum_{i=0}^{m-1}\sum_{i=0}^{j} i|\tilde{Y}_{i,j}| \prod_{l=j+1}^{m-1}(1 + \tilde{g}(l+1)).$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

Chen, C. C. and Kho, K. M. (1992). *Principles and Techniques in Combinatorics*. World Scientific Publishing Co. Pte. Ltd.

Corcino, R., Alquiza, M. and Caumeran, J. M. (2005). Counting Restricted Functions on the Formation of Points Having Integral Coordinates. *Mindanao Forum. 19*(1), 9-18.

Zehna, P. W. and Johnson, R. L. (1972). *Elements of Set Theory*. Allyn and Bacon, Inc., United States of America.