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Claw -free graphs and line graphs

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ABSTRACT

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The research of my dissertation is motivated by the conjecture of Thomassen that every 4-connected line graph is hamiltonian and by the conjecture of Tutte that every 4-edge-connected graph has a no-where-zero 3-flow. Towards the hamiltonian line graph problem, we proved that every 3-connected N_2-locally connected claw-free graph is hamiltonian, which was conjectured by Ryjacek in 1990; that every 4-connected line graph of an almost claw free graph is hamiltonian connected, and that every triangularly connected claw-free graph G with $|E(G)| \geq 3$ is vertex pancyclic. Towards the second conjecture, we proved that every line graph of a 4-edge-connected graph is $\mathbf{Z}_3$-connected.

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DEDICATION

To

my father Jinfu , my mother Luolan

and

my sister Honglian

Contents

Chapter 1

Introduction

We use [1] for terminology and notations not defined here. Let G be a graph. A graph with at least two vertices is called a **nontrivial** graph. For an integer $k > 0$, a k**-cycle**, denoted by C_k, is a 2-regular connected graph with k edges; similarly P_k denotes a path of length k. We use $\kappa(G)$, $\kappa'(G)$ to denote the **connectivity** and the **edge-connectivity** of G, respectively. The **degree** of a vertex $v \in V(G)$ and the **minimum degree** of G are respectively denoted by $d_G(x)$ and $\delta(G)$. An edge $e = uv$ is called a **pendant edge** if either $d_G(u) = 1$ or $d_G(v) = 1$. For a vertex or an edge subset X of G, $G[X]$ denotes the subgraph of G **induced** by X.

For a graph G and for $v \in V(G)$, **the neighborhood** $N_G(v)$ denotes the set of all vertices adjacent to v in G. A vertex $v \in V(G)$ is called a **locally connected vertex** if $G[N_G(v)]$ is connected. A graph G is **locally connected** if every vertex of G is locally connected.

A graph G is **pancyclic** if for each integer k with $3 \le k \le |V(G)|$, G has a k-cycle; G is **vertex pancyclic** if for every vertex $v \in V(G)$, G has a k-cycle C_k containing v as a vertex, for each k with $3 \le k \le |V(G)|$.

For a graph G and each $i = 0, 1, 2, \cdots$, we let $D_i(G) = \{v \in V(G) | d_G(v) = i\}$. For $H \subseteq G$ and $x \in V(G)$, we let $d_H(x) = |N_H(x)|$ and if $A \subseteq V(G)$, we let $G - A =$

$G[V(G) - A]$. When $A = \{v\}$, we use $G - v$ for $G - \{v\}$.

For a graph G and a vertex $v \in V(G)$, define

$$E_G(v) = \{e \in E(G) : \ e \text{ is incident with } v \text{ in } G\}.$$

An edge cut X of G is **peripheral** if for some $v \in V(G)$, $X = E_G(v)$; and is **essential** if each side of $G - X$ has an edge. A graph G is **essentially k-edge-connected** if $|E(G)| \geq k + 1$ and if for every $E_0 \subseteq E(G)$ with $|E_0| < k$, $G - E_0$ has exactly one component H with $E(H) \neq \emptyset$.

Let $X \subseteq E(G)$. The **contraction** G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. For convenience, we use G/e for $G/\{e\}$ and $G/\emptyset = G$; and if H is a subgraph of G, we write G/H for $G/E(H)$. Note that even if G is a simple graph, contracting some edges of G may result in a graph with multiple edges.

1.1 Line Graphs

The **line graph** of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are linked by k edges if and only if the corresponding edges in G share exactly k vertices in common. Note that when G is a simple graph, two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent.

Proposition 1.1.1 *Let G be a nontrivial simple graph. Then $L(G)$ is complete if and only if G is a K_3 or a $K_{1,n}$ for an integer $n \geq 1$.*

Proof The line graph $L(G)$ is complete if and only if any two edges in G are adjacent. If $|E(G)| = 1$, $G = K_2 = K_{1,1}$; if $|E(G)| = 2$, $G = P_2 = K_{1,2}$; if $|E(G)| = 3$, $G = K_3$ or $G = K_{1,3}$; if $|E(G)| \geq 4$, $G = K_{1,n}$. $\square$

Proposition 1.1.2 *(i) The graphs K_3 and $K_{1,n}$ do not have essential edge-cuts.*

(ii) Every essential edge cut in G corresponds to a vertex cut in $L(G)$. Moreover, if $L(G)$ is not complete, then X is a vertex cut in $L(G)$ if and only if the corresponding edge set is an essential edge cut in G.

Proof (i) It is straightforward by the definition of an essential edge cut.

(ii) Let X be an essential edge cut in G. Since X is essential, $G - X$ has at least two components each of which has at least one edge. So the set of the corresponding vertices of X in $L(G)$ is a vertex cut of $L(G)$. Conversely let X' be a vertex cut of $L(G)$. If $L(G)$ is not complete, $L(G) - X'$ has at least two components. Then the edge set of the corresponding edges of X' in G is an essential edge cut of G. $\square$

The following proposition follows directly from Proposition 1.1.2(ii).

Proposition 1.1.3 *If $L(G)$ is k-connected, then G is essentially k-edge-connected. Moreover, when $L(G)$ is not complete, G is essentially k-edge-connected if and only if $L(G)$ is k-connected.*

1.2 Hamiltonian and Hamilton-connected line Graphs

A subgraph H of a graph G is **dominating** if $E(G - V(H)) = \emptyset$. Let $O(G)$ denote the set of odd degree vertices of G. A graph G is **eulerian** if $O(G) = \emptyset$ and G is connected. A spanning closed trail of G is also called a **spanning eulerian subgraph** of G. If a closed trail C of G satisfies $E(G - V(C)) = \emptyset$, then C is called a **dominating eulerian subgraph**.

From the following theorem we can see that there is a close relationship between dominating eulerian subgraphs in G and hamilton cycles in its line graph $L(G)$.

Theorem 1.2.1 *(Harary and Nash-Williams, [11]) Let G be a graph with $|E(G)| \geq 3$. Then $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.*

A graph G is **hamilton-connected** if for $u, v \in V(G)(u \neq v)$, there exists a (u, v)-path containing all vertices of G.

We view a trail of G as a vertex-edge alternating sequence

$$v_0, e_1, v_1, e_2, \cdots, e_k, v_k \tag{1.1}$$

such that all the e_i's are distinct and such that for each $i = 1, 2, \cdots, k$, e_i is incident with both v_{i-1} and v_i. All the vertices in $\{v_1, v_2, \cdots, v_{k-1}\}$ are **internal vertices** of the trail in (1.1). For edges $e', e'' \in E(G)$, an (e', e'')-trail of G is a trail of G whose first edge is e' and whose last edge is e''. (Thus the trail in (1.1) is an (e_1, e_k)-trail). A **dominating** (e', e'')**-trail** of G is an (e', e'')-trail T of G such that every edge of G is incident with an internal vertex of T; and a **spanning** (e', e'')**-trail** of G is a dominating (e', e'')-trail T of G such that $V(T) = V(G)$.

With a similar argument in the proof of Theorem 1.2.1, one can obtain a theorem for hamilton-connected line graphs.

Theorem 1.2.2 *Let G be a graph with $|E(G)| \geq 3$. Then $L(G)$ is hamilton-connected if and only if for any pair of edges $e_1, e_2 \in E(G)$, G has a dominating (e_1, e_2)-trail.*

1.3 Subdivided graphs

We say that an edge $e \in E(G)$ is **subdivided** when it is replaced by a path of length 2 whose internal vertex, denoted by $v(e)$, has degree 2 in the resulting graph. The process of taking an edge e and replacing it by that length 2 path is called **subdividing** e (see Figure 1.1).

Figure 1.1

For a graph G and edges $e', e'' \in E(G)$, let $G(e')$ denote the graph obtained from G by subdividing e', and let $G(e', e'')$ denote the graph obtained from G by subdividing both e' and e''. Then,

$$V(G(e', e'')) - V(G) = \{v(e'), v(e'')\}.$$

Lemma 1.3.1 *For a graph G and edges $e', e'' \in E(G)$, each of the following holds.*

(i) if $G(e', e'')$ has a spanning $(v(e'), v(e''))$-trail, then G has a spanning (e', e'')-trail.

(ii) if $G(e', e'')$ has a dominating $(v(e'), v(e''))$-trail, then G has a dominating (e', e'')-trail.

Proof (i) Let $e' = u_1u_2, e'' = w_1w_2$ and T be a spanning $(v(e'), v(e''))$-trail of $G(e', e'')$. Since T is spanning in $G(e', e'')$, u_1, u_2, w_2, w_2 must be on T. So we can assume that $T = v(e')u_1 \cdots w_1v(e'')$. Delete $v(e'), v(e'')$ in T and join u_1u_2, w_1w_2 by an edge respectively and denote the resulting trail by T'. Then T' is a spanning (e', e'')-trail of G.

(ii) Let C be a dominating $(v(e'), v(e''))$-trail of $G(e', e'')$. By the definition of a dominating $(v(e'), v(e''))$-trail, u_1, u_2, w_2, w_2 must be on T. So we can construct C' from C the same way as constructing T' from T in the proof of (i). Then C' is a dominating (e', e'')-trail of G.

1.4 Core graphs

Let G be a graph such that $\kappa(L(G)) \geq 3$ and $L(G)$ is not complete. The **core** of this graph G, denoted by $\widetilde{G}$, is obtained from G by the following two operations until no vertices of degree 1 or 2 remain:

Operation 1 recursively delete the vertices of degree 1;

Operation 2 contract exactly one edge xy or yz for each path xyz in G with $d_G(y) = 2$ (see Figure 1.2).

Figure 1.2

Lemma 1.4.1 *Let G be a graph, $G' = G - D_1(G)$ and $\widetilde{G}$ the core graph of G. If $\kappa(L(G)) \geq 3$ and $L(G)$ is not complete, then*

(i) G' is nontrivial and $\delta(G') \geq \kappa'(G') \geq 2$, $D_2(G) = D_2(G')$.

(ii) $\widetilde{G}$ in nontrivial and $\delta(\widetilde{G}) \geq \kappa'(\widetilde{G}) \geq 3$.

(iii) for $v \in V(G)$ with $d_G(v) = 1$ or $d_G(v) = 2$, $N_G(v) \subseteq V(\widetilde{G})$.

(iv) $\widetilde{G}$ is well defined.

Proof Let $e = uv \in E(G)$ and $u \in D_1(G)$. Since $\kappa(L(G)) \geq 3$, $d_G(v) \geq 4$. If $d_{G'}(v) = 0$, then $G = K_{1,n}$ (for a positive integer n) and so $L(G)$ is complete, contrary to the assumption; if $1 \leq d_{G'}(v) \leq 2$, then $E_{G'}(v)$ is an essential $d_{G'}(v)$-edge-cut of G, contrary to the assumption that $\kappa(L(G)) \geq 3$. Hence $d_{G'}(v) \geq 3$. So G' is nontrivial.

And we also have $\delta(G') \geq 2$. If there exists some $v' \in V(G')$ such that $d_{G'}(v') = 1$, then there must be at least one pendent edge incident with v' in G. And so $E_{G'}(v')$ is an essential 1-edge-cut of G, contrary to the assumption that $\kappa(L(G)) \geq 3$. Note that every essential edge cut of G' is also an essential edge cut of G. So G' does not have essential 1-edge-cuts. This completes the proof of (i).

Now we show that $D_2(G) = D_2(G')$. If there exists some $v' \in V(G')$ such that $d_{G'}(v') = 2$ but $d_G(v') \neq 2$, then there must be at least one pendent edge incident with v' in G. Then $E_{G'}(v')$ is an essential 2-edge-cut of G, contrary to the assumption that $\kappa(L(G)) \geq 3$.

Suppose that there exists $x \in V(G')$ such that $d_{G'}(x) = 2$ and $N_{G'}(x) = \{x_1, x_2\}$. As

$D_2(G) = D_2(G'), d_G(x) = 2$. Since $\kappa(L(G)) \geq 3$, $d_G(x) + d_G(x_1) \geq 5$ and so $d_G(x_1) \geq 3$. If $d_{G'}(x_1) = 2$, then there must be at least one pendent edge incident with x_1 in G. Then $E_{G'}(x_1)$ is an essential 2-edge-cut of G, contrary to the assumption that $\kappa(L(G)) \geq 3$. Hence $d_{G'}(x_1) \geq 3$. Similarly we have $d_{G'}(x_2) \geq 3$. So $\widetilde{G}$ is nontrivial and $\delta(\widetilde{G}) \geq 3$ by the definition of $\widetilde{G}$. Note that every essential edge cut of $\widetilde{G}$ is also an essential edge cut of G. So $\widetilde{G}$ does not have essential edge-cuts of size less than 3. This completes the proof of (ii).

From the proofs of (i) and (ii), the proof for (iii) is straightforward.

(iv). By (iii), for $v \in V(G)$ with $d_G(v) \geq 3$, the degree of v will never change when we do the two operations in the definition of core graphs. So $\widetilde{G}$ is unique. □

Proposition 1.4.2 *Let G be a graph and $\widetilde{G}$ the core graph of G.*

(i) If $\widetilde{G}$ has a spanning eulerian subgraph, then G has a dominating eulerian subgraph.

(ii) If $\widetilde{G}(e', e'')$ has a spanning $(v(e'), v(e''))$-trail for any $e', e'' \in E(\widetilde{G})$, then for any $e', e'' \in E(G)$, $G(e', e'')$ has a dominating $(v(e'), v(e''))$-trail.

Proof If $G = \widetilde{G}$, there is nothing to prove. Therefore, we assume that $G \neq \widetilde{G}$ and consider the following cases for (i) and (ii) respectively.

(i) Let T be a spanning eulerian subgraph of $\widetilde{G}$. By the definition of core graphs, there are some edges of $\widetilde{G}$ which may not be in G and we know each of these edges (say xz) must be obtained by contracting exactly one edge of some edge xy or yz for some path xyz in G with $d_G(y) = 2$ (see Figure 1.2). So we can subdivide xz in $\widetilde{G}$ (see Figure 1.1). For all these edges we subdivide them in T, so we get T' and it is also a spanning eulerian subgraph of $G - D_1(G)$. So T' is a dominating eulerian subgraph of G.

(ii) **Case 1** For $e', e'' \in E(G)$, if they are both in $E(\widetilde{G})$, then by the assumption, $\widetilde{G}(e', e'')$ has a spanning $(v(e'), v(e''))$-trail T. By Lemma 1.4.1(iii), T can be adjusted to a dominating $(v(e'), v(e''))$-trail in $G(e', e'')$.

Case 2 At least one of e' and e'' is not in $E(\widetilde{G})$.

Without loss of generality, we can assume first that e'' is not a pendent edge of G and if $e'' \notin E(\widetilde{G})$, then by the definition of $\widetilde{G}$, there must be an edge $f'' \in E(G)$ such that for some vertex z of degree 2 in G, $E_G(z) = \{e'', f''\}$. In this case, by the definition of $\widetilde{G}$ we may contract f'' and so we may always assume that $e'' \in E(\widetilde{G})$. With this view point, if e' is not a pendent edge of G either, then we may assume that $e' \in E(\widetilde{G})$. That is, if neither of e', e'' is a pendent edge of G, then we can always assume that they are both in $E(\widetilde{G})$. And this is back to Case 1. So we may assume that e' is a pendent edge of G.

If $e'' \in E(\widetilde{G})$, let $h_1 \in E(\widetilde{G})$ be an edge adjacent to e' in G and e' and h_1 are both incident with a vertex v. By the assumption, $\widetilde{G}(h_1, e'')$ has a spanning $(v(h_1), v(e''))$-trail T'. Since e' is a pendent edge adjacent to h_1, By Lemma 1.4.1(iii), T' can be extended to a dominating $(v(e'), v(e''))$-trail of $G(e', e'')$.

If $e'' \notin E(\widetilde{G})$, then both e' and e'' are pendent edges of G. A similar argument indicates that $G(e', e'')$ also has a dominating $(v(e'), v(e''))$-trail. □

1.5 Claw free graphs

For a graph G, an induced subgraph H isomorphic to $K_{1,3}$ is called a **claw** of G, and the only vertex of degree 3 of H is the **center** of the claw (see Figure 1.3). A graph G is **claw free** if it does not contain a claw.

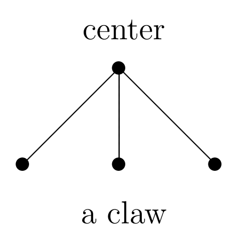


Figure 1.3

Chapter 2

Hamiltonian claw-free graphs

2.1 Background

A graph G is N_2**-locally connected** if for every vertex v in G, the edges not incident with v but having at least one end adjacent to v in G induce a connected graph.

The following theorems give the hamiltonicity of locally and N_2-locally connected graphs respectively.

Theorem 2.1.1 *(Oberly and Sumner, [21]) Every connected locally connected claw-free graph on at least three vertices is hamiltonian.*

Theorem 2.1.2 *(Ryjáček, [22]) Let G be a connected, N_2-locally connected claw-free graph without vertices of degree 1, which does not contain an induced subgraph H isomorphic to either G_1 or G_2(Figure 2.1) such that $N_G(x)$ of every vertex x of degree 4 in H is disconnected. Then G is hamiltonian.*

Theorem 2.1.3 *(Li, [20]) Let G be a connected, N_2-locally connected claw-free graph with $\delta(G) \geq 3$, which does not contain an induced subgraph H isomorphic to either G_1 or G_2 (Figure 2.1). Then G is vertex pancyclic.*

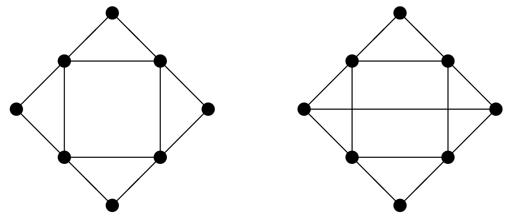


Figure 2.1

2.2 Main Result

The main purpose of this chapter is to prove the following theorem, conjectured by Ryjáček in [22].

Theorem 2.2.1 *(Lai, Shao and Zhan, [19])Every 3-connected N_2-locally connected claw-free graph is hamiltonian.*

In [23], Ryjáček defined the closure $cl(G)$ of a claw-free graph G by recursively completing the neighborhood of any locally connected vertex of G, as long as this is possible. The closure $cl(G)$ is a well-defined claw-free graph and its connectivity is at least equal to the connectivity of G.

In order to prove Theorem 2.2.1, we need the following Theorems 2.2.2, 2.2.3 and Lemma 2.2.4.

Theorem 2.2.2 *(Ryjáček, [23]) Let G be a claw-free graph and $cl(G)$ its closure. Then*
(i) there is a triangle-free graph H such that $cl(G)$ is the line graph of H,
(ii) both graphs G and $cl(G)$ have the same circumference.

Theorem 2.2.3 *(Lai, [15]) Let G be a 2-connected graph with $\delta(G) \geq 3$. If every edge of G is in an m-cycle of G ($m \leq 4$), then G has a spanning eulerian subgraph.*

Lemma 2.2.4 *Let G be an N_2-locally connected graph and let x be a locally connected vertex of G such that $G[N_G(x)]$ is not complete. Let $N' = \{uv : u, v \in N_G(x), uv \notin E(G)\}$ and let G' be the graph with vertex set $V(G') = V(G)$ and with edge set $E(G') = E(G) \cup N'$. Then G' is N_2-locally connected.*

The proof of Lemma 2.2.4 Let $w \in V(G')$. If $w = x$, then $N_2(w, G')$ is connected since $N_{G'}(x)$ is complete. So we may assume that $w \neq x$. Since G is N_2-locally connected, $N_2(w, G)$ is connected. If $E(N_2(w, G')) - E(N_2(w, G)) = \emptyset$, then $E(N_2(w, G')) = E(N_2(w, G))$ and $N_2(w, G')$ is connected. Thus we assume that

$$E(N_2(w, G')) - E(N_2(w, G)) \neq \emptyset.$$

Let $e = uv \in E(N_2(w, G')) - E(N_2(w, G))$. Since $e = uv \in E(N_2(w, G'))$, we have $w \notin \{u, v\}$, and so $uv \in E(G')$. Without loss of generality, we assume that $wu \in E(G')$.

Case 1. $uv \in E(G)$.

By $e = uv \notin E(N_2(w, G))$, we have $wu, wv \notin E(G)$. Since $wu \in E(G')$ by the assumption, $w, u \in N_G(x)$. So $xu \in E(N_2(w, G))$. Therefore adding a new edge uv to $N_2(w, G)$ does not change its connectivity, and so $N_2(w, G')$ is connected.

Case 2. $uv \notin E(G)$.

Since $uv \in E(G')$, we have $u, v \in N_G(x)$. If $w \in N_G(x)$, then $xu, xv \in E(N_2(w, G))$. Thus adding a new edge uv to $N_2(w, G)$ does not change its connectivity, and so $N_2(w, G')$ is connected. If $w \notin N_G(x)$, then we have $wu \in E(G)$ since $wu \in E(G')$(otherwise, $w \in N_G(x)$, a contradiction). Thus $xu \in E(N_2(w, G))$. So adding a new edge uv to $N_2(w, G)$ does not change its connectivity, and therefore $N_2(w, G')$ is connected. □

The proof of Theorem 2.2.1 We can assume that G is not complete. By Theorem 2.2.2(ii), the graph G is hamiltonian if and only if its closure $cl(G)$ is hamiltonian. By Lemma 2.2.4 and as $cl(G)$ is both 3-connected and claw free, the graph $cl(G)$ is also a 3-connected N_2-locally connected claw-free graph. By Theorem 2.2.2, we may assume that for a triangle free graph H, $G = cl(G) = L(H)$.

Claim Let $e = uv \in E(H)$. If e is not a pendant edge, then e is in some 4-cycle of H.

Proof. Since H is triangle-free, we have $N_H(u) \cap N_H(v) = \emptyset$. Let $v_e \in V(G)$ correspond to the edge $e \in E(H)$ in terms of the line graph. Since e is not a pendant edge and G is claw free, $N_G(v_e)$ is the union of two disjoint cliques. Suppose they are L_1, L_2. Since G is 3-connected, there exits at least one path $w_1 w_2 \cdots w_n$ which is edge disjoint with $G[V(L_1) \cup V(L_2) \cup \{v_e\}]$ in $G - v_e$ with $w_1 \in V(L_1), w_n \in V(L_2)$. Since G is N_2-locally connected, we have that $n = 3$. Thus $v_e w_1 w_2 w_3 v_e$ is an induced 4-cycle of G, which corresponds to a 4-cycle in H containing e. □

The proof of Theorem 2.2.1, continued. Let $\widetilde{H}$ be the core graph of H. Since G is 3-connected, $\widetilde{H}$ is 3-edge-connected. Let B be an arbitrary block of $\widetilde{H}$. Since $\widetilde{H}$ is 3-edge-connected, $\delta(B) \geq 3$. By the above Claim, every edge of B lies in a cycle of B of length at most 4. By Theorem 2.2.3 and since B is 2-connected, B has a spanning eulerian subgraph. Since every block of $\widetilde{H}$ has a spanning eulerian subgraph, $\widetilde{H}$ has a spanning eulerian subgraph. By Proposition 1.4.1(i), H has a dominating eulerian subgraph. By Theorem 1.2.1, $cl(G)$ is hamiltonian. □

Chapter 3

Hamiltonian connected almost claw free graphs

3.1 Background

In [25], Thomassen conjectured that every 4-connected line graph is hamiltonian, and in 1986, Zhan proved:

Theorem 3.1.1 *(Zhan, [27]) If G is a 4-edge-connected graph, then the line graph $L(G)$ is hamiltonian connected.*

In 2001, Kriesell presented a nice result.

Theorem 3.1.2 *(Kriesell, [14]) Every 4-connected line graph of a claw free graph is hamiltonian connected.*

Let C_4 denote a 4-cycle in K_5. The graph $K_5 - E(C_4)$ is called an **hourglass**. A graph G is **hourglass free** if G does not have an induced subgraph isomorphic to $K_5 - E(C_4)$.

Theorem 3.1.3 *(Broersma, Kriesell and Ryjáček, [2]) Every 4-connected hourglass free line graph is hamiltonian connected.*

It is well known that every hamiltonian connected graph with at least 4 vertices must be 3-connected. In this chapter, we investigate such graphs G that $L(G)$ is hamiltonian connected if and only if $L(G)$ is 3-connected. Our main result is the following

Theorem 3.1.4 *Let G be a connected graph with $|V(G)| \geq 4$ and $\tilde{G}$ be the core graph of G. If every 3-edge-cut of the core $\tilde{G}$ has at least one edge lying in a cycle of length at most 3 in $\tilde{G}$, then the following statements are equivalent.*
(i) $L(G)$ is hamiltonian connected;
(ii) $\kappa(L(G)) \geq 3$.

Theorem 3.1.4 clearly extends Theorems 3.1.1 and 3.1.2. The following corollaries of Theorem 3.1.4 also extend Theorem 3.1.3 and Theorem 3.1.2 respectively.

Corollary 3.1.5 *Let G be a graph with $|V(G)| \geq 4$. Suppose that $L(G)$ is hourglass free in which every 3-cut of $L(G)$ is not an independent set. Then $L(G)$ is hamiltonian-connected if and only if $\kappa(L(G)) \geq 3$.*

A set $B \subset V(G)$ is a **dominating** set if every vertex of G belongs to B or has a neighbor in B. The size of a minimum dominating set of G will be called **dominating number** of G and is denoted by $\gamma(G)$. If $\gamma(G) \leq k$, then G is k-**dominated**. A graph G is **almost claw free** if the vertices that are centers of claws in G are independent and if the neighborhoods of the center of each claw in G is 2-dominated. Note that every claw free graph is an almost claw free graph and there exist almost claw free graphs that are not claw-free.

Corollary 3.1.6 *Every 4-connected line graph of an almost claw free graph is hamiltonian-connected.*

The verification of Corollaries are in Section 3.4.

3.2 Catlin's Reduction Method

In this section we introduce Catlin's reduction method, and provide the mechanism needed in the proofs. The proofs of the main results are in Section 3. In the last section, we present some applications of our main results.

A graph G is **collapsible** if for any even subset X of $V(G)$, G has a spanning connected subgraph R_X of G such that $O(R_X) = X$. Catlin [5] showed that every graph G has a unique subgraph H each of whose components is a maximal collapsible subgraph of G. The contraction G/H is the **reduction** of G. A graph G is **reduced** if G has no nontrivial collapsible subgraphs; or equivalently, if G equals the reduction of G. We summarize some results on Catlin's reduction method and other related facts below.

Theorem 3.2.1 *Let G be a graph and let H be a collapsible subgraph of G. Let v_H denote the vertex onto which H is contracted in G/H. Each of the following holds.*
(i) (Catlin, Theorem 3 of [5]) G is collapsible if and only if G/H is collapsible. In particular, G is collapsible if and only if the reduction of G is K_1.
(ii) (Catlin, Theorem 8 of [5]) 2-cycles and 3-cycles are collapsible.
(iii) If G is collapsible, then for any pair of vertices $u, v \in V(G)$, G has a spanning (u, v)-trail.
(iv) For vertices $u, v \in V(G/H) - \{v_H\}$, if G/H has a spanning (u, v)-trail, then G has a spanning (u, v)-trail.
(v)(Catlin, Theorem 5 of [5])Any subgraph of a reduced graph is reduced.
(vi) If G is collapsible, and if $e \in E(G)$, then G/e is also collapsible.

Proof (iii) Let $X = \{u, v\}$. Then $|X| \equiv 0 \pmod 2$, and a spanning connected subgraph $R_X = X$ of G with $O(R_X) = \{u, v\}$ is a spanning (u, v)-trail.

(iv) Let Γ' be a spanning (u, v)-trail of G/H and let

$$X = \{w \in V(H) : w \text{ is incident with an odd number of edges in } \Gamma'\}.$$

Since v_H has even degree in Γ', $|X| \equiv 0 \pmod 2$. Let R'_X be a spanning connected subgraph of H with $O(R'_X) = X$. Then $\Gamma = G[E(\Gamma') \cup E(R'_X)]$ is a spanning (u, v)-trail in G.

(vi) follows by the definition of collapsible graphs. □

Let $\tau(G)$ denote the maximum number of edge-disjoint spanning trees of G. Catlin showed the relationship between $\tau(G)$ and the edge-connectivity $\kappa'(G)$. Part (ii) of the next theorem is an observation made in [4] and in [7].

Theorem 3.2.2 *Let G be a graph, H be a subgraph of G, and $k > 0$ be an integer.*
(i) (Catlin, [3]) $\kappa'(G) \geq 2k$ if and only if for any edge subset $X \subseteq E(G)$ with $|X| \leq k$, $\tau(G - X) \geq k$.
(ii) If $\tau(H) \geq k$ and if $\tau(G/H) \geq k$, then $\tau(G) \geq k$.

Theorem 3.2.3 *(Catlin and Lai, Theorem 4 of [8]) Let G be a graph with $\tau(G) \geq 2$ and let $e', e'' \in E(G)$. Then G has a spanning (e', e'')-trail if and only if $\{e', e''\}$ is not an essential edge cut of G.*

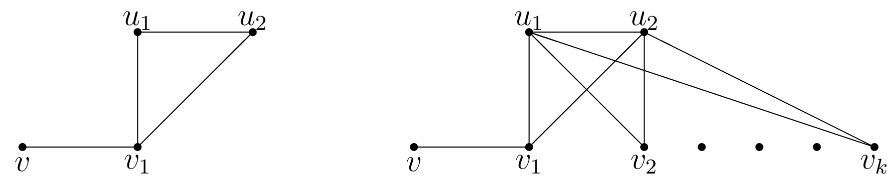
We define $F(G)$ be the minimum number of additional edges that must be added to G such that the resulting graph has two edge-disjoint spanning trees.

Theorem 3.2.4 *Let G be a graph.*
(i) (Catlin, Han and Lai, Lemma 2.3 of [6]) If for any $H \subset G$ with $|V(H)| < |V(G)|$, H is reduced, and if $|V(G)| \geq 3$, then $F(G) = 2|V(G)| - |E(G)| - 2$.
(ii) (Catlin, Theorem 7 of [5]) If $F(G) \leq 1$, then G is collapsible if and only if $\kappa'(G) \geq 2$.
(iii) (Catlin, Han and Lai, Theorem 1.3 of [6]) Let G be a connected graph and t an integer. If $F(G) \leq 2$, then G is collapsible if and only if G cannot be contracted to a member in $\{K_2\} \cup \{K_{2,t} : t \geq 1\}$.

3.3 Proof of Theorem 3.1.4

We start with a few more lemmas.

Lemma 3.3.1 *Let G be a graph, $v, v_1, u_1, u_2 \in V(G)$ be such that $d_G(v_1) = 3$ and $E_G(v_1) = \{v_1v, v_1u_1, v_1u_2\}$, and let $X' = \{u_1u_2, u_1v_i, u_2v_i : 1 \leq i \leq k\}$ be an edge subset of G (depicted in Figure 3.1) and $W = G[X']$. If $\tau((G - vv_1)/W) \geq 2$, then $\tau(G) \geq 2$.*



(a): $k = 1$ (b): $k \geq 2$

Figure 3.1 W with edge vv_1

Proof Let $H = (G - vv_1)/W$. For $k = 1$ (see Figure 3.1(a)), let T_1', T_2' be two edge-disjoint spanning trees of H. Then $T_1 = G[E(T_1') \cup \{vv_1, u_1u_2\}]$ and $T_2 = G[E(T_2') \cup \{v_1u_1, v_1u_2\}]$ are two edge-disjoint spanning trees of G.

For $k \geq 2$ (see Figure 3.1(b)), let

$$L = W/\{u_1u_2, u_1v_1, u_2v_1\}.$$

Then every edge of L is in a parallel class of two edges, and so $\tau(L) \geq 2$. Since $H = (G - vv_1)/W = ((G - vv_1)/\{u_1u_2, u_1v_1, u_2v_1\})/L$, and since $\tau(H) \geq 2$, it follows by Theorem 3.2.2(ii) that $\tau((G - vv_1)/\{u_1u_2, u_1v_1, u_2v_1\}) \geq 2$, and so $\tau(G) \geq 2$ by what we have just proved the case when $k = 1$. □

Lemma 3.3.2 *Let G be a 3-edge-connected graph, G_1 and G_2 be connected subgraphs of G such that $G = G_1 \cup G_2$, $|E(G_2)| \geq 1$ and $|V(G_1) \cap V(G_2)| \leq 2$. If for every 3-edge-cut X of G with $X \subseteq E(G_2)$, X has at least one edge lying in a cycle of length at most 3 in G_2, then either G has a peripheral 3-edge-cut or each of the following holds.*

(i) $G_2(e)$ is not reduced for any $e \in E(G_2)$.

(ii) G_2 is not reduced.

Proof If for an edge $e \in E(G_2)$, $G_2(e)$ is not reduced, then G_2 has a nontrivial subgraph H such that either $e \notin H$ and H is collapsible or $H(e)$ is collapsible. By Theorem 3.2.1(vi),

G_2 has a nontrivial collapsible subgraph, and so G_2 is not reduced. Therefore it suffices to prove (i).

Suppose that all 3-edge-cuts of G are non-peripheral and $G_2(e)$ is reduced for some $e \in E(G_2)$.

Assume first that $V(G_1) \cap V(G_2) = \{v\}$, then $d_{G_2(e)}(u) \geq 4$ for $u \in V(G_2(e)) - \{v(e), v\}$ by the assumption. As $G_2(e)$ is reduced, it follows by Theorem 3.2.4(i) that $F(G_2(e)) = 2|V(G_2(e))| - |E(G_2(e))| - 2 \leq 2|V(G_2(e))| - \frac{4(|V(G_2(e))|-2)+3+2}{2} - 2 = -\frac{1}{2} \leq 0$. So $G_2(e)$ is not reduced by Theorem 3.2.4(ii), contrary to the assumption that $G_2(e)$ is reduced.

Now assume that $V(G_1) \cap V(G_2) = \{v_1, v_2\}$, then $d_{G_2(e)}(v) \geq 4$ for $u \in V(G_2(e)) - \{v(e), v_1, v_2\}$ and $d_{G_2(e)}(v_1) + d_{G_2(e)}(v_2) \geq 3$. As $G_2(e)$ is reduced, it follows by Theorem 3.2.4(i) that $F(G_2(e)) = 2|V(G_2(e))| - |E(G_2(e))| - 2 \leq 2|V(G_2(e))| - \frac{4(|V(G_2(e))|-3)+3+2}{2} - 2 = \frac{3}{2}$. So $G_2(e)$ is not reduced by Theorem 3.2.4(ii), contrary to the assumption that $G_2(e)$ is reduced. □

Lemma 3.3.3 *If G is a graph with $\tau(G) \geq 2$ and $\kappa'(G) \geq 3$, then $G(e', e'')$ is collapsible for any $e', e'' \in E(G)$.*

Proof Since $\tau(G) \geq 2$, $F(G(e', e'')) \leq 2$. By Theorem 3.2.4(iii), $G(e', e'')$ is either collapsible, or the reduction of $G(e', e'')$ is a K_2 or a $K_{2,t}$ for some integer $t \geq 1$. Since $\kappa'(G) \geq 3$, $\kappa'(G(e', e'')) \geq 2$ and $G(e', e'')$ has at most two 2-edge-cuts. Thus $G(e', e'')$ can not be contracted to K_2 or $K_{2,t}$ for some integer $t \geq 1$, and so $G(e', e'')$ must be collapsible. □

Theorem 3.3.4 *Let G be a graph with $\kappa'(G) \geq 3$. If every 3-edge-cut of G has at least one edge in a 2-cycle or 3-cycle of G, then the graph $G(e', e'')$ is collapsible for any $e', e'' \in E(G)$.*

Proof By contradiction, we assume that

$$G \text{ is a counterexample to Theorem 3.3.4 with } |V(G)| \text{ minimized.} \tag{3.1}$$

Thus G satisfies the hypotheses of Theorem 3.3.4 but for some $e', e'' \in E(G)$, $G(e', e'')$ is not collapsible.

Let G_1 be the reduction of $G(e', e'')$. The following observations (I), (II) and (III) follow from the assumption that $\kappa'(G) \geq 3$, from (2) and Theorem 3.2.1(i), and from the definition of $G(e', e'')$.

(I) The only edge cuts of size 2 in $G(e', e'')$ are $E_{G(e',e'')}(v(e'))$ and $E_{G(e',e'')}(v(e''))$.
(II) $G_1 \neq K_1$ and so G_1 is not collapsible.
(III) For every 3-edge-cut X_1 of G_1, there is a 3-edge-cut X of G such that

$$X = \begin{cases} (X_1 - f') \cup e' & : \text{ if } X_1 \text{ contains } f' \in E_{G_1}(v(e')) \text{ and } E_{G_1}(v(e'')) \cap X_1 = \emptyset \\ (X_1 - f'') \cup e'' & : \text{ if } X_1 \text{ contains } f'' \in E_{G_1}(v(e'')) \text{ and } E_{G_1}(v(e')) \cap X_1 = \emptyset \\ (X_1 - \{f', f''\}) \cup \{e', e''\} & : \text{ if } X_1 \text{ contains } f' \in E_{G_1}(v(e')) \text{ and } f'' \in E_{G_1}(v(e'')) \\ X_1 & : \text{ otherwise} \end{cases}$$

In any case, we shall say that X is an edge-cut in G corresponding to the edge-cut X_1 in G_1, or vice versa.

Let X be a 3-edge-cut of G such that at least one edge of X lies in a cycle C_X of G with $|E(C_X)| \leq 3$. This C_X is called a **short cycle related to the edge-cut** X. If $e' \in E(C_X)$, then call X an e'**-cut**. Similarly, we define an e''**-cut**.

Since G_1 is the reduction of $G(e', e'')$, we have either $G_1 = G(e', e'')$ or $G_1 \neq G(e', e'')$. Next we show that neither of these two cases is possible.

Case 1 $G_1 \neq G(e', e'')$.

Then by the definition of reduction, $G_1 = G(e', e'')/H$ for a nontrivial subgraph H of $G(e', e'')$ each of whose components is a maximal collapsible subgraph of $G(e', e'')$.

If $v(e'), v(e'') \notin V(H)$, then $v(e'), v(e'') \in V(G_1)$ and $E_{G_1}(v(e')) \cup E_{G_1}(v(e'')) \subseteq E(G_1)$. Let $G_1' = (G_1 - \{v(e'), v(e'')\}) \cup \{e', e''\}$. Then $G_1' = G/H$ satisfies the conditions of Theorem 3.3.4 with $|V(G_1')| < |V(G)|$. By (3), $G_1 = G_1'(e', e'')$ must be collapsible, contrary to (II).

If $v(e'), v(e'') \in V(H)$, then $E_{G_1}(v(e')) \cup E_{G_1}(v(e'')) \subseteq E(H)$, as collapsible graphs are 2-edge-connected. Thus $e', e'' \notin E(G_1) = E(G(e', e'')) - E(H)$ and so by (I), $\kappa'(G_1) \geq 3$. If G_1 has a 3-edge-cut X, then as $X \cap E(H) = \emptyset$ and by (III), X must be a 3-edge-cut of G. It follows by the assumption of Theorem 3.3.4 that X has a related short cycle C_X in G with $|E(C_X)| \leq 3$ and with $|E(C_X) \cap X| = 2$. Since C_X is a collapsible subgraph by Theorem 3.2.1(ii), $C_X \subseteq H$, and so $X \cap E(H) \neq \emptyset$, a contradiction. Thus $\kappa'(G_1) \geq 4$, and so by Theorem 3.2.2(i) and 2.7(iii), G_1 is collapsible, contrary to (II).

Therefore we assume without loss of generality that $v(e') \notin V(H)$ and $v(e'') \in V(H)$. Let $H_1 = (H - v(e'')) \cup e''$. Thus each component of H_1 is collapsible by the definition of collapsible graphs. Since e' is not in H_1,

$$G_1 = G(e', e'')/H = (G/H_1)(e') \text{ and } \kappa'(G/H_1) \geq 3 \ . \tag{3.2}$$

Claim 1 *Each of the following holds for the graph G/H_1.*
(i). The graph G/H_1 must have 3-edge-cuts.
(ii). Every 3-edge-cut of G/H_1 is an e'-cut of G/H_1.
(iii). One of 3-edge-cuts of G/H_1 is peripheral.

Proof of Claim 1 (i). If G/H_1 has no 3-edge-cuts, then by (3), $\kappa'(G/H_1) \geq 4$. By Theorem 3.2.2(i), $F((G/H_1)(e')) \leq 1$, and so by Theorem 3.2.4(ii), $G_1 = (G/H_1)(e')$ is collapsible, contrary to (II).

(ii). Let X be a 3-edge-cut of G/H_1. Since $G_1 = (G/H_1)(e')$, G_1 has a 3-edge-cut X_1 corresponding to X. If X is not an e'-cut, then $C_{X_1} = C_X$ is a collapsible subgraph of G_1 by Theorem 3.2.1(ii), contrary to the assumption that G_1 is reduced.

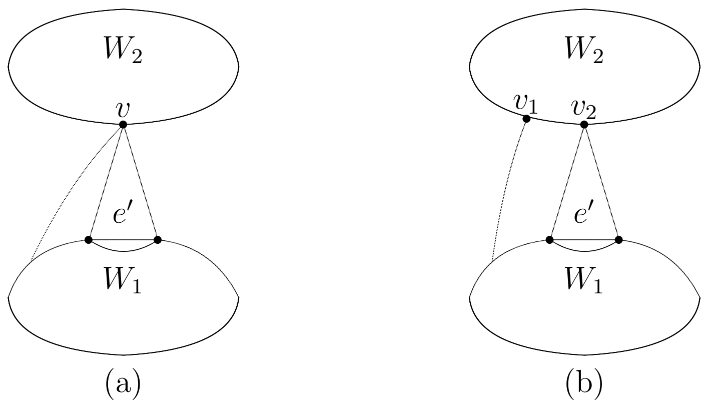


Figure 3.2 G/H_1

(iii). Now we assume that all 3-edge-cuts are non-peripheral and let X be a non-peripheral e'-cut of G/H_1. Let W_1, W_2 denote the two nontrivial components of $(G/H_1) - X$(see Figure 3.2). Since X is also a 3-edge-cut of G, some edge of X lies in a short cycle C_X of G with $|E(C_X)| \leq 3$, and so two edges in X must be adjacent. Thus, without loss of generality, we may assume that $|V((G/H_1)[X]) \cap V(W_2)| \leq 2$ and that $e' \notin E(W_2)$. As G_1 is reduced and $G_1 = (G/H_1)(e')$, W_2, being a subgraph of a reduced graph, is also reduced by Theorem 3.2.1(v). Apply Lemma 3.3.2 with G and G_2 in Lemma 3.3.2 replaced by G/H_1 and W_2 respectively to conclude that W_2 is not reduced, contrary to the assumption that W_2 is reduced.

This completes the proof for Claim 1. $\square$

By Claim 1(i), (ii) and (iii), G/H_1 must have a peripheral 3-edge-cut which is also an e'-cut, *i.e.*, whose related short cycle contains e'. Then G/H_1 has a subgraph S isomorphic to the graph in Figure 3.1(a), where $E_{G/H_1}(v_1)$ is a peripheral 3-edge-cut in G/H_1 and $e' = u_1u_2$.

Let M be a maximal edge subset of G/H_1 such that $E(S) - vv_1 \subseteq M$ and such that $V((G/H_1)[M]) - \{u_1, u_2\}$ equals the set of all the vertices adjacent to both u_1 and u_2 in G/H_1. By Claim 1(ii), the related short cycle of each 3-edge-cut contains e'. And so the subgraph $W = G[M \cup vv_1]$ is isomorphic to the graph in Figure 3.1(b). By Claim 1(ii), the related short cycle of any 3-edge-cut of G/H_1 must be contained in $(G/H_1)[M]$. If

$(G/H_1)/M = K_1$, then G/H_1 is spanned by M. Since M is obtained from a $K_{2,t}(t \geq 2)$ by adding one edge joining two vertices of degree t and e' joins a vertex of degree 2 to another vertex in this $K_{2,t}$, $\tau(G/H_1) \geq 2$ and so $G_1 = (G/H_1)(e')$ is collapsible by Lemma 3.3.3 and Theorem 3.2.1(vi), contrary to (II).

Therefore we may assume that $(G/H_1)/M$ is a nontrivial 4-edge-connected graph. By Theorem 3.2.2(i), $\tau((G/H_1 - vv_1)/M) \geq 2$. By Lemma 3.3.1, $\tau(G/H_1) \geq 2$, and so by Theorem 3.2.4(i), $F[(G/H_1)(e')] \leq 1$. Thus by Theorem 3.2.4(ii), $G_1 = (G/H_1)(e')$ is collapsible, contrary to (II). This contradiction precludes Case 1.

Case 2 $G_1 = G(e', e'')$.

Claim 2 *Each of the following must hold.*
(i). The graph G has at least three 3-edge-cuts.
(ii). Every 3-edge-cut of G is either an e'-cut or an e''-cut of G.
(iii). One of the 3-edge-cuts of G is peripheral.

Proof of Claim 2 (i). As $\kappa'(G) \geq 3$, if G has at most two 3-edge-cuts, then we can add two new edges f_1, f_2 to G such that $\kappa'(G + \{f_1, f_2\}) \geq 4$. It follows by Theorem 3.2.2(i) that $\tau(G) \geq 2$. Thus by Lemma 3.3.3, $G(e', e'')$ is collapsible, contrary to (II).

(ii). Let X be a 3-edge-cut of G and suppose that the short cycle C_X related to X does not contain e' or e''. Since $G_1 = G(e', e'')$, G_1 has a 3-edge-cut X_1 corresponding to X. Then by Theorem 3.2.1(ii), C_X is a collapsible subgraph of G_1, contrary to the assumption that G_1 is reduced.

(iii). Assume that all 3-edge-cuts are non-peripheral. By (i) and (ii), we can assume that G has e'-cuts and let X be an e'-cut of G.

Let W_1, W_2 denote the two nontrivial components of $G - X$. Since X is a 3-edge-cut of G, some edge of X lies in a short cycle C_X of G with $|E(C_X)| \leq 3$, and so two edges in X must be adjacent. Thus, without loss of generality, we may assume that $|V(G[X]) \cap V(W_2)| \leq 2$ and $e' \notin E(W_2)$.

Case 1 of Claim 2(iii) $e'' \notin E(W_2)$. As $G_1 = G(e', e'')$ is reduced, W_2 which does not contain e'' and so is a subgraph of the reduced graph G_1, is also reduced by Theorem 3.2.1(v). Apply Lemma 3.3.2 with G_2 in Lemma 3.3.2 replaced by W_2 to conclude that W_2 is not reduced, contrary to the assumption that W_2 is reduced.

Case 2 of Claim 2(iii) $e'' \in E(W_2)$. As $G_1 = G(e', e'')$ is reduced, $W_2(e'')$ which is a subgraph of the reduced graph G_1, is also reduced by Theorem 3.2.1(v). Apply Lemma 3.3.2 with G_2, e in Lemma 3.3.2 replaced by W_2, e'' respectively to conclude that $W_2(e'')$ is not reduced, contrary to the assumption that $W_2(e'')$ is reduced.

This completes the proof for Claim 2. □

By Claim 2(i), (ii) and (iii), we may assume that G has a peripheral e'-cut. Then G has a subgraph S_1 isomorphic to the graph in Figure 3.1(a), where $E_G(v_1)$ is a peripheral 3-edge-cut in G and $e' = u_1u_2$.

Let M_1 be a maximal edge subset of G such that $E(S_1) - vv_1 \subseteq M_1$ and such that $V(G[M_1]) - \{u_1, u_2\}$ equals the set of all the vertices adjacent to both u_1 and u_2 in G. And so the subgraph $W' = G[M_1 \cup vv_1]$ is isomorphic to the graph in Figure 3.1(b). With $z \mapsto z'$ being a graph isomorphism from W in Figure 3.1(b) to W', we may assume that

$$E(W') = M_1 \cup \{v'v'_1\} = \{u'_1u'_2, u'_1v'_i, u'_2v'_i : 1 \leq i \leq k\} \cup \{v'v'_1\} \text{ and } e' = u'_1u'_2.$$

Let $v'v'_1 = e_1$. By Claim 2(ii), the related short cycle of any e'-cut of G must be contained in $G[M_1]$. Define $G_{11} = G/M_1$. If $G_{11} = K_1$, then G is spanned by M_1. Since M_1 is obtained from a $K_{2,t} (t \geq 2)$ by adding one edge joining two vertices of degree t and since e_1 joins a vertex of degree 2 to another vertex in this $K_{2,t}$, $\tau(G) \geq 2$ and so $G_1 = G(e', e'')$ is collapsible by Lemma 3.3.3, contrary to (II). Thus we may assume that G_{11} is a nontrivial graph with $\kappa'(G_{11}) \geq 3$.

Claim 3 (i) The graph G_{11} must have 3-edge-cuts.

(ii) Every 3-edge-cut of G_{11} must be an e''-cut of G.

(iii) G_{11} has a peripheral e''-cut.

Proof of Claim 3 (i) If $\kappa'(G_{11}) \geq 4$, then by Theorem 3.2.2(i), $\tau(G_{11} - e_1) = \tau(G/M_1' - e_1) \geq 2$ and so by Lemma 3.3.1, $\tau(G) \geq 2$. It follows by Lemma 3.3.3 that $G(e', e'')$ is collapsible, contrary to (II).

(ii) As any edge-cut of G_{11} is also an edge-cut of G and $e' \notin E(G_{11})$, by Claim 2(ii), every 3-edge-cut of G_{11} must be an e''-cut of G.

(iii) By a similar argument as in the proof of Claim 1(iii), G_{11} has a peripheral e''-cut.

This completes the proof of Claim 3. □

By Claim 3(i), (ii) and (iii), we may assume that G_{11} has a peripheral e''-cut. Then G_{11} has a subgraph S_2 isomorphic to the graph in Figure 3.1(a), where $E_{G_{11}}(v_1)$ is a peripheral 3-edge-cut in G_{11} and $e'' = u_1u_2$.

Let M_2 be a maximal edge subset of G such that $E(S_2) - vv_1 \subseteq M_2$ and such that $V(G[M_2]) - \{u_1, u_2\}$ equals the set of all the vertices adjacent to both u_1 and u_2 in G_{11}. By Claim 3(ii), the related short cycle of each 3-edge-cut must contain e''. And so the subgraph $W'' = G[M_2 \cup vv_1]$ is isomorphic to the graph in Figure 3.1(b). With $z \mapsto z''$ being a graph isomorphism from W in Figure 3.1(b) to W'', we may assume that

$$E(W'') = M_2 \cup \{v''v_1''\} = \{u_1''u_2'', u_1''v_i'', u_2''v_i'' : 1 \leq i \leq k\} \cup \{v''v_1''\} \text{ and } e'' = u_1''u_2''.$$

Let $v''v_1'' = e_2$ and $L = G_{11}/M_2 = G/(M_1 \cup M_2)$. Then by Claim 3(ii) and as $W'' = G_{11}[M_2]$ is maximal, we must have $\kappa'(L) \geq 4$ (similar argument as $\kappa'(G11)$). Since

$$L - \{e_1, e_2\} = G_{11}/M_2 - \{e_1, e_2\} = (G_{11} - e_1) - e_2)/M_2,$$

it follows by Theorem 3.2.2(i) that $\tau(L - \{e_1, e_2\}) \geq 2$.

By applying Lemma 3.3.1 to e_2 and M_2, $\tau(G_{11} - e_1) \geq 2$. Since $G_{11} - e_1 = (G - e_1)/M_1$, by applying Lemma 3.3.1 again to e_1 and M_1, $\tau(G) \geq 2$. Thus by Lemma 3.3.3, $G(e', e'')$ must be collapsible, contrary to (II). This contradiction precludes Case 2. □

Proof of Theorem 3.1.4 Since Theorem 3.1.4(i) trivially implies Theorem 3.1.4(ii), it suffices to show that Theorem 3.1.4(ii) implies Theorem 3.1.4(i). Assume that $L(G)$ is not complete. By Lemma 1.4.1(ii), $\kappa'(\tilde{G}) \geq 3$. By Theorem 3.3.4 and Theorem 3.2.1(iii),

$\tilde{G}(e', e'')$ has a spanning $(v(e'), v(e''))$-trail for any $e', e'' \in E(\tilde{G})$. Then by Proposition 1.4.2(ii), $G(e', e'')$ has a dominating $(v(e'), v(e''))$-trail for any $e', e'' \in E(G)$. By Proposition 1.2.2, Theorem 3.1.4 is proved.

3.4 Applications

Let $\mathcal{F}$ denote the set of connected graphs such that a graph $G \in \mathcal{F}$ if and only if each of the following holds:

(F1) If X is an edge cut of G with $|X| \leq 3$, then there exists a vertex $v \in V(G)$ of degree $|X|$ such that $X = E_G(v)$, and

(F2) for every $v \in V(G)$ of degree 3, v lies in a k-cycle C_v of G, where $2 \leq k \leq 3$.

The next corollary follows from Theorem 3.3.4 and Theorem 3.2.1(iii).

Corollary 3.4.1 *([18]) Let $G \in \mathcal{F}$. If $\kappa'(G) \geq 3$, then for every pair of edges $e', e'' \in E(G)$, then*

(i). $G(e', e'')$ is collapsible.

(ii). $G(e', e'')$ has a spanning $(v(e'), v(e''))$-trail.

For convenience, we restate Corollaries 3.1.5 and 3.1.6 below.

Corollary 3.4.2 *Let G be a graph with $|V(G)| \geq 4$. Suppose that $L(G)$ is hourglass free in which every 3-cut of $L(G)$ is not an independent set. Then $L(G)$ is hamiltonian-connected if and only if $\kappa(L(G)) \geq 3$.*

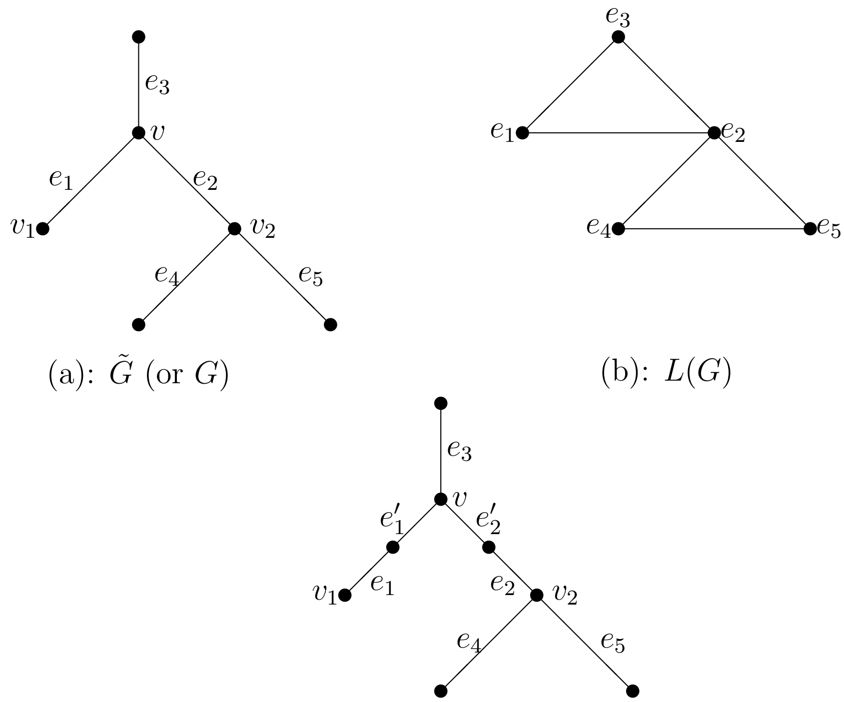
Proof If suffices to show that if $\kappa(L(G)) \geq 3$, then $L(G)$ is hamiltonian-connected. We may assume that $L(G)$ is not a complete graph.

Let $\tilde{G}$ denote the core of G. As $L(G)$ is not a complete graph and $\kappa(L(G)) \geq 3$, by Lemma 1.4.1(ii), $\tilde{G}$ is nontrivial and $\kappa'(\tilde{G}) \geq 3$. By Theorem 3.1.4, it suffices to show that every 3-edge-cut of $\tilde{G}$ has an edge lying in a cycle of length at most 3. Let $X = \{e_0, e_1, e_2\}$ be a 3-edge-cut of $\tilde{G}$. By the definition of $\tilde{G}$, we may assume that $X \subseteq E(G)$ and so X is an edge cut of G. Assume first that X is non-peripheral. Since every 3-cut of $L(G)$ is not an independent set, two of the corresponding vertices e_0, e_1, e_2 in $L(G)$ are adjacent. We may assume that e_1, e_2 are adjacent in $L(G)$ and so are in G. By the definition of $\tilde{G}$, e_1, e_2 are adjacent in $\tilde{G}$(see Figure 3.3). Since $\kappa'(\tilde{G}) \geq 3$, there is some edge e_3 incident with v and there are some edges e_4, e_5 incident with v_2 in $\tilde{G}$(see Figure 3.3(a)). By the definition of $\tilde{G}$, we may assume that $e_3, e_4, e_5 \in E(G)$ and e_3 is incident with v and e_4, e_5 are incident with v_2 in G.

Case 1 At least one of $\{e_1, e_2\}$ is not subdivided in G. Without loss of generality we assume that e_2 is not subdivided in G (see Figure 3.3(a)). Since $L(G)$ is hourglass free and without loss of generality, we may assume that e_4 is adjacent to e_1 in $L(G)$. Thus e_4 is either incident with v or v_1 in G. In any case, e_2 is in a cycle of length at most 3 in G, so is in $\tilde{G}$.

Case 2 Both e_1, e_2 are subdivided in G (see Figure 3.3(c)), then $\{e_0, e_1, e_2\}$ is a 3-edge-cut of G and so the corresponding vertex set in $L(G)$ is a 3-cut of $L(G)$. If $v_1 = v_2$, then e_1 lies in a 2-cycle in $\tilde{G}$; otherwise we may assume without loss of generality that e_0 is incident with v_1 and $X'' = \{e_0, e_1', e_2\}$ is a 3-edge-cut of G and so the corresponding vertex set in $L(G)$ is a 3-cut of $L(G)$. Since X'' is a 3-edge-cut of G, we must have that $v_2 = v_1$. In either case one edge of X lies in a cycle of length at most 3 in $\tilde{G}$.

Next we assume that $X' = \{e_1, e_2, e_3\}$ is a peripheral edge cut in $\tilde{G}$. Then there exists $v \in V(\tilde{G})$ such that $E_{\tilde{G}} = \{e_1, e_2, e_3\}$ and $e_i = vv_i, i = 1, 2, 3$. Since $\delta(\tilde{G}) \geq 3$ (Lemma 1.4.1(ii)), we may assume that $v_i (i = 1, 2, 3)$ is incident with e_{i1} and e_{i2} in $E(\tilde{G}) - \{e_1, e_2, e_3\}$. If at least one of $\{e_1, e_2, e_3\}$ is not subdivided in G, with the same argument as in Case 1, we can see that an edge in X' must be lying in a cycle of length at most 3 in $\tilde{G}$. If each of $\{e_1, e_2, e_3\}$ is subdivided in G, it's easy to see that X' has one edge lying in a cycle of length 2 in $\tilde{G}$. $\square$



(a): $\tilde{G}$ (or G)

(b): $L(G)$

(c): G

Figure 3.3

Corollary 3.4.3 *Every 4-connected line graph of an almost claw free graph is hamiltonian-connected.*

Proof Let G be an almost claw free graph such that $L(G)$ is 4-connected. By Theorem 3.1.4, it suffices to show that every 3-edge-cut of $\tilde{G}$ must have an edge lying in a cycle of length at most 3. Since $L(G)$ is 4-connected, G has no essential 3-edge-cuts. By the definition of $\tilde{G}$, $\tilde{G}$ has no essential 3-edge-cuts either. Let X be a peripheral 3-edge-cut of $\tilde{G}$. If there are no edges of X in a 2-cycle or 3-cycle of $\tilde{G}$, then $\tilde{G}[X]$ must be a claw of $\tilde{G}$. Let $v \in V(\tilde{G})$ be the center of the claw X. By the definition of $\tilde{G}$, $\tilde{G}[X]$ gives rise to a claw with center v in G. Since v is of degree 3 in G, the neighborhood of v in G can not be 2-dominated. So there must be at least one edge of X lying in a 2-cycle or a 3-cycle of $\tilde{G}$. By Theorem 3.1.4, $L(G)$ is hamiltonian connected. $\square$

Chapter 4

Triangularly connected claw-free graph

4.1 Introduction

Graphs considered in this chapter are finite and simple graphs. For subgraphs G_1 and G_2 of a graph G, $G_1 \triangle G_2$ is the subgraph of G induced by the edge set $E(G_1) \cup E(G_2) - (E(G_1) \cap E(G_2))$.

Theorem 4.1.1 *(D. J. Oberly and D. P. Sumner [21]) Every connected, locally connected claw-free graph is hamiltonian.*

Theorem 4.1.2 *(L. Clark [10], R. H. Shi [24], and C.-Q. Zhang [28]) Every connected, locally connected claw-free graph is vertex pancyclic.*

A graph G is **triangularly connected** if for every pair of edges $e_1, e_2 \in E(G)$, G has a sequence of 3-cycles $C_1, C_2, \cdots, C_l$ such that $e_1 \in C_1, e_2 \in C_l$ and such that $E(C_i) \cap E(C_{i+1}) \neq \emptyset$, $(1 \leq i \leq l-1)$. The following proposition follows from definitions immediately.

Proposition 4.1.3 *Every connected, locally connected graph is triangularly connected.*

One can easily construct graphs that are triangularly connected but not locally connected. For example, let $k > 3$ be an integer, and let $H_1, H_2, \cdots, H_k$ be complete graphs of order at least 4. Obtain a graph G by the following process: for each $i = 1, 2, ..., k-1$, identify an edge e'_i in H_i with an edge e''_i in H_{i+1} to form an edge e_i such that $H_i \cap H_{i+1} = G[\{e_i\}] \cong K_2$ and such that all these e_i's form a matching of G, and identify a vertex v_1 in H_1 with a vertex v_k in H_k to form a vertex v in G such that $V(H_1) \cap V(H_k) = \{v\}$ and such that v is not incident with any of the edges $e_1, e_2, \cdots, e_{k-1}$. Then G is triangularly connected, but $N(v)$ does not induce a connected subgraph of G.

We present another view point for triangularly connectedness. Let $\mathcal{C}_3(G)$ denote the graph whose vertex set is the set of all 3-cycles of G. For two 3-cycles C_1 and C_2 of G, If C_1 and C_2 have common edge in G, then C_1 is adjacent to C_2 in $\mathcal{C}_3(G)$. Again by definitions, we have

Proposition 4.1.4 *A graph G is triangularly connected if and only if both of the following hold:*
(i) $\forall e \in E(G)$, $\exists C_e \in V(\mathcal{C}_3(G))$ such that $e \in E(C_e)$, and
(ii) the graph $\mathcal{C}_3(G)$ is connected.

In view of Proposition 4.1.3, our main result of this paper, as stated below in Theorem 1.5, extends Theorems 4.1.1 and 4.1.2.

Theorem 4.1.5 *Every triangularly connected claw-free graph G with $|E(G)| \geq 3$ is vertex pancyclic.*

4.2 Proof of the Theorem 4.1.5

By the definition of triangularly connectedness, any triangularly connected graph must also be connected. Let $v \in V(G)$ be a vertex. Since G is connected and since $|E(G)| \geq 3$, G must have two distinct edges, one of which is incident with v. By the assumption that G is triangularly connected, G has a 3-cycle. Let $n = |V(G)|$.

Suppose that for some integer $r \geq 3$, and for each l with $3 \leq l \leq r$, G has an l-cycle containing v. If $r = n$, then done. Therefore we assume that $r < n$. Let C be an r-cycle containing v. It suffices to show that G has an $(r+1)$-cycle C' containing v.

Arguing by contradiction, we assume that

$$G \text{ does not have an } (r+1)\text{-cycle } C' \text{ containing } v. \tag{4.1}$$

Since $r < n$ and since G is connected,

$$\partial C = \{e \in E(G) : \ e \text{ is incident with exactly one vertex in } V(C)\} \neq \emptyset. \tag{4.2}$$

Denote $C = v_1 v_2 \cdots v_r v_1$, where the subscripts are integers modulo r. We shall prove Theorem 4.1.5 by verifying each of the following claims.

(2.1) For each $e = v_i u \in \partial C$, where $u \in V(G) - V(C)$. Then $v_{i-1}v_{i+1} \in E(G)$ and $uv_{i-1}, uv_{i+1} \notin E(G)$.

Proof: If $uv_{i-1} \in E(G)$ or $uv_{i+1} \in E(G)$, then $C \triangle G[\{u, v_{i-1}, v_i\}]$ or $C \triangle G[\{u, v_{i+1}, v_i\}]$, respectively, is an $(r+1)$-cycle containing v, contrary to statement (4.1). Therefore, both $uv_{i-1} \notin E(G)$ and $uv_{i+1} \notin E(G)$. But $G[\{u, v_{i-1}, v_i, v_{i+1}\}] \ncong K_{1,3}$, and so we must have $v_{i-1}v_{i+1} \in E(G)$. □

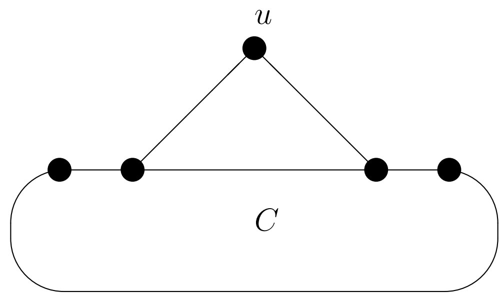


Figure 4.1: Proof of (2.1)

By Proposition 4.1.4(i), there exist 3-cycles $C_1, C_2, \cdots, C_m$ of G such that $E(C) \subseteq E(C_1) \cup \cdots \cup E(C_m)$ and such that m is minimized with this property. By the minimality of m,

$$E(C_i) \cap E(C) \neq \emptyset,\ 1 \leq i \leq m. \tag{4.3}$$

For each $e \in \partial C$, by Proposition 4.1.4(i), G has a 3-cycle C_0 such that $e \in E(C_0)$, and $\mathcal{C}_3(G)$ has a shortest path P in $\mathcal{C}_3(G)$ from C_0 to the vertex set $\{C_1, C_2, \cdots, C_m\}$. By Claim (2.1), the length of P is at least one. Without the loss of generality, we may assume $P = Z_0 Z_1 \cdots Z_k$ where $k \geq 1$, and where $Z_0 = C_0$ and $Z_k \in \{C_1, C_2, \cdots, C_m\}$. Note that P (and so the length of P) may depend on e. We choose an edge $e \in \partial C$ such that

$$k \text{ is as small as possible.} \tag{4.4}$$

Without loss of generality, we assume that $Z_k = C_1$. Since P is a shortest path, for each i with $0 \leq i \leq k-1$,

$$E(Z_i) \cap E(C) = \emptyset. \tag{4.5}$$

(2.2) $C_1 \neq C$.

Proof: If $C_1 = C$, then $r = 3$ and so $C \triangle Z_{k-1}$ is an $(r+1)$-cycle containing v, contrary to statement (4.1). □

By (4.5), $E(Z_0) \cap E(C) = \emptyset$. Since $e \in \partial C$ and by (4.2), $|V(Z_0) \cap V(C)| \leq 2$. Thus there must be a largest integer i_0 such that $|V(Z_{i_0}) \cap V(C)| = 2$.

(2.3) $i_0 = 0$.

Proof: Let $Z_{i_0} = u'v_iv_ju'$, for some $1 \leq i < j \leq r$, where $u' \in V(G) - V(C)$. Suppose that $i_0 > 0$. Then $e' = u'v_i$ corresponds to a shorter path $P' = Z_{i_0} \cdots Z_k$, contrary to (4.4). □

By the choice of Z_{i_0} and by (2.2), we may assume that $Z_0 = uv_iv_ju$ and $Z_1 = v_hv_iv_jv_h$, where $1 \leq h < i < j \leq r$. By (2.1), we have

$$v_{i-1}v_{i+1}, v_{j-1}v_{j+1} \in E(G). \tag{4.6}$$

(2.4) $i - h \not\equiv 1 \pmod r$ and $h - j \not\equiv 1 \pmod r$, and so $k > 1$.

Proof: Suppose that $i = h + 1$ (or $h \equiv j + 1 \pmod r$). By (4.6),

$$C' = v_hv_juv_iv_{i+1}v_{i+2} \cdots v_{j-1}v_{j+1}v_{j+2} \cdots v_h$$

is an $(r+1)$-cycle containing v, contrary to statement (4.1), (See Figure 4.2). The case when $h \equiv j + 1 \pmod r$ can be proved similarly.

If $k = 1$, then $E(Z_1) \cap E(C) \neq \emptyset$, and so we may assume that either $i = h + 1$ or $h \equiv j + 1 \pmod r$. As shown above, this leads to a contradiction to statement (4.1). □

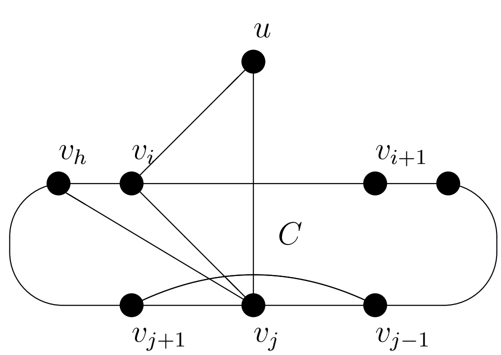


Figure 4.2: Proof of (2.4)

(2.5) $v_{h-1}v_{h+1} \in E(G)$.

Proof: By contradiction, we assume that $v_{h-1}v_{h+1} \notin E(G)$. Since $G[\{v_h, v_{h-1}, v_{h+1}, v_i\}] \not\cong K_{1,3}$, either $v_iv_{h-1} \in E(G)$, or $v_iv_{h+1} \in E(G)$.

Suppose first that $v_iv_{h-1} \in E(G)$. By (4.6),

$$C' = v_hv_{h+1}\cdots v_{i-1}v_{i+1}v_{i+2}\cdots v_{j-1}v_{j+1}\cdots v_{h-1}v_iuv_jv_h$$

is an $(r+1)$-cycle containing v, contrary to statement (4.1), (See Figure 4.3).

Therefore, we have $v_iv_{h+1} \in E(G)$. Then

$$C' = v_hv_juv_iv_{h+1}v_{h+2}\cdots v_{i-1}v_{i+1}\cdots v_{j-1}v_{j+1}v_{j+2}\cdots v_h$$

is an $(r+1)$-cycle containing v, contrary to statement (4.1), (See Figure 4.3). □

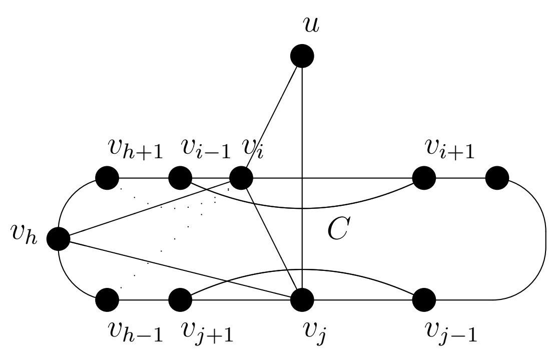


Figure 4.3: Proof of (2.5)

By Claim (2.4), $k \geq 2$. By (4.4), $E(Z_2) \cap E(Z_1) \neq \{v_iv_j\}$, for otherwise $Z_0Z_2\cdots Z_k$ is a shorter path, contrary to (4.4). Thus we may assume that $E(Z_2) \cap E(Z_1) = \{v_iv_h\}$.

Now we consider the induced subgraph $G[\{v_h, v_i, u, v_{i+1}\}]$. As this cannot be isomorphic to a $K_{1,3}$, either $uv_{i+1} \in E(G)$, contrary to Claim (2.1); or $uv_h \in E(G)$, whence $e' = uv_i$ and $Z_0' = uv_iv_hu$ would correspond to a path $Z_0'Z_2\cdots Z_k$, contrary to (4.4); or $v_hv_{i+1} \in E(G)$, whence by Claim (2.5) and (4.6),

$$C' = v_hv_{i+1}v_{i+2}\cdots v_{j-1}v_{j+1}v_{j+2}\cdots v_{h-1}v_{h+1}v_{h+2}\cdots v_{i-1}v_iuv_jv_h$$

is an $(r+1)$-cycle containing v, contrary to statement (4.1), (See Figure 4.4).

These contradictions establish Theorem 4.1.5.

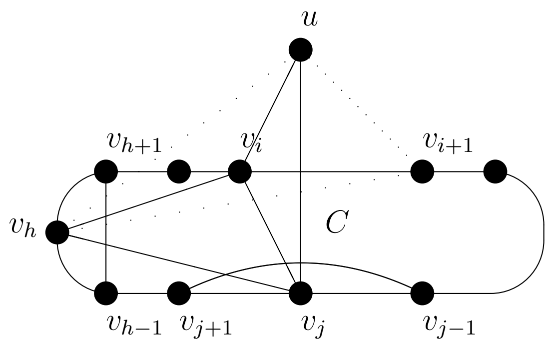


Figure 4.4: Proof of the last paragraph

For an integer $k \geq 2$, a graph G is k-cycle connected if for every pair of edges e_1, e_2 in $E(G)$, G has a sequence of l-cycles ($l \leq k$) $C_1, C_2, \cdots, C_r$ such that $e_1 \in E(C_1)$ and $e_2 \in E(C_r)$ and $E(C_i) \cap E(C_{i+1}) \neq \emptyset$ for $i = 1, 2 \cdots, r-1$. We complete this chapter with the following conjecture.

Conjecture 4.2.1 *Every 3-connected 4-cycle connected claw-free graph G with $|E(G)| \geq 3$ is vertex pancyclic.*

Chapter 5

$\mathbf{Z}_3$-connected Line Graphs

5.1 Introduction

Graphs considered in this chapter are finite graphs with possible loops and multiple edges. We use $\mathbf{Z}$ to denote the group of all integers, and for an integer $n > 1$, $\mathbf{Z}_n$ to denote the cyclic group of order n. For a graph G and a vertex $v \in V(G)$, define

$$E_G(v) = \{e \in E(G) : \ e \text{ is incident with } v \text{ in } G\}.$$

Let G be a digraph, A be a nontrivial additive Abelian group with additive identity 0, and $A^* = A - \{0\}$. For an edge $e \in E(G)$ oriented from a vertex u to a vertex v, u is referred as the **tail** of e, while v the **head** of e. For a vertex $v \in V(G)$, the set of all edges incident with v being the tail (or the head, respectively) is denoted by $E^+(v)$ (or $E^-(v)$, respectively). We define

$$F(G, A) = \{f \mid f : E(G) \mapsto A\} \quad \text{and} \quad F^*(G, A) = \{f \mid f : E(G) \mapsto A^*\}.$$

For each $f \in F(G, A)$, the **boundary** of f is a function $\partial f : V(G) \mapsto A$ defined by $\partial f = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e)$, for each vertex $v \in V(G)$, where "$\sum$" refers to the addition in A. We define

$$Z(G, A) = \{b \mid b : V(G) \mapsto A \ \text{ with } \sum_{v \in V(G)} b(v) = 0\}.$$

An undirected graph G is A**-connected**, if G has an orientation G' such that for every function $b \in Z(G, A)$, there is a function $f \in F^*(G', A)$ such that $\partial f = b$. For an Abelian group A, let $\langle A \rangle$ denote the family of graphs that are A-connected. It has been observed in [13] that that $G \in \langle A \rangle$ is independent of the orientation of G.

An A-**nowhere-zero-flow** (abbreviated as an A-NZF) of G is a function $f \in F^*(G, A)$ such that $\partial f = 0$. For an integer $k \geq 2$, a k-**nowhere-zero-flow** (abbreviated as a k-NZF) of G is a function $f \in F^*(G, \mathbf{Z})$ such that $\partial f = 0$ and such that for every $e \in E(G)$, $0 < |f(e)| < k$. Tutte ([26], also [12]) showed that a graph G has an A-NZF if and only if G has an $|A|$-NZF.

The concept of A-connectivity was introduced by Jaeger *et al* in [13], where A-NZF is successfully generalized to A-connectivity. For a graph G, define

$$\Lambda_g(G) = \min\{k : \text{if } A \text{ is an abelian group of order at least } k, \text{then } G \in \langle A \rangle\}.$$

From the definitions, if $\Lambda_g(G) \leq k$, then G has a k-NZF. The following conjectures have been proposed.

Conjecture 5.1.1 *(Tutte [26], and [12]) Every* 4*-edge connected graph has a* 3*-NZF.*

Conjecture 5.1.2 *(Jaeger et al [12]) If* G *is* 5*-edge connected graph, then* $\Lambda_g(G) \leq 3$*.*

Both conjectures are still open.

In [9], Chen *et al* adjusted the definition of line graphs to reflect the fact of multiple edges in the original graph and the following is proved.

Theorem 5.1.3 *(Chen et al.,[9]) If every 4-edge-connected line graph has a 3-NZF, then every 4-edge-connected graph has a 3-NZF.*

The main purpose of this chapter is to investigate when a line graph is $\mathbf{Z}_3$-connected or has a 3-NZF. By the definition of a line graph, for a vertex $v \in V(G)$, the edges incident with v in G induce a complete subgraph H_v in $L(G)$, and when $u, v \in V(G)$ with $u \neq v$, H_v and H_u are edge disjoint complete subgraphs of $L(G)$. Such an observation motivates the following definition.

For a connected graph G, a partition $(E_1, E_2, \cdots, E_k)$ of $E(G)$ is a **clique partition** of G if $G[E_i]$ is spanned by a complete graph for each $i \in \{1, 2, \cdots, k\}$. Furthermore, $(E_1, E_2, \cdots, E_k)$ is a (≥ 4)-**clique partition** of G, if for each $i \in \{1, 2, \cdots, k\}$, $G[E_i]$ is spanned by a K_{n_i} with $n_i \geq 4$; and a K_m-**partition** if for each $i \in \{1, 2, \cdots, k\}$, $G[E_i]$ is spanned by a K_m. Note that if G is simple, and if $(E_1, E_2, \cdots, E_k)$ of $E(G)$ is a clique partition of G, then $|V(G[E_i]) \cap V(G[E_j])| \leq 1$ where $i \neq j$ and $i, j \in \{1, 2, \cdots, k\}$.

Our main result is the following.

Theorem 5.1.4 *If G is 4-edge-connected and G has a (≥ 4)-clique partition, then $\Lambda_g(G) \leq 3$.*

The corollary below follows from Theorem 5.1.4 and from the observations made above.

Corollary 5.1.5 *Each of the following holds.*
(i) If $\kappa'(G) \geq 4$, then $\Lambda_g(L(G)) \leq 3$.
(ii) Every line graph of a 4-edge-connected graph has a 3-NZF.

We display some of the prerequisites in Section 5.2 and present the proof of the main result in Section 5.3.

5.2 Prerequisites

Throughout this section, we use the notation that $\mathbf{Z}_3 = \{0, 1, 2\}$ with the mod 3 addition.

Theorem 5.2.1 *Let G be a graph and A be an abelian group. Each of the following holds.*
(i) (Proposition 3.2 of [16]) Let H be a subgraph of G and $H \in \langle \mathbf{Z}_3 \rangle$, then $G/H \in \langle \mathbf{Z}_3 \rangle$ if and only if $G \in \langle \mathbf{Z}_3 \rangle$.
(ii) ([13] and Lemma 3.3 of [16]) For an integer $n \geq 1$ and an abelian group A, the n-cycle $C_n \in \langle A \rangle$ if and only if $|A| \geq n+1$. (Thus $\Lambda_g(K_n) = n+1$.)
(iii) (Corollary 3.5 of [16]) For $n \geq 5$, $\Lambda_g(K_n) = 3$.
(iv) (Lemma 2.1 of [17]) If for every edge e in a spanning tree of G, G has a subgraph $H_e \in \langle A \rangle$ with $e \in E(H_e)$, then $G \in \langle A \rangle$.

Lemma 5.2.2 *Let G be a graph, and let G' denote the graph obtained from G by contracting the 2-cycles of G (if there are any) and then contracting all loops of the resulting graph (if there are any). If $G' \in \langle \mathbf{Z}_3 \rangle$, then $G \in \langle \mathbf{Z}_3 \rangle$.*

Proof This follows from Theorem 5.2.1(ii) and (i). □

Lemma 5.2.3 *Let G be a graph and $H \cong K_4$ a subgraph of G and $v \in V(H)$ (see Figure 5.1, 5.2(a)).*
(i) If $d_G(v) = 6$ and G has another subgraph $H' \cong K_4$ with $V(H) \cap V(H') = \{v\}$, then let G_v be the graph obtained from G by splitting the vertex $v \in V(G)$ into v_1, v_2 (see Figure 1(b)), and by first deleting x_3v_1, y_3v_2 and then contracting v_1x_1, v_2y_1 (see Figure 1(c)); if $d_G(v) > 6$, let G_v be the graph obtained from G by splitting the vertex $v \in V(G)$ into v_1, v_2 and then contracting v_1x_1(see Figure 5.2(b)). If $G_v \in \langle \mathbf{Z}_3 \rangle$, then $G \in \langle \mathbf{Z}_3 \rangle$.

(ii) Suppose that G is simple and if $(E_1, E_2, \cdots, E_k)$ $(k \geq 2)$ is a K_4-partition of G. Define G_v as in (i), and obtain a graph G' by contracting repeatedly cycles of length ≤ 2 in G_v until no such cycles exist. Then G' has a K_4-clique partition. Moreover, if $\kappa'(G') \leq 3$, we must have $\kappa'(G') = 3$ and for any 3-edge-cut X of G', there exists $u \in V(G')$ such that $X \subseteq E_{G'}(u)$.

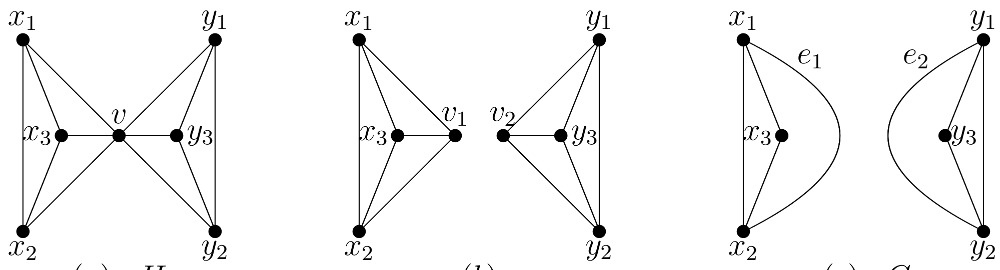


Figure 5.1

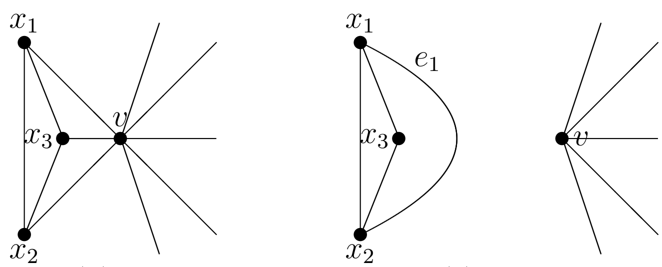


Figure 5.2

Proof (i) If $d_G(v) = 6$, using the notation in Figure 1(c), we may assume that G_v is so oriented and the edge e_1 is oriented from x_1 to x_2, e_2 is from y_1 to y_2 in G_v. Then restore G from G_v by preserving the orientation of G_v and by orienting the edges incident with v as follows: from v to x_2 and x_3, from v to y_2 and y_3, and from x_1, y_1 to v.

Let $b \in Z(G, \mathbf{Z}_3)$. We consider three cases below.

Case 1 $b(v) = 0$.

Let

$$b'(z) = \begin{cases} b(z) & , \text{ if } z \in V(G_v) - \{x_3, y_3\} \\ b(z) + 1 & , \text{ if } z = x_3 \\ b(z) + 2 & , \text{ if } z = y_3 \end{cases}$$

Then $b' \in Z(G_v, \mathbf{Z}_3)$. Since $G_v \in \langle \mathbf{Z}_3 \rangle$, there exits $f_1 \in F^*(G_v, \mathbf{Z}_3)$ such that $\partial f_1 = b'$ under the given orientation of G_v. Let $f \in F^*(G, \mathbf{Z}_3)$ be given by

$$f(e) = \begin{cases} f_1(e) & , \text{ if } e \in E(G) - \{x_1v, vx_3, vx_2, y_1v, vy_3, vy_2\} \\ f_1(e_1) & , \text{ if } e \in \{x_1v, vx_2\} \\ f_1(e_2) & , \text{ if } e \in \{y_1v, vy_2\} \\ 1 & , \text{ if } e = vx_3 \\ 2 & , \text{ if } e = vy_3 \end{cases}$$

Then, for each $z \in V(G)$,

$$\partial f(z) = \begin{cases} \partial f_1(z) = b'(z) = b(z) & , \text{ if } z \in V(G) - \{x_3, v, y_3\} \\ \partial f_1(x_3) - f(vx_3) = b(x_3) + 1 - 1 = b(x_3) & , \text{ if } z = x_3 \\ \partial f_1(y_3) - f(vy_3) = b(y_3) + 2 - 2 = b(y_3) & , \text{ if } z = y_3 \\ 1 + 2 = 0 = b(v) & , \text{ if } z = v \end{cases}$$

It follows that $\partial f = b$.

Case 2 $b(v) = 1$.

Let

$$b'(z) = \begin{cases} b(z) & , \text{ if } z \in V(G_v) - \{x_3, y_3\} \\ b(z) + 2 & , \text{ if } z = x_3 \\ b(z) + 2 & , \text{ if } z = y_3 \end{cases}$$

Then $b' \in Z(G_v, \mathbf{Z}_3)$. Since $G_v \in \langle \mathbf{Z}_3 \rangle$, there exits $f_1 \in F^*(G_v, \mathbf{Z}_3)$ such that $\partial f_1 = b'$ under the given orientation of G_v. Let $f \in F^*(G, \mathbf{Z}_3)$ be given by

$$f(e) = \begin{cases} f_1(e) & , \text{ if } e \in E(G) - \{x_1v, vx_3, vx_2, y_1v, vy_3, vy_2\} \\ f_1(e_1) & , \text{ if } e \in \{x_1v, vx_2\} \\ f_1(e_2) & , \text{ if } e \in \{y_1v, vy_2\} \\ 2 & , \text{ if } e = vx_3 \\ 2 & , \text{ if } e = vy_3 \end{cases}$$

Then, for each $z \in V(G)$,

$$\partial f(z) = \begin{cases} \partial f_1(z) = b'(z) = b(z) & , \text{ if } z \in V(G) - \{x_3, v, y_3\} \\ \partial f_1(x_3) - f(vx_3) = b(x_3) + 2 - 2 = b(x_3) & , \text{ if } z = x_3 \\ \partial f_1(y_3) - f(vy_3) = b(y_3) + 2 - 2 = b(y_3) & , \text{ if } z = y_3 \\ 2 + 2 = 1 = b(v) & , \text{ if } z = v \end{cases}$$

It follows that $\partial f = b$.

Case 3 $b(v) = 2$.

$$b'(z) = \begin{cases} b(z) & : \text{ if } z \in V(G_v) - \{x_3, y_3\} \\ b(z)+1 & : \text{ if } z = x_3 \\ b(z)+1 & : \text{ if } z = y_3 \end{cases}$$

Then $b' \in Z(G_v, \mathbf{Z}_3)$. Since $G_v \in \langle \mathbf{Z}_3 \rangle$, there exits $f_1 \in F^*(G_v, \mathbf{Z}_3)$ such that $\partial f_1 = b'$ under the given orientation of G_v. Let $f \in F^*(G, \mathbf{Z}_3)$ be given by

$$f(e) = \begin{cases} f_1(e) & , \text{ if } e \in E(G) - \{x_1v, vx_3, vx_2, y_1v, vy_3, vy_2\} \\ f_1(e_1) & , \text{ if } e \in \{x_1v, vx_2\} \\ f_1(e_2) & , \text{ if } e \in \{y_1v, vy_2\} \\ 1 & , \text{ if } e = vx_3 \\ 1 & , \text{ if } e = vy_3 \end{cases}$$

Then, for each $z \in V(G)$,

$$\partial f(z) = \begin{cases} \partial f_1(z) = b'(z) = b(z) & , \text{ if } z \in V(G) - \{x_3, v, y_3\} \\ \partial f_1(x_3) - f(vx_3) = b(x_3) + 1 - 1 = b(x_3) & , \text{ if } z = x_3 \\ \partial f_1(y_3) - f(vy_3) = b(y_3) + 1 - 1 = b(y_3) & , \text{ if } z = y_3 \\ 1 + 1 = 2 = b(v) & , \text{ if } z = v \end{cases}$$

Thus $\partial f = b$.

The proof for the case when $d_G(v) > 6$(see Figure 5.2) is similar to that for $d_G(v) = 6$ and so is omitted.

(ii) Since G is simple, when $i \neq j$,

$$|V(G[E_i]) \cap V(G[E_j])| \leq 1.$$

By the definition of G_v and G', if $d_G(v) = 6$, G' can be obtained by first splitting v into v_1 and v_2 and then contracting both K_4 cliques of the resulting graph containing v_1 or v_2; if $d_G(v) > 6$, G' can be obtained by first splitting v into v_1 and v_2 and then contracting the K_4 clique of the resulting containing v_1. Therefore, in either case, G' has a K_4-clique partition.

Suppose that $\kappa'(G') \le 3$. Let X be an edge cut of G' with $|X| \le 3$. Since every edge of G' must be in one of the K_4 cliques, X must contain an edge cut of a K_4, and so $|X| = 3$, and there exists $u \in V(G')$ such that $X \subseteq E_{G'}(u)$. □

Lemma 5.2.4 *Let G be a loopless graph spanned by a complete graph $K_n (n \ge 4)$ and R a nonempty subset of $E(G)$. Then $G/R \in \langle \mathbf{Z}_3 \rangle$.*

Proof Since G is loopless and R is not empty, G/R must have a 2-cycle. If $n = 4$, we contract this 2-cycle in G/R. Then the resulting graph has at most 2 vertices and so is $\mathbf{Z}_3$ connected. If $n > 4$, we can argue by Theorem 5.2.1(i) and by induction on n and contract the 2-cycle in G/R to reduce the order of G so that induction hypothesis can be applied. □

5.3 Proof of Theorem 5.1.4

Proof Note that by Theorem 5.2.1(ii) and (iii), Theorem 5.1.4 holds if $|V(G)| \le 5$, and so we assume that $|V(G)| \ge 6$. By Theorem 5.2.1(ii) and (iv), for each i with $1 \le i \le k$, $\Lambda_g(G[E_i]) \le 4$. Again by Theorem 5.2.1(iv), $\Lambda_g(G) \le 4$. It suffices to show that $\Lambda_g(G) \ne 4$.

We argue by contradiction. Suppose that there exists a graph G with $\kappa'(G) \ge 4$ and with a (≥ 4)-clique partition $(E_1, E_2, \cdots, E_k)$, such that $\Lambda_g(G) = 4$. Therefore we may choose such a graph that

$$G \text{ is not } \mathbf{Z}_3\text{-connected.} \tag{5.1}$$

and that

$$|V(G)| + |E(G)| \text{ is minimized.} \tag{5.2}$$

(3.1) G does not have a nontrivial subgraph H such that $H \in \langle \mathbf{Z}_3 \rangle$.

Proof Suppose that G has a nontrivial maximal subgraph $H \in \langle \mathbf{Z}_3 \rangle$. Then there must exist some E_i such that $E_i \cap E(H) \ne \emptyset$. Let $L = G[E(H) \cup E_i]$. Then $L/H \cong$

$G[E_i]/(E_i \cap E(H))$. Since $E_i \cap E(H) \neq \emptyset$ and since $G[E_i]$ is spanned by a complete graph, by Lemma 5.2.4, $L/H \in \langle \mathbf{Z}_3 \rangle$. Since $H \in \langle \mathbf{Z}_3 \rangle$, by Theorem 5.2.1(i), $L \in \langle \mathbf{Z}_3 \rangle$. But since H is a subgraph of L and since H is maximal, we must have $H = L$, and so $E_i \subseteq E(L) = E(H)$. Hence we may assume that there exists a smallest integer m with $0 \leq m < k$, such that $E_i \subseteq E(H)$ for each $i \geq m+1$ and $E_i \cap E(H) = \emptyset$ for each $i < m+1$. Therefore, $(E_1, E_2, \cdots, E_m)$ is a (≥ 4)-clique partition of G/H, and $\kappa'(G/H) \geq 4$. By (5.2) and since H is nontrivial, $G/H \in \langle \mathbf{Z}_3 \rangle$. By Theorem 5.2.1(i) and since $H \in \langle \mathbf{Z}_3 \rangle$, we conclude that $G \in \langle \mathbf{Z}_3 \rangle$, contrary to (5.1). □

By Theorem 5.2.1(ii) and (iii), loops, 2-cycles and K_m with $m \geq 5$ are in $\langle \mathbf{Z}_3 \rangle$. Therefore Claim (3.2) below follows immediately from (3.1).

(3.2) G is simple, and for each $i \in \{1, 2, \cdots, k\}$, $G[E_i] \cong K_4$.

By (3.2), G is simple and so any two distinct K_4 clique of G can have at most one vertex in common. By the assumption that $\kappa'(G) \geq 4$, we have

(3.3) $\delta(G) \geq 4$ and so $k \geq 4$.

If G has a cut vertex, then by (5.2), each block of G is in $\langle \mathbf{Z}_3 \rangle$ and so by Theorem 1.2(iv), $G \in \langle \mathbf{Z}_3 \rangle$, contrary to (5.1). Thus

(3.4) $\kappa(G) \geq 2$.

Furthermore, we have

(3.5) For any $v \in V(G)$, G has a vertex 2-cut (a vertex cut with 2 vertices) containing v.

Proof By (3.2) and (3.3), $(E_1, E_2, \cdots, E_k)$, $(k \geq 4)$ is a K_4-partition of G. Pick $v \in V(G)$ such that

$$v \in V(G[E_{l_1}]) \cap V(G[E_{l_2}]) \cap \cdots \cap V(G[E_{l_m}]),\ (m \geq 2).$$

Split v and perform the operation as in Lemma 5.2.3(i) to get graph G_v, and contract 2-cycles and loops in G_v. Denote the resulting graph by G'. Then G' also has a K_4-partition by Lemma 5.2.3(ii).

By (3.4), G' is connected. If $\kappa'(G') \geq 4$, then by (5.2), $G' \in \langle \mathbf{Z}_3 \rangle$. By Lemmas 5.2.1(i), 5.2.2, and 5.2.3, $G \in \langle \mathbf{Z}_3 \rangle$, contrary to (5.1).

Thus $\kappa'(G') \leq 3$, and so $\kappa'(G') = 3$. By Lemma 5.2.3(ii), if X is a 3-edge-cut of G', then there exists $u \in V(G')$ such that $X \subseteq E_{G'}(u)$. Since X is a 3-edge-cut of G', it follows that u is a cut vertex of G' and $u \neq v$, and so $\{u, v\}$ is a vertex 2-cut of G. □

Let $W = \{w_1, w_2\}$ be a vertex cut of G and $W_1', W_2', \cdots$, are components of $G - W$. Define $G_i = G[V(W_i') \cup W]$ to be the subgraph induced by $V(W_i') \cup W$ and we call each G_i a W**-component** of G . For each vertex 2-cut W of G, let $S(W)$ denote a specified W-component such that $|V(S(W))|$ is minimized, among all W-components of G.

Choose a subgraph $H \in \{S(W) : W \text{ is a 2-cut of } G\}$ such that $|V(H)|$ is the smallest among them. Then for some vertex 2-cut $W = \{w, w'\}$ of G, $H = S(W)$.

Since H is a W-component, we have $V(H) - W \neq \emptyset$ and so we can pick a vertex $v \in V(H) - W$. By (3.5) G has a vertex 2-cut $W' = \{v, v'\}$ where $v' \in V(G'')$.

Case 1 $v' \in V(H)$.

If $v' = w$ (or $v' = w'$, respectively), then $W'' = \{v, w\}$ (or $\{v, w'\}$, respectively) is a vertex 2-cut of G and $|S(W'')| < |V(H)|$, contrary to the choice of H. If $v' \in V(H) - \{w, w'\}$ and $\{v, v'\}$ separates w and w' in H, then $\{v, v'\}$ is not a 2-cut of G. Therefore, W' does not separate w, w' in H, and so a W'-component of G which does not contain w and w' would be a proper subgraph of H, contrary to the choice of H, (see Figure 5.3).

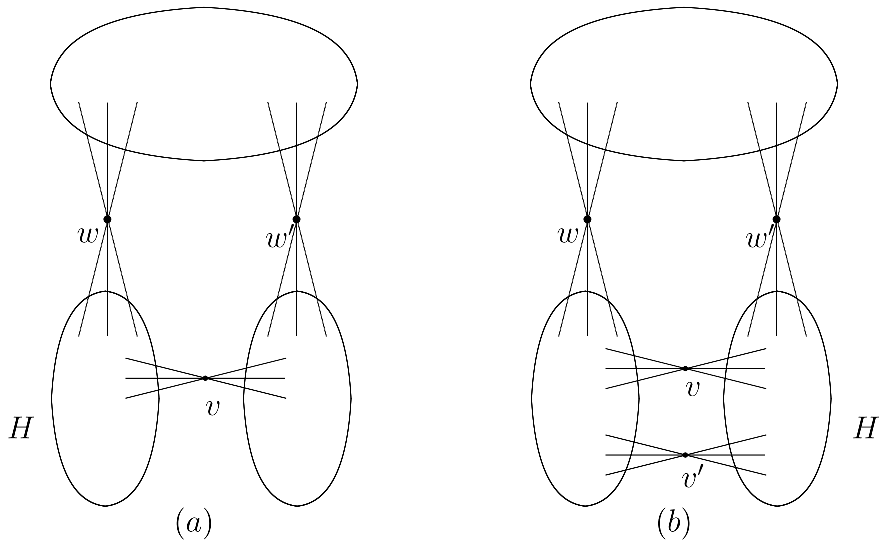


Figure 5.3

Case 2 $v' \notin V(H)$.

By (3.4), v must by a cut vertex of H separating w and w' in H, and so $W'' = \{v, w\}$ is also a vertex 2-cut of G, and a W''-component that does not contain w' is a violation to the choice of H (see Figure 5.4).

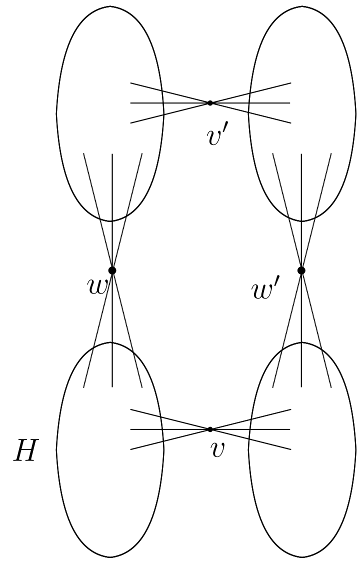


Figure 5.4

Thus neither of the cases is possible. The contradictions establish Theorem 5.1.4.

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