\textbf{\$p\$-EQUIVALENCE OF IMPULSE DIFFERENTIAL EQUATIONS}

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\textbf{Abstract.} By the Schauder's fixed point theorem the \$p\$-equivalence between two impulse differential equations is proved.

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1. \textbf{Introduction}

Impulse differential equations constitute a useful mathematical apparatus for the investigation of evolutionary processes in physics, chemistry, control theory and robotics which are subject to the action of short-time forces in the form of impulses. The work of Mil'man and Myshkis [1] marked the beginning of the mathematical theory of these equations.

In the present paper an \$p\$-equivalence between two arbitrary impulse differential equations is proved. That means that for every solution of the first equation there exists a solution of the second equation such that the difference of these solutions lies in the space \$p\$, and vice versa. Similar problems under other conditions are considered in [2], [3].

2. \textbf{Statement of the problem}

Let \( X = \mathbb{R}^n \) be the \( n \) -dimensional Euclidean space with identity operator \( I \) and norm \( \| \cdot \| \). By \( T = \{ t_n \} \) we denote a sequence of points \( 0 = t_0 < t_1 < t_2 < \ldots \) satisfying the condition \( \lim_{n \to \infty} t_n = \infty \).
Consider the impulse equation
\[
\frac{dx}{dt} = F(t, x) \quad \text{for } t \neq t_n
\]  
(1)
\[
x(t_n^+) = Q_n x(t_n) \quad \text{for } t = t_n
\]  
(2)
and
\[
\frac{dy}{dt} = G(t, y) \quad \text{for } t \neq t_n
\]  
(3)
\[
y(t_n^+) = D_n y(t_n) \quad \text{for } t = t_n,
\]  
(4)
where \(F(t, x), G(t, y) : \mathbb{R}_+ \times X \to X\) (\(\mathbb{R}_+ = [0, \infty)\)) are continuous functions and \(Q_n, D_n : X \to X\) (\(n = 1, 2, \ldots\)). Moreover, we assume that all considered functions are continuous from the left.

**Definition 1.** We shall say that the function \(\psi(t) (t \geq 0)\) is a solution of the equation \((1) - (2) ((3) - (4))\) if for \(t \neq t_n\), it satisfies equation \((1) ((3))\) and for \(t = t_n\) the condition of “jump” \((2) ((4))\).

Let \(1 \leq p < \infty\). By \(B_r\) we denote the closed ball in the space \(X\) with a center at zero and radius \(r\).

Let \(\Omega \subset \mathbb{R}_+\). By \(L_p(\Omega, X)\) we denote the space of all functions \(x : \Omega \to X\) for which \(\int_\Omega ||x(t)||^p dt < \infty\). When \(X = \mathbb{R}\) we shall write \(L_p(\Omega)\).

**Definition 2.** The equation \((3) - (4)\) is called \(L_p\)-equivalent to the equation \((1) - (2)\) in the ball \(B_r\) if there exists \(\rho > 0\) such that for any solution \(x(t)\) of \((1) - (2)\) lying in \(B_r\) there exists a solution \(y(t)\) of \((3) - (4)\) lying in the ball \(B_{r+\rho}\) and satisfying the relation \(y(t) - x(t) \in L_p(\mathbb{R}_+, X)\). If equation \((3) - (4)\) is \(L_p\)-equivalent to equation \((1) - (2)\) in the ball \(B_r\) and vice versa, we shall say that equations \((1) - (2)\) and \((3) - (4)\) are \(L_p\)-equivalent in the ball \(B_r\).

### 3. Main results

#### 3.1. Equivalent equations

Let
\[
w(t, s) = \prod_{j=n(t)}^{n(s)+1} Q_j \quad (0 \leq s < t)
\]  
(5)
and
\[
\tilde{w}(t, s) = \prod_{i=n(t)}^{n(s)+1} D_i \quad (0 \leq s < t),
\]  
(6)
where \(n(\tau) = \max\{n : t_n < \tau\}\).
Lemma 1. Each solution \( x(t) \) of equation (1) – (2) which lies in the ball \( B_r \) is a solution of the nonlinear integral equation

\[
x(t) = w(t,0)x(0) + \sum_{i=0}^{n(t)-1} w(t,t_i^+) \int_{t_i}^{t_{i+1}} F(s,x(s))ds + \int_{t_{n(t)}}^{t} F(s,x(s))ds \tag{7}
\]

and each solution \( y(t) \) of equation (3) – (4) which lies in the ball \( B_{r+\rho} \) is a solution of the nonlinear integral equation

\[
y(t) = \tilde{w}(t,0)y(0) + \sum_{i=0}^{n(t)-1} \tilde{w}(t,t_i^+) \int_{t_i}^{t_{i+1}} G(s,y(s))ds + \int_{t_{n(t)}}^{t} G(s,y(s))ds \tag{8}
\]

Lemma 1 is proved by straightforward verification.

Set

\[
z(t) = y(t) - x(t). \tag{9}
\]

Then the function \( z(t) \) is a solution of the nonlinear integral equation

\[
z(t) = \tilde{w}(t,0)y(0) - w(t,0)x(0) + \\
+ \sum_{i=0}^{n(t)-1} \left\{ \tilde{w}(t,t_i^+) \int_{t_i}^{t_{i+1}} G(s,x(s) + z(s))ds - w(t,t_i^+) \int_{t_i}^{t_{i+1}} F(s,x(s))ds \right\} + \\
+ \int_{t_{n(t)}}^{t} \left\{ G(s,x(s) + z(s)) - F(s,x(s)) \right\} ds.
\tag{10}
\]

Let

\[
H(x,z)(t) = \tilde{w}(t,0)y(0) - w(t,0)x(0) + \\
+ \sum_{i=0}^{n(t)-1} \left\{ \tilde{w}(t,t_i^+) \int_{t_i}^{t_{i+1}} G(s,x(s) + z(s))ds - w(t,t_i^+) \int_{t_i}^{t_{i+1}} F(s,x(s))ds \right\} + \\
+ \int_{t_{n(t)}}^{t} \left\{ G(s,x(s) + z(s)) - F(s,x(s)) \right\} ds.
\tag{11}
\]

Then

\[
z(t) = H(x,z)(t). \tag{12}
\]

From Definition 2 it follows that to establish the \( \mathcal{L}_p \) - equivalence of equation (3) – (4) to equation (1) – (2) it suffices to show that for each solution \( x(t) \) of equation (1) – (2) lying in the ball \( B_r \) the operator equation (12) has a fixed point \( z(t) \) such that \( x(t) + z(t) \in B_{r+\rho} \) for some \( \rho > 0 \) and which lies in \( \mathcal{L}_p(\mathbb{R}_+, X) \).
Let $S(\mathbb{R}_+, X)$ be the space of all functions which are continuous for $t \neq t_n$ ($n = 1, 2, ...$), have at the points $t_n$ limits on the left and right and are left continuous. The space $S(\mathbb{R}_+, X)$ is linear and locally convex. A metric can be introduced by

$$\rho(x, y) = \sum_{n=0}^{\infty} \frac{1}{2^n} \max_{t \in [t_n, t_{n+1}]} \| x(t) - y(t) \|,$$

The convergence with respect to this metric coincides with the uniform convergence on each bounded interval. For this space an analog of Arzella-Ascoli’s theorem is valid.

**Lemma 2.** [2] The set $M \subseteq S(\mathbb{R}_+, X)$ is relatively compact if and only if $M$ is equicontinuous on each interval $(t_{n-1}, t_n]$ ($n = 1, 2, ...$).

**Proof.** We apply the theorem of Arzella-Ascoli on each interval $(t_{n-1}, t_n]$ ($n = 1, 2, ...$) and constitute diagonal line sequence, which is converging on each from them.□

In the proof of the existence of a fixed point of the operator $H$ from the equation (12) we use a modification of Schauder’s classical principle.

**Lemma 3.** [2] Let the operator $H$ transform the set

$$C(r) = \{ x \in S(\mathbb{R}_+, X) : x(t) \in B_r, \ t \geq 0 \}$$

into itself and be continuous and compact.

Then $H$ has a fixed point in $C(r)$.

### 3.2. Conditions for $\mathcal{L}_p$-equivalence

**Theorem 1.** Let the following conditions are fulfilled.

1. The operator-valued functions $w(t, s)$ and $\tilde{w}(t, s)$ satisfy the condition

$$\| \tilde{w}(t, 0) \xi - w(t, 0) \eta \| \leq \chi_{r, \rho}(t) \quad (0 \leq t < \infty),$$

where $\xi \in B_{r+\rho}$, $\eta \in B_r$, $\chi_{r, \rho}(t) \in \mathcal{L}_p(\mathbb{R}_+)$ and $r, \rho > 0$.

2. The functions $F(t, x)$, $G(t, y)$ and $w(t, s)$, $\tilde{w}(t, s)$ satisfy the condition

$$\sup_{\| u \| \leq r, \| v \| \leq r + \rho} \sum_{i=0}^{n(t)-1} \| \tilde{w}(t, t_i^+) \int_{t_i}^{t_i+1} G(s, v) ds - w(t, t_i^+) \int_{t_i}^{t_i+1} F(s, u) ds \| \leq \psi_{r, \rho}(t),$$

$$\text{(14)}$$
Proof. We shall show that for any function $\psi_{r,\rho}(t) \in \mathcal{L}_p(\mathbb{R}_+)$ and

$$\sup_{\|u\| \leq r} \int_{t_n(t)}^t \|G(s, v) - F(s, u)\| ds \leq \varphi_{r,\rho}(t),$$

where $\varphi_{r,\rho}(t) \in \mathcal{L}_p(\mathbb{R}_+)$.

3. The function $G(t, y)$ satisfies the condition

$$\sup_{\|v\| \leq r+\rho} \|G(t, v)\| \leq \Phi_{r,\rho}(t),$$

where $\Phi_{r,\rho}(t)$ is integrable on each interval $(t_{n-1}, t_n)$ ($n = 1, 2, \ldots$).

4. The inequality

$$\chi_{r,\rho}(t) + \psi_{r,\rho}(t) + \varphi_{r,\rho}(t) \leq \rho$$

holds for each $t \geq 0$.

Then the equation (3) – (4) is $\mathcal{L}_p$-equivalent to the equation (1) – (2) in the ball $B_r$.

Proof. We shall show that for any function $x(t) \in B_r$ ($t \geq 0$) the operator $H(x, z)$ defined by (11) maps the set

$$C(\rho) = \{z \in S(\mathbb{R}_+, X) : z(t) \in B_\rho, t \geq 0\}$$

into itself.

Let $x(t) \in B_r$ ($t \geq 0$) and let $z \in C(\rho)$. Then from (11) we obtain the estimate:

$$\|H(x, z)(t)\| \leq \|\tilde{w}(t, 0)y(0) - w(t, 0)x(0)\| +$$

$$+ \sum_{i=0}^{n(t)-1} \|\tilde{w}(t, t_i^+) \int_{t_i}^{t_{i+1}} G(s, x(s) + z(s)) ds - w(t, t_i^+) \int_{t_i}^{t_{i+1}} F(s, x(s)) ds\| +$$

$$+ \int_{t_{n(t)}}^t \|G(s, x(s) + z(s)) - F(s, x(s))\| ds \leq \|\tilde{w}(t, 0)y(0) - w(t, 0)x(0)\| +$$

$$+ \sup_{\|u\| \leq r} \sum_{i=0}^{n(t)-1} \|\tilde{w}(t, t_i^+) \int_{t_i}^{t_{i+1}} G(s, v) ds - w(t, t_i^+) \int_{t_i}^{t_{i+1}} F(s, u) ds\| +$$

$$+ \sup_{\|u\| \leq r} \int_{t_{n(t)}}^t \|G(s, v) - F(s, u)\| ds \leq \chi_{r,\rho}(t) + \psi_{r,\rho}(t) + \varphi_{r,\rho}(t) \leq \rho$$

for each $t \geq 0$.

We obtain $\|H(x, z)(t)\| \leq \rho$, i.e., $H(x, z) \in C(\rho)$. Hence, for any $x \in C(\rho)$, the set $C(\rho)$ is invariant with respect to $H(x, z)$.
Let be \( L = \{ u(t) = H(x, z)(t) : \| z \| \leq \rho \} \).

First we shall establish that the set \( L \) is compact in \( S(\mathbb{R}_+, X) \).

We shall show the equicontinuity of the functions of the set \( L \). In fact, for \( t', t'' \in (t_{n-1}, t_n] \) following equalities hold:

\[
\begin{align*}
w(t', s) &= w(t'', s) = w(t_n, s) \\
\tilde{w}(t', s) &= \tilde{w}(t'', s) = \tilde{w}(t_n, s) \\
n(t') &= n(t'') = n - 1
\end{align*}
\]

For \( t', t'' \in (t_{n-1}, t_n] \) we obtain

\[
\begin{align*}
\| u(t') - u(t'') \| &= \\
&= \| (\tilde{w}(t', 0)y(0) - w(t', 0)x(0)) - (\tilde{w}(t'', 0)y(0) - w(t'', 0)x(0)) + \\
&+ \sum_{i=0}^{n(t')-1} \{ \tilde{w}(t', t_i^+ \int_{t_i}^{t_{i+1}} G(s, x(s) + z(s))ds - w(t', t_i^+) \int_{t_i}^{t_{i+1}} F(s, x(s))ds \} ds - \\
&- \sum_{i=0}^{n(t'')-1} \{ \tilde{w}(t'', t_i^+ \int_{t_i}^{t_{i+1}} G(s, x(s) + z(s))ds - w(t'', t_i^+) \int_{t_i}^{t_{i+1}} F(s, x(s))ds \} ds + \\
&+ \int_{t'_{n(t')}}^{t'} \{ G(s, x(s) + z(s)) - F(s, x(s)) \} ds - \\
&- \int_{t''_{n(t'')}}^{t''} \{ G(s, x(s) + z(s)) - F(s, x(s)) \} ds \leq \\
&\leq \sup_{\| u \| \leq r} \int_{t'}^{t''} \| G(s, v) - F(s, u) \| ds
\end{align*}
\]

The equicontinuity of the functions of the set \( L \) follows from the last estimate.

From Lemma 2 the compactness of the set \( L \) follows.

We shall show that the operator \( H(x, z) \) is continuous in \( S(\mathbb{R}_+, X) \).

Let the sequence \( \{ z_n(t) \} \subset C(\rho) \) be convergent in the metric of the space \( S(\mathbb{R}_+, X) \) (i.e., uniformly converges on each bounded interval) to the function \( z(t) \in C(\rho) \). Then, for \( t \in \mathbb{R}_+ \) the sequence \( G(t, x(t) + z_n(t)) \) converges to \( G(t, x(t) + z(t)) \). From conditions 3 of Theorem 1 it follows that the convergent sequence of functions \( G(t, x(t) + z_n(t)) \) is majorized by the intergrable function
The fixed point $\Phi_{r,\rho}(t)$. That’s why within the integral in formula
\[
H(x, z_n)(t) = \tilde{w}(t, 0)y(0) - w(t, 0)x(0) + \\
+ \sum_{i=0}^{n(t)-1} \sum_{i=0}^{t_i} \left\{ \int_{t_i}^{t_{i+1}} G(s, x(s) + z_n(s))ds - w(t, t_i) \int_{t_i}^{t_{i+1}} F(s, x(s))ds \right\} + \\
+ \int_{t_{n(t)}}^{t} \left\{ G(s, x(s) + z_n(s)) - F(s, x(s)) \right\}ds
\]
we may pass to the limit. Hence $H(x, z_n)(t)$ tends to $H(x, z)(t)$ for $t \in \mathbb{R}_+$. Since $H(x, z)$ maps $C(\rho)$ into a compact set, $H(x, z_n)$ tends to $H(x, z)$ in $S(\mathbb{R}_+, X)$ as well.

From Lemma 3 it follows that for any $x \in C(\rho)$ the operator $H(x, z)$ has a fixed point $z$ in $C(\rho)$, i.e. $z = H(x, z)$.

We shall show that this fixed point $z(t)$ lies in $\mathcal{L}_p(\mathbb{R}_+, X)$.
\[
\|z(t)\| \leq \|\tilde{w}(t, 0)y(0) - w(t, 0)x(0)\| + \\
+ \sup_{\|u\| \leq r} \sum_{i=0}^{n(t)-1} \|\tilde{w}(t, t_i) \int_{t_i}^{t_{i+1}} G(s, v)ds - w(t, t_i) \int_{t_i}^{t_{i+1}} F(s, u)ds\| + \\
+ \sup_{\|u\| \leq r} \int_{t_{n(t)}}^{t} \|G(s, v) - F(s, u)\|ds \leq \chi_{r,\rho}(t) + \psi_{r,\rho}(t) + \varphi_{r,\rho}(t)
\]
\[
\|z\|_p = \left( \int_0^t \|z(t)\|^p dt \right)^{\frac{1}{p}} \leq \left( \int_0^t \|\chi_{r,\rho}(t) + \psi_{r,\rho}(t) + \varphi_{r,\rho}(t)\|^p dt \right)^{\frac{1}{p}} \leq \\
\leq \|\chi_{r,\rho}\|_p + \|\psi_{r,\rho}\|_p + \|\varphi_{r,\rho}\|_p
\]

Hence this fixed point belongs to the space $\mathcal{L}_p(\mathbb{R}_+, X)$, i.e., the equations (3) – (4) are $\mathcal{L}_p$-equivalent to the equations (1) – (2) in ball $B_r$.

Theorem 1 is proved. □

**Remark 1.** Condition (13) means that the “impulse difference” of the two equations belongs in the space $\mathcal{L}_p(\mathbb{R}_+)$. Condition (14) means that the sum of the “integral differences” of $G$ and $F$ with weights $\tilde{w}$ and $w$ on the balls $B_{r+\rho}$ and $B_r$ respectively on any interval $[t_i, t_{i+1}]$ lies in the space $\mathcal{L}_p(\mathbb{R}_+)$. Condition (15) means that the “integral difference” of the ordinary parts on any interval $[t_i, t_{i+1}]$ lies in the space $\mathcal{L}_p(\mathbb{R}_+)$. 

**Remark 2.** It may be noted that the condition (18) in [2] is not fulfilled if one of the equations is an ordinary. Let the equation (3) – (4) be ordinary i.e. $D_n = I$. Then for any solution of the impulse equation there exists a solution
of the ordinary equation. If we have evidently or numerical representation of
the solution of the ordinary equation, then the solution of the impulse equation
will be $\Sigma_p$-near to this solution.

**Corollary 1.** Let the operators $Q_n, D_n \ (n = 1, 2, \ldots)$ are linear and the
following conditions are fulfilled.

1. The operator-valued function $w(t, s)$ and $\tilde{w}(t, s)$ satisfy the conditions

$$||\tilde{w}(t, s)|| \leq M \quad (0 \leq s < t < \infty),$$

where $M$ is a positive number and

$$||\tilde{w}(t, 0) - w(t, 0)|| \leq \chi_{r, \rho}(t) \quad (0 \leq t < \infty),$$

where $\xi \in B_{r+\rho}, \eta \in B_r, \chi_{r, \rho}(t) \in \Sigma_p(\mathbb{R}_+)$ and $r, \rho > 0$.

2. The functions $F(t, x)$ and $G(s, y)$ satisfy the condition

$$\sup_{\|u\| \leq r} \int_{0}^{t} \|\tilde{w}(t, s)G(s, v) - w(t, s)F(s, u)\| ds \leq \psi_{r, \rho}(t),$$

where $\psi_{r, \rho}(t) \in \Sigma_p(\mathbb{R}_+)$.  

3. The function $G(t, y)$ satisfies the condition

$$\sup_{\|v\| \leq r+\rho} \|G(t, v)\| \leq \varphi_{r, \rho}(t) \in \Sigma_1(\mathbb{R}_+)$$

4. The inequality

$$\chi_{r, \rho}(t) + \psi_{r, \rho}(t) \leq \rho$$

holds for each $t \geq 0$.

Then the equation (3) - (4) is $\Sigma_p$-equivalent to the equation (1) - (2) in
the ball $B_r$.

**Proof.** The corollary follows immediately from the relations

$$x(t) = \tilde{w}(t, 0)x(0) + \int_{0}^{t} \tilde{w}(t, s)F(s, x(s)) ds,$$

$$y(t) = w(t, 0)y(0) + \int_{0}^{t} w(t, s)G(s, y(s)) ds.$$
Example. Consider the impulse equations

\[
\frac{dx}{dt} = F(t, x) \quad \text{for} \quad t \neq n
\]

\[
x(n^+) = 5^{-n}(2 - \sin n) \quad \text{for} \quad n = 1, 2, \ldots
\]

and

\[
\frac{dy}{dt} = G(t, y) \quad \text{for} \quad t \neq n
\]

\[
y(n^+) = 5^{-n}\sin y(n) \quad \text{for} \quad n = 1, 2, \ldots
\]

where \(F(t, x), G(t, y) : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}\) are continuous functions. Let for some \(0 < r < \Delta\) the functions \(G(t, y)\) and \(F(t, x)\) satisfy the conditions

\[
\sup_{|u| \leq r} \int_{|v| \leq \Delta} |G(s, v) - F(s, u)| ds \leq \varphi(t) \in \mathcal{L}_p(\mathbb{R}_+)
\]

\[
\sup_{|v| \leq \Delta} |G(t, v)| \leq \Phi(t)
\]

The function \(\varphi(t)\) satisfies the condition

\[
4.5^{-[t]} + 4.5^{-[t]} + \varphi(t) \leq \Delta - r
\]

The function \(\Phi(t)\) is integrable on each interval \((n-1, n)\) \((n = 1, 2, \ldots)\).

We note that the conditions \((27) - (29)\) are fulfilled for example by

\[
F(t, x) = \frac{\ln 5}{4} 5^{-t} \frac{x}{1 + x^2}
\]

\[
G(t, y) = \frac{\ln 5}{4} 5^{-t} y \sin^2 y
\]

Indeed in this case we have

\[
\sup_{|u| \leq r} \int_{|v| \leq \Delta} \frac{\ln 5}{4} 5^{-s} |v \sin^2 v - \frac{u}{1 + u^2}| ds = \frac{1}{4} (5^{-[t]} - 5^{-t}) \sup_{|u| \leq r} |v \sin^2 v - \frac{u}{1 + u^2}| \leq \frac{1}{2} 5^{-[t]} \Delta + r
\]

Set

\[
\varphi(t) = \frac{1}{2} 5^{-[t]} (\Delta + r) \in \mathcal{L}_p(\mathbb{R}_+)
\]
The function $\varphi(t)$ satisfies (29)

$$4.5^{-|t|} + 4. t. 5^{-|t|} + \frac{1}{2} 5^{-|t|} (\Delta + r) < \Delta - r,$$

for each $t \geq 0$.

Otherwise

$$\sup_{|v| \leq \Delta} |G(t, v)| = \sup_{|v| \leq \Delta} \left| \frac{\ln 5}{4} - t v \sin^2 v \right| \leq \frac{ln 5}{4} \Delta 5^{-t} \in \mathcal{L}_p(\mathbb{R}^+).$$

We shall show that the conditions of Theorem 1 are fulfilled.

We have

$$Q_n x = 5^{-n}(2 - \sin x), \quad D_n y = 5^{-n} \sin y.$$ 

Then for any $\xi \in B_\Delta, \eta \in B_r (0 < r < \Delta), t \in (t_n, t_{n+1}]$ we obtain

$$|\tilde{w}(t, 0)\xi - w(t, 0)\eta| = |\prod_{i=[t]} D_i \xi - \prod_{i=[t]} Q_i \eta| =$$

$$= |D_n \xi_{n-1} - Q_n \eta_{n-1}| = |5^{-|t|} \sin \xi_{n-1} - 5^{-|t|} (2 - \sin \eta_{n-1})| \leq 4.5^{-|t|},$$

where

$$\xi_{n-1} = D_{n-1} D_{n-2} \ldots D_1 \xi, \quad \eta_{n-1} = Q_{n-1} Q_{n-2} \ldots Q_1 \eta.$$ 

Set $\chi(t) = 4.5^{-|t|}$.

We shall show that $\chi(t) \in \mathcal{L}_p(\mathbb{R}^+)$

$$\int_0^\infty |\chi(t)|^p dt = \int_0^\infty |4.5^{-|t|}|^p dt = 4^p \int_0^{5^{1-t}} 5^{1-t} p dt = 20^p \int_0^\infty 5^{-p} dt < \infty.$$ 

Hence $\chi(t) \in \mathcal{L}_p(\mathbb{R}^+)$.

We shall show that the condition 2 of Theorem 1 is fulfilled. Let $t \in (t_n, t_{n+1}]$. Then

$$\sup_{|u| \leq t} \left| \sum_{i=0}^{[t]-1} |\tilde{w}(t, i^+) \int_i^G(s, v) ds - w(t, i^+) \int_i^F(s, u) ds| =$$

$$= \sup_{|u| \leq t} \left| \sum_{i=0}^{[t]-1} |5^{-|t|} \sin \xi_{n-1,i+1} - 5^{-|t|} (2 - \sin \eta_{n-1,i+1})| \leq$$

$$\leq \sum_{i=0}^{[t]-1} 4.5^{-|t|} \leq 4. t. 5^{-|t|},$$

where

$$\xi_{n-1,i+1} = D_{n-1} D_{n-2} \ldots D_{i+1} \int_i^{i+1} G(s, v) ds$$

$$\eta_{n-1,i+1} = Q_{n-1} Q_{n-2} \ldots Q_{i+1} \int_i^{i+1} F(s, u) ds.$$
Set $\psi(t) = 4t.5^{-|t|}$.

We shall show that $\psi(t) \in L_p([0, \infty))$.

\[
\int_0^\infty |\psi(t)|^p dt = \int_0^\infty |4t.5^{-|t|}|^p dt = 4^p \int_0^\infty t^p.5^{-p|t|} dt \\
\leq 4^p \int_0^\infty t^p.5^{(1-t)p} dt = 20^p \int_0^\infty t^p.5^{-p} dt < \infty.
\]

Hence $\psi(t) \in L_p([0, \infty))$.

For the condition 4 of Theorem 1 we obtain

\[
\chi(t) + \psi(t) + \varphi(t) = 4.5^{-|t|} + 4t.5^{-|t|} + \varphi(t) \leq \Delta - r
\]

for each $t \geq 0$. Hence the equation (25) – (26) is $L_p$-equivalent to the equation (23) – (24) in the ball $B_r (0 < r < \Delta)$.

\textbf{References}


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