THE EXPONENTIAL INTEGRAL AND THE CONVOLUTION

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Abstract. The exponential integral $e_i(\lambda x)$ and its associated functions $e_{i+}(\lambda x)$ and $e_{i-}(\lambda x)$ are defined as locally summable functions on the real line and their derivatives are found as distributions. Some convolution products of these distributions and other distributions are then found.

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The exponential integral $e_i(x)$ is defined for $x > 0$ by

$$e_i(x) = \int_x^\infty u^{-1} e^{-u} du,$$  \hspace{1cm} (1)

see Sneddon [3], the integral diverging for $x \leq 0$. It was pointed out in [1] that equation (1) can be rewritten in the form

$$e_i(x) = \int_x^\infty u^{-1} [e^{-u} - H(1-u)] du - H(1-x) \ln |x|,$$

where $H$ denotes Heaviside’s function. The integral in this equation is convergent for all $x$ and so was used to define $e_i(x)$ on the real line.

More generally, if $\lambda \neq 0$, $e_i(\lambda x)$ was defined in the obvious way by

$$e_i(\lambda x) = \int_{\lambda x}^\infty u^{-1} [e^{-u} - H(1-u)] du - H(1-\lambda x) \ln |\lambda x|.$$  \hspace{1cm} (2)

Further, $e_{i+}(\lambda x)$ and $e_{i-}(\lambda x)$ were defined by

$$e_{i+}(\lambda x) = H(x) e_i(\lambda x), \quad e_{i-}(\lambda x) = H(-x) e_i(\lambda x).$$
so that
\[ \text{ei}(\lambda x) = \text{ei}_+(\lambda x) + \text{ei}_-(\lambda x). \]  
\tag{3}

In particular, if \( \lambda > 0 \), we have
\[ \text{ei}(\lambda x) = \int_x^\infty u^{-1}[e^{-\lambda u} - H(1 - \lambda u)] \, du - H(1 - \lambda x) \ln |\lambda x|, \]  
\tag{4}
and
\[ \text{ei}_+(\lambda x) = \int_x^\infty u^{-1}e^{-\lambda u} \, du, \quad x > 0, \]  
\tag{5}
\[ \text{ei}_-(\lambda x) = -\gamma - \ln |\lambda| + \int_0^x u^{-1}(e^{-\lambda u} - 1) \, du - \ln x_-, \quad x < 0, \]  
\tag{6}
where
\[ \gamma = -\int_0^\infty u^{-1}[e^{-\lambda u} - H(1 - \lambda u)] \, du \]
is Euler’s constant.

If \( \lambda < 0 \), we have
\[ \text{ei}(\lambda x) = -\int_{-\infty}^x u^{-1}[e^{-\lambda u} - H(1 - \lambda u)] \, du - H(1 - \lambda x) \ln |\lambda x|, \]  
\tag{7}
and
\[ \text{ei}_+(\lambda x) = -\gamma - \ln |\lambda| - \int_0^x u^{-1}(e^{-\lambda u} - 1) \, du - \ln x_+, \quad x > 0, \]  
\tag{8}
\[ \text{ei}_-(\lambda x) = -\int_{-\infty}^x u^{-1}e^{-\lambda u} \, du, \quad x < 0. \]  
\tag{9}

The derivatives of these functions were found as
\[ [\text{ei}(\lambda x)]' = -e^{-\lambda x}x^{-1} = -x^{-1} - \sum_{i=1}^\infty \frac{(-\lambda)^i}{i!} x^{i-1}, \]  
\tag{10}
\[ [\text{ei}_+(\lambda x)]' = -e^{-\lambda x}x_+^{-1} - (\gamma + \ln |\lambda|)\delta(x) \]  
\[ = -x_+^{-1} - \sum_{i=1}^\infty \frac{(-\lambda)^i}{i!} x_+^{i-1} - (\gamma + \ln |\lambda|)\delta(x), \]  
\tag{11}
\[ [\text{ei}_-(\lambda x)]' = e^{-\lambda x}x_-^{-1} + (\gamma + \ln |\lambda|)\delta(x) \]  
\[ = x_-^{-1} - \sum_{i=1}^\infty \frac{\lambda^i}{i!} x_-^{i-1} + (\gamma + \ln |\lambda|)\delta(x), \]  
\tag{12}
for all \( \lambda \neq 0 \).

We now note the following results obtained by replacing \( x \) by \(-x\) in the functions \( \text{ei}(\lambda x) \), \( \text{ei}_+(\lambda x) \) and \( \text{ei}_-(\lambda x) \).
\[ \text{ei}(\lambda(-x)) = \text{ei}((-\lambda)x), \]  
\tag{13}
\[ \text{ei}_+(\lambda(-x)) = H(-x)\text{ei}(\lambda(-x)) = \text{ei}_-((-\lambda)x), \]  
\tag{14}
\[ \text{ei}_-(\lambda(-x)) = H(x)\text{ei}(\lambda(-x)) = \text{ei}_+((-\lambda)x). \]  
\tag{15}
These results will be used to deduce results for \( \lambda < 0 \) from results proved for \( \lambda > 0 \).

The classical definition of the convolution product of two functions \( f \) and \( g \) is as follows:

**Definition 1.** Let \( f \) and \( g \) be functions. Then the convolution product \( f \ast g \) is defined by

\[
(f \ast g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t) \, dt
\]

for all points \( x \) for which the integral exist.

It follows easily from the definition that if \( f \ast g \) exists then \( g \ast f \) exists and

\[
f \ast g = g \ast f
\]

and if \( (f \ast g)' \) and \( f \ast g' \) (or \( f' \ast g \)) exists, then

\[
(f \ast g)' = f \ast g' \quad \text{(or } f' \ast g).\]

**Definition 2.** Let \( f \) and \( g \) be distributions in \( \mathcal{D}' \). Then the convolution product \( f \ast g \) is defined by the equation

\[
\langle (f \ast g)(x), \phi \rangle = \langle f(y), \langle g(x), \phi(x+y) \rangle \rangle
\]

for arbitrary \( \phi \) in \( \mathcal{D} \), provided \( f \) and \( g \) satisfy either of the conditions

(a) either \( f \) or \( g \) has bounded support,

(b) the supports of \( f \) and \( g \) are bounded on the same side.

It follows that if the convolution product \( f \ast g \) exists by this definition then equations (16) and (17) are satisfied. In the following, the locally summable functions \( e^{\lambda x}_+ \) and \( e^{\lambda x}_- \) are defined for \( \lambda \neq 0 \) by

\[
e^{\lambda x}_+ = H(x)e^{\lambda x} \quad e^{\lambda x}_- = H(-x)e^{\lambda x}.
\]

Note that

\[
e^{\lambda(-x)} = e^{(-\lambda)x}, \quad e^{\lambda(-x)}_+ = e^{(-\lambda)x}_-, \quad e^{\lambda(-x)}_- = e^{(-\lambda)x}_+.
\]

These results will also be used to deduce results for \( \lambda < 0 \) from results proved for \( \lambda > 0 \).

We now prove the following theorem.
Theorem 1. If \( \lambda \neq 0 \) and \( \mu \neq 0 \), then the convolution product \( e_+(\lambda x) * e_+^{\mu x} \) exists and
\[
e_+(\lambda x) * e_+^{\mu x} = \mu^{-1} \left\{ e_+^{\mu x} \left[ (\lambda + \mu)x \right] + \ln |1 + \mu/\lambda|e_+^{\mu x} - e_+(\lambda x) \right\} \quad (19)
\]
if \( \lambda + \mu \neq 0 \) and
\[
e_+(\lambda x) * e_-^{\lambda x} = \lambda^{-1} \left[ e_+(\lambda x) + (\gamma + \ln |\lambda|)e_-^{\lambda x} + e^{-\lambda x} \ln x_+ \right]. \quad (20)
\]
if \( \lambda + \mu = 0 \).

Proof. The convolution product \( e_+(\lambda x) * e_+^{\mu x} = 0 \) if \( x < 0 \) and so we suppose that \( x > 0 \). There are four cases to consider to prove equation (19).

Case (i). \( \lambda > 0 \), \( \lambda + \mu > 0 \).

We first of all prove that
\[
e_+(\lambda x) * e_+^{\mu x} = \mu^{-1} e_+^{\mu x} \int_0^x u^{-1} [e^{-\lambda u} - e^{-(\lambda + \mu)u}] du +
+ \mu^{-1} (e^{\mu x} - 1) e_+(\lambda x). \quad (21)
\]
We have
\[
e_+(\lambda x) * e_+^{\mu x} = \int_0^x e^{\mu(x-t)} \int_0^\infty u^{-1} e^{-\lambda u} du dt
= \int_0^x u^{-1} e^{-\lambda u} \int_0^u e^{\mu(x-t)} dt du + \int_x^\infty u^{-1} e^{-\lambda u} \int_0^x e^{u(x-t)} dt du
= \mu^{-1} e_+^{\mu x} \int_0^x u^{-1} [e^{-\lambda u} - e^{-(\lambda + \mu)u}] du +
+ \mu^{-1} (e^{\mu x} - 1) e_+(\lambda x),
\]
giving equation (21).

Further,
\[
\int_0^x u^{-1} [e^{-\lambda u} - H(1 - \lambda u)] du = \int_0^\infty u^{-1} [e^{-\lambda u} - H(1 - \lambda u)] du +
- \int_x^\infty u^{-1} e^{-\lambda u} du + \int_x^\infty u^{-1} H(1 - \lambda u) du
= -\gamma - e_+(\lambda x) + \int_x^\infty u^{-1} H(1 - \lambda u) du. \quad (22)
\]
Similarly
\[
\int_0^x u^{-1} [e^{-(\lambda + \mu)u} - H(1 - (\lambda + \mu)u)] du = -\gamma - e_+[(\lambda + \mu)x] +
+ \int_x^\infty u^{-1} H[1 - (\lambda + \mu)u] du. \quad (23)
\]
It follows from equations (22) and (23) that
\[
\int_0^x u^{-1}(e^{-\lambda u} - e^{-(\lambda + \mu)u}) \, du = e^{i\mu}[(\lambda + \mu)x] - e^{i\lambda}x + \\
+ \int_0^\infty u^{-1}[H(1 - \lambda u) - H(1 - (\lambda + \mu)u)] \, du \\
= e^{i\mu}[(\lambda + \mu)x] - e^{i\lambda}x + \ln(1 + \mu / \lambda). \tag{24}
\]
Equation (19) now follows from equations (21) and (24) for Case (i).

Case (ii). \( \lambda > 0, \ \lambda + \mu < 0 \).

Equations (21) and (22) again hold in this case but since \( \lambda + \mu < 0 \), we have from equation (8)
\[
\int_0^x u^{-1}(e^{-(
\lambda + \mu)u} - 1) \, du = -\gamma - \ln |\lambda + \mu| - e^{i\mu}[(\lambda + \mu)x] - \ln x_+. \tag{25}
\]
It follows from equations (22) and (25) that equation (24) again holds. Equation (19) follows for Case (ii).

Case (iii). \( \lambda < 0, \ \lambda + \mu < 0 \).

This time we have
\[
e^{i\lambda}x \ast e^{i\mu} = -\gamma + \ln |\lambda| \int_0^x e^{\mu t} dt - \int_0^x e^{i\lambda(x-t)} \int_0^t u^{-1}(e^{-\lambda u} - 1) \, du \, dt + \\
- \int_0^x e^{i\lambda(x-u)} \ln u \, du \\
= -\mu^{-1}(\gamma + \ln |\lambda|)(e^{\mu x} - 1) - \int_0^x u^{-1}(e^{-\lambda u} - 1) \int_0^x e^{i\mu(x-t)} \, dt \, du + \\
+ \mu^{-1}e^{i\mu x} \int_0^x \ln u \, d(e^{-\mu u} - 1) \\
= -\mu^{-1}(\gamma + \ln |\lambda|)(e^{\mu x} - 1) + \\
+ \mu^{-1}e^{i\mu x} \int_0^x u^{-1}(e^{-\lambda u} - 1)(1 - e^{\mu(x-u)}) \, du + \\
+ \mu^{-1}(1 - e^{i\mu x}) \ln x - \mu^{-1}e^{i\mu x} \int_0^x u^{-1}(e^{-\mu u} - 1) \, du \\
= -\mu^{-1}e^{i\mu}[(\lambda x) - \mu^{-1}e^{i\mu x} \int_0^x u^{-1}(e^{-(\lambda + \mu)u} - e^{-\mu u}) \, du + \\
- \mu^{-1}(\gamma + \ln |\lambda|)e^{\mu x} - \mu^{-1}e^{i\mu x} \ln x - \mu^{-1}e^{i\mu x} \int_0^x u^{-1}(e^{-\mu u} - 1) \, du \\
= -\mu^{-1}e^{i\mu}[(\lambda x) - \mu^{-1}e^{i\mu x} \int_0^x u^{-1}(e^{-(\lambda + \mu)u} - 1) \, du + \\
- \mu^{-1}(\gamma + \ln |\lambda|)e^{\mu x} - \mu^{-1}e^{i\mu x} \ln x. \tag{26}
\]
and equation (19) follows for Case (iii).
Case (iv). \( \lambda < 0, \lambda + \mu > 0 \).

Equation (26) still holds for this case but this time we have
\[
\int_0^x u^{-1}(e^{-\lambda u} - 1) du = \int_0^\infty u^{-1}[e^{-(\lambda+\mu)u} - H(1-(\lambda+\mu)u)] du + \int_x^\infty u^{-1}e^{-(\lambda+\mu)u} du + \int_x^{(\lambda+\mu)^{-1}} u^{-1} du = \gamma - e^{i\lambda}(\lambda + \mu) - \ln((\lambda + \mu)x)
\]
and equation (19) now follows from this equation and equation (26) for Case (iv).

We now have a further two cases to consider when \( \lambda + \mu = 0 \).

Case (v). \( \lambda > 0, \lambda + \mu = 0 \).

Equation (21) holds for this case. Further, replacing \( \lambda + \mu \) by \( \mu \) in equation (25) we have
\[
\int_0^x u^{-1}(e^{-\lambda u} - 1) du = -\gamma - e^{i\lambda}(\lambda x) - \ln(\lambda x)
\]
and equation (20) now follows from equation (21) for Case (v).

Case (vi). \( \lambda < 0, \lambda + \mu = 0 \).

Equation (26) holds when \( \mu = -\lambda \) but it reduces to
\[
e^{i\lambda}(\lambda x) * e_{-\lambda}^{-\lambda} = \lambda e^{i\lambda}(\lambda x) + \lambda^{-1}(\gamma + \ln|\lambda|)e^{-\lambda x} + e^{i\lambda}(-\lambda x) \ln x
\]
and equation (20) follows for Case (vi).

Corollary 1.1. If \( \lambda \neq 0 \) and \( \mu \neq 0 \), then the convolution product \( (e^{-\lambda x}x^{-s}) * e_{+}^{\mu x} \) exists for \( s = 1, 2, \ldots \). In particular, if \( \lambda + \mu \neq 0 \), then
\[
(e^{-\lambda x}x^{-1}) * e_{+}^{\mu x} = -e^{i\mu x} e^{i\lambda}(\lambda x) + (\gamma + \ln|\lambda + \mu|)e_{+}^{\mu x}
\]
and if \( \lambda + \mu = 0 \), then
\[
(e^{-\lambda x}x^{-1}) * e_{+}^{-\lambda} = e^{-\lambda x} \ln x. \tag{29}
\]

Proof. The convolution product \( (e^{-\lambda x}x^{-s}) * e_{+}^{\mu x} \) exists by Definition 2 for \( s = 1, 2, \ldots \) since \( e^{-\lambda x}x^{-s} \) and \( e_{+}^{\mu x} \) are both bounded on the left. In particular, we have from equations (11), (17) and (19)
\[
[-e^{-\lambda x}x^{-1} - (\gamma + \ln|\lambda|)\delta(x)] * e_{+}^{\mu x} = e^{i\lambda}(\lambda x) * [\mu e_{+}^{\mu x} + \delta(x)] = e^{i\lambda}(\lambda x) * [(\lambda + \mu)x] + \ln|1 + \mu/\lambda|e_{+}^{\mu x}
\]
and equation (28) follows.

Similarly, using equations (11), (17) and (20), we have
\[-e^{-\lambda x} x_+^{-1} - (\gamma + \ln |\lambda|) \delta(x) \ast e_+^\lambda x = e_+^\lambda (\lambda x) \ast \left[ -\lambda e_+^\lambda x + \delta(x) \right]
\]
\[= - e_+^\lambda (\lambda x) - (\gamma + \ln |\lambda|) e_+^\lambda x - e^{-\lambda x} \ln x_+ + e_+^\lambda (\lambda x)
\]
and equation (29) follows. □

**Theorem 2.** If \( \lambda \neq 0 \) and \( \mu \neq 0 \), then the convolution product \( e_{i-}^\lambda (\lambda x) \ast e_{i-}^{\mu x} \)
exists and
\[e_{i-}^\lambda (\lambda x) \ast e_{i-}^{\mu x} = -\mu^{-1} \{ e^{\mu x} e_{i-}^\lambda [(\lambda + \mu)x] + \ln(1 + \mu/\lambda)e_{i-}^{\mu x} - e_{i-}^\lambda (\lambda x) \} \] (30)
if \( \lambda + \mu \neq 0 \), and
\[e_{i-}^\lambda (\lambda x) \ast e_{i-}^{-\lambda x} = -\lambda^{-1} \{ e_{i-}^\lambda (\lambda x) + (\gamma + \ln |\lambda|) e_{i-}^{-\lambda x} + e^{-\lambda x} \ln x_- \} \] (31)
if \( \lambda + \mu = 0 \).

**Proof.** Replacing \( \lambda \) by \(-\lambda \) and \( \mu \) by \(-\mu \) in equation (19) we get
\[e_{i+}^\lambda ((-\lambda)x) \ast e_{i+}^{(-\mu)x} = -\mu^{-1} \{ e^{(-\mu)x} e_{i+}^\lambda [(-\lambda + \mu)x] + \ln(1 + \mu/\lambda)e_{i+}^{(-\mu)x} + e_{i+}^\lambda ((-\lambda)x) \}
\]
and equation (30) follows on replacing \( x \) by \(-x \) in this equation.

Equation (31) follows similarly. □

**Corollary 2.1.** If \( \lambda \neq 0 \) and \( \mu \neq 0 \), then the convolution product \( e_{i-}^{-\lambda x} x_s^{-1} \ast e_{i-}^{\mu x} \)
exists for \( s = 1, 2, \ldots \). In particular, if \( \lambda + \mu \neq 0 \), then
\[(e_{i-}^{-\lambda x} x_+^{-1}) \ast e_{i-}^{\mu x} = -e^{\mu x} e_{i-}^\lambda [(\lambda + \mu)x] - (\gamma + \ln |\lambda + \mu|) e_{i-}^{\mu x} \] (32)
and if \( \lambda + \mu = 0 \) then
\[(e_{i-}^{-\lambda x} x_+^{-1}) \ast e_{i-}^{-\lambda x} = e^{-\lambda x} \ln x_- \] (33)

**Proof.** The existence of convolution product \( e_{i-}^{-\lambda x} x_s^{-1} \ast e_{i-}^{\mu x} \) follows from equations (11), (17) and (30). In particular, we have from equations (11), (17) and (30)
\[e_{i-}^{-\lambda x} x_+^{-1} + (\gamma + \ln |\lambda|) \delta(x) \ast e_{i-}^{\mu x} = e_{i-}^\lambda (\lambda x) \ast [\mu e_{i-}^{\mu x} - \delta(x)]
\]
\[= -e^{\mu x} e_{i-}^\lambda [(\lambda + \mu)x] - \ln(1 + \mu/\lambda)e_{i-}^{\mu x}
\]
and equation (32) follows. Similarly, using equations (11), (17) and (31), we have
\[e_{i-}^{-\lambda x} x_+^{-1} + (\gamma + \ln |\lambda|) \delta(x) \ast e_{i-}^{-\lambda x} = e_{i-}^\lambda (\lambda x) \ast [-\lambda e_{i-}^{-\lambda x} - \delta(x)]
\]
\[= (\gamma + \ln |\lambda|) e_{i-}^{-\lambda x} + e^{-\lambda x} \ln x_- \]
and equation (33) follows. □
Theorem 3. If \( \lambda, \lambda + \mu > 0 \) and \( \mu \neq 0 \), then the convolution product \( \text{ei}_+(\lambda x) * e^{\mu x} \) exists and

\[
\text{ei}_+(\lambda x) * e^{\mu x} = \mu^{-1} \ln(1 + \mu/\lambda)e^{\mu x}.
\] (34)

Proof. We have

\[
\text{ei}_+(\lambda x) * e^{\mu x} = \int_0^\infty e^{\mu(x-t)} \int_0^t u^{-1} e^{-\lambda u} \, du \, dt
\]

\[
= \int_0^\infty u^{-1} e^{-\lambda u} \int_0^u e^{\mu(x-t)} \, dt \, du
\]

\[
= \mu^{-1} e^{\mu x} \int_0^\infty u^{-1} [e^{-\lambda u} - e^{-(\lambda+\mu)u}] \, du.
\]

Now

\[
\int_0^\infty u^{-1} [e^{-\lambda u} - e^{-(\lambda+\mu)u}] \, du = \int_0^\infty u^{-1} [e^{-\lambda u} - H(1 - \lambda u)] \, du +
\]

\[
- \int_0^\infty u^{-1} [e^{-(\lambda+\mu)u} - H(1 - (\lambda + \mu)u)] \, du +
\]

\[
+ \int_0^\infty u^{-1} [H(1 - \lambda u) - H(1 - (\lambda + \mu)u)] \, du
\]

\[
= -\gamma + \gamma + \ln(1 + \mu/\lambda)
\]

and equation (34) follows. \( \square \)

Note 1. Theorem 3 is equivalent to the van der Pol formula [4]

\[
\int_0^\infty e^{-px} \text{ei}(\lambda x) \, dx = p^{-1} \ln(1 + p/\lambda).
\]

Corollary 3.1. If \( \lambda, \lambda + \mu > 0 \) and \( \mu \neq 0 \), then the convolution product

\( (e^{-\lambda x}x_+^{-1}) * e^{\mu x} \) exists and

\[
(e^{-\lambda x}x_+^{-1}) * e^{\mu x} = -(\gamma + \ln|\lambda + \mu|)e^{\mu x}.
\] (35)

Proof. Differentiating equation (34) we get

\[
[-e^{-\lambda x}x_+^{-1} - (\gamma + \ln|\lambda|)\delta(x)] * e^{\mu x} = \ln(1 + \mu/\lambda)e^{\mu x}
\]

and equation (35) follows. \( \square \)

Note 2. Corollary 3.1 is equivalent to

\[
\int_{-\infty}^\infty e^{-px}x_+^{-1} \, dx = -\gamma - \ln p,
\]

due to Gel’fand and Shilov [2].
Corollary 3.2. If \( \lambda, \lambda + \mu > 0 \) and \( \mu \neq 0 \), then the convolution products \( \text{ei}_+(\lambda x) \ast e_\mu^- \) and \( (e^{-\lambda x} x_+^{-1}) \ast e_\mu^- \) exist and
\[
ei_+(\lambda x) \ast e_\mu^- = \mu^{-1} \{ \text{ei}_+(\lambda x) - \text{ei}_+(\lambda + \mu) x + \ln(1 + \mu/\lambda)e_\mu^- \} \tag{36}
\]
\[
(e^{-\lambda x} x_+^{-1}) \ast e_\mu^- = e_\mu^- \text{ei}_+[(\lambda + \mu) x] - (\gamma + \ln |\lambda + \mu|)e_\mu^- . \tag{37}
\]

Proof. Equation (36) follows from equations (19) and (34). Equation (37) then follows from equations (28) and (35). \( \square \)

Theorem 4. If \( \lambda, \lambda + \mu < 0 \) and \( \mu \neq 0 \), then the convolution product \( \text{ei}_-(\lambda x) \ast e_\mu^- \) exists and
\[
\text{ei}_-(\lambda x) \ast e_\mu^- = -\mu^{-1} \ln(1 + \mu/\lambda)e_\mu^- . \tag{38}
\]

Proof. Replacing \( \lambda \) by \( -\lambda \) and \( \mu \) by \( -\mu \) in equation (34) we get
\[
\text{ei}_+[(\lambda) x] \ast e^{-\mu x} = -\mu^{-1} \ln(1 + \mu/\lambda)e^{-\mu x}
\]
and equation (38) follows on replacing \( x \) by \( -x \) in this equation. \( \square \)

The results of the corollaries follow easily.

Corollary 4.1. If \( \lambda, \lambda + \mu < 0 \) and \( \mu \neq 0 \), then the convolution product \( (e^{-\lambda x} x_+^{-1}) \ast e_\mu^- \) exists and
\[
(e^{-\lambda x} x_+^{-1}) \ast e_\mu^- = -(\gamma + \ln |\lambda + \mu|)e_\mu^- .
\]

Corollary 4.2. If \( \lambda, \lambda + \mu < 0 \) and \( \mu \neq 0 \), then the convolution products \( \text{ei}_-(\lambda x) \ast e_\mu^+ \) and \( (e^{-\lambda x} x_+^{-1}) \ast e_\mu^+ \) exist and
\[
\text{ei}_-(\lambda x) \ast e_\mu^+ = \mu^{-1} \{ \text{ei}_+(\lambda x) - \text{ei}_+(\lambda + \mu) x + \ln(1 + \mu/\lambda)e_\mu^+ \}
\]
\[
(e^{-\lambda x} x_+^{-1}) \ast e_\mu^+ = e_\mu^+ \text{ei}_-[(\lambda + \mu) x] - (\gamma + \ln |\lambda + \mu|)e_\mu^+ .
\]

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