A CELLULAR SIMPLEX WITH PRESCRIBED NUMBERS OF POINTS IN REGIONS DETERMINED BY ITS FACETS

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(Received November 9, 1995)

Abstract. Let $P$ be a finite set of points in the 3-dimensional Euclidean space $\mathbb{R}^3$ in general position. For $x_0, x_1, x_2, x_3 \in P$, let $H^+(x_0; x_1, x_2, x_3)$ (resp. $H^-(x_0, x_1, x_2, x_3)$) denote the open half space containing $x_0$ (resp. not containing $x_0$) and bounded by the plane containing $x_1, x_2, x_3$. Further let

$$P(x_0; x_1, x_2, x_3) := P \cap H^+(x_1; x_0, x_2, x_3) \cap H^+(x_2; x_0, x_1, x_3) \cap H^+(x_3; x_0, x_1, x_2).$$

In this paper, we show the following statement: if $|P| \geq 4$, and if $k_1, k_2, k_3, k_4$ are integers with $k_1 + k_2 + k_3 + k_4 = |P| - 4$ and $k_1, k_2, k_3, k_4 \leq \frac{|P| - 2}{2}$ and $k_1 + k_2 \leq \frac{|P| - 2}{2}$, then for any $p_1, p_2 \in P (p_1 \neq p_2)$, there exist $q_1, q_2 \in P$ such that the convex hull of $\{p_1, p_2, q_1, q_2\}$ is a 3-simplex (tetrahedron) containing no point of $P$ in its interior and such that

$$|P(p_1; p_2, q_1, q_2)| \leq k_1 \leq P \cap H^-(p_1; p_2, q_1, q_2),$$
$$|P(p_2; p_1, q_1, q_2)| \leq k_2 \leq P \cap H^-(p_2; p_1, q_1, q_2),$$
$$|P(q_1; q_2, p_1, p_2)| \leq k_3 \leq P \cap H^-(q_1; q_2, p_1, p_2),$$
$$|P(q_2; q_1, p_1, p_2)| \leq k_4 \leq P \cap H^-(q_2; q_1, p_1, p_2).$$

AMS 1991 Mathematics Subject Classification. 52A37.

Key words and phrases. Finite set of points in the Euclidean space, simplex, affine flat.

§1. Introduction.

For a subset $V$ of the $d$-dimensional Euclidean space $\mathbb{R}^d$, let $\text{conv}(V)$ denote the convex hull of $V$, and let $\text{aff}(V)$ denote the affine flat spanned by $V$. For $d + 1$ points $x_0, x_2, \cdots, x_d$ not lying in the same (affine) $(d - 1)$-flat in $\mathbb{R}^d$,
Let $H^+(x_0; x_1, \ldots, x_d)$ (resp. $H^-(x_0; x_1, \ldots, x_d)$) denote the open half-space which is bounded by $\text{aff}(\{x_1, \ldots, x_d\})$ and contains $x$ (resp. does not contain $x$). Now let $P$ be a fixed set of points in $\mathbb{R}^d$. We say that $P$ is in general position if no $d + 1$ points of $P$ lie in the same $(d - 1)$-flat. For $d + 1$ points $x_0, x_1, \ldots, x_d$ not lying in the same $(d - 1)$-flat, let

$$P(x_0; x_1, \ldots, x_d) := P \cap \bigcap_{1 \leq i \leq d} H^+(x_i; x_0, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d).$$

If a subset $V$ of $\mathbb{R}^3$ contains no point of $P$ in its interior, $V$ is said to be vacuum. Further, following Kupitz\[2\], we call a polyhedron $D$ cellular if $D$ is vacuum and all vertices of $D$ are points of $P$. In this paper, we show the following theorem as a 3-dimensional version of Lemma 3 in [1]:

**Theorem 1.** Let $P$ be a finite set of points in $\mathbb{R}^3$ in general position. Suppose that $|P| \geq 4$, and let $k_1, k_2, k_3, k_4$ be integers such that $k_1 + k_2 + k_3 + k_4 = |P| - 4$, $0 \leq k_1, k_2, k_3, k_4 \leq \frac{|P| - 2}{2}$ and $k_1 + k_2 \leq \frac{|P| - 2}{2}$. Further let $p_1, p_2$ be specified points of $P$ with $p_1 \neq p_2$. Then there exist two points $q_1, q_2$ of $P$ such that $\text{conv}(\{p_1, p_2, q_1, q_2\})$ is a cellular 3-simplex and the following inequalities hold:

$$
\begin{align*}
(1.1) & \quad |P(p_1; p_2, q_1, q_2)| \leq k_1 \leq |P \cap H^-(p_1; p_2, q_1, q_2)|, \\
(1.2) & \quad |P(p_2; p_1, q_1, q_2)| \leq k_2 \leq |P \cap H^-(p_2; p_1, q_1, q_2)|, \\
(1.3) & \quad |P(q_1; q_2, p_1, p_2)| \leq k_3 \leq |P \cap H^-(q_1; q_2, p_1, p_2)|, \\
(1.4) & \quad |P(q_2; q_1, p_1, p_2)| \leq k_4 \leq |P \cap H^-(q_2; q_1, p_1, p_2)|.
\end{align*}
$$

S2. Proof of Theorem 1.

Let $P, k_1, k_2, k_3, k_4, p_1, p_2$ be as in Theorem 1. Let $\pi: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the orthogonal projection in the direction of $\overrightarrow{p_1p_2}$. We use the following result in the plane case (a slight modification of Claim 1 in [1]):

**Proposition 1.** Let $P'$ be a finite set of points in $\mathbb{R}^2$, and let $r_0'$ be a specified point of $P'$. Suppose that $P' \geq 3$ and any line passing through $r_0'$ contains at most one point of $P'$ other than $r_0'$. Let $k_1', k_2', k_3'$ be integers satisfying $0 \leq k_1', k_2', k_3' \leq \frac{|P'|-1}{2}$ and $k_1' + k_2' + k_3' = |P'|-3$. Then there exist $x' \in \mathbb{R}^2 - P$ and $r_1', r_2' \in P' - \{r_0'\}$ such that

$$P' = \{r_0', r_1', r_2'\} \cup P'(r_0'; x', r_1') \cup P'(r_0'; r_1', r_2') \cup P'(r_0'; r_2', x')$$

and

$$|P'(r_0'; x', r_1')| = k_1', \quad |P'(r_0'; r_1', r_2')| = k_2', \quad |P'(r_0'; r_2', x')| = k_3'.$$
Since \(k_1 + k_2 \leq \frac{|P| - 2}{2}\), we can apply Proposition 1 to \(\pi(P) = \{\pi(p) \mid p \in P\}\) with \(r'_0 = \pi(p_1) = \pi(p_2)\) and \(k'_1 = k_3, k'_2 = k_1 + k_2, k'_3 = k_4\). Let \(x', r'_1, r'_2\) be as in the conclusion of the Proposition 1. We use the same technique as in the proof of Lemma 3 in [1]. Let \(l_0\) be a line passing through \(r'_0\) and \(x'\). Take \(s'_1, s'_2 \in P(r'_0; r'_1, r'_2) \cup \{r'_1, r'_2\}\) so that for \(i = 1, 2\), \(s'_i\) lies in the same side of \(l_0\) as \(r'_i\), and

\[
(2.5) \quad \text{the line segment } s'_1s'_2 \text{ is an edge of } \text{conv}(P(r'_0; r'_1, r'_2) \cup \{r'_1, r'_2\}) \text{ satisfying } \text{conv}\{r'_0, s'_1, s'_2\} \cap H^-(r'_0; s'_1, s'_2) = \emptyset.
\]

Now we return to \(\mathbb{R}^3\). Let \(x, r_i, s_i (i = 1, 2)\) be the points of \(P\) such that \(x, \pi(r_i) = r'_i, \pi(s_i) = s'_i\), respectively. Let

\[
K_1 := H^+(x; r_1, p_1, p_2) \cap H^+(r_1; x, p_1, p_2),
K_2 := H^+(r_1; r_2, p_1, p_2) \cap H^+(r_2; r_1, p_1, p_2),
K_3 := H^+(r_2; x, p_1, p_2) \cap H^+(x; r_2, p_1, p_2).
\]

Then the conclusion of Proposition 1 implies that \(K_i \cap K_j = \emptyset\) if \(i \neq j\), and

\[
(2.6) \quad |P \cap K_1| = k'_1 = k_3,
(2.7) \quad |P \cap K_2| = k'_2 = k_1 + k_2,
(2.8) \quad |P \cap K_3| = k'_3 = k_4.
\]

Let \(H_0 := \pi^{-1}(l_0)\) and let \(S = (P \cap K_2) \cup \{r_1, r_2\}\). By (2.5),

\[
\Delta := H^+(r'_1; r'_0, r'_2) \cap H^+(r'_2; r'_0, r'_1) \cap H^+(r'_0, r'_1, r'_2)
\]

is vacuum. Since \(K_2 \cap H^+(p_1; p_2, s_1, s_2) \cap H^+(p_2; p_1, s_1, s_2) \subset \pi^{-1}(\Delta)\), this implies that \(S \cap H^+(p_1; p_2, s_1, s_2) \cap H^+(p_2; p_1, s_1, s_2) = \emptyset\). Thus by (2.7), \(|S \cap H^+(p_2; p_1, s_1, s_2)| \leq k_1 + 2\) or \(|S \cap H^+(p_1; p_2, s_1, s_2)| \leq k_2 + 2\) holds. By symmetry, we may assume

\[
(2.9) \quad |S \cap H^+(p_2; p_1, s_1, s_2)| \leq k_1 + 2.
\]

For a plane \(H\) and a point \(x \notin H\), let \(H^+(x)\) (resp. \(H^-(x)\)) denote the open (resp. closed) half-space which is bounded by \(H\) and contains \(x\), and let \(H^{-}(x)\) (resp. \(H^-(x)\)) denote the open (resp. closed) half-space which is bounded by \(H\) and does not contain \(x\). Let \(H_1\) be a plane containing \(p_1\) such that

\[
(2.10) \quad |S \cap H_1^+(p_2)| = k_1 + 2,
(2.11) \quad S \cap H_1^+(p_2) \cap H_0^+(r_i) \neq \emptyset \text{ for } i = 1, 2.
\]
Note that by (2.9), there exists a plane satisfying (2.10) and (2.11). We choose \( H_3 \) so that the angle between \( H_0 \cap H_1 \cap K_2 \) and \( \overline{p_1p_2} \) is as small as possible. Take \( q_1, q_2 \) so that

\[
q_1 \in S \cap H_1^+(p_2) \cap H_0^+(r_1),
\]
\[
q_2 \in S \cap H_1^+(p_2) \cap H_0^+(r_2),
\]
and

\[
\triangle p_2q_1q_2 \text{ is a facet of } \text{conv}\left((S \cup \{p_2\}) \cap H_1^+(p_2)\right) \text{ satisfying } \text{conv}\left(\{p_1, p_2, q_1, q_2\}\right) \cap H^-(p_1; p_2, q_1, q_2) = \emptyset.
\]

By (2.14), \( \text{conv}\left(\{p_1, p_2, q_1, q_2\}\right) \) is vacuum. We now proceed to verify the inequalities in the conclusion of Theorem 1. By (2.12) and (2.13),

\[
P(q_1; q_2, p_1, p_2) \subseteq P \cap K_1 \subseteq P \cap H^-(q_1; q_2, p_1, p_2) \quad \text{and}
\]
\[
P(q_2; q_1, p_1, p_2) \subseteq P \cap K_3 \subseteq P \cap H^-(q_2; q_1, p_1, p_2)
\]
hold, and hence (2.6), (2.8) imply (1.3), (1.4), respectively. Similarly by (2.14),

\[
P(p_1; p_2, q_1, q_2) \subseteq S \cap H_1^+(p_2) - \{q_1, q_2\} \subseteq P \cap H^-(p_1; p_2, q_1, q_2)
\]
holds, and hence (2.10) implies (1.1). Further, it also follows from the choice of \( q_1, q_2 \) that

\[
P(p_2; p_1, q_1, q_2) \subseteq S \cap H_1^-(p_2).
\]

Since

\[
|S \cap H_1^-(p_2)| = (k_1 + k_2 + 2) - (k_1 + 2) = k_2
\]
by (2.7) and (2.10), this immediately implies the first inequality in (1.2).

We are now left with the verification of the second inequality in (1.2). Suppose

\[
|P \cap H^-(p_2; p_1, q_1, q_2)| < k_2.
\]

Then clearly

\[
|S \cap H^-(p_2; p_1, q_1, q_2)| < k_2.
\]

On the other hand, by (2.7) and (2.9),

\[
|S \cap H^-(p_2; p_1, s_1, s_2)| \geq (k_1 + k_2 + 2) - (k_1 + 2) = k_2
\]
holds. Let \( y, z \) be the intersection points of the line passing through \( s_1, s_2 \) and \( \text{aff}(\{p_1, p_2, r_1\}), \text{aff}(\{p_1, p_2, r_2\}) \), respectively. Then (2.15) and (2.16) imply that \( S \cap H^-(p_2; p_1, s_1, s_2) \not\subseteq S \cap H^-(p_2; p_1, q_1, q_2) \), which implies that at least one of \( y, z \) belongs to \( H^+(p_2; p_1, q_1, q_2) \). We may assume

\[
y \in H^+(p_2; p_1, q_1, q_2)
\]
without loss of generality. We now show the existence of a plane containing \( p_0 \) which gives rise to a contradiction to the choice of \( H_1 \). Toward this end, we divide the situation into two cases according to the location of \( q_2 \).
Case 1 \( q_2 = s_2 \) or \( q_2 \in H^+(p_2; p_1, s_1, s_2) \)
In this case,
\[
S \cap H^-(p_2; p_1, y, q_2) \supseteq S \cap H^-(p_2; p_1, s_1, s_2)
\]
holds and hence by (2.16),
\[
|S \cap H^-(p_2; p_1, y, q_2)| \geq |S \cap H^-(p_2; p_1, s_1, s_2)| \geq k_2.
\]

Let \( l_1 \) be the line passing through \( p_1, q_2 \), and let \( H \) be a (movable) plane containing \( l_1 \). If we gradually rotate \( H \) with \( l_1 \) as the axis, the value of \( |S \cap H^-(p_2)| \) changes by one at each moment when \( H \) hits a point of \( P \). Therefore by (2.15) and (2.18), there exists \( H_2 \in l_1 \cup H^+(p_2, p_1, q_1, q_2) \cup H^+(p_2, p_1, y, q_2) \) such that \( l_1 \in H_2 \) and \( |S \cap H^+_2(p_2)| = k_2 \), or equivalently, \( |S \cap H^+_2(p_2)| = k_1 + 2 \).

Now to get a contradiction, we let \( K'_2 := H^+(q_1; q_2, p_1, p_2) \cap H^+(q_2; q_1, p_1, p_2) \) (note that by (2.12) and (2.13), \( H_0 \) intersects with \( K'_2 \)). Then by (2.17), it is easy to see that
\[
H_0 \cap K'_2 \cap H^+_2(p_2) \subset H_0 \cap K'_2 \cap H^+(p_2; p_1, q_1, q_2)
\]
\[
\subseteq H_0 \cap K'_2 \cap H^+_1(p_2),
\]
which yields a contradiction to the minimality of the angle between \( H_0 \cap H_1 \cap K_2 \) and \( p_1p_2 \).

Case 2 \( q_2 \in H^-(p_2; p_1, s_1, s_2) \)
If (2.18) holds, a contradiction can be derived in the same way as in Case 2. Thus we may assume
\[
|S \cap H^-(p_2; p_1, y, q_2)| < k_2.
\]

Let \( l_2 \) be the line passing through \( p_1, y \). Then again as in Case 1, (2.16) and (2.19) imply that we can find a plane \( H_3 \in l_2 \cup H^+(p_2, p_1, y, q_2) \cup H^+(p_2, p_1, s_1, s_2) \) such that \( l_2 \in H_3 \) and \( |S \cap H^+_3(p_2)| = k_1 + 2 \) by considering the rotation of a plane containing \( l_2 \) with \( l_2 \) as the axis. Thus again it is easy to see that
\[
H_0 \cap K'_2 \cap H^+_3(p_2) \subset H_0 \cap K'_2 \cap H^+(p_2; p_1, q_1, q_2)
\]
\[
\subseteq H_0 \cap K'_2 \cap H^+_1(p_2),
\]
which yields a contradiction. \( \square \)
Acknowledgment

The author is grateful to Professor Yoshimi Egawa for his valuable remarks and helpful advice.

References


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