A periodic projective bimodule resolution of an algebra associated with a cyclic quiver and a separable algebra, and the Hochschild cohomology ring

Manabu Suda

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Abstract. Let ∆ be a separable algebra over a commutative ring $R$ and $f(x)$ a monic polynomial over the center of ∆. We deal with the $R$-algebra $Λ = \Delta \Gamma / (f(X^s))$, where $\Delta \Gamma$ is the path algebra of the cyclic quiver $\Gamma$ with $s$ vertices and $s$ arrows, and $X$ is the sum of all arrows. We show that $Λ$ has a periodic projective bimodule resolution of period 2. Moreover, by using the resolution, we describe the structure of the Hochschild cohomology ring of $Λ$ by means of the Yoneda product.

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§1. Introduction

The Hochschild cohomology rings of path algebras of an oriented cyclic quiver with relations have been studied by some authors. Let $A$ be the algebra $K\Gamma / (h(X))$ over a commutative ring $K$, where $K\Gamma$ is the path algebra of the oriented cyclic quiver $\Gamma$ with $s$ vertices and $s$ arrows, $h(x)$ is a monic polynomial over $K$ and $X$ is the sum of all arrows in $K\Gamma$. If $K$ is a field and $h(x) = x^k$ for an integer $k \geq 2$, then $A = K\Gamma / (X^k)$ is a basic self-injective Nakayama algebra and the Hochschild cohomology ring of the algebra is determined by Erdmann and Holm [EH]. Also, if $s = 1$, then $A$ is equal to $K[x] / (h(x))$ and the structure of the Hochschild cohomology ring of $A$ is described by Holm [H]. Furthermore, if $s \geq 2$ and $h(x) = f(x^s)$ with a monic polynomial $f(x)$ over $K$, then the Hochschild cohomology ring of $A = K\Gamma / (f(X^s))$ is determined by Furuya and Sanada [FS].
On the other hand, $\Delta \Gamma / (X^s - \alpha)$, a path algebra over a noncommutative ring $\Delta$ with a relation, is isomorphic to a subalgebra $B = \Delta [E_{11}, E_{22}, \ldots, E_{ss}, C]$ of the full matrix ring $M_s(\Delta)$ (see Lemma 6.1). We are interested in the Hochschild cohomology for a class of matrix algebras including the above $B$ and basic hereditary orders which we studied in [SS]. Thus we will consider a general case that the coefficient rings of path algebras are noncommutative.

In this paper, we deal with the algebra $\Lambda = \Delta \Gamma / (f(X^s))$ over $R$, where $\Delta$ is a separable algebra over a commutative ring $R$, $s$ a positive integer and $\Gamma$ the oriented cyclic quiver with $s$ vertices $e_1, e_2, \ldots, e_s$ and $s$ arrows $a_1, a_2, \ldots, a_s$ such that $a_i$ starts at $e_i$ and ends at $e_{i+1}$. We consider the path algebra $\Delta \Gamma := \Delta \otimes_R \Lambda$ and are used to give the resolution of $\Lambda$, where $\Lambda^e$ denotes the enveloping algebra of $\Lambda$. In Section 3, by using the $\Lambda^e$-projective modules, we construct a periodic $\Lambda^e$-projective resolution of period 2 of $\Lambda$ (Theorem 3.2). In Section 4, we compute the Hochschild cohomology groups of $\Lambda$. The complex which is obtained by the $\Lambda^e$-projective resolution and is used to give the Hochschild cohomology groups of $\Lambda$ has a difference between the case $s \geq 2$ and the case $s = 1$. Hence, we deal with the case $s \geq 2$ in Section 4.2 (Theorem 4.4) and the case $s = 1$ in Section 4.3 (Theorem 4.5). In Section 5, we describe the structure of the Hochschild cohomology ring of $\Lambda$ by means of the Yoneda product. We deal with the case $s \geq 2$ in Section 5.1 (Theorems 5.2 and 5.4) and the case $s = 1$ in Section 5.2 (Theorems 5.11 and 5.13). In Section 6, we give some applications (Propositions 6.2 and 6.3).

We remark that if $\Delta = R$ then the results of Propositions 6.2 and 6.3 coincide with [KSS, Theorem 1.1] and [H, Theorem 7.1], respectively.

§2. Preliminaries

Let $\Delta$ be an algebra over a commutative ring $R$, $s$ a positive integer and $\Gamma$ the oriented cyclic quiver with $s$ vertices $e_1, e_2, \ldots, e_s$ and $s$ arrows $a_1, a_2, \ldots, a_s$ such that $a_i$ starts at $e_i$ and ends at $e_{i+1}$. We consider the path algebra $\Delta \Gamma := \Delta \otimes_R \Lambda$ over $R$, where $\Lambda$ is the path algebra of $\Gamma$ over $R$. Hence $a_i = e_{i+1} a_i e_i$ holds for each $1 \leq i \leq s$, where the subscripts $i$ of $e_i$ are considered to be modulo $s$. We put $X = a_1 + a_2 + \cdots + a_s$ and $f(x) = x^n + z_{n-1} x^{n-1} + \cdots + z_1 x + z_0 \in Z(\Delta)[x]$, where $f(x)$ is a monic polynomial.
over the center $Z(\Delta)$ of $\Delta$. Note that $Xe_i = e_{i+1}X$ for $1 \leq i \leq s$. In this paper, we deal with the $R$-algebra $\Lambda := \Delta \Gamma/(f(X^s))$, where $(f(X^s))$ is the two-sided ideal of $\Delta \Gamma$ generated by $f(X^s)$. Note that $f(X^s)$ is an element of $Z(\Delta \Gamma)$, so $(f(X^s)) = f(X^s)\Delta \Gamma$. Thus we have $\Lambda = \bigoplus_{i=1}^{s} \bigoplus_{k=0}^{ns-1} \Delta X^k e_i$ and rank$_\Lambda \Lambda = ns^2$. We identify $\Lambda$ with $\Delta[x]/(f(x))$ in the case $s = 1$.

Throughout the paper, we denote $\otimes_R$ by $\otimes$ and the enveloping algebra $\Lambda \otimes \Lambda$ of $\Lambda$ by $\Lambda^e$. We assume that $\Delta$ is a separable $R$-algebra which is projective as an $R$-module from now on. Then $\Delta$ is a finitely generated $R$-module. If $s = 1$ and $n = 1$ then $\Lambda = \Delta$ has trivial cohomology, so we assume $n \geq 2$ in the case $s = 1$.

It is well known that $\Delta$ is a separable $R$-algebra if and only if there exist $(x_\nu)_{1 \leq \nu \leq m}$ and $(y_\nu)_{1 \leq \nu \leq m}$ in $\Delta$ such that

\begin{equation}
\sum_{\nu=1}^{m} x_\nu y_\nu = 1 \tag{2.1}
\end{equation}

and

\begin{equation}
\sum_{\nu=1}^{m} (ax_\nu) \otimes y_\nu^o = \sum_{\nu=1}^{m} x_\nu \otimes (y_\nu a)^o \quad \text{for all } a \in \Delta. \tag{2.2}
\end{equation}

We set $\delta^e = \sum_{\nu=1}^{m} x_\nu \otimes y_\nu^o \in \Delta^e$, which is called a separating idempotent for $\Delta$ (cf. [P]). Note that $\delta^e \delta^e = \delta^e$ and $\delta^e \Delta := \{ \sum_{\nu=1}^{m} x_\nu ay_\nu \mid a \in \Delta \} = Z(\Delta)$. We regard elements in $\Delta$ as elements in $\Lambda$ by the natural embedding $\Delta \to \Lambda$. Since there exists the left $\Lambda^e$-isomorphism $\Lambda^e \xrightarrow{\sim} \Lambda \otimes \Lambda$: $a \otimes b^o \mapsto a \otimes b$, if we denote the image of $\delta^e$ by $\delta$, i.e., $\delta = \sum_{\nu=1}^{m} x_\nu \otimes y_\nu \in \Lambda \otimes \Lambda$, then

\begin{equation}
\delta a = a \delta \quad \text{for all } a \in \Delta \tag{2.3}
\end{equation}

holds by (2.2). Moreover, since $(e_i \otimes e_j^o)\delta^e$ is an idempotent for $\Lambda^e$, we have that $\Lambda^e((e_i \otimes e_j^o)\delta^e)$ is a left $\Lambda^e$-projective module for each $1 \leq i, j \leq s$, hence we can define the following left $\Lambda^e$-projective modules which are direct summands of $\Lambda \otimes \Lambda$:

\[
P_0 = \bigoplus_{i=1}^{s} \Lambda e_i \delta e_i \Lambda, \quad P_1 = \bigoplus_{i=1}^{s} \Lambda e_{i+1} \delta e_i \Lambda.
\]

Note that $P_0 = P_1 = \Lambda \delta \Lambda$ in the case $s = 1$.

§3. A periodic $\Lambda^e$-projective resolution of $\Lambda$

In this section, we will construct a periodic $\Lambda^e$-projective resolution of period 2 of $\Lambda$ by using the left $\Lambda^e$-projective modules $P_0$ and $P_1$ defined in Section 2.
Lemma 3.1. There exist the left $\Lambda^e$-homomorphisms $\phi : P_1 \to P_0$ and $\kappa : \Lambda \to P_1$ which satisfy the following:

$$
\phi(e_{i+1}\delta e_i) = e_{i+1}(X\delta - \delta X)e_i,
$$

$$
\kappa(e_i) = e_i \left( \sum_{j=1}^n z_j \left( \sum_{l=0}^{js-1} X^l \delta X^{js-l-1} \right) \right) e_i,
$$

for $1 \leq i \leq s$, where we set $z_n = 1$.

Proof. We define the left $\Lambda^e$-homomorphism $\tilde{\phi} : \Lambda \otimes \Lambda \to \Lambda \otimes \Lambda$ by $\tilde{\phi}(1 \otimes 1) = X\delta - \delta X$. Then, by (2.1), (2.3) and $Xe_i = e_{i+1}X$ for $1 \leq i \leq s$, we have

$$
\tilde{\phi}(e_{i+1}\delta e_i) = ((e_{i+1} \otimes e^*_i)\delta) \tilde{\phi}(1 \otimes 1) = ((e_{i+1} \otimes e^*_i)\delta)(X\delta - \delta X)
$$

$$
= (e_{i+1} \otimes e^*_i) \sum_{i=1}^m (Xx_\nu y_\nu - x_\nu y_\nu X)
$$

$$
= (e_{i+1} \otimes e^*_i) \left( X\delta \left( \sum_{\nu=1}^m x_\nu y_\nu \right) - \left( \sum_{\nu=1}^m x_\nu y_\nu \right) \delta X \right)
$$

$$
= e_{i+1}(X\delta - \delta X)e_i \in P_0.
$$

Hence, if we set $\tilde{\phi}|_{P_1} = \phi$ then $\phi$ is the desired left $\Lambda^e$-homomorphism.

Next, we define the left $\Lambda$-homomorphism $\kappa : \Lambda = \bigoplus_{i=1}^s \Lambda e_i \to P_1$ by

$$
\kappa(e_i) = e_i \left( \sum_{j=1}^n z_j \left( \sum_{l=0}^{js-1} X^l \delta X^{js-l-1} \right) \right) e_i,
$$

since $X^k e_i = e_{i+k} X^k$ holds for $1 \leq i \leq s$ and $k \geq 0$. We will show that $\kappa$ is a right $\Lambda$-homomorphism. First, note that $\kappa(e_i e_j) = \kappa(e_i) e_j$ for $1 \leq i, j \leq s$. Second, by (2.3), we have

$$
\kappa(e_i X) - \kappa(e_i) X = X\kappa(e_{i-1}) - \kappa(e_i) X
$$

$$
= Xe_{i-1} \left( \sum_{j=1}^n z_j \left( \sum_{l=0}^{js-1} X^l \delta X^{js-l-1} \right) \right) e_{i-1}
$$

$$
- e_i \left( \sum_{j=1}^n z_j \left( \sum_{l=0}^{js-1} X^l \delta X^{js-l-1} \right) \right) e_i X
$$

$$
= e_i \left( \sum_{j=1}^n z_j \left( \sum_{l=0}^{js-1} X^{l+1} \delta X^{js-l-1} \right) \right) e_{i-1}
$$
Hence, \( \kappa(e_iX) = \kappa(e_i)X \) holds. Finally, we show that \( \kappa(e_i\lambda) = \kappa(e_i)\lambda \) for all \( \lambda \in \Lambda \). Note that \( \kappa(ae_i) = a\kappa(e_i) = \kappa(e_i)a \) for all \( a \in \Delta \), since \( z_1, z_2, \ldots, z_{n-1}, z_n \) are elements of \( Z(\Delta) \). If we set \( \lambda = \sum_{j=1}^{s} \sum_{k=0}^{n-1} a_{jk} X^k e_j \in \Lambda \) (\( a_{jk} \in \Delta \)) then it follows that

\[
\kappa(e_i)\lambda = \kappa(e_i)e_i\lambda = \kappa(e_i) \sum_{j=1}^{s} \sum_{k=0}^{n-1} a_{jk} X^k e_{i-k}e_j = \sum_{j=1}^{s} \sum_{k=0}^{n-1} \kappa(a_{jk}e_i) X^k e_{i-k}e_j
\]

\[
= \kappa \left( \sum_{j=1}^{s} \sum_{k=0}^{n-1} a_{jk} e_i X^k e_{i-k} e_j \right) = \kappa \left( e_i \left( \sum_{j=1}^{s} \sum_{k=0}^{n-1} a_{jk} X^k e_j \right) \right) = \kappa(e_i \lambda).
\]

This completes the proof of the lemma. \( \square \)

**Theorem 3.2.** There exists the following exact sequence of left \( \Lambda^e \)-modules which is \( (\Lambda,\Delta) \)-split:

\[
0 \longrightarrow \Lambda \xrightarrow{\kappa} P_1 \xrightarrow{\phi} P_0 \xrightarrow{\pi} \Lambda \longrightarrow 0,
\]

where \( \pi : P_0 \rightarrow \Lambda \) is the multiplication map. Hence we have the periodic left \( \Lambda^e \)-projective resolution of period 2:

\[
\cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} \Lambda \longrightarrow 0,
\]

where \( d_1 \) and \( d_0 \) are left \( \Lambda^e \)-homomorphisms given by

\[
d_1(e_{i+1}\delta e_i) = \phi(e_{i+1}\delta e_i) = e_{i+1}(X\delta - \delta X)e_i,
\]

\[
d_0(e_i\delta e_i) = (\kappa\pi)(e_i\delta e_i) = e_i \left( \sum_{j=1}^{n} \sum_{l=0}^{j-1} X^l \delta X^{j-l-1} \right) e_i
\]

for \( 1 \leq i \leq s \).
To prove Theorem 3.2, we prepare the following lemmas.

**Lemma 3.3.** The sequence (3.1) is a complex of left $\Lambda^e$-modules.

**Proof.** Since $\pi(\delta) = \sum_{\nu=1}^m x_{\nu} y_{\nu} = 1$, we have

$$(\pi \phi)(e_{i+1} \delta e_i) = \pi(e_{i+1}(X \delta - \delta X)e_i) = e_{i+1}(X - X)e_i = 0$$

and

$$(\phi \kappa)(e_i) = \phi \left( e_i \left( \sum_{j=1}^n z_j \left( \sum_{l=0}^{j-1} X^l \delta X^{j-1-l} \right) \right) e_i \right)$$

$$= \phi \left( \sum_{j=1}^n z_j \left( \sum_{l=0}^{j-1} X^l e_i \delta e_i - l X^{j-1-l} \right) \right)$$

$$= \sum_{j=1}^n z_j \left( \sum_{l=0}^{j-1} X^l e_i \delta e_i - l X^{j-1-l} \right)$$

$$= e_i \left( \sum_{j=1}^n z_j \left( X^j \delta - \delta X^j \right) \right) e_i$$

$$= e_i \left( \left( \sum_{j=1}^n z_j X^j \right) \delta - \delta \left( \sum_{j=1}^n z_j X^j \right) \right) e_i$$

$$= e_i \left( (-z_0) \delta - \delta (-z_0) \right) e_i = 0$$

for $1 \leq i \leq s$. This completes the proof of the lemma. □

**Lemma 3.4.** There exist the $(\Lambda, \Delta)$-homomorphisms $h_{-1} : \Lambda \rightarrow P_0$, $h_0 : P_0 \rightarrow P_1$ and $h_1 : P_1 \rightarrow \Lambda$ which satisfy the following:

$$h_{-1}(1) = \sum_{j=1}^s e_j \delta e_j,$$

$$h_0(e_i \delta e_i X^k) = \begin{cases} 0 & \text{if } k = 0, \\ -e_i \left( \sum_{j=0}^{k-1} X^j \delta X^{k-j-1} \right) e_i \text{ if } 1 \leq k \leq ns - 1, \end{cases}$$
For $1 \leq i \leq s$, where we denote a left $\Lambda$- and right $\Delta$-homomorphism by a $(\Lambda, \Delta)$-homomorphism. Then $\{h_{-1}, h_0, h_1\}$ is a contracting homotopy of (3.1).

Proof. If we define the left $\Lambda$-homomorphism $h_{-1} : \Lambda \to P_0$ by $h_{-1}(1) = \sum_{j=1}^{s} e_j \delta e_j$, then it is clear that $h_{-1}$ is a $(\Lambda, \Delta)$-homomorphism by (2.3). Next, since $X^k e_i = e_{i+k} X^k$ holds for $1 \leq i \leq s$ and $k \geq 0$, we define the $(\Lambda, \Delta)$-homomorphisms $\tilde{h}_0 : \Lambda \otimes \Lambda \to P_1$ and $\tilde{h}_1 : \Lambda \otimes \Lambda \to \Lambda$ by

\[
\tilde{h}_0(1 \otimes e_i X^k) = \begin{cases} 
0 & \text{if } k = 0, \\
- \left( \sum_{j=0}^{k-1} X^j \delta X^{k-j-1} \right) e_{i-k} & \text{if } 1 \leq k \leq ns - 1,
\end{cases}
\]

\[
\tilde{h}_1(1 \otimes e_i X^k) = \begin{cases} 
0 & \text{if } 0 \leq k \leq ns - 2, \\
e_{i+1} & \text{if } k = ns - 1,
\end{cases}
\]

for $1 \leq i \leq s$. If we set $\tilde{h}_0|_{P_0} = h_0$ and $\tilde{h}_1|_{P_1} = h_1$, then it easily follows that $h_0$ and $h_1$ are the desired $(\Lambda, \Delta)$-homomorphisms by (2.1) and (2.3).

(1) $\pi h_{-1} = \text{id}_\Lambda$; For all $\lambda \in \Lambda$, we have

\[
(\pi h_{-1})(\lambda) = \pi \left( \lambda \left( \sum_{j=1}^{s} e_j \delta e_j \right) \right) = \lambda \left( \sum_{j=1}^{s} e_j \right) = \lambda.
\]

Hence we get the desired equation.

(2) $h_{-1} \pi + \phi h_0 = \text{id}_{P_0}$;

(a) Case $k = 0$: For $1 \leq i \leq s$, we have

\[
(h_{-1} \pi + \phi h_0)(e_i \delta e_i) = h_{-1}(e_i) + \phi(0) = e_i \left( \sum_{j=1}^{s} e_j \delta e_j \right) = e_i \delta e_i.
\]

(b) Case $1 \leq k \leq ns - 1$: For $1 \leq i \leq s$, we have

\[
(h_{-1} \pi + \phi h_0)(e_i \delta e_i X^k) = h_{-1}(e_i X^k) - \phi \left( e_i \left( \sum_{j=0}^{k-1} X^j \delta X^{k-j-1} e_{i-k} \right) \right)
\]
\[ e_i X^k \left( \sum_{j=1}^{s} e_j \delta e_j \right) - e_i \left( \sum_{j=0}^{k-1} X^j (X \delta - \delta X) X^{k-j-1} \right) e_{i-k} \]

\[ = X^k e_{i-k} \delta e_{i-k} - e_i \left( \sum_{j=0}^{k-1} (X^{j+1} \delta X^{k-j-1} - X^j \delta X^{k-j}) \right) e_{i-k} \]

\[ = e_i X^k \delta e_{i-k} - e_i (X^k \delta - \delta X^k) e_{i-k} = e_i \delta e_i X^k. \]

Hence we get the desired equation.

(3) \( h_0 \phi + \kappa h_1 = \text{id}_{P_1}; \)

(a) Case \( k = 0 \): For \( 1 \leq i \leq s \), we have

\[ (h_0 \phi + \kappa h_1)(e_{i+1} \delta e_i) = h_0(e_{i+1}(X \delta - \delta X)e_i) + \kappa(0) \]

\[ = h_0(X e_i \delta e_i - e_{i+1} \delta e_{i+1} X) = e_{i+1} \delta e_i. \]

(b) Case \( 1 \leq k \leq ns - 2 \): For \( 1 \leq i \leq s \), we have

\[ (h_0 \phi + \kappa h_1)(e_{i+1} \delta e_i X^k) \]

\[ = h_0(e_{i+1}(X \delta - \delta X)e_i X^k) + \kappa(0) = h_0(X e_i \delta e_i X^k - e_{i+1} \delta e_{i+1} X^{k+1}) \]

\[ = -X e_i \left( \sum_{j=0}^{k-1} X^j \delta X^{k-j-1} \right) e_{i-k} + e_{i+1} \left( \sum_{j=0}^{k} X^j \delta X^{k-j} \right) e_{i-k} \]

\[ = -e_{i+1} \left( \sum_{j=0}^{k-1} X^{j+1} \delta X^{k-j-1} \right) e_{i-k} + e_{i+1} \left( \sum_{j=0}^{k} X^j \delta X^{k-j} \right) e_{i-k} \]

\[ = e_{i+1} \delta X^k e_{i-k} = e_{i+1} \delta e_i X^k. \]

(c) Case \( k = ns - 1 \): For \( 1 \leq i \leq s \), we have

\[ (h_0 \phi + \kappa h_1)(e_{i+1} \delta e_i X^{ns-1}) \]

\[ = h_0(e_{i+1}(X \delta - \delta X)e_i X^{ns-1}) + \kappa(e_{i+1}) \]

\[ = h_0(X e_i \delta e_i X^{ns-1} - e_{i+1} \delta e_{i+1} X^{ns}) + \kappa(e_{i+1}) \]

\[ = -X e_i \left( \sum_{j=0}^{ns-2} X^j \delta X^{ns-j-2} \right) e_{i+1} \]

\[ + h_0 \left( e_{i+1} \delta e_{i+1} \left( \sum_{j=0}^{n-1} \tilde{z}_j X^{js} \right) \right) + \kappa(e_{i+1}) \]

\[ = -e_{i+1} \left( \sum_{j=0}^{ns-2} X^{j+1} \delta X^{ns-j-2} \right) e_{i+1} + \sum_{j=0}^{n-1} \tilde{z}_j h_0(e_{i+1} \delta e_{i+1} X^{js}) + \kappa(e_{i+1}) \]
\[
- e_{i+1} \left( \sum_{j=0}^{n-2} X^{j+1} \delta X^{n-2-j} \right) e_{i+1} + \sum_{j=1}^{n-1} z_j e_{i+1} \left( \sum_{l=0}^{j-1} X^l \delta X^{j-1} \right) e_{i+1}
\]

\[
+ e_{i+1} \left( \sum_{j=1}^{n} z_j \left( \sum_{l=0}^{j-1} X^l \delta X^{j-1} \right) \right) e_{i+1}
\]

\[
= - e_{i+1} \left( \sum_{j=0}^{n-2} X^{j+1} \delta X^{n-2-j} \right) e_{i+1} + e_{i+1} \left( \sum_{l=0}^{n-1} X^l \delta X^{n-1-l} \right) e_{i+1}
\]

\[
= e_{i+1} \delta X^{n-1} e_{i+1} = e_{i+1} \delta e_{i} X^{n-1}.
\]

Hence we get the desired equation.

(4) \( h_1 \kappa = \text{id}_\Lambda \); For \( 1 \leq i \leq s \), we have

\[
(h_1 \kappa)(e_i) = h_1 \left( e_i \left( \sum_{j=1}^{n} z_j \left( \sum_{l=0}^{j-1} X^l \delta X^{j-1} \right) \right) e_i \right)
\]

\[
= h_1 \left( \sum_{j=1}^{n} z_j \left( \sum_{l=0}^{j-1} X^l \delta e_{i-l} X^{j-1} \right) \right)
\]

\[
= h_1(e_i \delta e_{i-1} X^{n-1}) = e_i.
\]

Hence we get the desired equation.

These complete the proof of the lemma. \( \square \)

**Proof of Theorem 3.2.** We have the exact sequence (3.1) of left \( \Lambda^e \)-modules which is \( (\Lambda, \Delta) \)-split by means of Lemmas 3.3 and 3.4. Then the latter statement is clear. \( \square \)

**§4. The Hochschild cohomology groups of \( \Lambda \)**

In this section, we compute the Hochschild cohomology group \( \text{HH}^t(\Lambda) := \text{Ext}^t_{\Lambda^e}(\Lambda, \Lambda) \) of \( \Lambda \) for each \( t \geq 0 \) by means of the projective \( \Lambda^e \)-resolution (3.2). We regard \( \text{HH}^t(\Lambda) \) as a \( Z(\Lambda) \)-module. Since the resolution (3.2) is periodic of period 2, we have a \( Z(\Lambda) \)-isomorphism \( \text{HH}^{t+2}(\Lambda) \simeq \text{HH}^t(\Lambda) \) for each \( i \geq 1 \). Therefore, it suffices to compute \( \text{HH}^t(\Lambda) \) for \( t = 0, 1, 2 \).
4.1. Some lemmas

In this subsection, we give some lemmas in order to calculate the Hochschild cohomology groups of $\Lambda$.

**Lemma 4.1.** We have $Z(\Delta \Gamma) = Z(\Delta)[X^s]$. Also we have

$$Z(\Lambda) = \left( Z(\Delta)[X^s] + (f(X^s)) \right) / (f(X^s)) \cong Z(\Delta)[X^s] / \left( Z(\Delta)[X^s] \cap (f(X^s)) \right)$$

as rings, where $Z(\Delta)[X^s] \cap (f(X^s))$ is equal to the ideal of $Z(\Delta)[X^s]$ generated by $f(X^s)$. So we have $Z(\Lambda) \cong Z(\Delta)[x] / (f(x))$ as rings.

**Proof.** First, we will show $Z(\Delta \Gamma) = Z(\Delta)[X^s]$. Let

$$y = \sum_{i=1}^{s} \sum_{j=0}^{N} b_{i,j} X^j e_i \in Z(\Delta \Gamma), \quad \text{where } b_{i,j} \in \Delta \text{ and } N \geq 0.$$ 

Then we have

$$y = \sum_{i=1}^{s} \sum_{l=0}^{q} b_{1,ls} X^l s e_i, \quad \text{where } N = sq + r \text{ and } 0 \leq r \leq s - 1,$$

since $y e_p = y e_p e_p = e_p y e_p$ for $1 \leq p \leq s$. Next, we have $b_{1,ls} = b_{2,ls} = \cdots = b_{s,ls}$, since $y (X e_p) = (X e_p) y$ for $1 \leq p \leq s$. So it follows that

$$y = \sum_{i=1}^{s} \sum_{l=0}^{q} b_{1,ls} X^l s e_i = \sum_{l=0}^{q} b_{1,ls} X^l s \in \Delta[X^s].$$

Moreover, we have $b_{1,ls} \in Z(\Delta)$ for $0 \leq l \leq q$, since $ay = ya$ for all $a \in \Delta$. Hence $Z(\Delta \Gamma) \subset Z(\Delta)[X^s]$ holds. The converse inclusion follows from the fact that $X^s \in Z(\Delta \Gamma)$ and $Z(\Delta) \subset Z(\Delta \Gamma)$. Therefore we have the desired equation.

Second, we will show $Z(\Lambda) = \left( Z(\Delta)[X^s] + (f(X^s)) \right) / (f(X^s))$. Let

$$y = \sum_{i=1}^{s} \sum_{j=0}^{n s - 1} b_{i,j} X^j e_i \in Z(\Delta), \quad \text{where } b_{i,j} \in \Delta.$$

By similar calculation, we have

$$y = \sum_{l=0}^{n-1} b_{1,ls} X^l s \in (\Delta[X^s] + (f(X^s))) / (f(X^s)).$$
hence \( Z(\Lambda) \subset (Z(\Delta)[X^s] + (f(X^s)))/(f(X^s)) \). The converse inclusion follows from the fact that \( X^s \in Z(\Lambda) \) and \( (Z(\Delta) + (f(X^s)))/(f(X^s)) \subset Z(\Lambda) \). Therefore we have the desired equation. It is clear that the ring isomorphism
\[
(Z(\Delta)[X^s] + (f(X^s)))/(f(X^s)) \simeq Z(\Delta)[X^s]/(Z(\Delta)[X^s] \cap (f(X^s)))
\]
exists.

Third, let \( I \) be the ideal of \( Z(\Delta)[X^s] \) generated by \( f(X^s) \). We will show \( I = Z(\Delta)[X^s] \cap (f(X^s)) \). Since \( f(X^s) \in Z(\Delta \Gamma) \), we set
\[
y = f(X^s)v \in Z(\Delta)[X^s] \cap (f(X^s)), \quad \text{where } v \in \Delta \Gamma.
\]
Then we have \( vu = uv \) for all \( u \in \Delta \Gamma \), hence it follows that \( f(X^s)(vu - uv) = 0 \). Now we will show that \( f(X^s) \) is not a zero divisor in \( \Delta \Gamma \). Let
\[
0 \neq w = \sum_{i=1}^{s} \sum_{j=0}^{N} b_{i,j}X^i e_i \in \Delta \Gamma, \quad \text{where } b_{i,j} \in \Delta \text{ and } N \geq 0,
\]
i.e., \( b_{i_0,N} \neq 0 \) for some \( 1 \leq i_0 \leq s \). If \( f(X^s)w = 0 \), then \( b_{i_0,N} = 0 \) since \( f(X^s)w e_{i_0} = 0 \). This contradicts the assumption. So \( f(X^s) \) is not a zero divisor. Hence we have \( vu = uv \) for all \( u \in \Delta \Gamma \), i.e., \( v \in Z(\Delta \Gamma) = Z(\Delta)[X^s] \). Therefore \( y = f(X^s)v \in I \), so \( Z(\Delta)[X^s] \cap (f(X^s)) \subset I \). The converse inclusion follows from \( f(X^s) \in Z(\Delta)[X^s] \). Hence we have \( I = Z(\Delta)[X^s] \cap (f(X^s)) \) as required.

Finally, we will show \( Z(\Lambda) \simeq Z(\Delta)[x]/(f(x)) \) as rings. It is clear that the map
\[
Z(\Delta)[X^s]/I \longrightarrow Z(\Delta)[x]/(f(x)); \quad X^s \mapsto x
\]
is a ring isomorphism. Therefore we have the ring isomorphism as required. This completes the proof of the lemma. \( \square \)

By this lemma, we also regard \( \text{HH}^t(\Lambda) \) as a \( Z(\Delta)[x]/(f(x)) \)-module for \( t \geq 0 \).

**Lemma 4.2.** We have \( e_{i+k\Lambda} e_i = (\Delta[X^s] X^k e_i + (f(X^s)))/(f(X^s)) \) for \( 1 \leq i \leq s \) and \( 0 \leq k \leq s - 1 \). Moreover, we have \( \delta^s(e_{i+k\Lambda} e_i) = Z(\Lambda) X^k e_i \) which is a free \( Z(\Lambda) \)-module of rank 1.

**Proof.** For \( 0 \leq k \leq s - 1 \) and \( 1 \leq i \leq s \), let
\[
y = \sum_{p=1}^{s} \sum_{j=0}^{n_s-1} b_{p,j}X^j e_p \in e_{i+k\Lambda} e_i, \quad \text{where } b_{p,j} \in \Delta.
\]
Then we have
\[
y = e_{i+k}y e_i = \sum_{j=0}^{n-1} b_{i,j} X^j e_{i+k-j} e_i
\]
\[
= \sum_{l=0}^{n-1} b_{l,k+l} X^{k+l} e_i \in (\Delta[X^s]X^k e_i + (f(X^s)))/(f(X^s)),
\]
hence \(e_{i+k} \Lambda e_i \subset (\Delta[X^s]X^k e_i + (f(X^s)))/(f(X^s))\). It is clear that the converse inclusion holds. Moreover we have
\[
\delta^e(e_{i+k} \Lambda e_i) = \delta^e(\Delta[X^s]X^k e_i + (f(X^s)))/(f(X^s))
\]
\[
= ((\delta^e(\Delta)[X^s]X^k e_i + (f(X^s)))/(f(X^s))
\]
\[
= (Z(\Delta)[X^s]X^k e_i + (f(X^s)))/(f(X^s))
\]
\[
= Z(\Lambda)X^k e_i
\]
by Lemma 4.1. We will show that \(Z(\Lambda)X^k e_i\) is a free \(Z(\Lambda)\)-module of rank 1. Let \(z = \sum_{l=0}^{n-1} b_l X^l e_i \in Z(\Lambda)\) where \(b_l \in \Delta\). If \(z X^k e_i = 0\), then we have \(b_l = 0\) for \(0 \leq l \leq n - 1\), hence \(z = 0\) follows.

By this lemma, for \(1 \leq i \leq s\) and \(0 \leq k \leq s - 1\), there exist the following \(Z(\Lambda)\)-isomorphisms:
\[
\text{Hom}_{\Lambda^e}(\Lambda e_{i+k} \delta e_i \Lambda, \Lambda) \xrightarrow{\sim} ((e_{i+k} \otimes e_i^0) \delta^e) \Lambda = Z(\Lambda)X^k e_i;
\]
\[
\phi \mapsto \phi(e_{i+k} \delta e_i),
\]

since \((e_{i+k} \otimes e_i^0) \delta^e\) are idempotents in \(\Lambda^e\), where we regard \(\text{Hom}_{\Lambda^e}(\Lambda e_{i+k} \delta e_i \Lambda, \Lambda)\) as \(Z(\Lambda)\)-modules by setting
\[
(z \phi)(y) := z(\phi(y))
\]
for \(z \in Z(\Lambda)\), \(\phi \in \text{Hom}_{\Lambda^e}(\Lambda e_{i+k} \delta e_i \Lambda, \Lambda)\) and \(y \in \Lambda e_{i+k} \delta e_i \Lambda\). Note that the inverse maps of the above isomorphisms are
\[
\Phi_{i,k} : ((e_{i+k} \otimes e_i^0) \delta^e) \Lambda \longrightarrow \text{Hom}_{\Lambda^e}(\Lambda e_{i+k} \delta e_i \Lambda, \Lambda);
\]
\[
((e_{i+k} \otimes e_i^0) \delta^e) \lambda \longmapsto (e_{i+k} \delta e_i) \mapsto ((e_{i+k} \otimes e_i^0) \delta^e) \lambda
\]
respectively. By means of these isomorphisms, we have the following \(Z(\Lambda)\)-isomorphisms:
\[
u_0 : \text{Hom}_{\Lambda^e}(P_0, \Lambda) \xrightarrow{\sim} \bigoplus_{i=1}^s \text{Hom}_{\Lambda^e}(\Lambda e_i \delta e_i \Lambda, \Lambda) \xrightarrow{\sim} \bigoplus_{i=1}^s Z(\Lambda)e_i;
\]
\[
\phi \longmapsto (\phi_i)_i \longmapsto \sum_i \phi_i(e_i \delta e_i)\]
for \( s \geq 1 \), and

\[
\begin{align*}
    u_1 : \Hom_{\Lambda^e}(P_1, \Lambda) & \xrightarrow{=} \bigoplus_{i=1}^s \Hom_{\Lambda^e}(\Lambda e_{i+1} \delta e_i \Lambda, \Lambda) \xrightarrow{=} \bigoplus_{i=1}^s Z(\Lambda)xe_i; \\
    \psi & \longmapsto \sum_i \psi_i(e_{i+1} \delta e_i)
\end{align*}
\]

for \( s \geq 2 \), where we set \( \phi_i = \phi|_{\Lambda e_i \delta e_i \Lambda} \) and \( \psi_i = \psi|_{\Lambda e_{i+1} \delta e_i \Lambda} \).

### 4.2. The Hochschild cohomology groups of \( \Lambda \) in the case \( s \geq 2 \)

In this subsection, we assume that \( s \geq 2 \). By means of the resolution (3.2) and Lemma 4.2, we have the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \Hom_{\Lambda^e}(P_0, \Lambda) & \xrightarrow{d_1^\#} & \Hom_{\Lambda^e}(P_1, \Lambda) & \xrightarrow{d_0^\#} & \Hom_{\Lambda^e}(P_0, \Lambda) & \longrightarrow & \cdots \\
\downarrow u_0 & & \downarrow u_1 & & \downarrow u_0 \\
0 & \longrightarrow & \bigoplus_{i=1}^s Z(\Lambda)e_i & \xrightarrow{d_1^\#} & \bigoplus_{i=1}^s Z(\Lambda)xe_i & \xrightarrow{d_0^\#} & \bigoplus_{i=1}^s Z(\Lambda)e_i & \longrightarrow & \cdots ,
\end{array}
\]

where we set \( d_1^\# = \Hom_{\Lambda^e}(d_1, \Lambda) \), \( d_0^\# = \Hom_{\Lambda^e}(d_0, \Lambda) \), \( d_1^* = u_1 d_1^\# u_0^{-1} \) and \( d_0^* = u_0 d_0^\# u_1^{-1} \). The inverse maps of \( u_0 \) and \( u_1 \) are given by the following:

\[
\begin{align}
    u_0^{-1}(\lambda e_i)(e_j \delta e_j) &= \begin{cases} 
     \Phi_{i,0}(\lambda e_i) = \lambda e_i & \text{if } j = i, \\
     0 & \text{if } j \neq i,
    \end{cases} \\
    u_1^{-1}(\lambda xe_i)(e_{j+1} \delta e_j) &= \begin{cases} 
     \Phi_{i,1}(\lambda xe_i) = \lambda xe_i & \text{if } j = i, \\
     0 & \text{if } j \neq i
    \end{cases}
\end{align}
\]

for \( \lambda \in Z(\Lambda) \) and \( 1 \leq i, j \leq s \).

**Lemma 4.3.** In the case \( s \geq 2 \), we have

\[
\begin{align*}
    d_1^*(\lambda e_i) &= \lambda X(e_i - e_{i-1}), \\
    d_0^*(\lambda xe_i) &= \lambda X^*f'(X^*)
\end{align*}
\]

for \( \lambda \in Z(\Lambda) \) and \( 1 \leq i \leq s \), where \( f'(x) \) denotes the derivative of \( f(x) \).

**Proof.** Let \( \lambda \in Z(\Delta) \) and \( 1 \leq i \leq s \). Then, by (4.1), we have

\[
d_1^*(\lambda e_i) = (u_1 d_1^\#)(u_0^{-1}(\lambda e_i)) = u_1 (u_0^{-1}(\lambda e_i)d_1)
\]
\[
\begin{align*}
= & \sum_{j=1}^{s} (u_0^{-1}(\lambda e_i) d_1) (e_{j+1} \delta e_j) \\
= & \sum_{j=1}^{s} u_0^{-1}(\lambda e_i)(e_{j+1}(X\delta - \delta X)e_j) \\
= & \sum_{j=1}^{s} u_0^{-1}(\lambda e_i)(Xe_j \delta e_j - e_{j+1} \delta e_{j+1}X) \\
= & Xu_0^{-1}(\lambda e_i)(e_i \delta e_i) - u_0^{-1}(\lambda e_i)(e_i \delta e_i)X \\
= & X\lambda e_i - \lambda e_i X = \lambda X(e_i - e_{i-1}).
\end{align*}
\]

We also have
\[
\begin{align*}
d^*_0(\lambda X e_i) &= (u_0d_0^*)(u_1^{-1}(\lambda X e_i)) = u_0(u_1^{-1}(\lambda X e_i)d_0) \\
= & \sum_{k=1}^{s} (u_1^{-1}(\lambda X e_i)d_0) (e_k \delta e_k) \\
= & \sum_{k=1}^{s} u_1^{-1}(\lambda X e_i) \left( e_k \left( \sum_{j=1}^{n} z_j \left( \sum_{l=0}^{js-1} X^l \delta X^{js-l-1} \right) \right) e_k \right) \\
= & \sum_{k=1}^{s} u_1^{-1}(\lambda X e_i) \left( \sum_{j=1}^{n} z_j \left( \sum_{l=0}^{js-1} X^l e_{k-l} \delta e_{k-l-1} X^{js-l-1} \right) \right) \\
= & \sum_{k=1}^{s} \left( \sum_{j=1}^{n} z_j \left( \sum_{l=0}^{js-1} X^l u_1^{-1}(\lambda X e_i)(e_{k-l} \delta e_{k-l-1}) X^{js-l-1} \right) \right) \\
= & \sum_{k=1}^{s} \sum_{j=1}^{n} z_j \left( \sum_{0 \leq l \leq js-1} \text{s.t. } i \equiv k-l-1 \text{ (mod } s) \right) \left( X^l (\lambda X e_i) X^{js-l-1} \right) \\
= & \lambda \sum_{k=1}^{s} \sum_{j=1}^{n} z_j \left( \sum_{0 \leq l \leq js-1} \text{s.t. } i \equiv k-l-1 \text{ (mod } s) \right) X^{js} e_k \\
= & \lambda X^s \left( \sum_{j=1}^{n} jz_j X^{(j-1)s} \right) \left( \sum_{k=1}^{s} e_k \right) = \lambda X^s f'(X^s),
\end{align*}
\]

by means of (4.2). \qed 

The results of Lemmas 4.1, 4.2 and 4.3 are similar to those of [FS, Lemmas
2.1, 2.2 and 2.3. Thus the following theorem is easily shown by a similar proof to that given in [FS, Theorem 2 and Corollary 2.4], so we omit the details.

**Theorem 4.4.** In the case \( s \geq 2 \), there exist the following isomorphisms of \( Z(\Delta)[x]/(f(x)) \)-modules:

\[
\text{HH}^t(\Lambda) \simeq \begin{cases} 
Z(\Delta)[x]/(f(x)) & \text{if } t = 0, \\
\text{Ann}_{Z(\Delta)[x]/(f(x))}(xf'(x)) & \text{if } t \text{ is odd}, \\
Z(\Delta)[x]/(xf'(x), f(x)) & \text{if } t \text{ is even}.
\end{cases}
\]

Moreover, if \( Z(\Delta) \) is a field then \( \text{HH}^t(\Lambda) \simeq Z(\Delta)[x]/(xf'(x), f(x)) \) for \( t \geq 1 \).

### 4.3. The Hochschild cohomology groups of \( \Lambda \) in the case \( s = 1 \)

In this subsection, we assume that \( s = 1 \) (i.e., \( \Lambda = \Delta[x]/(f(x)) \)) and \( n \geq 2 \).

In this case, we recall that \( P_0 = P_1 = \Lambda \delta \delta \Lambda \). By Theorem 3.2, we have the periodic left \( \Lambda^e \)-projective resolution:

\[
\cdots \rightarrow \Lambda \delta \delta \Lambda \xrightarrow{d_0} \Lambda \delta \delta \Lambda \xrightarrow{d_1} \Lambda \delta \delta \Lambda \xrightarrow{\pi} \Lambda \xrightarrow{0},
\]

where \( \pi \) is the multiplication map, and \( d_1, d_0 \) are the left \( \Lambda^e \)-homomorphisms given by

\[
d_1(\delta) = x\delta - \delta x, \quad d_0(\delta) = \sum_{j=1}^{n} z_j \left( \sum_{l=0}^{j-1} x^l \delta x^{j-l-1} \right),
\]

since \( X \) is identified with \( x \). So, by Lemma 4.2, we have the following commutative diagram:

\[
\begin{array}{cccc}
0 & \rightarrow & \text{Hom}_{\Lambda^e}(\Lambda \delta \delta \Lambda, \Lambda) & \xrightarrow{d_1^\#} \text{Hom}_{\Lambda^e}(\Lambda \delta \delta \Lambda, \Lambda) \\
\gamma \downarrow & & \gamma \downarrow & & \gamma \downarrow \\
0 & \rightarrow & \text{Z}(\Lambda) & \xrightarrow{d_1^*} \text{Z}(\Lambda) & \xrightarrow{d_0^*} \text{Z}(\Lambda) & \xrightarrow{d_1^*} \cdots,
\end{array}
\]

where we set \( d_1^\# = \text{Hom}_{\Lambda^e}(d_1, \Lambda), \ d_0^\# = \text{Hom}_{\Lambda^e}(d_0, \Lambda), \ d_1^* = u_0 d_1^\# u_0^{-1}, \) and \( d_0^* = u_0 d_0^\# u_0^{-1} \). Since

\[
u_0 : \text{Hom}_{\Lambda^e}(\Lambda \delta \delta \Lambda, \Lambda) \xrightarrow{\sim} \text{Z}(\Lambda); \quad \phi \mapsto \phi(\delta)
\]

and \( u_0^{-1}(\lambda)(\delta) = \lambda \) for all \( \lambda \in \text{Z}(\Lambda) \), we have \( d_1^* = 0 \) and \( d_0^* = \lambda f'(x) \). Therefore the following theorem follows.
Theorem 4.5. In the case $s = 1$, i.e., $\Lambda = \Delta[x]/(f(x))$, there exist the following isomorphisms of $Z(\Lambda)$-modules:

$$\text{HH}^t(\Lambda) \simeq \begin{cases} 
Z(\Lambda) = Z(\Delta)[x]/(f(x)) & \text{if } t = 0, \\
\text{Ann}_{Z(\Delta)}(f'(x)) = \text{Ann}_{Z(\Delta)[x]/(f(x))}(f'(x)) & \text{if } t \text{ is odd}, \\
Z(\Lambda)/(f'(x)) \simeq Z(\Delta)[x]/(f'(x), f(x)) & \text{if } t \text{ is even}.
\end{cases}$$

Moreover, if $Z(\Delta)$ is a field then $\text{HH}^t(\Lambda) \simeq Z(\Delta)[x]/(f'(x), f(x))$ for $t \geq 1$.

§5. The Hochschild cohomology ring of $\Lambda$

In this section, we determine the ring structures of the even Hochschild cohomology ring $\text{HH}^e(\Lambda) := \bigoplus_{i \geq 0} \text{HH}^{2i}(\Lambda)$ of $\Lambda$ and the Hochschild cohomology ring $\text{HH}^*(\Lambda) := \bigoplus_{t \geq 0} \text{HH}^t(\Lambda)$ of $\Lambda$, where the multiplication is given by the Yoneda product $\times$ (cf. [FS, Section 3]). We deal with the case $s \geq 2$ in Section 5.1 and the case $s = 1$ in Section 5.2.

5.1. The Hochschild cohomology ring of $\Lambda$ in the case $s \geq 2$

In this subsection except Remark 5.5, we assume that $s \geq 2$. The following results in this subsection are easily shown by similar proofs to those given in [FS]. Therefore, we will describe the results only and omit the detailed proof.

Proposition 5.1. There exists the following isomorphism of $Z(\Delta)$-algebras:

$$\text{HH}^e(\Lambda) \simeq Z(\Delta)[u, w]/(f(u), uf'(u)w),$$

where $\deg u = 0$ and $\deg w = 2$.

Proof. By using Theorem 4.4, we can prove the proposition by similar arguments to [FS, Proposition 3.2]. \qed

We consider the case $f'(x) = 0$. Then we identify $\text{HH}^t(\Lambda)$ with $Z(\Delta)[x]/(f(x))$ for $t \geq 0$, by Theorem 4.4.

Theorem 5.2. Let $Z(\Delta)$ be an integral domain, $\text{char} Z(\Delta) = p > 0$ and $f(x) \in Z(\Delta)[x]$ a monic polynomial with $f'(x) = 0$, so we set $f(x) = \sum_{j=0}^{n_0} z_{jp^j} x^j$ for some positive integer $n_0$.

(i) If $p = 2$, then we have the following isomorphism of $Z(\Delta)$-algebras:

$$\text{HH}^*(\Lambda) \simeq Z(\Delta)[u, v, w]/\left(f(u), v^2 - \left(\sum_{0 \leq j \leq n_0 \text{ s.t. } j \text{ is odd}} z_{2j} u^{2j}\right) w\right),$$

where $\deg u = 0$, $\deg v = 1$ and $\deg w = 2$. 
(ii) If \( p \neq 2 \), then we have the following isomorphism of \( Z(\Delta) \)-algebras:

\[
\text{HH}^*(\Lambda) \simeq Z(\Delta)[u, v, w]/(f(u), v^2),
\]

where \( \deg u = 0 \), \( \deg v = 1 \) and \( \deg w = 2 \).

**Proof.** We can prove the theorem by similar arguments to [FS, Theorem 3]. □

Now we consider the case \( f'(x) \neq 0 \). So, from now on, we assume that \( f'(x) \neq 0 \) in this subsection except Remark 5.5. We treat the elementary case \( f(x) = g^k(x) \) with a monic irreducible polynomial \( g(x) \in Z(\Delta)[x] \) and a positive integer \( k \). Then, since \( 0 \neq f'(x) = kg'(x)g^{k-1}(x) \), it follows that \( \text{char } Z(\Delta) \nmid k \).

First, we consider the case \( g(x) = x \). In this case, we note that if \( \Delta = R \) is a field then the ring structure of \( \text{HH}^*(\Lambda) \) is determined in [EH, Proposition 5.6].

**Proposition 5.3.** Let \( f(x) = x^k \) with a positive integer \( k \) and \( f'(x) \neq 0 \). Then we have the following isomorphism of \( Z(\Delta) \)-algebras:

\[
\text{HH}^*(\Lambda) \simeq Z(\Delta)[u, v, w]/(u^k, v^2),
\]

where \( \deg u = 0 \), \( \deg v = 1 \) and \( \deg w = 2 \).

**Proof.** By Theorem 4.4, we identify \( \text{HH}^t(\Lambda) \) with \( Z(\Delta)[x]/(x^k) = Z(\Lambda) \) for \( t \geq 0 \). Let \( u = x + (x^k) \in \text{HH}^0(\Lambda) \), \( v = 1 + (x^k) \in \text{HH}^1(\Lambda) \) and \( w = 1 + (x^k) \in \text{HH}^2(\Lambda) \). Since we have the results which are similar to [FS, Lemmas 3.1, 3.3 and 3.4], the following follows. For \( i \geq 0 \), \( \text{HH}^{2i+1}(\Lambda) \) is the \( Z(\Lambda) \)-module generated by \( w^i \) and \( \text{HH}^{2i+1}(\Lambda) \) is the \( Z(\Lambda) \)-module generated by \( w^i v \). We obtain the relation \( w^k = 0 \) in degree 0. We also obtain the relation \( v^2 = 0 \) in degree 2. Indeed, if \( k = 1 \) then the relation is clear, and if \( k \geq 2 \) then we have \( v^2 = \sum_{j=2}^k \sum_{l=1}^{j-1} l \left( \sum_{l=1}^{k-1} l \right) x^j + (x^k) = \left( \sum_{l=1}^{k-1} l \right) x^k + (x^k) = 0 \). Therefore we get the desired isomorphism. □

Second, we consider the case \( g(x) \neq x \) and \( Z(\Delta) \) is a unique factorization domain. Then we have

\[
\text{HH}^1(\Lambda) = \text{Ann}_{Z(\Delta)[x]/(g^k(x))}(xkg'(x)g^{k-1}(x)) = (g(x))/(g^k(x)),
\]

\[
\text{HH}^2(\Lambda) = Z(\Delta)[x]/(g^k(x), xkg'(x)g^{k-1}(x))
\]

for \( k \geq 1 \). If \( k = 1 \) then \( \text{HH}^1(\Lambda) = 0 \), and hence the Hochschild cohomology ring of \( \Lambda \) has been calculated by Proposition 5.1.
Theorem 5.4. Let $\Omega$ be a unique factorization domain, $p = \text{char} \, \Omega \geq 0$ and $f(x) = g^k(x) = \sum_{j=0}^{n} z_j x^j \in \Omega[x]$ with $f'(x) \neq 0$, where $g(x) \in \Omega[x]$ is monic irreducible, $g(x) \neq x$ and $k \geq 2$.

(i) If $p = 2$, then there exists the following isomorphism of $\Omega$-algebras:

$$\text{HH}^*(\Omega) \simeq \Omega[u, v, w]/I,$$

where $I$ is the ideal of $\Omega[u, v, w]$ generated by

$$g^k(u), g^{k-1}(u)v, v^2 - g^2(u) \left( \sum_{0 \leq j \leq n \text{ s.t. } j \equiv 2 \text{ or } 3 \pmod{4}} z_j u^j \right) w, kug^{k-1}(u)g'(u)w,$$

and $\deg u = 0$, $\deg v = 1$, $\deg w = 2$.

(ii) If $p \neq 2$ (including the case $p = 0$), then there exists the following isomorphism of $\Omega$-algebras:

$$\text{HH}^*(\Omega) \simeq \Omega[u, v, w]/(g^k(u), g^{k-1}(u)v, v^2, kug^{k-1}(u)g'(u)w),$$

where $\deg u = 0$, $\deg v = 1$ and $\deg w = 2$.

**Proof.** We can prove the theorem by similar arguments to [FS, Theorem 4].

**Remark 5.5.** Suppose that $\Omega$ is a field and $s \geq 1$. Let $f(x) = g_1^{k_1}(x) \cdots g_l^{k_l}(x)$ be a factorization of $f(x)$ into irreducible factors in $\Omega[x]$. Since the result of [FS, Lemma 3.6] holds in the case $s \geq 1$, we have the following decomposition of $\Omega$-algebras:

$$\Lambda = \Delta \oplus \Omega \bigl( Z(\Delta) \bigr) (\Omega/\Omega(x^s)) \simeq \Delta \oplus \Omega \bigl( Z(\Delta) \bigr) \bigl( g_1^{k_1}(\Delta) \bigr) \oplus \cdots \oplus \Delta \oplus \Omega \bigl( Z(\Delta) \bigr) \bigl( g_l^{k_l}(\Delta) \bigr) \oplus \Delta \oplus \Omega \bigl( Z(\Delta) \bigr) \bigl( X^s \bigr).$$

Then there exists the following isomorphism of $\Omega$-algebras:

$$\text{HH}^*(\Delta \oplus \Omega/\Omega(x^s)) \simeq \text{HH}^*(\Delta \oplus \Omega(g_1^{k_1}(\Delta)) \oplus \cdots \oplus \Delta \oplus \Omega(g_l^{k_l}(\Delta)) \oplus \Delta \oplus \Omega(X^s)).$$

Hence, it suffices to consider the case $f(x) = g^k(x)$ for an irreducible polynomial $g(x) \in \Omega[x]$ and a positive integer $k$ in order to determine the ring structure of $\text{HH}^*(\Lambda)$. 

5.2. The Hochschild cohomology ring of $\Lambda$ in the case $s = 1$

In this subsection, we assume that $s = 1$ (i.e., $\Lambda = \Delta[x]/(f(x))$) and $n \geq 2$. Note that the isomorphisms of Theorem 4.5 are given explicitly as follows:

\[ Z(\Delta)[x]/(f(x)) \sim HH^0(\Lambda); \quad q(x) + (f(x)) \mapsto \phi, \]
\[ \text{Ann}_{Z(\Delta)[x]/(f(x))}(f'(x)) \sim HH^1(\Lambda); \quad q(x) + (f(x)) \mapsto \phi, \]
\[ Z(\Delta)[x]/(f'(x), f(x)) \sim HH^2(\Lambda); \quad q(x) + (f'(x), f(x)) \mapsto \phi + \text{Im} d^R_0, \]

where $\phi : \Lambda \delta \Lambda \to \Lambda$ is the $\Lambda'$-homomorphism given by $\phi(\delta) = q(x) + (f(x))$. Thus we will identify

\[ HH^0(\Lambda) = Z(\Delta)[x]/(f(x)), \quad HH^1(\Lambda) = \text{Ann}_{Z(\Delta)[x]/(f(x))}(f'(x)) \]
\[ \text{and } HH^2(\Lambda) = Z(\Delta)[x]/(f'(x), f(x)) \]

by these isomorphisms.

We denote the resolution (4.3) by

\[ \cdots \xrightarrow{d_4} P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} \Lambda \to 0, \]

where $P_i = P_0 = \Lambda \delta \Lambda$, $d_{2i} = d_0$ and $d_{2i+1} = d_1$ for $i \geq 1$. Let $w$ be the coset in $HH^2(\Lambda)$ with $1 \in Z(\Delta)[x]$: $w = 1 + (f'(x), f(x)) \in HH^2(\Lambda)$. Then $w$ is represented by the multiplication map $\pi : P_2(= P_0) \to \Lambda$. In this subsection, we will use $w$ in the meaning above.

**Lemma 5.6.** If $Q = q(x) + (f(x)) \in HH^0(\Lambda)$, where $q(x) \in Z(\Delta)[x]$, then we have $Q \times w = q(x) + (f'(x), f(x)) \in HH^2(\Lambda)$. Also, we have $w \times w = 1 + (f'(x), f(x)) \in HH^4(\Lambda)$. Hence $HH^{2i}(\Lambda)$ is the $Z(\Lambda)$-module generated by $w^i$ in $HH^{2i}(\Lambda)$ for $i \geq 1$.

**Proof.** The element $Q = q(x) + (f(x)) \in HH^0(\Lambda)$ where $q(x) \in Z(\Delta)[x]$ is represented by the $\Lambda'$-homomorphism $\phi : P_0 \to \Lambda$ given by $\phi(\delta) = q(x) + (f(x))$.

First, we compute the product $Q \times w \in HH^2(\Lambda)$. It is clear that $\text{id}_{\Lambda \delta \Lambda} : P_2 \to P_0$ is a lifting of $\pi : P_2 \to \Lambda$. Hence $Q \times w$ is the element in $HH^2(\Lambda)$ represented by $\phi : P_2 \to \Lambda$. Therefore we have $Q \times w = q(x) + (f'(x), f(x)) \in HH^2(\Lambda)$.

Second, we compute the product $w \times w \in HH^4(\Lambda)$. It is clear that $\text{id}_{\Lambda \delta \Lambda} : P_2 \to P_0, P_3 \to P_1, P_4 \to P_2$ are liftings of $\pi : P_2 \to \Lambda$. Hence $w \times w$ is the element in $HH^4(\Lambda)$ represented by $\pi : P_4 \to \Lambda$. Therefore we have $w \times w = 1 + (f'(x), f(x)) \in HH^4(\Lambda)$. \qed

By this Lemma, we have the structure of the even Hochschild cohomology ring of $\Lambda$. 
Proposition 5.7. There exists the following isomorphism of $Z(\Delta)$-algebras:

$$\text{HH}^0(\Lambda) \cong Z(\Delta)[u, w]/(f(u), f'(u)w),$$

where $\deg u = 0$ and $\deg w = 2$.

Proof. Let $u = x + (f(x)) \in Z(\Delta)[x]/(f(x)) = \text{HH}^0(\Lambda)$. Then we have the relation $f(u) = 0$ in degree 0. Moreover, by Lemma 5.6, $\text{HH}^2(\Lambda)$ is the $\text{HH}^0(\Lambda)$-module generated by $w^i$ and there is the relation $f'(u)w^i = 0$ in degree $2i$ for $i \geq 1$. Therefore we have the desired isomorphisms of $Z(\Delta)$-algebras. □

Now we calculate the Yoneda product in odd degree.

Lemma 5.8. If $Q_0 = q_0(x) + (f(x)) \in \text{HH}^0(\Lambda)$ where $q_0(x) \in Z(\Delta)[x]$, and $Q_1 = q_1(x) + (f(x)) \in \text{HH}^1(\Lambda)$ where $q_1(x)$ is an element in $Z(\Delta)[x]$ such that $f'(x)q_1(x) \in (f(x))$, then we have $Q_0 \times Q_1 = q_0(x)q_1(x) + (f(x)) \in \text{HH}^1(\Lambda)$. Also, we have $Q_1 \times w = q_1(x) + (f(x)) \in \text{HH}^1(\Lambda)$.

Proof. The elements $Q_0$ and $Q_1$ are represented by the $\Lambda^e$-homomorphisms $\phi_0 : P_0 \to \Lambda$ and $\phi_1 : P_1 \to \Lambda$ given by $\phi_0(\delta) = q_0(x) + (f(x))$ and $\phi_1(\delta) = q_1(x) + (f(x))$, respectively. Then the $\Lambda^e$-homomorphism $\sigma : P_1 \to P_0$ given by $\sigma(\delta) = \delta q_1(x)$ is a lifting of $\phi_1$ and $\phi_0 \sigma : P_1 \to \Lambda$ satisfies $(\phi_0 \sigma)(\delta) = q_0(x)q_1(x) + (f(x))$. Therefore we have $Q_0 \times Q_1 = q_0(x)q_1(x) + (f(x))$.

Next we compute $Q_1 \times w$. It is clear that $\text{id}_\Delta : P_2 \to P_0$, $P_3 \to P_1$ are liftings of of $\pi : P_2 \to \Lambda$. Hence $Q_1 \times w$ is the element in $\text{HH}^0(\Lambda)$ represented by $\phi_1 : P_3 \to \Lambda$. Therefore we have $Q_1 \times w = q_1(x) + (f(x)) \in \text{HH}^0(\Lambda)$. □

Lemma 5.9. If $Q = q(x) + (f(x))$, $\tilde{Q} = \tilde{q}(x) + (f(x)) \in \text{HH}^1(\Lambda)$ where $q(x), \tilde{q}(x)$ are elements in $Z(\Delta)[x]$ such that $f'(x)q(x), f'(x)\tilde{q}(x) \in (f(x))$, then we have

$$Q \times \tilde{Q} = q(x)\tilde{q}(x) + \sum_{j=2}^{n} z_j \left( \sum_{l=1}^{j-1} \frac{1}{l} \right) x^{j-2} + (f'(x), f(x)).$$

Proof. The elements $Q$ and $\tilde{Q}$ are represented by the $\Lambda^e$-homomorphisms $\phi : P_1 \to \Lambda$ and $\tilde{\phi} : P_1 \to \Lambda$ given by $\phi(\delta) = q(x) + (f(x))$ and $\tilde{\phi}(\delta) = \tilde{q}(x) + (f(x))$ respectively. It is clear that the $\Lambda^e$-homomorphism $\sigma_0 : P_1 \to P_0$ given by $\sigma_0(\delta) = \delta \tilde{q}(x)$ is a lifting of $\tilde{\phi} : P_1 \to \Lambda$. Define the $\Lambda^e$-homomorphism $\sigma_1 : P_2 \to P_1$ by

$$\sigma_1(\delta) = \sum_{j=2}^{n} z_j \left( \sum_{l=1}^{j-1} \sum_{k=0}^{l-1} x^k \delta x^{j-k-2} \right) \tilde{q}(x).$$
Then we have that $\sigma_1$ is a lifting of $\tilde{\phi}$, i.e., $\sigma_0d_0 = \sigma_0d_2 = d_1\sigma_1$. Indeed, by means of the equation $f'(x)\tilde{q}(x) = 0$ in $\Lambda$, we can calculate as follows. First, note that

$$(\sigma_0d_0)(\delta) = \sigma_0 \left( \sum_{j=1}^{n} z_j \left( \sum_{l=0}^{j-1} x^l \delta x^{j-l-1} \right) \right) = \sum_{j=1}^{n} z_j \left( \sum_{l=0}^{j-1} x^l \delta x^{j-l-1} \right) \tilde{q}(x).$$

We also have

$$(d_1\sigma_1)(\delta) = d_1 \left( \sum_{j=2}^{n} z_j \left( \sum_{l=0}^{j-1} x^l \delta x^{j-l-1} \right) \tilde{q}(x) \right)$$

$$= \sum_{j=2}^{n} z_j \left( \sum_{l=0}^{j-1} x^l \delta x^{j-l-1} \tilde{q}(x) \right)$$

$$= \sum_{j=2}^{n} z_j \left( \sum_{l=0}^{j-1} x^l \delta x^{j-l-1} \tilde{q}(x) \right)$$

$$= \sum_{j=2}^{n} z_j \left( \sum_{l=0}^{j-1} x^l \delta x^{j-l-1} \tilde{q}(x) \right)$$

$$= \sum_{j=2}^{n} z_j \left( \sum_{l=0}^{j-1} x^l \delta x^{j-l-1} \tilde{q}(x) \right)$$

Hence $\sigma_0d_0 = d_1\sigma_1$ holds, so $\sigma_1$ is a lifting of $\tilde{\phi} : P_1 \to \Lambda$. Then, we have

$$(\phi\sigma_1)(\delta) = \phi \left( \sum_{j=2}^{n} z_j \left( \sum_{l=0}^{j-1} x^l \delta x^{j-l-1} \tilde{q}(x) \right) \right)$$

$$= \sum_{j=2}^{n} z_j \left( \sum_{l=0}^{j-1} x^l \delta x^{j-l-1} \tilde{q}(x) \right)$$
\[ \sum_{j=2}^{n} z_j \left( \sum_{l=1}^{j-2} x^2 \right) q(x) \tilde{q}(x). \]

This completes the proof of the lemma. \( \square \)

From now on, let \( Z(\Delta) \) be an integral domain in this subsection.

We consider the case \( f'(x) = 0 \), that is, char \( Z(\Delta) = p > 0 \) and \( f(x) = \sum_{j=0}^{n_0} z_{jp}x^{jp} \) for some positive integer \( n_0 \). Then, by Theorem 4.5, we identify \( HH^t(\Lambda) \) with \( Z(\Delta)[x]/(f(x)) \) for \( t \geq 0 \).

**Lemma 5.10.** Let \( Z(\Delta) \) be an integral domain, char \( Z(\Delta) = p > 0 \) and \( f(x) \in Z(\Delta)[x] \) a monic polynomial with \( f'(x) = 0 \), i.e., \( f(x) = \sum_{j=0}^{n_0} z_{jp}x^{jp} \) for some positive integer \( n_0 \). If \( i \) and \( k \) are odd, then we have

\[ Q \times \tilde{Q} = \begin{cases} 
q(x)\tilde{q}(x) \left( \sum_{1 \leq j \leq n_0 \text{ s.t. } j \text{ is odd}} z_{2j}x^{2j-2} \right) + (f(x)) & \text{if } p = 2, \\
0 & \text{if } p \neq 2,
\end{cases} \]

for \( Q = q(x) + (f(x)) \in HH^i(\Lambda) \) and \( \tilde{Q} = \tilde{q}(x) + (f(x)) \in HH^k(\Lambda) \) where \( q(x), \tilde{q}(x) \in Z(\Delta)[x] \).

**Proof.** For \( Q = q(x) + (f(x)) \in HH^i(\Lambda) \) and \( \tilde{Q} = \tilde{q}(x) + (f(x)) \in HH^k(\Lambda) \) where \( q(x) \) and \( \tilde{q}(x) \) are in \( Z(\Delta)[x] \), by Lemma 5.9, we have

\[ Q \times \tilde{Q} = q(x)\tilde{q}(x) \sum_{j=1}^{n_0} z_{jp} \left( \sum_{l=1}^{jp-1} l \right) x^{jp-2} + (f(x)). \]

If \( p = 2 \), then we have \( Q \times \tilde{Q} = q(x)\tilde{q}(x) \left( \sum_{1 \leq j \leq n_0 \text{ s.t. } j \text{ is odd}} z_{2j}x^{2j-2} \right) + (f(x)) \), since

\[ \sum_{1 \leq j \leq n_0 \text{ s.t. } j \text{ is odd}} z_{2j}x^{2j-2} \equiv \begin{cases} 
0 \pmod 2 & \text{if } j \text{ is even}, \\
1 \pmod 2 & \text{if } j \text{ is odd}.
\end{cases} \]

If \( p \neq 2 \), then we have \( Q \times \tilde{Q} = 0 \), since \( \sum_{l=1}^{jp-1} l \equiv 0 \pmod p \) for all \( j \geq 1 \). \( \square \)

**Theorem 5.11.** Let \( Z(\Delta) \) be an integral domain, char \( Z(\Delta) = p > 0 \) and \( f(x) \in Z(\Delta)[x] \) a monic polynomial with \( f'(x) = 0 \), i.e., \( f(x) = \sum_{j=0}^{n_0} z_{jp}x^{jp} \) for some positive integer \( n_0 \).
(i) If $p = 2$, then there exists the following isomorphism of $Z(\Delta)$-algebras:

\[
\text{HH}^*(\Lambda) \cong Z(\Delta)[u, v, w] \left/ \left( f(u), v^2 - \left( \sum_{1 \leq j \leq n \atop \text{s.t. } j \text{ is odd}} z_j u^{2j-2} \right) w \right) \right.,
\]

where $\deg u = 0$, $\deg v = 1$ and $\deg w = 2$.

(ii) If $p \neq 2$, then there exists the following isomorphism of $Z(\Delta)$-algebras:

\[
\text{HH}^*(\Lambda) \cong Z(\Delta)[u, v, w]/(f(u), v^2),
\]

where $\deg u = 0$, $\deg v = 1$ and $\deg w = 2$.

Proof. Let $u = x + (f(x)) \in \text{HH}^0(\Lambda)$, $v = 1 + (f(x)) \in \text{HH}^1(\Lambda)$ and $w = 1 + (f(x)) \in \text{HH}^2(\Lambda)$. By Lemmas 5.6 and 5.8, $\text{HH}^{2i+1}(\Lambda)$ is the $Z(\Lambda)$-module generated by $w^i v$ for $i \geq 0$. If $p \neq 2$, then we obtain the relation $v^2 = 0$ in degree 2 by Lemma 5.10. If $p = 2$, then $v \times v$ is the coset in $\text{HH}^2(\Lambda)$ represented by $\sum_{1 \leq j \leq n \atop \text{s.t. } j \text{ is odd}} z_j u^{2j-2} \in Z(\Delta)[x]$ by Lemma 5.10, so we have the relation $v^2 - \sum_{1 \leq j \leq n \atop \text{s.t. } j \text{ is odd}} z_j u^{2j-2} = 0$ in degree 2. Therefore we have the desired isomorphisms. \hfill \Box

Next we consider the case $f'(x) \neq 0$. So, from now on, we assume that $f'(x) \neq 0$ and $Z(\Delta)$ is a unique factorization domain in this subsection. We treat the elementary case $f(x) = g^k(x)$ with a monic irreducible polynomial $g(x) \in Z(\Delta)[x]$ and $k \geq 1$. Then, since $0 \neq f'(x) = kg'(x)g^{k-1}(x)$, it follows that char $Z(\Delta) \mid k$. By Theorem 4.5, we also have

\[
\text{HH}^1(\Lambda) = \text{Ann}_{Z(\Delta)[x]/(g^k(x))}(kg'(x)g^{k-1}(x)) = (g(x))/(g^k(x)),
\]

\[
\text{HH}^2(\Lambda) = Z(\Delta)[x]/(g^k(x), kg'(x)g^{k-1}(x)).
\]

If $k = 1$ then $\text{HH}^1(\Lambda) = 0$, and hence the Hochschild cohomology ring of $\Lambda$ has been calculated by Proposition 5.7. So we assume $k \geq 2$.

Lemma 5.12. Let $Z(\Delta)$ be a unique factorization domain, $p = \text{char } Z(\Delta) \geq 0$ and $f(x) = g^k(x) = \sum_{j=0}^n z_j x^j \in Z(\Delta)[x]$ with $f'(x) \neq 0$, where $g(x) \in Z(\Delta)[x]$ is monic irreducible and $k \geq 2$. If $i$ and $t$ are odd, then we have

\[
Q \times \hat{Q} = \begin{cases} 
q(x)\hat{q}(x)g^2(x) \left( \sum_{2 \leq j \leq n \atop \text{s.t. } j \equiv 2 \text{ or } 3 \mod 4} z_j x^{j-2} \right) + (f(x), f'(x)) & \text{if } p = 2, \\
0 & \text{if } p \neq 2,
\end{cases}
\]
for \( Q = q(x)g(x) + (f(x)) \in \text{HH}^i(\Lambda) \) and \( \tilde{Q} = \tilde{q}(x)g(x) + (f(x)) \in \text{HH}^i(\Lambda) \) where \( q(x), \tilde{q}(x) \in \mathbb{Z}(\Delta)[x] \).

**Proof.** By Lemma 5.9, we have
\[
Q \times \tilde{Q} = q(x)\tilde{q}(x)g^2(x)\sum_{j=2}^{n} z_j \left( \sum_{l=1}^{j-1} \right) x^{j-2} + (f(x), f'(x)).
\]
If \( p = 2 \), then we have
\[
Q \times \tilde{Q} = q(x)\tilde{q}(x)g^2(x)\left( \sum_{2 \leq j \leq n \text{ s.t. } j \equiv 2 \text{ or } 3 \pmod{4}} z_j x^{j-2} \right) + (f(x), f'(x)),
\]

since
\[
\sum_{l=1}^{j-1} l = \begin{cases} 
0 \pmod{2} & \text{if } j \equiv 0 \text{ or } 1 \pmod{4}, \\
1 \pmod{2} & \text{if } j \equiv 2 \text{ or } 3 \pmod{4}.
\end{cases}
\]
If \( p \neq 2 \), then
\[
\sum_{j=2}^{n} z_j \left( \sum_{l=1}^{j-1} \right) x^{j-2} = \sum_{j=2}^{n} z_j \frac{j(j-1)}{2} x^{j-2} = \frac{1}{2} \sum_{j=2}^{n} j(j-1)z_j x^{j-2} = \frac{1}{2}f''(x) = \frac{1}{2}kg^{k-2}(x) \left( (k-1)(g'(x))^2 + g(x)g''(x) \right),
\]
so we have \( Q \times \tilde{Q} = 0 \). \( \square \)

**Theorem 5.13.** Let \( \mathbb{Z}(\Delta) \) be a unique factorization domain, \( p = \text{char} \mathbb{Z}(\Delta) \geq 0 \) and \( f(x) = g^k(x) = \sum_{j=0}^{n} z_j x^j \in \mathbb{Z}(\Delta)[x] \) with \( f'(x) \neq 0 \), where \( g(x) \in \mathbb{Z}(\Delta)[x] \) is monic irreducible and \( k \geq 2 \).

(i) If \( p = 2 \), then there exists the following isomorphism of \( \mathbb{Z}(\Delta) \)-algebras:
\[
\text{HH}^\bullet(\Lambda) \simeq \mathbb{Z}(\Delta)[u, v, w]/I,
\]
where \( I \) is the ideal of \( \mathbb{Z}(\Delta)[u, v, w] \) generated by
\[
g^k(u), g^{k-1}(u)v, v^2 - g^2(u)\left( \sum_{2 \leq j \leq n \text{ s.t. } j \equiv 2 \text{ or } 3 \pmod{4}} z_j u^{j-2} \right) w, kg^{k-1}(u)g'(u)w,
\]
and \( \deg u = 0, \deg v = 1, \deg w = 2 \).
(ii) If $p \neq 2$ (including the case $p = 0$), then there exists the following isomorphism of $\mathbb{Z}(\Delta)$-algebras:

$$\text{HH}^*(\Lambda) \simeq \mathbb{Z}(\Delta)[u, v, w]/(g^k(u), g^{k-1}(u)v, v^2, kg^{k-1}(u)g'(u)w),$$

where $\deg u = 0$, $\deg v = 1$ and $\deg w = 2$.

**Proof.** Let $u = x + (g^k(x)) \in \text{HH}^0(\Lambda)$, $v = g(x) + (g^k(x)) \in \text{HH}^1(\Lambda)$ and $w = 1 + (g^k(x), kg^{k-1}(x)g'(x)) \in \text{HH}^2(\Lambda)$. Then we have the relation $g^k(u) = 0$ in degree 0. By Lemma 5.6, for $i \geq 1$, $\text{HH}^{2i}(\Lambda)$ is the $\mathbb{Z}(\Lambda)$-module generated by $w^i$, and we have the relation $kg^{k-1}(u)g'(u)w = 0$ in degree 2. Moreover, by Lemmas 5.6 and 5.8, for $i \geq 0$, $\text{HH}^{2i+1}(\Lambda)$ is the $\mathbb{Z}(\Lambda)$-module generated by $vw^i$, and we have the relation $g^{k-1}(u)v = 0$ in degree 1.

If $p \neq 2$, then by Lemma 5.12 we have the relation $v^2 = 0$ in degree 2. If $p = 2$, then by Lemma 5.12 $v \times v$ is the coset in $\text{HH}^2(\Lambda)$ represented by

$$g^2(x) \left( \sum_{\substack{2 \leq j \leq n \text{ s.t. } j \equiv 2\text{ or } 3 \pmod{4}}} z_j x^{j-2} \right).$$

So we have the relation

$$v^2 - g^2(u) \left( \sum_{\substack{2 \leq j \leq n \text{ s.t. } j \equiv 2\text{ or } 3 \pmod{4}}} z_j u^{j-2} \right) w = 0$$

in degree 2. Therefore we get the desired isomorphisms. □

We remark that the argument of Remark 5.5 holds in the case $s = 1$.

§6. Applications

In this section, we will give some applications of the results of Section 5. Let $\Delta$ be a separable $R$-algebra as usual.

Let $s$ be an integer with $s \geq 2$ and $\alpha_1, \alpha_2, \cdots, \alpha_s$ be nonzero elements of $Z(\Delta)$ such that $\alpha_i$ is not a zero divisor in $\Delta$ for each $1 \leq i \leq s$. Let $E_{ij}$ be the matrix unit in the $s \times s$ matrix ring $M_s(\Delta)$ for $1 \leq i, j \leq s$ and

$$C := \begin{bmatrix}
0 & \cdots & 0 & 0 \\
\alpha_1 & 0 & & \\
0 & \alpha_2 & & \\
& & \ddots & \ddots & \\
& & & \ddots & 0 \\
0 & \cdots & 0 & \alpha_{s-1} & 0
\end{bmatrix}.$$
Define the $R$-subalgebra $B$ of $M_s(\Delta)$ as follows:

$$B = \Delta[E_{11}, E_{22}, \ldots, E_{ss}, C].$$

Note that, in particular, if $\alpha_1 = \alpha_2 = \cdots = \alpha_{s-1} = 1$ then the algebra has the form

$$\begin{bmatrix}
\Delta & \alpha_s \Delta & \cdots & \alpha_s \Delta \\
\vdots & \Delta & \ddots & \vdots \\
\vdots & & \ddots & \alpha_s \Delta \\
\Delta & \cdots & \cdots & \Delta
\end{bmatrix}_{s \times s}
$$

which is similar to a basic hereditary order (cf. [SS]). We calculate the Hochschild cohomology ring of $B$. The following lemma shows that $B$ is isomorphic to $\Delta \Gamma/(X^s - \alpha)$ for some $f(x) \in Z(\Delta)[x]$, where we note that $\Delta$ needs not to be $R$-separable.

**Lemma 6.1.** Let $B$ be the $R$-algebra as above. Then $B$ is isomorphic to $\Delta \Gamma/(X^s - \alpha)$ as $R$-algebras, where we set $\alpha = \alpha_1 \alpha_2 \cdots \alpha_s$.

**Proof.** We have

$$aC = Ca \quad \text{for all } a \in \Delta \quad \text{and} \quad C^s = \alpha E,$$

where $E$ denotes the identity matrix. We also have

$$C^i E_{ii} = E_{i+i,j+i} C^i \quad \text{for } 1 \leq i \leq s \quad \text{and} \quad 0 \leq j \leq s-1,$$

where we regard the subscripts of matrix units modulo $s$. Since $\alpha_i$ is not a zero divisor in $\Delta$ for each $1 \leq i \leq s$, the set $\{C^i E_{ii} | 1 \leq i \leq s, 0 \leq j \leq s-1\}$ gives a $\Delta$-basis of $B$. Therefore there exists the following isomorphism of $\Delta$-modules:

$$\Delta \Gamma/(X^s - \alpha) \xrightarrow{\sim} B; \quad X^i e_i \mapsto C^i E_{ii}.$$

Moreover, it is clear that the isomorphism is an isomorphism of $R$-algebras. This completes the proof of the lemma. \qed

**Proposition 6.2.** Let $\Delta$ be a separable $R$-algebra and $B$ the $R$-algebra as above. Then there exists the following isomorphism of $Z(\Delta)$-algebras:

$$\text{HH}^*(B) \simeq Z(\Delta)[w]/(\alpha w),$$

where $\deg w = 2$ and $\alpha = \alpha_1 \alpha_2 \cdots \alpha_s$. 

Proof. By Lemma 6.1 and Theorem 4.4, we have
\[ HH_t(B) \simeq \text{Ann}_{Z(\Delta)}(x - \alpha)(x) \simeq \text{Ann}_{Z(\Delta)}(\alpha) = 0 \]
for \( t \) odd, since \( \alpha \) is not a zero divisor in \( \Delta \). Hence \( HH^*(B) \simeq HH^e(B) \) holds. Moreover, by Proposition 5.1, we have
\[ HH^e(B) \simeq Z(\Delta)[u,v]/(u - \alpha,uw) \simeq Z(\Delta)[w]/(\alpha w), \]
where \( \deg u = 0 \) and \( \deg w = 2 \). □

We remark that if \( \Delta = R \) then the result of Proposition 6.2 coincides with [KSS, Theorem 1.1].

Next, we calculate the Hochschild cohomology ring of the truncated polynomial \( R \)-algebra \( A_n := \Delta[x]/(x^n) \) with \( n \geq 2 \).

Proposition 6.3. Let \( \Delta \) be a separable \( R \)-algebra, \( Z(\Delta) \) a unique factorization domain with \( \text{char } Z(\Delta) = p \geq 0 \), and \( A_n \) the truncated polynomial \( R \)-algebra as above. Then there exists the following isomorphism of \( Z(\Delta) \)-algebras:
\[
HH^*(A_n) \simeq \begin{cases} 
Z(\Delta)[u,v,w]/(u^n,v^{n-1}v,v^2nw^{n-1}w) & \text{if } p \nmid n, \\
Z(\Delta)[u,v,w]/(u^n,v^2) & \text{if } 2 \neq p \mid n \text{ or } 2 = p \mid n \text{ and } 4 \nmid n, \\
Z(\Delta)[u,v,w]/(u^n,v^2 - u^{n-2}w) & \text{if } 2 = p \mid n \text{ and } 4 \nmid n,
\end{cases}
\]
where \( \deg u = 0 \), \( \deg v = 1 \) and \( \deg w = 2 \).

Proof. Let \( s = 1 \) and \( f(x) = x^n \) for \( n \geq 2 \), then \( \Lambda = \Delta[x]/(x^n) = A_n \), \( z_n = 1 \) and \( z_j = 0 \) for \( 0 \leq j \leq n - 1 \) in our previous notation.

First, we consider the case \( p \nmid n \). Then, since \( f'(x) \neq 0 \), we can apply Theorem 5.13 to \( A_n \). If \( p = 2 \), then we have
\[ HH^*(A_n) \simeq Z(\Delta)[u,v,w]/(u^n,v^{n-1}v,v^2nw^{n-1}w) \]
where \( \deg u = 0 \), \( \deg v = 1 \) and \( \deg w = 2 \), since \( \sum_{2 \leq j \leq n \text{ s.t. } j \equiv 2 \text{ or } 3 \text{ (mod 4)}\} z_ju^{j-2} \) is equal to \( u^{n-2} \) or 0. If \( p \neq 2 \), then we also have the same isomorphism.

Second, we consider the case \( p \mid n \). Then, since \( f'(x) = 0 \), we can apply Theorem 5.11 to \( A_n \). If \( p \neq 2 \), then \( HH^*(A_n) \simeq Z(\Delta)[u,v,w]/(u^n,v^2) \). If \( p = 2 \), then we have the desired isomorphisms, since the sum \( \sum_{1 \leq j \leq n/2 \text{ s.t. } j \text{ is odd}} z_ju^{2j-2} \) is equal to \( u^{n-2} \) if \( n/2 \) is odd and 0 if \( n/2 \) is even. □

We remark that if \( \Delta = R \) then the result of Proposition 6.3 coincides with [H, Theorem 7.1].
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References


Manabu Suda
Department of Mathematics, Tokyo University of Science
Wakamiya 26, Shinjuku, Tokyo 162-0827, Japan
E-mail: suda@ms.kagu.tus.ac.jp