Quasi-conformally flat manifolds satisfying certain condition on the Ricci tensor

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Abstract. The object of the present paper is to study a non-flat quasi-conformally flat Riemannian manifold whose Ricci tensor $S$ satisfies the condition $S(X,Y) = \gamma T(X)T(Y)$, where $\gamma$ is the scalar curvature and $T$ is a 1-form defined by $T(X) = g(X,\xi)$, $\xi$ is a unit vector field.

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§1. Introduction

The notion of a quasi-conformal curvature tensor was given by Yano and Sawaki [10]. According to them a quasi-conformal curvature tensor $C^*$ is defined by

$$C^*(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] - \frac{\gamma}{n}[\frac{a}{n-1} + 2b][g(Y,Z)X - g(X,Z)Y], \quad (1.1)$$

where $a$ and $b$ are constants and $R$, $Q$ and $\gamma$ are the Riemannian curvature tensor of type $(1, 3)$, the Ricci operator defined by $g(QX,Y) = S(X,Y)$ and the scalar curvature, respectively. If $a = 1$ and $b = -\frac{1}{n-2}$, then (1.1) takes the...
form

\[C^*(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX
- g(X,Z)QY] + \frac{C}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]\]

\[= C(X,Y)Z,\]

where \(C\) is the conformal curvature tensor [4]. Thus the conformal curvature tensor \(C\) is a particular case of the tensor \(C^*\). For this reason \(C^*\) is called the quasi-conformal curvature tensor. A manifold \((M^n, g)\) \((n > 3)\) shall be called quasi-conformally flat if \(C^* = 0\). It is known [1] that a quasi-conformally flat manifold is either conformally flat if \(a \neq 0\) or Einstein if \(a = 0\) and \(b \neq 0\). Since they give no restrictions for manifolds if \(a = 0\) and \(b = 0\), it is essential for us to consider the case of \(a \neq 0\) or \(b \neq 0\).

A Riemannian manifold of quasi-constant curvature was given by B. Y. Chen and K. Yano [3] as a conformally flat manifold with the curvature tensor \(\tilde{R}\) of type \((0, 4)\) satisfies the condition

\[\tilde{R}(X,Y,Z,W) = p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]
+ q[g(X,W)T(Y)T(Z) + g(Y,Z)T(X)T(W)
- g(X,Z)T(Y)T(W) - g(Y,W)T(X)T(Z)],\]  

(1.2)

where \(\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)\), \(R\) is the curvature tensor of type \((1, 3)\), \(p, q\) are scalar functions and \(T\) is a non-zero 1-form defined by

\[g(X, \tilde{\xi}) = T(X),\]  

(1.3)

where \(\tilde{\xi}\) is a unit vector filed. It can be easily seen that if the curvature tensor \(\tilde{R}\) is of the form (1.2), then the manifold is conformally flat. On the other hand, G. Vrânceanu [8] defined the notion of almost constant curvature by the same expression (1.2). Later A. L. Mocanu [6] pointed out that the manifold introduced by Chen and Yano and the manifold introduced by Vrânceanu are the same. Hence a Riemannian manifold is said to be of quasi-constant curvature if the curvature tensor \(\tilde{R}\) satisfies the relation (1.2). If \(q = 0\), then the manifold reduces to a manifold of constant curvature.

The present paper deals with the quasi-conformally flat manifold \((M^n, g)\) \((n > 3)\) whose Ricci tensor \(S\) satisfies

\[S(X,Y) = \gamma T(X)T(Y),\]  

(1.4)

where \(T\) is a non-zero 1-form defined by \(g(X, \xi) = T(X)\), \(\xi\) is a unit vector field. For the scalar curvature \(\gamma\) we suppose that \(\gamma \neq 0\) for each point of
M. Under the assumption above we know that $M$ is not Einstein. Hence we consider the case of $a \neq 0$ (See §3). We shall prove the following:

**Theorem 1.** A quasi-conformally flat manifold satisfying the condition (1.4) under the assumption of $\gamma \neq 0$ is a manifold of quasi-constant curvature.

**Theorem 2.** In a quasi-conformally flat Riemannian manifold satisfying the condition (1.4) under the same assumption as Theorem 1, the integral curves of the vector field $\xi$ are geodesic.

**Theorem 3.** In a quasi-conformally flat manifold satisfying (1.4) under the same assumption as Theorem 1, the vector field $\xi$ is a proper concircular vector field (See §4).

**Theorem 4.** If a quasi-conformally flat manifold satisfies (1.4) under the same assumption as Theorem 1, then the manifold is a locally product manifold.

**Theorem 5.** A quasi-conformally flat manifold satisfying (1.4) under the same assumption as Theorem 1 can be expressed as a locally warped product $I \times_{e^\gamma} M^*$ where $M^*$ is an Einstein manifold (See §4).

### §2. Preliminaries

From (1.1) we obtain

$$ (\nabla_W C^*)(X, Y)Z = a(\nabla_W R)(X, Y)Z + b[(\nabla_W S)(Y, Z)X - (\nabla_W S)(X, Z)Y + g(Y, Z)(\nabla_W Q)(X) - g(X, Z)(\nabla_W Q)(Y)] $$

$$ - \frac{d\gamma(W)}{n} \left[ a \frac{n}{n-1} + 2b[g(Y, Z)X - g(X, Z)Y] \right], $$

(2.1)

where $\nabla$ is the covariant differentiation with respect to the Riemannian metric $g$. We know that $(\text{div } R)(X, Y)Z = (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)$. Hence contracting (2.1) we obtain

$$ (\text{div } C^*)(X, Y)Z = (a + b)((\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)) $$

$$ + \frac{1}{n} \left[ \frac{n-4}{2} - \frac{a}{n-1} \right] (g(Y, Z)d\gamma(X) - g(X, Z)d\gamma(Y)). $$

(2.2)

Here we consider quasi-conformally flat manifold i.e., $C^* = 0$. Hence $\text{div } C^* = 0$, where 'div' denotes the divergence. If $a + b \neq 0$, then from (2.2) it follows that

$$ (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) $$

$$ = \frac{1}{n(a + b)} \left[ a \frac{n}{n-1} - \frac{(n-4)b}{2} \right] [g(Y, Z)d\gamma(X) - g(X, Z)d\gamma(Y)]. $$
This can be written as
\[(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = \alpha [g(Y, Z)d\gamma(X) - g(X, Z)d\gamma(Y)], \quad (2.3)\]
where \(\alpha = \frac{1}{n(a + b)}\left[\frac{a}{n - 1} - \frac{(n - 4)b}{2}\right] = \text{constant.}\)

§3. Quasi-conformally flat manifold satisfying the condition (1.4)

From (1.1) we get
\[\tilde{C}^* (X, Y, Z, W) = a\tilde{R}(X, Y, Z, W) + b[S(Y, Z)g(X, W) - S(X, Z)g(Y, W)]
+ S(X, W)g(Y, Z) - S(Y, W)g(X, Z)]
- \frac{\gamma}{n}\left[\frac{a}{n - 1} + 2b][g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]\right]. \quad (3.1)\]
If the manifold is quasi-conformally flat under the assumption of \(\gamma \neq 0\), then we get
\[\gamma(a + (n - 2)b) = 0.\]
Then we note that \(\frac{(n - 4)b}{2} - \frac{a}{n - 1} = \frac{3na}{2(n - 1)(n - 2)}\). Since \(a \neq 0\) under the assumption of \(\gamma \neq 0\), we know that \(a + b \neq 0\) and \(\alpha \neq 0\). Moreover, from (1.4) we have
\[\tilde{R}(X, Y, Z, W)
= \frac{b}{a}[S(X, Z)g(Y, W) - S(Y, Z)g(X, W) + S(Y, W)g(X, Z) - S(X, W)g(Y, Z)]
+ \frac{\gamma}{na}\left[\frac{a}{n - 1} + 2b][g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]\right]. \quad (3.2)\]
Using (1.4) in (3.2), we obtain
\[\tilde{R}(X, Y, Z, W)
= \frac{\gamma b}{a}[g(Y, W)T(X)T(Z) - g(X, W)T(Y)T(Z) + g(X, Z)T(Y)T(W)
- g(Y, Z)T(X)T(W)] + \frac{\gamma}{na}\left[\frac{a}{n - 1} + 2b][g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]\right],\]
which implies that the manifold is a manifold of quasi-constant curvature.

Hence we can state that

**Theorem 1.** A quasi-conformally flat manifold satisfying the condition (1.4) under the assumption of \(\gamma \neq 0\) is a manifold of quasi-constant curvature.
§4. The results concerning the product manifold

From (1.4) we have

\[ (\nabla_Z S)(X, Y) = d\gamma(Z)T(X)T(Y) + \gamma[\nabla_Z T(X)T(Y) + T(X)(\nabla_Z T)(Y)]. \]  

(4.1)

Substituting (4.1) in (2.3), we get

\[ d\gamma(Z)T(X)T(Y) + \gamma[\nabla_Z T(X)T(Y) + T(X)(\nabla_Z T)(Y)] - d\gamma(X)T(Z)T(Y) - \gamma[\nabla_X T(Z)T(Y) + T(Z)(\nabla_X T)(Y)] = \alpha[g(X, Y)d\gamma(Z) - g(Z, Y)d\gamma(X)]. \]  

(4.2)

Putting \( Y = Z = e_i \) in the above expression where \( \{e_i\} \) is an orthonormal basis of the tangent space at each point of the manifold and taking summation over \( i, 1 \leq i \leq n \), we get

\[ \alpha(1 - n)d\gamma(X) = d\gamma(\xi)T(X) + \gamma(\nabla_\xi T)(X) + \gamma T(X)(\delta T) - d\gamma(X), \]  

(4.3)

where we put \( \delta T = \sum_{i=1}^{n}(\nabla_{e_i} T)(e_i) \). Again putting \( Y = Z = \xi \) in (4.2), it yields

\[ \gamma(\nabla_\xi T)(X) = (\alpha - 1)[d\gamma(\xi)T(X) - d\gamma(X)]. \]  

(4.4)

Substituting (4.4) in (4.3), we get

\[ \alpha(n - 2)d\gamma(X) - \alpha d\gamma(\xi)T(X) + \gamma\delta T = 0. \]  

(4.5)

Now putting \( X = \xi \) in (4.5), it yields

\[ \alpha(n - 3)d\gamma(\xi) + \gamma\delta T = 0. \]  

(4.6)

From (4.5) and (4.6) it follows that

\[ \alpha d\gamma(X) = \alpha d\gamma(\xi)T(X). \]

Since \( \alpha \neq 0 \), we have

\[ d\gamma(X) = d\gamma(\xi)T(X). \]  

(4.7)

Putting \( Y = \xi \) in (4.2) and using (4.7), we obtain

\[ (\nabla_X T)(Z) - (\nabla_Z T)(X) = 0, \]  

(4.8)

since \( \gamma \neq 0 \). This means that the 1-form \( T \) defined by \( g(X, \xi) = T(X) \) is closed, i.e., \( dT(X, Y) = 0 \). Hence it follows that

\[ g(\nabla_X \xi, Y) = g(\nabla_Y \xi, X) \]  

(4.9)
for all \(X, Y\). Now putting \(Y = \xi\) in (4.9), we get
\[
g(\nabla_X \xi, \xi) = g(\nabla_\xi \xi, X). \tag{4.10}
\]
Since \(g(\nabla_X \xi, \xi) = 0\), from (4.10) it follows that \(g(\nabla_\xi \xi, X) = 0\) for all \(X\). Hence \(\nabla_\xi \xi = 0\). This means that the integral curves of the vector field \(\xi\) are geodesic. Therefore we can state the following:

**Theorem 2.** In a quasi-conformally flat Riemannian manifold satisfying the condition (1.4) under the assumption of \(\gamma \neq 0\), the integral curves of the vector field \(\xi\) are geodesic.

From (4.4), by virtue of (4.7) we get
\[
(\nabla_\xi T)(Z) = 0, \tag{4.11}
\]
since \(\gamma \neq 0\). Now we consider the scalar function
\[
f = \frac{\alpha d\gamma(\xi)}{\gamma}.
\]
We have
\[
\nabla_X f = \frac{\alpha}{\gamma^2} \left[ d\gamma(\xi) T(\nabla_X \xi) \gamma - d\gamma(X) d\gamma(\xi) \right] + \frac{\alpha}{\gamma} d^2\gamma(\xi, X), \tag{4.12}
\]
where the Hessian \(d^2\gamma\) is defined by \(d^2\gamma(X, Y) = X(Y \gamma) - (\nabla_X Y) \gamma\). On the other hand, (4.7) implies that
\[
d^2\gamma(Y, X) = d^2\gamma(\xi, Y) T(X) + d\gamma(\xi) T(Y) T(X) + d\gamma(\xi) (\nabla_Y T)(X),
\]
from which we get
\[
d^2\gamma(\xi, Y) T(X) = d^2\gamma(\xi, X) T(Y), \tag{4.13}
\]
since \((\nabla_X T)(Y) = (\nabla_Y T)(X)\) and \(d^2\gamma(Y, X) = d^2\gamma(X, Y)\). Putting \(X = \xi\) in (4.13), it follows that
\[
d^2\gamma(\xi, Y) = d^2\gamma(\xi, \xi) T(Y),
\]
since \(T(\xi) = 1\). Thus
\[
\nabla_X f = \mu T(X), \tag{4.14}
\]
where \(\mu = \frac{\alpha}{\gamma} \left[ d^2\gamma(\xi, \xi) - \frac{d\gamma(\xi)}{\gamma} d\gamma(\xi) \right] \) and we used (4.7). Using (4.14), it is easy to show that
\[
\omega(X) = \frac{\alpha}{\gamma} d\gamma(\xi) T(X) = fT(X)
\]
is closed. In fact,
\[ d\omega(X, Y) = 0. \]

Using (4.7) and (4.8) in (4.2), we get
\[ \gamma [T(Z)(\nabla_X T)(Y) - T(X)(\nabla_Z T)(Y)] = \alpha d\gamma(\xi)[g(Y, Z)T(X) - g(X, Y)T(Z)]. \]

Now putting \( Z = \xi \) in the above expression it yields
\[ -(\nabla_X T)(Y) = \alpha \frac{d\gamma(\xi)}{\gamma} [T(X)T(Y) - g(X, Y)], \]
by (4.11). Thus (4.15) can be rewritten as follows:
\[ (\nabla_X T)(Y) = -fg(X, Y) + \omega(X)T(Y), \]
where \( \omega \) is closed. But this means that the vector field \( \xi \) defined by \( g(X, \xi) = T(X) \) is a proper concircular vector field ([7], [9]). Hence we can state the following:

**Theorem 3.** In a quasi-conformally flat manifold satisfying (1.4) under the assumption of \( \gamma \neq 0 \), the vector field \( \xi \) is a proper concircular vector field.

From (4.16) it follows that
\[ \nabla_X \xi = -fX + \omega(X)\xi. \]

Let \( \xi^\perp \) denote the \((n-1)\)-dimensional distribution in a quasi-conformally flat manifold orthogonal to \( \xi \). If \( X \) and \( Y \) belong to \( \xi^\perp \), then
\[ g(X, \xi) = 0 \]
and
\[ g(Y, \xi) = 0. \]

Since \( (\nabla_X g)(Y, \xi) = 0 \), it follows from (4.17) and (4.19) that
\[ g(\nabla_X Y, \xi) = g(\nabla_X \xi, Y) = -fg(X, Y). \]

Similarly, we get
\[ g(\nabla_Y X, \xi) = g(\nabla_Y \xi, X) = -fg(X, Y). \]

Hence
\[ g(\nabla_X Y, \xi) = (\nabla_Y X, \xi). \]
Now \([X, Y] = \nabla_X Y - \nabla_Y X\) and therefore by (4.20) we obtain
\[
g([X, Y], \xi) = g(\nabla_X Y - \nabla_Y X, \xi) = 0.
\]
Hence \([X, Y]\) is orthogonal to \(\xi\). That is, \([X, Y]\) belongs to \(\xi^\perp\). Thus the distribution \(\xi^\perp\) is involutive [2]. Hence from Frobenius’ theorem [2] it follows that \(\xi^\perp\) is integrable. This implies that if a quasi-conformally flat manifold satisfies (1.4), then it is a product manifold. We can therefore state the following theorem:

**Theorem 4.** If a quasi-conformally flat manifold satisfies (1.4) under the assumption of \(\gamma \neq 0\), then the manifold is a locally product manifold.

If a quasi-conformally flat manifold satisfies (1.4) under the assumption of \(\gamma \neq 0\), then in view of Theorem 3, \(\xi\) is a concircular vector field. Also, \(M\) is a quasi-constant curvature manifold and satisfies (1.2) and from Theorem 4 we know that \(\xi^\perp\) is integrable and it holds
\[
g(\nabla_X Y, \xi) = -(\nabla_X T)(Y)
\]
for the local vector fields \(X, Y\) belonging to \(\xi^\perp\). Thus from (4.15) the second fundamental form \(k\) for each leaf satisfies
\[
k(X, Y) = -\alpha \frac{d\gamma(\xi)}{\gamma} g(X, Y) \xi.
\]
Hence we know that each leaf is totally umbilic. Therefore each leaf is a manifold of constant curvature. Hence it must be a warped product \(I \times_{e^\gamma} M^*\) where \(M^*\) is an Einstein manifold. Thus we can state the following result (See [9], [5]):

**Theorem 5.** A quasi-conformally flat manifold satisfying (1.4) under the assumption of \(\gamma \neq 0\) can be expressed as a locally warped product \(I \times_{e^\gamma} M^*\) where \(M^*\) is an Einstein manifold.

**References**


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