Magic covering of chain of an arbitrary 2-connected simple graph

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Abstract. A simple graph $G = (V, E)$ admits an $H$-covering if every edge in $E$ belongs to a subgraph of $G$ isomorphic to $H$. We say that $G$ is $H$-magic if there is a total labeling $f : V \cup E \to \{1, 2, 3, \ldots, |V| + |E|\}$ such that for each subgraph $H' = (V', E')$ of $G$ isomorphic to $H$, $\sum_{v \in V'} f(v) + \sum_{e \in E'} f(e)$ is constant. When $f(V) = \{1, 2, \ldots, |V|\}$, then $G$ is said to be $H$-supermagic. In this paper we show that a chain of any 2-connected simple graph $H$ is $H$-supermagic.

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§1. Introduction

The concept of $H$-magic graphs was introduced in [2]. An edge-covering of a graph $G$ is a family of different subgraphs $H_1, H_2, \ldots, H_k$ such that each edge of $E$ belongs to at least one of the subgraphs $H_i$, $1 \leq i \leq k$. Then, it is said that $G$ admits an $(H_1, H_2, \ldots, H_k)$-edge covering. If every $H_i$ is isomorphic to a given graph $H$, then we say that $G$ admits an $H$-covering.

Suppose that $G = (V, E)$ admits an $H$-covering. We say that a bijective function $f : V \cup E \to \{1, 2, 3, \ldots, |V| + |E|\}$ is an $H$-magic labeling of $G$ if there is a positive integer $m(f)$, which we call magic sum, such that for each subgraph $H' = (V', E')$ of $G$ isomorphic to $H$, we have, $f(H') = \sum_{v \in V'} f(v) + \sum_{e \in E'} f(e) = m(f)$. In this case we say that the graph $G$ is $H$-magic. When $f(V) = \{1, 2, \ldots, |V|\}$, we say that $G$ is $H$-supermagic and we denote its supermagic-sum by $s(f)$.

We use the following notations. For any two integers $n < m$, we denote by $[n, m]$, the set of all consecutive integers from $n$ to $m$. For any set $I \subset \mathbb{N}$ we write $\sum I = \sum_{x \in I} x$ and for any integers $k$, $\mathbb{I} + k = \{x + k : x \in \mathbb{I}\}$. Thus
$k + [n, m]$ is the set of consecutive integers from $k + n$ to $k + m$. It can be easily verified that $\sum (\mathbb{I} + k) = \sum \mathbb{I} + k\mathbb{I}$. Finally, given a graph $G = (V, E)$ and a total labeling $f$ on it we denote by $f(G) = \sum f(V) + \sum f(E)$.

In [2], A. Gutierrez, and A. Llado studied the families of complete and complete bipartite graphs with respect to the star-magic and star-supermagic properties and proved the following results.

- The star $K_{1,n}$ is $K_{1,h}$-supermagic for any $1 \leq h \leq n$.
- Let $G$ be a $d$-regular graph. Then $G$ is not $K_{1,h}$-magic for any $1 < h < d$.
- (a) The complete graph $K_n$ is not $K_{1,h}$-magic for any $1 < h < n - 1$.
  (b) The complete bipartite graph $K_{n,n}$ is not $K_{1,h}$-magic for any $1 < h < n$.
- The complete bipartite graph $K_{n,n}$ is $K_{1,n}$-magic for $n \geq 1$.
- The complete bipartite graph $K_{n,n}$ is not $K_{1,n}$-supermagic for any integer $n > 1$.
- For any pair of integers $1 < r < s$, the complete bipartite graph $K_{r,s}$ is $K_{1,h}$-supermagic if and only if $h = s$.

The following results regarding path-magic and path-supermagic coverings are also proved in [2].

- The path $P_n$ is $P_h$-supermagic for any integer $2 \leq h \leq n$.
- Let $G$ be a $P_h$-magic graph, $h > 2$. Then $G$ is $C_h$-free.
- The complete graph $K_n$ is not $P_h$-magic for any $2 < h \leq n$.
- The cycle $C_n$ is $P_h$-supermagic for any integer $2 \leq h < n$ such that $\gcd(n, h(h - 1)) = 1$.

Also in [2], the authors constructed some families of $H$-magic graphs for a given graph $H$ by proving the following results.

- Let $H$ be any graph with $|V(H)| + |E(H)|$ even. Then the disjoint union $G = kH$ of $k$ copies of $H$ is $H$-magic.

Let $G$ and $H$ be two graphs and $e \in E(H)$ a distinguished edge in $H$. We denote by $G \ast eH$ the graph obtained from $G$ by gluing a copy of $H$ to each edge of $G$ by the distinguished edge $e \in E(H)$.

- Let $H$ be a 2-connected graph and $G$ an $H$-free supermagic graph. Let $k$ be the size of $G$ and $h = |V(H)| + |E(H)|$. Assume that $h$ and $k$ are not both even. Then, for each edge $e \in E(H)$, the graph $G \ast eH$ is $H$-magic.
In [3], P. Selvagopal and P. Jeyanthi proved that for any positive integer \( n \), a \( k \)-polygonal snake of length \( n \) is \( C_k \)-supermagic.

In this paper we construct a chain graph \( H_n \) of 2-connected graph \( H \) of length \( n \), and prove that a chain graph \( H_n \) is \( H \)-supermagic.

§2. Preliminary Results

Let \( P = \{X_1, X_2, \ldots, X_k\} \) be partition set of a set \( X \) of integers. When all sets have the same cardinality we say then \( P \) is a \( k \)-equipartition of \( X \). We denote the set of subsets sums of the parts of \( P \) by \( \sum P = \{\sum X_1, \sum X_2, \ldots, \sum X_k\} \).

The following lemmas are proved in [2].

Lemma 1. Let \( h \) and \( k \) be two positive integers and let \( n = hk \). For each integer \( 0 \leq t \leq \lfloor \frac{h}{2} \rfloor \) there is a \( k \)-equipartition \( P \) of \([1, n] \) such that \( \sum P \) is an arithmetic progression of difference \( d = h - 2t \).

Lemma 2. Let \( h \) and \( k \) be two positive integers and let \( n = hk \). In the two following cases there exists a \( k \)-equipartition \( P \) of a set \( X \) such that \( \sum P \) is a set of consecutive integers.

(i) \( h \) or \( k \) are not both even and \( X = [1, hk] \).

(ii) \( h = 2 \) and \( k \) is even and \( X = [1, hk + 1] - \{\frac{k}{2} + 1\} \).

We have the following four results from the above two lemmas.

(a) If \( h \) is odd, then there exists a \( k \)-equipartition \( P = \{X_1, X_2, \ldots, X_k\} \) of \( X = [1, hk] \) such that \( \sum P \) is a set of consecutive integers and \( \sum P = \frac{h(hk + k + 1)}{2} + [1, k] \).

(b) If \( h \) is even, then there exists a \( k \)-equipartition \( P = \{X_1, X_2, \ldots, X_k\} \) of \( X = [1, hk] \) such that subsets sum are equal and is equal to \( \frac{h(hk + 1)}{2} \).

(c) If \( h \) is even and \( k \) is odd, then there exists a \( k \)-equipartition \( P = \{X_1, X_2, \ldots, X_k\} \) of \( X = [1, hk] \) such that \( \sum P \) is a set of consecutive integers and \( \sum P = \frac{h(hk + k + 1)}{2} + [\frac{k-1}{2}, \frac{k}{2}] \).

(d) If \( h = 2 \) and \( k \) is even, and \( X = [1, 2k + 1] - \{\frac{k}{2} + 1\} \) then there exists a \( k \)-equipartition \( P = \{X_1, X_2, \ldots, X_k\} \) of \( X \) such that \( \sum P \) is a set of consecutive integers and \( \sum P = \left[\frac{3k}{2} + 3, \frac{5k}{2} + 2\right] \).

We generalise the second part of Lemma 2.

Corollary 1. Let \( h \) and \( k \) be two even positive integers and \( h \geq 4 \). If \( X = [1, hk + 1] - \{\frac{k}{2} + 1\} \), there exists a \( k \)-equipartition \( P \) of \( X \) such that \( \sum P \) is a set of consecutive integers.
Proof. Let $Y = [1, 2k + 1] - \{ \frac{k}{2} + 1 \}$ and $Z = (2k + 1) + [1, (h - 2)k]$. Then $X = Y \cup Z$. By (d), there exists a $k$-equipartition $P_1 = \{ Y_1, Y_2, \ldots, Y_k \}$ of $Y$ such that 

$$\sum P_1 = \left[ \frac{3k}{2} + 3, \frac{5k}{2} + 2 \right].$$

As $h - 2$ is even, by (b) there exists a $k$-equipartition $P'_2 = \{ Z'_1, Z'_2, \ldots, Z'_k \}$ of $[1, (h - 2)k]$ such that 

$$\sum P'_2 = \left\{ \frac{(h - 2)(hk - 2k + 1)}{2} \right\}.$$

Hence, there exists a $k$-equipartition $P_2 = \{ Z_1, Z_2, \ldots, Z_k \}$ of $Z$ such that 

$$\sum P_2 = \left\{ (h - 2)(2k + 1) + \frac{(h - 2)(hk - 2k + 1)}{2} \right\}.$$

Let $X_i = Y_i \cup Z_i$ for $1 \leq i \leq k$. Then $P = \{ X_1, X_2, \ldots, X_k \}$ is a $k$-equipartition of $X$ such that $\sum P$ is a set of consecutive integers and 

$$\sum P = (h - 2)(2k + 1) + \frac{(h - 2)(hk - 2k + 1)}{2} + \left[ \frac{3k}{2} + 3, \frac{5k}{2} + 2 \right].$$

\[\square\]

§3. Chain of an arbitrary simple connected graph

Let $H_1, H_2, \ldots, H_n$ be copies of a graph $H$. Let $u_i$ and $v_i$ be two distinct vertices of $H_i$ for $i = 1, 2, \ldots, n$. We construct a chain graph $H_n$ of $H$ of length $n$ by identifying two vertices $u_i$ and $v_{i+1}$ for $i = 1, 2, \ldots, n - 1$. See Figures 1 and 2.

§4. Main Result

**Theorem 1.** Let $H$ be a 2-connected $(p, q)$ simple graph. Then $H_n$ is $H$-supermagic if any one of the following conditions is satisfied.

(i) $p + q$ is even

(ii) $p + q + n$ is even

**Proof.** Let $G = (V, E)$ be a chain of $n$ copies of $H$. Let us denote the $i$th copy of $H$ in $H_n$ by $H_i = (V_i, E_i)$. Note that $|V| = np - n + 1$ and $|E| = nq$. Moreover, we remark that by $H$ is a 2-connected graph, $H_n$ does not contain a subgraph $H$ other than $H_i$. 
Let $v_i$ be the vertex in common with $H_i$ and $H_{i+1}$ for $1 \leq i \leq n - 1$. Let $v_0$ and $v_n$ be any two vertices in $H_1$ and $H_n$ respectively so that $v_0 \neq v_1$ and $v_n \neq v_{n-1}$. Let $V_i' = V_i - \{v_{i-1}, v_i\}$ for $1 \leq i \leq n$.

**Case (i):** $p + q$ is even

Suppose $p$ and $q$ are odd. As $p - 2$ is odd, by (a) there exists an $n$-equipartition $P'_1 = \{X'_1, X'_2, \ldots, X'_n\}$ of $[1, n(p - 2)]$ such that

$$\sum P'_1 = \frac{(p - 3)(np - n + 1)}{2} + [1, n].$$

Adding $n+1$ to $[1, n(p - 2)]$, we get an $n$-equipartition $P_1 = \{X_1, X_2, \ldots, X_n\}$ of $[n + 2, np - n + 1]$ such that

$$\sum P_1 = (p - 2)(n + 1) + \frac{(p - 3)(np - n + 1)}{2} + [1, n]$$

Similarly, since $q$ is odd there exists an $n$-equipartition $P_2 = \{Y_1, Y_2, \ldots, Y_n\}$ of $(np - n + 1) + [1, nq]$ such that

$$\sum P_2 = q(np - n + 1) + \frac{(q - 1)(nq + n + 1)}{2} + [1, n]$$

Define a total labeling $f : V \cup E \rightarrow \{1, 2, 3, \ldots, np + nq - n + 1\}$ as follows:

(i) $f(v_i) = i + 1$ for $0 \leq i \leq n$.

(ii) $f(V'_i) = X_{n-i+1}$ for $1 \leq i \leq n$.

(iii) $f(E_i) = Y_{n-i+1}$ for $1 \leq i \leq n$.

Then for $1 \leq i \leq n$,

$$f(H_i) = f(v_{i-1}) + f(v_i) + \sum f(V'_i) + \sum f(E_i)$$

$$= f(v_{i-1}) + f(v_i) + \sum X_{n-i+1} + \sum Y_{n-i+1}$$

$$= \frac{n(p + q)^2 + 3(p + q) - 2n(p + q) + 2n - 2}{2}$$

As $H_i \cong H$ for $1 \leq i \leq n$, $Hn$ is $H$-supermagic.

Suppose both $p$ and $q$ are even. As $p$ is even, by Lemma 1, there exists an $n$-equipartition $P'_1 = \{X'_1, X'_2, \ldots, X'_n\}$ of $[1, n(p - 2)]$ such that $\sum P'_1$ is arithmetic progression of difference 2 and

$$\sum P'_1 = \left\{ n \left[(p - 2)^2 - 2\right] + p - 4 + 2r : 1 \leq r \leq n \right\}.$$
Adding \( n+1 \) to \([1, n(p-2)]\), we get an \( n\)-equipartition \( P_1 = \{X_1, X_2, \ldots, X_n\}\) of \([n + 2, np - n + 1]\) such that
\[
\sum P_1 = \left\{(p-2)(n+1) + \frac{n[(p-2)^2 - 2] + p - 4}{2} + 2i : 1 \leq i \leq n\right\}
\]

As \( q \) is even, by (b), there exists an \( n\)-equipartition \( P'_2 = \{Y'_1, Y'_2, \ldots, Y'_n\}\) of \([1, nq]\) such that \( \sum P'_2 = \left\{\frac{q(nq+1)}{2}\right\}\).

Adding \( np - n + 1 \) to \([1, nq]\) there exists an \( n\)-equipartition \( P_2 = \{Y_1, Y_2, \ldots, Y_n\}\) of \((np - n + 1) + [1, nq]\) such that
\[
\sum P_2 = \left\{q(np - n + 1) + \frac{q(nq+1)}{2}\right\}
\]

Define a total labeling \( f : V \cup E \rightarrow \{1, 2, 3, \ldots, np + nq - n + 1\}\) as follows:

(i) \( f(v_i) = i + 1 \) for \( 0 \leq i \leq n \).

(ii) \( f(V'_i) = X_{n-i+1} \) for \( 1 \leq i \leq n \).

(iii) \( f(E_i) = Y_{n-i+1} \) for \( 1 \leq i \leq n \).

Then for \( 1 \leq i \leq n \),
\[
f(H_i) = f(v_{i-1}) + f(v_i) + \sum f(V'_i) + \sum f(E_i)
= f(v_{i-1}) + f(v_i) + \sum X_{n-i+1} + \sum Y_{n-i+1}
= \frac{n(p+q)^2 + 3(p+q) - 2n(p+q) + 2n - 2}{2}
\]

As \( H_i \cong H \) for \( 1 \leq i \leq n \), \( Hn \) is \( H\)-supermagic.

Case (ii): \( p + q + n \) is even: Suppose \( p \) is odd, \( q \) is even and \( n \) is odd. Since \( p \) is odd as in proof of Case (i), there exists an \( n\)-equipartition \( P_1 = \{X_1, X_2, \ldots, X_n\}\) of \([n + 2, np - n + 1]\) such that
\[
\sum P_1 = (p-2)(n+1) + \frac{(p-3)(np - n + 1)}{2} + [1, n]
\]

Since \( q \) is even and \( n \) is odd, by (c) there exists an \( n\)-equipartition \( P'_2 = \{Y'_1, Y'_2, \ldots, Y'_n\}\) of \([1, nq]\) such that
\[
\sum P'_2 = \frac{q(nq+1)}{2} + \left[-\frac{n-1}{2}, \frac{n-1}{2}\right].
\]

Adding \( np-n+1 \) to \([1, nq]\) there exists an \( n\)-equipartition \( P_2 = \{Y_1, Y_2, \ldots, Y_n\}\) of \((np - n + 1) + [1, nq]\) such that
\[
\sum P_2 = q(np - n + 1) + \frac{q(nq+1)}{2} + \left[-\frac{n-1}{2}, \frac{n-1}{2}\right]
\]

Define a total labeling \( f : V \cup E \rightarrow \{1, 2, 3, \ldots, np + nq - n + 1\}\) as follows:
(i) \( f(v_i) = i + 1 \) for \( 0 \leq i \leq n \).

(ii) \( f(V'_i) = X_{n-i+1} \) for \( 1 \leq i \leq n \).

(iii) \( f(E_i) = Y_{n-i+1} \) for \( 1 \leq i \leq n \).

Then for \( 1 \leq i \leq n \),
\[
f(H_i) = f(v_{i-1}) + f(v_i) + \sum f(V'_i) + \sum f(E_i)
= f(v_{i-1}) + f(v_i) + \sum X_{n-i+1} + \sum Y_{n-i+1}
= \frac{n(p + q)^2 + 3(p + q) - 2n(p + q) + 2n - 2}{2}
\]

As \( H_i \cong H \) for \( 1 \leq i \leq n \), \( H_n \) is \( H \)-supermagic.

Suppose \( p \) is even, \( q \) is odd and \( n \) is odd. Since \( p - 2 \) is even and \( n \) is odd, by (c) there exists an \( n \)-equipartition \( P'_1 = \{X'_1, X'_2, \ldots, X'_n\} \) of \([1, n(p-2)]\) such that
\[
\sum P'_1 = (p-2)[n(p-2)+1] + \left[ -\frac{n-1}{2}, \frac{n-1}{2} \right].
\]

Adding \( n+1 \) to \([1, n(p-2)]\), we get an \( n \)-equipartition \( P_1 = \{X_1, X_2, \ldots, X_n\} \) of \([n+2, np-n+1]\) such that
\[
\sum P_1 = (p-2)(n+1) + \frac{(p-2)[n(p-2)+1]}{2} + \left[ -\frac{n-1}{2}, \frac{n-1}{2} \right].
\]

Since \( q \) is odd, as in Case (i) there exists an \( n \)-equipartition \( P_2 = \{Y_1, Y_2, \ldots, Y_n\} \) of \((np-n+1) + [1, nq]\) such that
\[
\sum P_2 = q(np-n+1) + \frac{(q-1)(nq+n+1)}{2} + [1, n]
\]

Define a total labeling \( f : V \cup E \to \{1, 2, 3, \ldots, np+nq-n+1\} \) as follows:

(i) \( f(v_i) = i + 1 \) for \( 0 \leq i \leq n \).

(ii) \( f(V'_i) = X_{n-i+1} \) for \( 1 \leq i \leq n \).

(iii) \( f(E_i) = Y_{n-i+1} \) for \( 1 \leq i \leq n \).

Then for \( 1 \leq i \leq n \),
\[
f(H_i) = f(v_{i-1}) + f(v_i) + \sum f(V'_i) + \sum f(E_i)
= f(v_{i-1}) + f(v_i) + \sum X_{n-i+1} + \sum Y_{n-i+1}
= \frac{n(p + q)^2 + 3(p + q) - 2n(p + q) + 2n - 2}{2}
\]

As \( H_i \cong H \) for \( 1 \leq i \leq n \), \( H_n \) is \( H \)-supermagic. \( \square \)
§5. Illustrations

A chain of a 2-connected \((5, 7)\) simple graph \(H\) of length 5 is shown in Figure 1 and a chain of a 2-connected \((6, 9)\) simple graph \(H\) of length 3 is shown in Figure 2.

Figure 1. \(p = 5, q = 7, s(f) = 322\).

Figure 2. \(p = 6, q = 9, s(f) = 317\).
References


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