Generalized finite operators and orthogonality

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Abstract. In this paper we prove that a spectraloid operator is finite, we present some generalized finite operators and we give a new class of finite operators. Also, the orthogonality of some operators is studied.

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§1. Introduction

Let $H$ be a separable infinite dimensional complex Hilbert space, and let $\mathcal{L}(H)$ denote the algebra of all bounded linear operators on $H$. For $A, B \in \mathcal{L}(H)$, the generalized derivation $\delta_{A,B} : \mathcal{L}(H) \to \mathcal{L}(H)$ is defined by

$$\delta_{A,B}(X) = AX - XB.$$  

We denote $\delta_{A,A}$ by $\delta_A$. Let $E$ be a complex Banach space. We say [1] that $b \in E$ is orthogonal to $a \in E$ if for all complex $\lambda$ there holds $\|a + \lambda b\| \geq \|a\|$. An operator $A \in \mathcal{L}(H)$ is called finite by J. P. Williams [12] if $\|AX - XA - I\| \geq 1$ for all $X \in \mathcal{L}(H)$, i.e. the range of $\delta_A$ is orthogonal to the identity operator. The pair $(A, B) \in \mathcal{L}(H) \times \mathcal{L}(H)$ is said to be generalized finite operators [7] if $\|AX - XB - I\| \geq 1$ for all $X \in \mathcal{L}(H)$, $\mathcal{F}(H)$ and $\mathcal{G}(H)$ denote the class of finite operators and the class of generalized finite operators respectively.

For $A \in \mathcal{L}(H)$ the set $W(A) = \{(Ax, x) : x \in H \text{ and } \|x\| = 1\}$ is called the numerical range of $A$.

In the following we will denote the spectrum, the point spectrum, the approximate spectrum and the approximate reducing spectrum of $A \in \mathcal{L}(H)$ by $\sigma(A)$, $\sigma_p(A)$, $\sigma_a(A)$ and $\sigma_{ar}(A)$ respectively.

An operator $A \in \mathcal{L}(H)$ is said to be spectraloid if $\omega(A) = r(A)$, where $r(A)$ (resp. $\omega(A)$) is the spectral radius (resp. numerical radius) of $A$, convexoid.
if $W(A) = \cos(A)$, where $\cos(A)$ is the convex hull of $\sigma(A)$, and transaloid
if $r\left((A - \lambda I)^{-1}\right) = \| (A - \lambda I)^{-1} \|$ for all $\lambda \notin \sigma(A)$. We have the following
inclusions:

- paranormal $\rightarrow$ normaloid
- hyponormal $\rightarrow$ spectraloid
- transaloid $\rightarrow$ convexoid

A bounded linear operator $A$ is in the class $\mathcal{Y}_\alpha$ for certain $\alpha \geq 1$ if there
exists a positive number $k_\alpha$ such that

$$|AA^* - A^*A|^{\alpha} \leq k_\alpha^2 ( (A - \lambda I)^* (A - \lambda I) ) \quad \text{for all } \lambda \in \mathbb{C}.$$

It is known that $\mathcal{Y}_\alpha \subseteq \mathcal{Y}_\beta$ for each $\alpha, \beta$ such as $1 \leq \alpha \leq \beta$ [11], where
$\mathcal{Y} = \cup_{\alpha \geq 1} \mathcal{Y}_\alpha$.

In this paper we prove that a spectraloid operator is finite and that the
operator of the form $A + K$ is also finite, where $A$ is convexoid and $K$ is
compact. We present some generalized finite operators and we give a new
class of finite operators. Also we study the orthogonality of certain operators.

§2. Preliminaries

**Lemma 1.** Let $A \in \mathcal{L}(H)$. If $\sigma_{ar}(A)$ is not empty, then $A$ is finite.

Proof. Let $\lambda \in \sigma_{ar}(A)$ and $\{x_n\}$ be a normalized sequence such that $(A - \lambda I) x_n \rightarrow 0$ and $(A - \lambda I)^* x_n \rightarrow 0$. If $X \in \mathcal{L}(H)$, then we have

$$\| AX - X A - I \| = \| (A - \lambda I) X - X (A - \lambda I) - I \| \geq | \langle (A - \lambda I) X x_n, x_n \rangle - \langle X (A - \lambda I) x_n, x_n \rangle - 1 \|.$$

Letting $n \rightarrow \infty$, we obtain $\| AX - X A - I \| \geq 1$. \qed

**Lemma 2.** Let $A \in \mathcal{L}(H)$. If $\Re A \geq 0$, then $\sigma_{a}(A) \subset \sigma_{ar}(A)$.

Proof. For $\lambda \in \sigma_a(A)$, there exists a sequence $\{x_n\}$ such that $(A - \lambda I) x_n \rightarrow 0$, and then

$$B = \Re (A - \lambda I) = \frac{1}{2} [(A - \lambda I) + (A - \lambda I)^*]$$

satisfies $\langle Bx_n, x_n \rangle \rightarrow 0$. Since $B \geq 0$, it results that $Bx_n \rightarrow 0$, i.e,

$$\frac{1}{2} [(A - \lambda I) x_n + (A - \lambda I)^* x_n] \rightarrow 0.$$

Since $(A - \lambda I) x_n \rightarrow 0$, we have $(A - \lambda I)^* x_n \rightarrow 0$. \qed
Lemma 3. For $A \in \mathcal{L}(H)$, $\partial W(A) \cap \sigma(A) \subset \sigma_{ar}(A)$.

Proof. By the transformation $A \mapsto \alpha A + \beta$ the hypothesis $\lambda \in \partial W(A) \cap \sigma(A)$ can be replaced by $0 \in \partial W(A) \cap \sigma(A)$ with $\Re e A \geq 0$. Since $0 \in \partial \sigma(A) \subset \sigma_{a}(A)$, it results from the previous lemmas that $0 \in \sigma_{ar}(A)$, hence $\partial W(A) \cap \sigma(A) \subset \sigma_{ar}(A)$.

§3. Main results

Theorem 1. Let $A \in \mathcal{L}(H)$ be convexoid. Then $A$ is finite.

Proof. If $A$ is convexoid, then $W(A) = \text{co} \sigma(A)$. Hence $\partial W(A) \cap \sigma(A) \neq \emptyset$. It follows immediately from the previous lemmas that $A$ is finite.

Remark 1. It is known that transaloid operators are convexoid operators, and then $\mathcal{F}(H)$ contains all the transaloid operators.

Theorem 2. Let $A \in \mathcal{L}(H)$ be spectraloid. Then $A$ is finite.

Proof. We have $\omega(A) = r(A)$. This implies that there exists $\lambda \in \sigma(A) \subset W(A)$ such that $|\lambda| = \omega(A)$, hence $\lambda \in \partial W(A)$, then $\partial W(A) \cap \sigma(A) \neq \emptyset$, which implies that $A \in \mathcal{F}(H)$.

As a consequence of the previous theorem we obtain:

Corollary 1. The following operators are finite.

(1) Hyponormal operators,
(2) Transaloid operators,
(3) Paranormal operators,
(4) Normaloid operators.

Lemma 4. [9] For $A \in \mathcal{L}(H)$, the following holds

$$W(A) = \text{co} \sigma(A) \iff \forall \lambda \notin \sigma(A) : \left\| (A - \lambda I)^{-1} \right\| \leq \left[ \text{dist}(\lambda, \text{co} \sigma(A)) \right].$$

Hence a convexoid element on a $C^*$-algebra $\mathcal{A}$, may be defined as an element $a \in \mathcal{A}$ satisfying

$$\forall \lambda \notin \sigma(a) : \left\| (a - \lambda e)^{-1} \right\| \leq \left[ \text{dist}(\lambda, \sigma(a)) \right]^{-1}.$$
Proof. It is known [6, p. 97] that there exist a Hilbert space $H$ and a *-isometric homomorphism $\varphi (\varphi : A \rightarrow \mathcal{L}(H))$. Then $\varphi (a)$ is convexoid. Since $\varphi$ is isometric it results from Theorem 1 that $a$ is finite. 

\begin{corollary}
Let $A \in \mathcal{L}(H)$ be convexoid. Then $T = A + K$ is finite, where $K$ is a compact operator.
\end{corollary}

Proof. Since the Calkin algebra $\mathcal{L}(H) / \mathcal{K}(H)$ is a $C^*$-algebra (where $\mathcal{K}(H)$ is the set of compact operators), $[A] = \{ A + K : K \in \mathcal{K}(H) \}$ is convexoid. Hence it follows from Theorem 3 $[A]$ is finite and we have, for all $X \in \mathcal{L}(H)$

$$\|TX - XT - I\| = \||[TX - XT - I]\| = \||[T][X] - [X][T] - [I]\| = \||[A][X] - [X][A] - [I]\| \geq 1.$$ 

\begin{lemma}
For $A, T \in \mathcal{L}(H)$, if $A \in \mathcal{Y}$ and $T$ is a normal operator such as $AT = TA$, then for all $\lambda \in \sigma_p(T)$

$$\|AX - XA - T\| \geq |\lambda|, \text{ for all } X \in \mathcal{L}(H).$$

\end{lemma}

Proof. Let $\lambda \in \sigma_p(T)$ and $M_\lambda$ be the eigenspace associated with $\lambda$. Since $AT = TA$, we have $AT^* = T^*A$ by the Fuglede's theorem [4]. Hence $M_\lambda$ reduces both $A$ and $T$. According to the decomposition $H = M_\lambda \oplus M_\lambda^\perp$, we can write $A$, $T$ and $X \in \mathcal{L}(H)$ as follows:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad T = \begin{bmatrix} \lambda & 0 \\ 0 & T_2 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.$$ 

Since the restriction of a class $\mathcal{Y}$ operator to a reduced subspace is a class $\mathcal{Y}$ operator and since $\mathcal{Y} \subset \mathcal{F}(H)$ [2], we have

$$\|AX - XA - T\| = \||[A_1X_1 - X_1A_1 - \lambda \quad * \quad *]|| \\
\geq \|A_1X_1 - X_1A_1 - \lambda\| \\
\geq |\lambda| \|A_1(X_1/\lambda) - (X_1/\lambda)A_1 - I\| \\
\geq |\lambda|.$$ 

\begin{lemma}
For $A, T \in \mathcal{L}(H)$, if $A \in \mathcal{Y}$ and $T$ is a normal operator such as $AT = TA$, then for all $\lambda \in \sigma_p(T)$

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Since the restriction of a class $\mathcal{Y}$ operator to a reduced subspace is a class $\mathcal{Y}$ operator and since $\mathcal{Y} \subset \mathcal{F}(H)$ [2], we have

$$\|AX - XA - T\| = \||[A_1X_1 - X_1A_1 - \lambda \quad * \quad *]|| \\
\geq \|A_1X_1 - X_1A_1 - \lambda\| \\
\geq |\lambda| \|A_1(X_1/\lambda) - (X_1/\lambda)A_1 - I\| \\
\geq |\lambda|.$$ 

\end{lemma}
In the sequel, we need the Berberian technique, and it allows us to construct a Hilbert space which contains a given Hilbert space $H$ on which we could speak about "approached eigenvectors" and those as regarded as eigenvectors.

**Proposition 1** (Berberian technique). Let $H$ be a complex Hilbert space, then there exist a Hilbert space $bH \supset H$ and an $*$-isometric homomorphism $\varphi : \mathcal{L}(H) \rightarrow \mathcal{L}(bH)$ ($A \mapsto \hat{A}$) preserving the order, i.e. for all $A, B \in \mathcal{L}(H)$ and for all $\alpha, \beta \in \mathbb{C}$ we have:

1. $\hat{A}^* = \hat{A}^*$,
2. $\hat{I} = I$,
3. $\alpha \hat{A} + \beta \hat{B} = \alpha \hat{A} + \varphi \hat{B}$,
4. $\hat{A} \hat{B} = \hat{A} \hat{B}$,
5. $\|\hat{A}\| = \|A\|$,
6. $\sigma(\hat{A}) = \sigma(A)$, $\sigma_p(\hat{A}) = \sigma_a(\hat{A}) = \sigma_a(A)$,
7. If $A$ is positive, then $\hat{A}$ is positive and $\hat{A}^\alpha = \hat{A}^\alpha$ for all $\alpha > 0$.

**Theorem 4.** Let $A \in \mathcal{Y}$. Then for every normal operator $T$ such that $AT = TA$, we have

$$\|AX -XA - T\| \geq \|T\|, \text{ for all } X \in \mathcal{L}(H).$$

**Proof.** Let $\lambda \in \sigma(T) = \sigma_a(T)$ [5]. Then it follows from Proposition 1 that $\hat{T}$ is normal, $\hat{A} \in \mathcal{Y}$, $\hat{A} \hat{T} = \hat{T} \hat{A}$ and $\lambda \in \sigma_p(\hat{T})$. By applying Lemma 5, we get

$$\|AX -XA - T\| = \|\hat{A} X - \hat{X} \hat{A} - \hat{T}\| \geq |\lambda|,$$

for all $X \in \mathcal{L}(H)$. Hence

$$\|AX -XA - T\| \geq \sup_{\lambda \in \sigma(\hat{T})} |\lambda| = r(\hat{T}) = \|\hat{T}\| = \|T\|, \text{ for all } X \in \mathcal{L}(H).$$

**Theorem 5.** Let $A, B \in \mathcal{L}(H)$. If $A, B^* \in \mathcal{Y}$, then

$$\|AX - XB - T\| \geq \|T\|, \text{ for all } X \in \mathcal{L}(H) \text{ and for all } T \in \ker \delta_{A,B}. $$
Proof. Let $T \in \ker \delta_{A,B}$. Then $T \in \ker \delta_{A^*,B^*}$ [11, Theorem 2]. Therefore, $ATT^* = TBT^* = TT^*A$. Since $A \in \mathcal{Y}$, $TT^*$ is normal and $A(TT^*) = (TT^*)A$, the previous theorem implies that

$$
\|T\|^2 = \|TT^*\| \leq \|TT^* - (AXT^* - XT^*A)\|
= \|TT^* - (AXT^* - XBT^*)\|
\leq \|T\| \|T - (AX - XB)\|.
$$

Thus

$$
\|AX - XB - T\| \geq \|T\|.
$$

\[\Box\]

Theorem 6. Let $A, B \in \mathcal{L}(H)$ be $A = \sum_{i=1}^{n} A_i$, $B = \sum_{i=1}^{n} B_i$. If there exists $j \leq n$ such that $(A_j, B_j) \in \mathcal{GF}(H_j)$, then $(A, B) \in \mathcal{GF}(H)$.

Proof. Let $j \leq n$ such that $(A_j, B_j) \in \mathcal{GF}(H_j)$. Then for all $X \in \mathcal{L}(H)$

$$
\|AX - XB - I\| = \left\| \begin{pmatrix}
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & *
\end{pmatrix}
\right\|
\geq \|A_jX_{jj} - X_{jj}B_j - I_j\|
\geq 1.
$$

\[\Box\]

Proposition 2. For $(A, B) \in \mathcal{GF}(H)$, the following assertions hold:

1. $(\alpha A + \beta, \alpha B + \beta) \in \mathcal{GF}(H)$, for each $\alpha, \beta \in \mathbb{C}$.
2. $(A^{-1}, B^{-1}) \in \mathcal{GF}(H)$, if $A$ and $B$ are invertible.
3. $(R, T) \in \mathcal{GF}(H)$, if $R$ and $T$ are simultaneously unitarily equivalent to $A$ and $B$ respectively.
4. $(B^*, A^*) \in \mathcal{GF}(H)$.
5. $(A^{2m}, B^{2m}) \in \mathcal{GF}(H)$, for all $m \in \mathbb{N}$.
6. $\sigma(A) \cap \sigma(B) \neq \emptyset$. 

Proof. See [11, Theorem 2].
Then we have
\[ f((\alpha A + \beta)X) = f(X(\alpha B + \beta)) \]
for all \( X \in \mathcal{L}(H) \).

(2) Let \( f \) be a state on \( \mathcal{L}(H) \) such that \( f(AX) = f(XB) \) for all \( X \in \mathcal{L}(H) \).
Then we have
\[ f(A^{-1}X) = f((A^{-1}XB^{-1}) B) = f(A(A^{-1}XB^{-1})) = f(XB^{-1}) \]
for all \( X \in \mathcal{L}(H) \).

(3) Let \( U \) be a unitary operator. Then by [7, Theorem 18] we have
\[ (A, B) \in \mathcal{GF}(H) \iff 0 \in W(AX - XB), \forall X \in \mathcal{L}(H) \]
\[ \iff 0 \in W(U^*(AX - XB)U), \forall X \in \mathcal{L}(H) \]
\[ \iff 0 \in W(U^*(AUU^*X - XUU^*B)U), \forall X \in \mathcal{L}(H) \]
\[ \iff 0 \in W((U^*AU)Y - Y(U^*BU)), \forall Y \in \mathcal{L}(H) \]
\[ \iff (U^*AU, U^*BU) \in \mathcal{GF}(H). \]

(4) Let \( f \) be a state on \( \mathcal{L}(H) \) such that \( f(AX) = f(XB) \) for all \( X \in \mathcal{L}(H) \).
Then we have
\[ f^*(B^*X) = \overline{f(B^*X)} = \overline{f(XB)} \]
\[ = \overline{f(AX^*)} = f(XA^*)^* \]
\[ = f^*(XA^*), \]
for all \( X \in \mathcal{L}(H) \). Since the adjoint of a state is a state, we have \( (B^*, A^*) \in \mathcal{GF}(H) \).

(5) Let \( f \) be a state on \( \mathcal{L}(H) \) such that \( f(AX - XB) = 0 \) for all \( X \in \mathcal{L}(H) \).
By recurrence we have:
For \( m = 0 \), \( (A^{2^0}, B^{2^0}) = (A, B) \in \mathcal{GF}(H) \). Suppose that, for all \( m \in \mathbb{N} \),
there exists a state \( f \) on \( \mathcal{L}(H) \) such that
\[ f(A^{2^m}X - XB^{2^m}) = 0, \text{ for all } X \in \mathcal{L}(H). \]
Then
\[ f(A^{2^m}(A^{2^m}X) - (A^{2^m}X)B^{2^m}) = 0 \]
and
\[ f(A^{2^m}(XB^{2^m}) - (XB^{2^m})B^{2^m}) = 0, \]
hence
\[ f(A^{2^{m+1}}X - XB^{2^{m+1}}) = 0. \]

(6) Suppose that \( \sigma(A) \cap \sigma(B) = \phi \). In [10] M. Rosenblum proved that
\( \sigma(\delta_{A,B}) \subset \sigma(A) - \sigma(B) \), and then \( \delta_{A,B} \) is invertible, hence there exists \( X \in \mathcal{L}(H) \) for which \( ||\delta_{A,B}(X) - I|| < 1. \) \( \square \)
Theorem 7. Let $A, B \in \mathcal{L}(H)$. If there exist a normed sequence $(f_n)_{n \geq 1}$ in $H$ and a scalar $\lambda$ such that

$$
\|(A - \lambda I)^* f_n\| \longrightarrow 0 \quad \text{and} \quad \|(B - \lambda I) f_n\| \longrightarrow 0,
$$

then $(A, B) \in \mathcal{GF}(H)$.

Proof. If $X \in \mathcal{L}(H)$. Then

$$
\|AX - XB - I\| = \|(A - \lambda I) X - X (B - \lambda I) - I\|
\geq \|[(A - \lambda I) X - X (B - \lambda I) - I] f_n, f_n]\|
= \|(X f_n, (A - \lambda I)^* f_n) - ((B - \lambda I) f_n, X^* f_n) - 1\|.
$$

By passage to the limit, we get $\|AX - XB - I\| \geq 1$, for all $X \in \mathcal{L}(H)$. □

Corollary 3. Let $A \in \mathcal{L}(H)$. Then, for all $\lambda \in \sigma_a(A)$ and for all $C \in \mathcal{L}(H)$,

$$
((A - \lambda I)^*, C (A - \lambda I)) \in \mathcal{GF}(H).
$$

Proof. Let $\lambda \in \sigma_a(A)$, then there exists a normed sequence $(f_n)_{n \geq 1}$ in $H$ such that $\|(A - \lambda I) f_n\| \longrightarrow 0$. If $T = A - \lambda I$ and $R = CT$ with $C \in \mathcal{L}(H)$, then

$$
\|[(T - 0)^*] f_n\| = \|(A - \lambda I) f_n\| \longrightarrow 0
$$

and

$$
\|(R - 0) f_n\| = \|C (A - \lambda I) f_n\| \longrightarrow 0,
$$

hence

$$
((A - \lambda I)^*, C (A - \lambda I)) = (T^*, R) \in \mathcal{GF}(H).
$$

□

Corollary 4. For all $A \in \mathcal{L}(H)$, there exists $B \in \mathcal{L}(H)$ for which $(A, B)$ is a generalized finite operator.

Proof. We say that the approximate spectrum is never empty. Let $\lambda \in \sigma_a(A^*)$, hence it follows from the previous corollary that

$$
((A^* - \lambda I)^*, C (A^* - \lambda I)) = ((A - \overline{\lambda I}), C (A^* - \lambda I)) \in \mathcal{GF}(H),
$$

for all $C \in \mathcal{L}(H)$, and by applying (1) of Proposition 2 we get

$$
(A, B) \in \mathcal{GF}(H),
$$

where $B = C (A^* - \lambda I) + \overline{\lambda I}$. □
Corollary 5. $\mathcal{F}(H)$ contains the following class:

$$S(H) = \left\{ A \in \mathcal{L}(H) : A - \lambda I = C(A^* - \lambda I) \text{ with } \lambda \in \sigma_a(A^*) \text{ and } C \in \mathcal{L}(H) \right\}.$$  

Proof. It follows from the previous corollary that, if $A \in S(H)$, then $(A, A) \in \mathcal{GF}(H)$ i.e. $A \in \mathcal{F}(H)$.

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