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TOTALLY ACYCLIC COMPLEXES

by

HOLLY MARIE ZOLT

(Under the Direction of Alina Iacob)

ABSTRACT

We consider the following question: when is every exact complex of injective modules a totally acyclic one? It is known, for example, that over a commutative Noetherian ring of finite Krull dimension this condition is equivalent with the ring being Iwanaga-Gorenstein. We give equivalent characterizations of the condition that every exact complex of injective modules (over arbitrary rings) is totally acyclic. We also give a dual result giving equivalent characterizations of the condition that every exact complex of flat modules is F-totally acyclic over an arbitrary ring.

INDEX WORDS: Totally acyclic complexes, Gorenstein injective modules, Gorenstein projective modules, Gorenstein flat modules

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TOTALLY ACYCLIC COMPLEXES

by

HOLLY MARIE ZOLT

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Fulfillment of the Requirements for the Degree

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TOTALLY ACYCLIC COMPLEXES

by

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DEDICATION

Grandpa, this one is for you. Thank you for always supporting me, loving me, and encouraging me in the pursuit of my dreams. Even though you have not been here since my freshman year of college, I know that you are up in heaven smiling down. I love and miss you. Until I see you again, save a place for me.

Dennis Riddle

September 3, 1941 - April 7, 2015

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LIST OF SYMBOLS

$\forall$	for all
$\in$	in, element of
$\rightarrow$	maps to
$\otimes$	tensor product
$\mathbb{Z}$	Integers
$\mathbb{Q}$	Rational numbers
$\mathbb{C}$	Complex numbers
$\oplus$	direct sum
$\Longrightarrow$	implies
$\cong$	isomorphic to
$\perp$	orthogonal
$\Longleftrightarrow$	if and only if
$\mathcal{GI}$	Class of Gorenstein Injective modules
$\mathcal{GF}$	Class of Gorenstein Flat modules
$\mathcal{GP}$	Class of Gorenstein Projective modules
$\mathcal{GC}$	Class of Gorenstein Cotorsion modules
$\widetilde{\mathcal{A}}$	Denotes the class of exact complexes X having all cycles in $\mathcal{A}$
$Z_n(X)$	denotes $ker\, \delta_n$
$\mathcal{P}roj$	Class of projective modules

CHAPTER 1

INTRODUCTION

According to Charles A. Weibel [23], homological algebra can trace its origins back to the mid 19th century and the work of two famous mathematicians: Bernahard Riemann and Enrico Betti, who were working together on homology numbers. However, rigorous development of the notion is credited to Poincaré when he began his work on homology numbers in 1895. At this point in time, homology was considered to be apart of topology. It possibly would have remained this way, had it not been for Emmy Noether. While speaking in Göttingen in 1925, she made the statement that homology was an abelian group [21]. This statement was incredibly important as it showed that she saw the algebraic structure behind the invariant. While homology was still considered to be topological in nature, her statement began to shift the perspective of the research until it became more algebraic in nature. By 1945 homology had become a part of the realm of algebra.

By 1929, L. Mayer introduced a purely algebraic version of a chain complex [20] (Poincaré is credited with the first appearance of the idea [23]), and in 1938 Hassler Whitney discovered the general construction of tensor products for abelian groups and modules [24]. Until this point, only special cases had been indirectly known and they had dealt with vector spaces. Baer introduced complete abelian groups over a ring R [1]. However, thanks to R. Eilenberg [3] we now refer to them as injective modules. Exact sequences were discovered in 1941 by Hurewucz [16], however they were not called exact sequences until 1947 in a paper by Kelley and Pitcher [19]. These developments ultimately led to the first major textbook in the area entitled *Homological Algebra*. In this text, Cartan and Eilenberg introduced the idea of projective modules and the notions of both projective and injective resolutions [4]. This textbook drastically changed homological algebra as it consolidated all the theorems and ideas of the past decade into one place, as well as introduced many of the modern notations used today.

Gorenstein homological algebra is a relative version of homological algebra and was started in the late 1960's by Auslander. The area continued to grow, and, in the 1990's, was revolutionized when Enoch and Jenda introduced the Gorenstein counterparts to injective and projective modules [8]. Since its development, it has been shown to be useful in commutative algebra, noncommutative algebra, representation theory, and model category theory [13]. That brings us to where we are today, so now to the thesis.

CHAPTER 2

PRELIMINARIES

2.1 MODULES AND TENSORS

A module, simply put, is a generalization of a vector space. However, as opposed to using scalars from field, a module uses scalars from a ring instead. When the ring is a field, the properties of a module are the same as those of a vector space.

Definition 2.1.1. *Let R be any ring. A* ***left R-module*** *or a* ***left module over R*** *is a set M together with:*

1. *a binary operation + on M under which M is an abelian group.*
2. *an action of R on M (i.e. a map $R \times M \to M$) denoted by $rm \; \forall \, r, s \in R$, $m, n \in M$ which satisfies:*
   a. $(r+s)m = rm + sm$
   b. $(rs)m = r(sm)$
   c. $r(m+n) = rm + rn$
   d. $Im = m$ *(if the ring has identity this condition is imposed)*

A *right R-module* is defined similarly. The action is denoted by mr. Should R be commutative, then every left R-module is also a right R-module. Throughout this section R will be considered to be commutative and have identity unless other wise stated.

Definition 2.1.2. *Let R be a ring, and let M,N be R-modules.*

1. $\varphi : M \to N$ *is an* ***R-module homomorphism*** *if*
   a. $\varphi(x+y) = \varphi(x) + \varphi(y) \; \forall \, x, y \in M$
   b. $\varphi(rx) = r\varphi(x) \; \forall \, x \in M, r \in R$

2. *An R-module homomorphism is an* ***isomorphism*** *(of R-modules) if it is both injective an surjective.*

3. $\varphi : M \to N$ *is an R-module homomorphism, let* $ker\,\varphi = \{m \in M | \varphi(m) = 0\}$

4. *Let M and N be R-modules. We define* $Hom_R(M, N)$ *to be the set of all R-module homomorphisms from M to N.*

$Hom(M, N)$ forms an abelian group under addition. Also, every R-module homomorphism is a group homomorphism, but not every group homomorphism is an R-module homomorphism. Moreover, when R is a field, R-module homomorphisms are called linear transformations.

Proposition 2.1.1.

1. *If* $\varphi \in Hom_R(L, M)$ *and* $\psi \in Hom_R(M, N)$ *then* $\psi \circ \varphi \in Hom_R(L, N)$

2. $Hom_R(M, M)$ *is a ring with identity, denoted with 1. When R is commutative* $Hom_R(M, M)$ *is an R-algebra.*

Proof. Let φ, ψ be as described, and let $r \in R$ and $x, y \in L$. Then,

$$(\psi \circ \varphi)(rx + y) = \psi(\varphi(rx + y))$$

as R-module homomorphisms are linear.

$$= \psi(r\varphi(x) + \varphi(y))$$

$$= r\psi(\varphi(x)) + \psi(\varphi(y))$$

$$= r(\psi \circ \varphi)(x) + (\psi \circ \varphi)(y)$$

For the proof of 2, see page 346 in [6] □

Definition 2.1.3. *Let R be a ring. A module M is **finitely generated** if there is a finite subset A of M such that $M = RA$. That is $M = \{r_1a_1 + r_2a_2 + \cdots + r_ma_m | r_i \in R, a_j \in A \quad \forall i, j \in \mathbb{Z}\}$.*

Should R also be a field, the generating set is a basis of the vector space so long as the vector space is finite dimensional. A module is *cyclic* should the module be generated by a single element in A (i.e. $M = Ra = \{ra | r \in R\}$. As an example, let $R = \mathbb{Z}$ and let A be any abelian group. The group can be finite or infinite. For any $n \in \mathbb{Z}$ and $a \in A$ define

$$na = \begin{cases} a + a + \cdots + a \quad (n\ times) & if \quad n > 0 \\ 0 & if \quad n = 0 \\ -a - a \cdots - a \quad (n\ times) & if \quad n < 0 \end{cases}$$

noting here that 0 is the additive identity of the abelian group. By this definition any abelian group A becomes a $\mathbb{Z}$-module. By the module axioms, this is the only possible action of $\mathbb{Z}$ on A to create a $\mathbb{Z}$-module. In this additive notation the cyclic subgroup can be written as $< a >= na$. If we were to define this multiplicatively, we would write $< a >= a^n$.

Definition 2.1.4. *An R-module F is said to be **free** on the subset A of F if for every $x \in F, x \neq 0$ there exists unique nonzero elements $r_1, \ldots, r_n$ of R and $a_1, \ldots a_n$ of A such that $x = r_1a_1 + \ldots r_na_n$ where n is an integer.*

Should our ring be a field, this definition is equivalent to that of a basis when looking at a vector space.

From Enochs and Jenda's book [9] we have the following definition of a tensor product.

Definition 2.1.5. *A **tensor product** of a right R-module M and a left R-module N is an abelian group T together with a universally balanced map $\sigma : M \times N \to T$*

If $\sigma : M \times N \to T$ and $\sigma' : M \times N \to T'$ are both universally balanced maps (that is $\sigma' = h\sigma$ where $h : T \to T'$ is unique), then we can complete the diagram.

$$\begin{tikzcd}
 & T \arrow[d, "h"] \\
M \otimes N \arrow[ur, "\sigma"] \arrow[r, "\sigma'"] \arrow[dr, "\sigma"'] & T' \arrow[d, "f"] \\
 & T
\end{tikzcd}$$

For an example, consider the rings R and S. Let $R = \mathbb{Z}$, $S = \mathbb{Q}$, and let A be some finite abelian group with order n. Consider $\mathbb{Q} \otimes_{\mathbb{Z}} A = 0$. We know that $1 \otimes 0 = 1 \otimes (0 + 0) = 1 \otimes 0 + 0 \otimes 0 = 0$. Now, if we consider $q \otimes a$, we know that q is rational and can be written as $\frac{q}{n}n$ and we know that $na = 0$ in A. Thus, by Lagrange's Theorem $q \otimes a = \frac{q}{n}n \otimes a = \frac{q}{n} \otimes (na) = \frac{q}{n} \otimes 0 = \frac{q}{n}(1 \otimes 0) = 0$

2.2 CATEGORIES AND FUNCTORS

Definition 2.2.1. *A category C consists of the following.*

1. *A class of objects denoted $Ob(C)$*

2. *For any pair $A, B \in Ob(C)$, a set $Hom_C(A, B)$ with the property that $Hom_C(A, B) \cap Hom_C(A', B') = \emptyset$ whenever $(A, B) \neq (A', B')$. $Hom_C(A, B)$ is called the **set of morphisms** from A to B.*

3. *A **compostion** $Hom_C(B, C) \times Hom_C(A, B) \to Hom_C(A, C)$ for all objects A, B, C denoted $(g, f) \mapsto gf$ (or $g \circ f$) satisfying the properties:*

    a. *for each $A \in Ob(C)$ there is an **identity homomorphism** $id_A \in Hom_C(A, A)$ such that $f \circ id_A = id_B \circ f = f$ for all $f \in Hom(A, B)$*

b. $h(gf) = (hg)f$ *for all* $f \in Hom_C(A, B)$, $g \in Hom_C(B, C)$, *and* $h \in Hom_C(C, D)$

Some examples of categories are sets, abelian groups (**Ab**), and left R-modules ($_R$**Mod**)

Definition 2.2.2. *If C and D are categories, then we say we have a* ***functor*** $F : C \to D$ *if we have:*

1. *a function* $Ob(C) \to Ob(D)$ *(denoted F)*
2. *funtions* $Hom_C(A, B) \to Hom_D(F(A), F(B))$ *also denoted F such that*
   a. *if* $f \in Hom_C(A, B)$, $g \in Hom_C(B, C)$ *then* $F(gf) = F(g)F(f)$ *and*
   b. $F(id_A) = id_{F(A)}$ *for each* $A \in Ob(C)$

We will primarily be concerned with the abelian categories of $_RMod$ and Mod_R.

Definition 2.2.3. *If* ***C*** *and* ***D*** *are abelian categories, then a functor* $F : C \to D$ *is said to be* ***left exact*** *if for every short exact sequence* $0 \to A \to B \to C \to 0$ *in* ***C*** *the sequence* $0 \to F(A) \to F(B) \to F(C)$ *is exact in D. Similarly, F is said to be* ***right exact*** *if* $F(A) \to F(B) \to F(C) \to 0$ *is exact. F is said to be an* ***exact functor*** *if it is left and right exact.*

We can note that $Hom(M, -)$ and $Hom(-, N)$ are left exact and the the tensor product of the functors are right exact. The proof of this is given on page 22 of [9].

2.3 EXACT SEQUENCES AND PROJECTIVE, INJECTIVE, AND FLAT MODULES

Definition 2.3.1. 1. *Let* α, β *be a pair of homomorphisms. The pair* $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ *is said to be* ***exact*** *(at Y) if* $image\, \alpha = ker\, \beta$

2. *A sequence* $\cdots \to X_{n-1} \to X_n \to X_{n+1} \to \cdots$ *of homomorphisms is said to be an* ***exact sequence*** *if it is exact at every* X_n *between a pair of homomorphisms.*

Proposition 2.3.1. *Let A, B, and C be R-modules over some ring R. Then*

1. *The sequence* $0 \to A \xrightarrow{\psi} B$ *is exact (as A) if and only if* ψ *is injective.*

2. *The sequence* $B \xrightarrow{\varphi} C \to 0$ *is exact at C if and only if* φ *is surjective.*

Proof. The homomorphism $0 \to A$ has the image 0 in A. This is the kernel of ψ if and only if ψ is injective. Similarly, the kernel of the zero homomorphism $C \to 0$ is all of C, which is the image of φ if and only if C is surjective. □

Definition 2.3.2. *The exact sequence* $0 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 0$ *is called a* ***short exact sequence****.*

Definition 2.3.3. *Let* $0 \to A \to B \to C \to 0$ *and* $0 \to A' \to B' \to C' \to 0$ *be short exact sequences of modules. Then a* ***homomorphism of short exact sequences*** *is a triple* α, β, γ *of module homomorphisms such that the following diagram commutes:*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

The homomorphism is called an isomorphism of short exact sequences if α, β, γ *are all isomorphisms. In that case the extensions B and B' are said to be isomorphic extensions.*

Proposition 2.3.2. *Let* α, β, γ *be homomophisms of short exact sequences and consider the diagram in definition 7.*

1. *If* α *and* γ *are injective then so is* β*.*

2. *If* α *and* γ *are surjective then so is* β*.*

3. *If α and γ are isomorphisms then so is β and then the two sequences are isomorphic.*

Proof. See [6] □

Definition 2.3.4. *An exact sequence $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ of R-modules is said to be **split exact**, or we say the sequence **splits**, if $Im\, f$ is a direct summand of M.*

Proposition 2.3.3. *The short exact sequence $0 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 0$ of R-modules is split if and only if there is an R-module homomorphism $\mu : C \to B$ such that $\varphi \circ \mu$ is the identity map on C.*

Proof. The proof follows directly from definitions. □

Throughout the rest of this section, we will assume that R is a ring with identity unless otherwise stated. We will also suppose that M, which is an R-module, is an extension of N by L. This corresponds to the short exact sequence of R-modules.

$$0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$$

Proposition 2.3.4. *Let D, L, and M be R-modules and let $\psi : L \to M$ be an R-module homomorphism. Then the map*

$$\psi' : Hom_R(D, L) \to Hom_R(D, M)$$

$$f \mapsto f' = \psi \circ f$$

is a homomorphism of abelian groups. If ψ is injective then ψ' is also injective, i.e., if

$$0 \to L \xrightarrow{\psi} M$$

is exact, then

$$0 \to Hom_R(D, L) \xrightarrow{\psi'} Hom_R(D, M)$$

is also exact.

Proof. ψ' being a homomorphism follows immediately. If ψ is injective, then there are distinct homomorphisms f and g from D into L that give distinct homomorphisms $\psi \circ f$ and $\psi \circ g$ from D into M, which then says that ψ' is injective. □

Theorem 2.3.1. *Let D, L, M, and N be R-modules. If*

$$0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$$

is exact, then the associated sequence

$$0 \to Hom_R(D, L) \xrightarrow{\psi'} Hom_R(D, M) \xrightarrow{\varphi'} Hom_R(D, N) \to 0$$

is also exact.

Proof. See [6] □

Definition 2.3.5. *An R-module P is said to be* ***projective*** *if given an exact sequence* $A \xrightarrow{\varphi} B \to 0$ *of R-modules and an R-homomorphism* $f : P \to B$*, then there exists an R-homomorphism* $\mu : P \to A$ *such that* $f = \varphi \circ \mu$*.*

$$\begin{tikzcd} & P \arrow[dl, dashed, "\mu"'] \arrow[d, "f"] & \\ A \arrow[r, "\varphi"] & B \arrow[r] & 0 \end{tikzcd}$$

This is equivalent to saying that our module P is projective if given that we have an exact sequence $A \to B \to 0$, then $Hom_R(P, A) \to Hom_R(P, B) \to 0$ is exact.

There is also an equivalent definition dealing with free modules.

Definition 2.3.6. *P is projective if it is a direct summand of a free R-module*

Proposition 2.3.5. *Let P be an R-module. Then the following are equivalent:*

1. *P is projective.*

2. *$Hom(P, -)$ is right exact.*

3. *Every short exact sequence $0 \to A \to B \to P \to 0$ is split exact.*

4. *P is a direct summand of a free R-module.*

Proof. See proof on page 40 of [9] □

Proposition 2.3.6. *Every free R-module is projective.*

Proof. The proof is immediate from the previous proposition. □

For any ring R, we have that R and R^n are projective R-modules . More generally, every vector space is a free module and thus projective. However the converse is not necessarily true. If we consider $\mathbb{Z}$ as a module over the ring $R = \mathbb{Z} \oplus \mathbb{Z}$, it is projective, but it is not a free module.

Definition 2.3.7. *An R-module F is said to be **flat** if given any exact sequence $0 \to A \to B$ of right R-modules, the tensored sequence $0 \to A \otimes_R F \to B \otimes_R F$ is exact.*

Proposition 2.3.7. *The direct sum $\oplus_{i \in I} F_i$ is flat if and only if each F_i is flat.*

Corollary 2.3.1. *Every projective module is flat.*

Proof. Let P be projective. P is a summand of a free module by theorem 2.7. But R is a flat R-module. Thus every free module is flat. Therefore, P is a direct summand of a flat module. By the previous proposition, P must be flat. □

The third type of module in this section is an injective module.

Definition 2.3.8. *An R-module E is said to be* ***injective*** *if given R-modules $A \subset B$ and a homomorphism $f : A \to E$, there exists a homomorphism $g : B \to E$ such that $g|_A = f$*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[dl, dashed, "g"] \\ & E & \end{tikzcd}$$

Theorem 2.3.2. *Let E be an R-module. The following are equivalent.*

1. *E is injective.*
2. *$Hom(-, E)$ is right exact.*
3. *E is a direct summand of every R module containing E.*

Proof. $1 \implies 2$: is obvious.

$2 \implies 3$: Consider the exact sequence $0 \to E \to B \to C \to 0$ of R-modules. Then $Hom(B, E) \to Hom(E, E) \to 0$ is exact, so E is a direct summand of B.

$3 \implies 1$: Let $A \subset B$ and consider the pushout diagram:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[d, "f'"] \\ & A' \arrow[r, "j"] & B' \end{tikzcd}$$

j is one to one, so $0 \to E \xrightarrow{j} C$ is split exact. So there is a map $s : C \to E$ such that $s \circ j = id_E$. Then $g = s \circ f'$ is an extension of f since $g \circ i = s \circ f' \circ i = s \circ f \circ j = f$. $\square$

2.4 COMPLEXES, RESOLUTIONS, EXT, AND TOR

For this thesis, it is important to understand the notion of a complex.

Definition 2.4.1. *A* ***(Chain) Complex C*** *of R-modules is a sequence*

$$\boldsymbol{C} : \cdots \to C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} C_{-1} \xrightarrow{\delta_{-1}} \cdots$$

of R-modules and R-homomorphisms such that $\delta_{n-1} \circ \delta_n = 0$ *for all* $n \in \mathbb{Z}$.

Also, one can note that in a complex **C**, the $Im(\delta_{n+1}) \subset ker(\delta_n)$. Typically, $ker(\delta_n)$ and $Im(\delta_{n+1})$ are denoted $Z_n(\mathbf{C})$ and $B_n(\mathbf{C})$ respectively. Their elements are called n-cycles and n-boundaries respectively. Finally, we can also define the nth homology module of **C**. This module is defined as $\frac{ker(\delta_n)}{Im(\delta_{n+1})}$ and is denoted by $H_n(\mathbf{C})$. So $H_n(\mathbf{C}) = 0$ if and only if **C** is exact at C_n.

Theorem 2.4.1. *If* $0 \to$ ***C'*** $\to$ ***C*** $\to$ ***C"*** $\to 0$ *is an exact sequence of complexes then there is an exact sequence*

$$\cdots \to H_{n+1}(\boldsymbol{C'}) \to H_{n+1}(\boldsymbol{C}) \to H_{n+1}(\boldsymbol{C''}) \to H_n(\boldsymbol{C'}) \to H_n(\boldsymbol{C}) \to \cdots$$

for each $n \in \mathbb{Z}$.

An *acyclic* complex is simply an exact complex. Using this we will define totally acyclic and F-totally acyclic complexes.

Definition 2.4.2. *Let* $P = \cdots \to P_1 \to P_0 \to P_{-1} \to \cdots$ *be an acyclic complex of projective modules and let* $I = \cdots \to I_1 \to I_0 \to I_{-1} \to \cdots$ *be an acyclic complex of injective modules.* P *is said to be* ***totally acyclic*** *if for any projective R-module* P*, the complex* $\cdots \to Hom(P_{-1}, P) \to Hom(P_0, P) \to Hom(P_1, P) \to \cdots$ *is still acyclic.* I *is said to be* ***totally acyclic*** *if for any injective R-module* I*, the complex* $\cdots \to Hom(I, I_1) \to Hom(I, I_0) \to Hom(I, I_{-1}) \to \cdots$ *is still acyclic.*

Definition 2.4.3. *Let* $F = \cdots \to F_1 \to F_0 \to F_{-1} \to \cdots$ *be an acyclic complex of flat modules. F is said to be* ***F-totally acyclic*** *if for any injective right R-module* I*, the complex* $\cdots \to I \otimes F_1 \to I \otimes F_0 \to I \otimes F_{-1} \to \cdots$ *is still acyclic.*

In an Iwanaga Gorenstein ring, every acyclic complex of flat modules is F-totally acyclic. Thus over this ring, the class of Gorenstein projective (injective, flat) modules coincide with that of the cycles of acyclic complexes projectives (injectives, flats, respectively) [11].

Definition 2.4.4. *Let M be any R-module. A* ***projective resolution*** *of M is an exact sequence*

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0 \tag{2.1}$$

such that each P_i is a projective R-module.

Definition 2.4.5. *Every R-module N has an exact sequence $0 \to N \to E^0 \to E^1 \to \cdots$ with E^i injective. This sequence is called an* ***injective resolution*** *of N.*

Definition 2.4.6. *Every R-module has a* ***flat resolution*** *that is an exact sequence $\cdots \to F_1 \to F_0 \to M \to 0$ with each F_i flat.*

This follows from the definition of a projective resolution as well as the fact that every projective module is flat.

Every R-module has a projective resolution, Now that projective resolutions have been defined, we can also define the Ext group and the cohomology group derived from the functor $Hom_R(-, D)$. Consider the deleted projective resolution of (2.1).

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0$$

If we apply the functor $Hom(-, A)$ to the deleted resolution, we obtain the following exact complex:

$$0 \to Hom_R(P_0, A) \xrightarrow{\delta_0} Hom_R(P_1, A) \xrightarrow{\delta_1} Hom_R(P_2, A) \xrightarrow{\delta_2} \cdots \tag{2.2}$$

The complex defined in (2.2) is not normally an exact complex. Thus, we define the *nth cohomology module* of the complex denoted $Ext^i_R(M, A)$.

$$Ext^i_R(M, A) = Ker\, \delta_{i+1} / \, Im\, \delta_i$$

Proposition 2.4.1. *For any R-module A we have* $Ext^0_R(A, D) \cong Hom_R(A, D)$

Proof. See page 780 of [6] □

Theorem 2.4.2. *Let* $0 \to L \to M \to N \to 0$ *be a short exact sequence of R-modules. Then there is a long exact sequence of abelian groups*

$$0 \to Hom_R(N, D) \to Hom_R(M, D) \to Hom_R(L, D) \xrightarrow{\delta_0} Ext^1_R(N, D) \to$$
$$Ext^1_R(M, D) \to Ext^1_R(L, D) \xrightarrow{\delta_1} Ext^2_R(N, D) \to \cdots$$

.

Proof. See page 784 of [6]. □

Proposition 2.4.2. *If P is projective, then* $Ext^n_R(P, B) = 0$ *for all R-modules B and all* $n \geq 1$

Proof. If P is projective R-module, then the simple exact sequence

$$0 \to P \xrightarrow{1} P \to 0$$

is a projective resolution of P. Taking homomorphism into B gives the simple cochain complex

$$0 \to Hom_R(P, B) \xrightarrow{1} Hom_R(P, B) \to 0 \cdots 0 \to \cdots$$

from which it follows by definition of $Ext^n_R(P, B)$ for all $n \geq 1$ as desired. □

The converse of this statement is also true (see [9] prop 8.4.3), which allows the proposition to be if and only if. There is also a similar proposition dealing with injective R-modules in [6] on page 784. The proposition in essence equates a module Q being injective to the $Ext^n_R(A, Q) = 0$ for all R-module A and for all n greater than or equal to 1. We can also similarly define injective resolutions.

By Theorem 3.1.7 in Enochs and Jenda [9], every R-module can be embedded in an injective R-module. Thus it follows that R-module N has an exact sequence $0 \to N \to E^0 \to E^1 \to \cdots$ This sequence is called an *injective resolution*. The propositions and theorems for projective resolutions also hold for injective resolutions.

Now that we have defined resolutions, we can define the Tor functor. For this functor we will be consider the following two sequences

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} B \to 0 \tag{2.3}$$

$$\cdots \to D \otimes P_n \xrightarrow{1\otimes d_n} D \otimes P_{n-1} \to \cdots \xrightarrow{1\otimes d_1} D \otimes P_0 \xrightarrow{1\otimes \epsilon} D \otimes B \to 0 \tag{2.4}$$

Definition 2.4.7. *Let D be a right R-module and let B be a left R-module. For any projective resolution of B by left R-modules as in (2.3) let* $1 \otimes d_n : D \otimes P_n \to D \otimes P_{n-1}$ *for all* $n \geq 1$ *as in (2.4). Then*

$$Tor_n^R(D, B) = \frac{ker(1\otimes d_n)}{image(1\otimes d_{n+1})}$$

where $Tor_0^R(D, B) = (D \otimes P_0)/\, image(1 \otimes d_{n+1})$.

The group $Tor_n^R(D, B)$ is called the nth Homology group derived from the functor $D \otimes -$.

Proposition 2.4.3. *For any left R-module B, we have* $Tor_0^R(D, B) \cong D \otimes B$

Proposition 2.4.4. *For every R-module homomorphism* $f : B \to B'$ *there are induced maps* $\psi_n : Tor_n^R(D, B) \to Tor_n^R(D, B')$ *on homology groups*

Theorem 2.4.3. *Let* $0 \to L \to M \to N \to 0$ *be a short exact sequence of left R-modules. Then there is a long exact sequence of abelian groups*

$$\cdots \to Tor_2^R(D, N) \xrightarrow{\delta_1} Tor_1^R(D, L) \to Tor_1^R(D, M) \to Tor_1^R(D, N) \xrightarrow{\delta_0} D \otimes L \to$$
$$D \otimes M \otimes N \to 0$$

Proposition 2.4.5. *For a right R-module D, the following are equivalent.*

1. *D is a flat R-module*

2. $Tor_n^R(D, B) = 0$ *for all left R-modules B and for all* $n \geq 1$

Proof. See page 790 of [6] □

Definition 2.4.8. *An R-module M is said to have* ***projective dimension*** *at most* n*, denoted* $pd_R M \leq n$*, if there is a projective resolution* $0 \to P^n \to \cdots \to P^1 \to P^0 \to M \to 0$

We can similarly define both injective dimension and flat dimension. Respectively, they are denoted inj dim M and flat dim M.

2.5 GORENSTEIN COUNTERPARTS

In this section, we will define Iwanaga-Gorenstein rings and modules. We will also define the Gorenstein counterparts to projective, injective, and flat modules. However, in order to do that, we will first define Noetherian rings and modules.

Definition 2.5.1. *Let R be a commutative ring with identity. R is* ***Noetherian*** *if every ideal is finitely generated.*

Recall that an ideal, in essence is simply a subgroup of a ring. Ideals allow us to form the quotient ring R/I where R is a ring and I an ideal of R. By defining Noetherian modules, we can also obtain an alternative definition for a Noetherian ring.

Definition 2.5.2. *Let M be an R-module*

1. *M is* ***Noetherian*** *if every chain of ascending chain of submodules of M terminates.*

2. *A ring R is* ***Noetherian*** *if it is Noetherian as a left R-module over itself.*

If we consider a Principal Ideal Domain, an integral domain in which every ideal is principal (generated by a single element), we can see that every PID is Noetherian. This is because every submodule is the same as its ideals.

Definition 2.5.3. *A ring R is said to be* ***Iwanaga-Gorenstein*** *or simply* ***Gorenstein*** *if R is both left and right Noetherian and if R has finite self-injective dimension as a left and as a right R-module.*

Proposition 2.5.1. *If R is left (right) noetherian and the left (right) self injective dimension of R is $n < \infty$, then inj dim F $\leq n$ for every flat left (right) R-module. Also, if flat dim $M < \infty$ for a left (right) R-module M, then the projective dimension of $M \leq n$.*

Proof. See page 211 of [9]. □

Now that we have defined both Noetherian and Iwanaga-Gorenstein rings, we can move on to defining the Gorenstein counterparts for projective, injective, and flat modules.

Definition 2.5.4. *A module M is said to be* ***Gorenstein Projective*** *if there is a $Hom(-, \mathcal{P}roj)$ exact-exact sequence*

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

of projective modules such that $M = ker(P^0 \to P^1)$.

Definition 2.5.5. *A module N is said to be* ***Gorenstein Injective*** *if there exists a $Hom(\mathcal{I}nj, -)$ exact-exact sequence*

$$\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$$

of injective modules such that $N = ker(E^0 \to E^1)$.

Thus in other words, a Gorenstein projective module is the kernel of a totally acyclic complex of projective modules. Similarly a Gorenstein injective module is the kernel of a totally acyclic complex of injective modules. If we consider, for example, a projective

module it will be Gorenstein. In fact, all projective modules are Gorenstein projective. However, the converse is not true. Consider the ring $\mathbb{Z}/4\mathbb{Z}$ and the module $M = \mathbb{Z}/2\mathbb{Z}$. M is Gorenstein projective, but it is not projective. Similarly, every injective module is is Gorenstein injective. Again, the converse of the statement is not true. This can be demonstrated using the same ring and module as above.

Proposition 2.5.2. *Let R be Noetherian and let $0 \to M' \to M \to M''$ be an exact sequence of finitely generated right R-modules. If M' and M'' are Gorenstein projective, then M is Gorenstein projective as well. If M'' and M are Gorenstein projective , then so is M'. If M and M' are Gorenstein projective, then M'' is Gorenstein projective if and only if $Ext^1(M'', P) = 0$ for all finitely generated projective R-modules P.*

Both the remark and proposition can be dually defined for Gorenstein injective modules.

Proof. See page 249 of [9] □

The final Gorenstein counterpart is for flat modules.

Definition 2.5.6. *A module M is said to be **Gorenstein flat** if there exists and $\mathcal{I}nj \otimes -$ exact-exact sequence*

$$\cdots \to F_1 \to F_0 \to F^0 \xrightarrow{\delta} F^1 \to \cdots$$

of flat modules such that $M = ker(\delta)$.

Similar to the previous definitions, a Gorenstein flat module is the kernel of an F-totally acyclic complex of flat modules. It follows from the definition that $Tor_i(E, M) = 0$ for all $i \geq 1$ and any injective module E.

Proposition 2.5.3. *Let R be Noetherian. Then every finitely generated Gorenstein projective module is Gorenstein flat.*

Theorem 2.5.1. *Let R be Noetherian and let* $0 \to M' \to M \to M''$ *be an exact sequence of finitely generated right R-modules. If* M' *and* M'' *are Gorenstein flat, then M is Gorenstein flat as well. If* M'' *and* M *are Gorenstein flat, then so is* M'*. If* M *and* M' *are Gorenstein flat, then* M'' *is Gorenstein flat if and only if* $0 \to E \otimes M' \to E \otimes M$ *is exact for any injective module E.*

Proof. See page 257 of [9]. □

2.6 DIMENSIONS

Projective, injective, and flat dimensions, also known as homological dimensions, are defined in terms of resolutions.

Definition 2.6.1. *The minimal length of a finite projective resolution* $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ *of an R-module M is called the* ***projective dimension of*** M *and is denoted* $pd_R M$ *If* M *does not admit a finite projective resolution, then the projective dimension is infinite.*

Similarly, we can define both injective and flat dimensions. Respectively, they are denoted injdim M and flatdim M. We can also define the Krull dimension.

Definition 2.6.2. *The* ***Krull dimension*** *of R, denoted dim R, is the supremum of the number of strict inclusions in a chain of prime ideals.*

If we consider a field for example, the Krull dimension is 0. If we use a PID, then the dimension is 1, given that it is not a field. For example, $\mathbb{Z}$ and $k[x]$ both have Krull dimension 1 becasue they are PIDs.

Definition 2.6.3. *The minimal length of a finite exact sequence of an R-module* M

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

with each G_j being Gorenstein projective is called the ***Gorenstein projective dimension of M*** *and is denoted Gpd$_R M$.*

Again, we can similarly define *Gorenstein injective dimension*, denoted Gid$_R M$, and *Gorenstein flat dimension*, denoted Gfd$_R M$.

CHAPTER 3

MAIN RESULTS

3.1 PRELIMINARIES

Before stating and proving the two main theorems of this thesis, there are several definitions and some prerequisite knowledge that will first need to be addressed. Note that $\mathcal{GI}, \mathcal{GP}$, and $\mathcal{GF}$ refer to the classes of Gorenstein injective, projective, and flat modules respectively. Also, $\mathcal{P}roj$ and $\mathcal{I}nj$ refer to the classes of projective and injective modules respectively. Recall that for a complex $X = \cdots \to X_{n+1} \xrightarrow{d_n+1} X_n \xrightarrow{d_n} X_{n-1} \to \cdots$ we use the notation $Z_n(X) = ker\, d_n$

Definition 3.1.1. *Let $\mathcal{A}$ be a class of R-modules. An acyclic complex X is in $\tilde{\mathcal{A}}$ if $Z_j(X) \in \mathcal{A}$ for all integers j.*

Definition 3.1.2. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in R-mod. A complex Y is a dg-$\mathcal{A}$ complex if each $Y_n \in \mathcal{A}$ and if each map $Y \to U$ is null homotopic for each complex $U \in \tilde{\mathcal{B}}$.*

Definition 3.1.3. *Let $\mathcal{C}$ be a class of modules and let $\mathcal{D}$ be an abelian category. The **right orthogonal class**, $\mathcal{C}^{\perp}$ is defined to be the class of objects $Y \in \mathcal{D}$ such that $Ext^1(A, Y) = 0$ for all $A \in \mathcal{C}$.*

The *left orthogonal class* of $\mathcal{C}$ can be defined similarly and is denoted ${}^{\perp}\mathcal{C}$.

Definition 3.1.4. *Let $\mathcal{D}$ be an abelian category and let $(\mathcal{A}, \mathcal{B})$ be a pair of classes in $\mathcal{D}$. $(\mathcal{A}, \mathcal{B})$ is called a **cotorsion pair** if $\mathcal{B} = \mathcal{A}^{\perp}$ and $\mathcal{A} = {}^{\perp}\mathcal{B}$.*

Definition 3.1.5. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair. The cotorsion pair is called **hereditary** if $Ext^i(A, B) = 0$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$, and for all $i \geq 1$.*

Definition 3.1.6. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair. The cotorsion pair is called **complete** if for each object M in $\mathcal{D}$ there exists short exact sequences $0 \to M \to B \to A \to 0$ and $0 \to B' \to A' \to M \to 0$ with $A \in \mathcal{A}, B \in \mathcal{B}$.*

According to Gillespie ([14], Definition 3.3 and Proposition 3.6), there are four classes of complexes in $Ch(R)$ that are associated with a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in R-mod:

1. An acyclic complex X is an $\mathcal{A}$-complex if $Z_j(X) \in \mathcal{A}$ for all integers j. $\widetilde{\mathcal{A}}$ denotes the class of all acyclic $\mathcal{A}$-complexes.

2. An acyclic complex X is a $\mathcal{B}$-complex if if $Z_j(X) \in \mathcal{B}$ for all integers j. $\widetilde{\mathcal{B}}$ denotes the class of all acyclic $\mathcal{B}$-complexes.

3. A complex Y is a dg-$\mathcal{A}$ complex if each $Y_n \in \mathcal{A}$ and each map $Y \to U$ is null homotopic, for each complex $U \in \widetilde{\mathcal{B}}$. dg($\mathcal{A}$) denotes the class of all dg-$\mathcal{A}$ complexes.

4. A complex W is a dg-$\mathcal{B}$ complex if each $W_n \in \mathcal{A}$ and each map $V \to W$ is null homotopic, for each complex $V \in \widetilde{\mathcal{A}}$. dg($\mathcal{B}$) denotes the class of all dg-$\mathcal{A}$ complexes.

It was shown by Yang and Liu ([25]) that when $(\mathcal{A}, \mathcal{B})$ is a complete hereditary cotorsion pair in R-mod, the pairs $(dg(\mathcal{A}), \widetilde{\mathcal{B}})$ and $(\widetilde{\mathcal{A}}, dg(\mathcal{B}))$ are also complete and hereditary cotorsion pair. Moreover, by Gillespie ([14]), $\widetilde{\mathcal{A}} = dg(\mathcal{A}) \cap \mathcal{E}$ and $\widetilde{\mathcal{B}} = dg(\mathcal{B}) \cap \mathcal{E}$, where $\mathcal{E}$ is the class of all acyclic complexes. Two other known complete hereditary cotorsion pairs are $(\mathcal{P}roj, R-mod)$ and $(R-mod, \mathcal{I}nj)$. Using these pairs, you can obtain the complete and hereditary cotorsion pairs: $(\mathcal{E}, dg(\mathcal{I}nj))$ and $(dg(\mathcal{P}roj), \mathcal{E})$. Due to Krause ([18]), over a left Noetherian ring, the pair $({}^{\perp}\mathcal{GI}, \mathcal{GI})$ is a complete and hereditary cotorsion pair, and therefore $(dg({}^{\perp}\mathcal{GI}), \widetilde{\mathcal{GI}})$ is a complete and hereditary cotorsion pair in $Ch(R)$. Saroch and Stovicek ([22]) showed that over any ring R that $({}^{\perp}\mathcal{GI}, \widetilde{\mathcal{GI}})$ is a complete hereditary cotorsion pair, and therefore $(dg({}^{\perp}\mathcal{GI}), \widetilde{\mathcal{GI}})$ and $({}^{\perp}\mathcal{GI}, \widetilde{\mathcal{GI}})$ are complete cotorsion pair in $Ch(R)$. Utilizing this information and the definitions above from Gillespie, we can more specifically say that $X \in dg(\mathcal{GI}) \iff X_n \in \mathcal{GI}$ and any $f : A \to X$ is homotopic to 0 $\forall A \in \widetilde{{}^{\perp}\mathcal{GI}}$.

Should R be a right n-perfect ring, that is each flat right R-module has projective dimension $\leq n$, then $(\mathcal{GP}, \mathcal{GP}^{\perp})$ and $({}^{\perp}\mathcal{GI}, \widetilde{\mathcal{GI}})$ are complete and hereditary cotorsion pairs (see Estrada, Iacob, and Odabaşi ([12] Proposition 7). Finally, due to Enochs, Jenda, and Ramos [10], if R is a left coherent ring, then $(\mathcal{GF}, \mathcal{GF}^{\perp})$ and $(\widetilde{\mathcal{GF}}, dg(\mathcal{GF}^{\perp}))$. $\mathcal{GF}^{\perp}$ is known as the class of *Gorenstein cotorsion modules* and is often times denoted by $\mathcal{GC}$.

Definition 3.1.7. *Let $\mathcal{A}$ be a class of modules. $dw(\mathcal{A})$ denotes the class of complexes of modules, X, such that each component, X_n, is in $\mathcal{A}$*

3.2 MAIN RESULTS

It is known that over a Gorenstein ring every complex of injective modules is totally acyclic. Similarly, every complex of projective modules and flat modules is totally acyclic and F-totally acyclic respectively. Over these rings, Gorenstein injective (and respectively projective and flat) modules are just the cycles of the exact complexes of injective (projective, flat) modules. Therefore, these questions "Can these conditions characterize Gorenstein rings?" and, more generally, "Is it possible to characterize Gorenstein rings in in terms of exact complexes of Gorenstein injective, Gorenstein projective, and Gorenstein flat modules?" would be normal questions to ask. Our results give equivalent characterizations of the statement "Every exact complex of injective modules is totally acyclic"; we also prove a dual result for complexes of flat modules. Both results hold over any ring (associative with identity).

Theorem 3.2.1. *Let R be any ring, then the following are equivalent:*

1. *Every exact complex of injective R-modules is totally acyclic.*
2. *Every exact complex of Gorenstein injective R-modules is in $\widetilde{\mathcal{GI}}$.*
3. *Every complex of Gorenstein injective R-modules is a dg-Gorenstein injective complex.*

Proof. 1. $\implies$ 2. Let Y be an exact complex of Gorenstein injective modules. Since $({}^{\perp}\mathcal{GI}, \mathcal{GI})$ is a complete hereditary cotorsion pair, we have that the pair $(\widetilde{{}^{\perp}\mathcal{GI}}, dg(\mathcal{GI}))$ is a complete cotorsion pair in $Ch(R)$. So there exists a short exact sequence $0 \to A \to B \to Y \to 0$ with $B \in \widetilde{{}^{\perp}\mathcal{GI}}$ and with $A \in dg(\mathcal{GI})$. Both B and Y are exact complexes, so A is also an exact complex. Since A is exact and in $dg(\mathcal{GI})$, it is in $\widetilde{\mathcal{GI}}$. For each n we have a short exact sequence $0 \to A_n \to B_n \to Y_n \to 0$ with both Y_n and A_n in $\mathcal{GI}$. Therefore $B_n \in \mathcal{GI}$ for all n. Since $B_n \in^{\perp} \mathcal{GI} \cap \mathcal{GI}$, it follows that B_n is injective for each integer n. Thus B is an exact complex of injective modules, and then by (1), B is totally acyclic. Thus $Z_n(B)$ is Gorenstein injective for all n.

The exact sequence $0 \to Z_n(A) \to Z_n(B) \to Z_n(Y) \to 0$ with both $Z_n(A)$ and $Z_n(B)$ Gorenstein injective gives us that $Z_n(Y)$ is Gorenstein injective. Therefore $Y \in \widetilde{\mathcal{GI}}$

2. $\implies$ 3. Let X be a complex of Gorenstein injective R-modules. Since $(\mathcal{E}, dg(\mathcal{I}nj))$ is a complete cotorsion pair, there exists an exact sequence $0 \to A \to B \to X \to 0$ with A being a DG-injective complex and with B being an exact complex. Then for each n there is an exact sequence $0 \to A_n \to B_n \to X_n \to 0$ with A_n injective and with X_n Gorenstein injective. It follows that each B_n is Gorenstein injective. So B is an exact complex of Gorenstein injective modules. By (2), B is in $\widetilde{\mathcal{GI}}$, and therefore in $dg(\mathcal{GI})$.

Let $Y \in \widetilde{{}^{\perp}\mathcal{GI}}$. The exact sequence $0 \to A \to B \to X \to 0$ gives an exact sequence $0 = Ext^1(Y, B) \to Ext^1(Y, X) \to Ext^2(Y, A) = 0$ (since Y is exact and A is a DG-injective complex). It follows that $Ext^1(Y, X) = 0$ for any $Y \in \widetilde{{}^{\perp}\mathcal{GI}}$, so $X \in dg(\mathcal{GI})$. Therefore, we have that $dw(\mathcal{GI}) \subseteq dg(\mathcal{GI})$. The other inclusion always holds, thus $dg(\mathcal{GI}) = dw(\mathcal{GI})$.

3. $\implies$ 1. Let X be an exact complex of injective left R-modules. In particular, let $X \in dw(\mathcal{GI})$, by (3), $X \in dg(\mathcal{GI})$. Since $X \in dg(\mathcal{GI})$ and X is exact, it follows that $X \in \widetilde{\mathcal{GI}}$, and therefore $Z_n(X) \in \mathcal{GI}$ for all n. Thus X is a totally acyclic complex. $\square$

Theorem 3.2.2. *Let R be any ring, then the following are equivalent.*

1. *Every exact complex of flat right R-modules if F-totally acyclic*

2. *Every exact complex of cotorsion-flat right R-modules is F-totally acyclic*

3. *Every exact complex of Gorenstein flat right R-modules is in* $\widetilde{\mathcal{GF}}$.

4. *Every complex of Gorenstein flat right R-modules is a dg-Gorenstein flat complex.*

Proof. 1. $\implies$ 2 This is immediate.

2. $\implies$ 3. Let Y be an exact complex of cotorsion flat right R-modules that is F-totally acyclic. $(\mathcal{GF}, \mathcal{GC})$ is a complete hereditary cotorsion pair in R-mod, so we have that $(\widetilde{\mathcal{GF}}, dg(\mathcal{GC}))$ is a complete hereditary cotorsion pair in R-mod. Thus, there exists a short exact sequence $0 \to B \to A \to Y \to 0$ with $A \in \widetilde{\mathcal{GF}}$ and $B \in dg(\mathcal{GC})$. Both Y and A are exact so B is also exact. Since B is exact and B is in $dg(\mathcal{GC})$, $B \in \widetilde{\mathcal{GC}}$. For each $n \in \mathbb{Z}$ we have a short exact sequence $0 \to B_n \to A_n \to Y_n \to 0$ with $B_n \in \mathcal{GC}$ and $Y_n \in \mathcal{GF}$. So we have a split exact sequence and $A_n \simeq B_n \oplus Y_n$ and $A_n \in \mathcal{GF}$. Thus, B_n is in both $\mathcal{GF}$ and $\mathcal{GC}$. Therefore, $B_n \in Flat$. $(Flat, Cot)$ and $(\mathcal{GF}, \mathcal{GC})$ are cotorsion pairs. $Flat \subseteq \mathcal{GF}$ implies that $\mathcal{GC} \subseteq Cot$. Thus B_n is also cotorsion, making it cotorsion flat. This implies that B is an exact complex of cotorsion flat modules, so by (2) we have that B is F-totally acyclic, and therefore $Z_n(B) \in \mathcal{GF}\ \forall\ n$.

The exact sequence $0 \to Z_n(B) \to Z_n(A) \to Z_n(Y) \to 0$ with both $Z_n(B)$ and $Z_n(A)$ Gorenstein flat yields that $GfdZ_n(Y) \leq 1$. However, Y is an exact complex of flat modules, $Y = \cdots \to Y_{n+1} \to Y_n \to Y_{n-1} \to \cdots$. So there exists the exact sequence $0 \to Z_{n+1}(Y) \to Y_n \to Z_n(Y) \to 0$ with $Gfd(Z_n(Y)) \leq 1$ and $Y_n \in Flat$ implies $Z_{n+1}(Y) \in \mathcal{GF}$ for all $n \in \mathbb{Z}$. Thus $Y \in \widetilde{\mathcal{GF}}$.

3. $\implies$ 4. Let Y be an exact complex of Gorenstein flat R-modules. $(dg(\mathcal{P}roj), \mathcal{E})$ is a complete hereditary cotorsion pair in $Ch(R)$. Therefore, there exists a short exact sequence $0 \to Y \to B \to A \to 0$ with $A \in dg(\mathcal{P}roj)$ and B an exact complex. For each $n \in \mathbb{Z}$ we have the short exact sequence $0 \to Y_n \to B_n \to A_n \to 0$ with Y_n Gorenstein flat and A_n

projective and therefore flat. Thus B_n is Gorenstein flat, which implies $B \in \widetilde{\mathcal{GF}}$ (by 3) and in $dg(\mathcal{GF})$.

Let $D \in \widetilde{\mathcal{GC}}$. Our short exact sequence $0 \to Y \to B \to A \to 0$ gives an exact sequence $0 = Ext^1(B, D) \to Ext^1(Y, D) \to Ext^2(A, D) = 0$ (since A is a DG-projective complex and Y is exact). It follows that $Ext^1(Y, D) = 0$ for any $D \in \widetilde{\mathcal{GC}}$, so Y is in $dg(\mathcal{GF})$. Therefore $dw(\mathcal{GF}) \subseteq dg(\mathcal{GF})$. By definition, the other direction always holds. Thus $dg(\mathcal{GF}) = dw(\mathcal{GF})$.

4. $\implies$ 1 This follows from the fact that $dg(\mathcal{GF}) \cap \mathcal{E} = \widetilde{\mathcal{GF}}$. □

With more hypothesis on the ring, it is known that Theorem 1 and Theorem 2 are equivalent, and moreover, they characterize Gorenstein rings.

Corollary 3.2.1. *Let R be a commutative noetherian ring of finite Krull dimension. The following statements are equivalent:*

1. *Every exact complex of injective R-modules is totally acyclic.*
2. *Every exact complex of Gorenstein injective R-modules is in $\widetilde{\mathcal{GI}}$.*
3. *Every complex of Gorenstein injective R-modules is a dg-Gorenstein injective complex.*
4. *Every exact complex of flat right R-modules if F-totally acyclic*
5. *Every exact complex of cotorsion-flat right R-modules is F-totally acyclic*
6. *Every exact complex of Gorenstein flat right R-modules is in $\widetilde{\mathcal{GF}}$.*
7. *Every complex of Gorenstein flat right R-modules is a dg-Gorenstein flat complex.*

Proof. This is true by [11]. See corollary 2. □

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