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# GORENSTEIN PROJECTIVE PRECOVERS IN THE CATEGORY OF MODULES

by

**KATELYN ANNA COGGINS**

(Under the Direction of Alina Iacob)

## ABSTRACT

It was recently proved that if  $R$  is a coherent ring such that  $R$  is also left  $n$ -perfect, then the class of Gorenstein projective modules,  $\mathcal{GP}$ , is precovering. We will prove that the class of Gorenstein projective modules is special precovering over any left GF-closed ring  $R$  such that every Gorenstein projective module is Gorenstein flat and every Gorenstein flat module has finite Gorenstein projective dimension. This class of rings includes that of right coherent and left  $n$ -perfect rings.

*Key Words:* Gorenstein projective, Gorenstein flat, precover.

*2010 Mathematics Subject Classification:* 18G25;18G35.

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MODULES**

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B.S., Georgia Southern University, 2014

A Thesis Submitted to the Graduate Faculty of Georgia Southern University in Partial  
Fulfillment  
of the Requirement for the Degree

MASTER OF SCIENCE

STATESBORO, GEORGIA

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Electronic Version Approved:

July 2016

## DEDICATION

This thesis is dedicated to every student who struggled and questioned their abilities, every student that was told they would not make it, every student that almost gave up but found a way to press on. It is also dedicated to the teachers, mentors, and peers that helped these students to keep going and not lose confidence. Without the latter, this thesis and many others like it would not be completed.

“If I have seen further, it is by standing on the shoulders of giants.”

- Sir Isaac Newton

## ACKNOWLEDGMENTS

I wish to acknowledge the incredible guidance and support of my advisor, Dr. Alina Iacob, who has been such a joy to work with throughout my undergraduate and graduate years. I admire her intelligence and quick wit, and I truly value all that she has taught me.

I would also like to acknowledge Drs. David Stone and Andrew Sills. I am thankful for the helpful words and suggestions they gave that have led to making this thesis complete. I have enjoyed the time spent learning from each of them and have the utmost respect for their kindness and intelligence. I am also forever grateful to the numerous faculty members at Georgia Southern University that have contributed to my education and life in so many ways, including but most certainly not limited to Dr. Hua Wang, Dr. Sharon Taylor, and Mrs. Nikki Collins.

To my fellow students, you are too numerous to mention by name, but I am grateful to have had such wonderful minds I could lean on and I admire your passions for academia. To my office mate Nicolas Smoot, I cannot fully express enough thanks for all of the support and advice on anything from teaching to classwork to life.

Thank you to Michael Poor, my high school teacher and friend who first introduced me to the beautiful world of calculus and sparked my excitement for math. I am glad he saw potential and challenged it and always had a great sense of humor.

I am thankful for my family and all of the love and support they have given me. They taught me to always strive for excellence and I would not be where I am today without each and every one of them.

Finally, I am forever grateful to my adoring husband William Anthony Coggins. He has kept my head above water throughout this entire process when I felt overwhelmed in a sea of theorems and propositions. Without him, this thesis may not have been finished.

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## CHAPTER 1

### INTRODUCTION

Homological algebra is at the root of modern techniques in many areas of mathematics including commutative and non commutative algebra, algebraic geometry, algebraic topology and representation theory. Not only that all these areas make use of the homological methods but homological algebra serves as a common language and this makes interactions between these areas possible and fruitful. A relative version of homological algebra is the area called Gorenstein homological algebra. This newer area started in the late 60s but really took off in the mid 90s, with the introduction of the Gorenstein (projective, injective, flat) modules. It has proved to be very useful in characterizing various classes the rings. Also, methods and results from Gorenstein homological algebra have successfully been used in algebraic geometry, as well as in representation theory. But the main problem in using the Gorenstein homological methods is that they can only be applied when the corresponding Gorenstein resolutions exist. So the main open problems in this area concern identifying the type of rings over which Gorenstein homological algebra works. Of course one hopes that this is the case for any ring. But so far only the existence of the Gorenstein flat resolutions was proved over arbitrary rings (in 2014). The existence of the Gorenstein projective resolutions and the existence of the Gorenstein injective resolutions are still open problems. And they have been studied intensively in recent years.

We consider the problem of the existence of the Gorenstein projective precovers. Their existence over Gorenstein rings is known (Enochs-Jenda, 2000). Then Peter Jorgensen (2007) proved their existence over commutative noetherian rings with dualizing complexes. In 2011, D. Murfet and S. Salarian extended his result to commutative noetherian rings of finite Krull dimension. Recently Murfets and Salarians result was extended to right coherent and left  $n$ -perfect rings in the work of Gillespie

([7]) and Estrada-Iacob-Odabasi ([5]).

Our main result shows that the class of Gorenstein projective modules is special precovering over any left GF-closed ring  $R$  such that every Gorenstein projective module is Gorenstein flat and every Gorenstein flat module has finite Gorenstein projective dimension. This class of rings includes that of right coherent and left  $n$ -perfect rings. But the inclusion is a strict one: in [6], section 4, we give examples of left GF-closed rings that have the desired properties (every Gorenstein projective is Gorenstein flat and every Gorenstein flat has finite Gorenstein projective dimension), and that are not right coherent.

## CHAPTER 2

### PRELIMINARIES

#### 2.1 Modules

We begin our study by looking at modules, which generalize the concept of vector spaces. A vector space is a special type of module in which the underlying ring is a field. The definition of a module is also similar to that of a group action, with some additional requirements on the set  $M$ . We recall the following definition for a module  $M$  over a ring  $R$ .

**Definition 2.1.** *Let  $R$  be a ring (not necessarily commutative nor with 1). A **left  $R$ -module** or a **left module over  $R$**  is a set  $M$  together with*

1. *a binary operation  $+$  on  $M$  under which  $M$  is an abelian group, and*
2. *an action of  $R$  on  $M$  (that is, a map  $R \times M \rightarrow M$ ) denoted by  $rm$ , for all  $r \in R$  and for all  $m \in M$  which satisfies*

$$(a) \ (r + s)m = rm + sm, \text{ for all } r, s \in R, m \in M$$

$$(b) \ (rs)m = r(sm), \text{ for all } r, s \in R, m \in M, \text{ and}$$

$$(c) \ r(m + n) = rm + rn, \text{ for all } r \in R, m, n \in M.$$

*If the ring  $R$  has a 1 we impose the additional axiom:*

$$(d) \ 1m = m, \text{ for all } m \in M$$

We use the term left  $R$ -module or left module over  $R$  since the ring elements appear on the left. Similarly, we can define a *right  $R$ -module* or a *right module over  $R$*  with the ring elements appearing on the right. If the ring  $R$  is a commutative ring and  $M$  is a left  $R$ -module, we can make  $M$  a right  $R$ -module by defining  $mr = rm$  for any  $m \in M$  and  $r \in R$ . In this case, we just call  $M$  an  $R$ -module. However, this

does not hold in general, hence not every left  $R$ -module is a right  $R$ -module and vice versa. We will denote a left  $R$ -module  $M$  by  ${}_R M$ , and similarly a right  $R$ -module  $N$  by  $N_R$ . Modules over a ring  $R$  with 1 are called *unital* modules. Generally, we will assume that the ring  $R$  is commutative with 1, unless otherwise noted.

For example, if  $R = \mathbb{Z}$  and  $A$  is any abelian group, then  $A$  is a  $\mathbb{Z}$ -module. For any  $n \in \mathbb{Z}$  and  $a \in A$ , we define:

$$na = \begin{cases} a + a + \dots + a \text{ (} n \text{ times)} & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ -a - a - \dots - a \text{ (} -n \text{ times)} & \text{if } n < 0 \end{cases}$$

Thus, we see by this definition of an action of  $\mathbb{Z}$  on  $A$  that  $A$  is in fact a  $\mathbb{Z}$ -module. The converse also holds. That is, for any  $\mathbb{Z}$ -module  $M$ ,  $M$  is an abelian group. Hence,  $\mathbb{Z}$ -modules and abelian groups are one and the same.

A submodule of  $M$  is simply a subset of  $M$  which also satisfies the same restricted conditions of a module (over the same ring  $R$ ). If the ring  $R$  is a field, submodules are the same as subspaces. Every  $R$ -module  $M$  has at least two submodules,  $M$  and the trivial submodule 0.

**Definition 2.2.** Let  $R$  be a ring and let  $M$  and  $N$  be  $R$ -modules. A map  $\varphi : M \rightarrow N$  is an  **$R$ -module homomorphism** if:

$$(a) \varphi(x + y) = \varphi(x) + \varphi(y), \text{ for all } x, y \in M \text{ and}$$

$$(b) \varphi(rx) = r\varphi(x), \text{ for all } r \in R, x \in M.$$

Additionally, we denote the set of all  $R$ -module homomorphisms from  $M$  into  $N$  by  $\text{Hom}_R(M, N)$ .

The set  $\text{Hom}_R(M, N)$  is an abelian group under addition. If  $R$  is also commutative,  $\text{Hom}_R(M, N)$  is an  $R$ -module. For  $\varphi \in \text{Hom}_R(M, N)$ , the kernel (denoted  $\text{Ker}\varphi$ ) and image (denoted  $\text{Im}\varphi$ ) are defined as usual.

A *complex* of  $R$ -modules is a sequence of  $R$ -modules and  $R$ -homomorphisms such as

$$\cdots \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} C_{n-1} \xrightarrow{f_{n-1}} \cdots$$

such that  $f_{i-1} \circ f_i = 0$  for every integer  $i$ . In other words,  $\text{Im}(f_i) \subseteq \text{Ker}(f_{i-1})$ .

**Definition 2.3.** A pair of homomorphisms  $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$  is said to be **exact** (at  $Y$ ) if  $\text{Im}(\alpha) = \text{Ker}(\beta)$ . Further, a sequence  $\cdots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_n} X_n \xrightarrow{f_{n+1}} X_{n+1} \xrightarrow{f_{n+2}} \cdots$  of homomorphisms is said to be an **exact sequence** if it is exact at every  $X_i$ .

Using the definition of exactness, can obtain the following result.

**Proposition 2.4.** Let  $A$ ,  $B$ , and  $C$  be  $R$ -modules over some ring  $R$ . Then

1. The sequence  $0 \rightarrow A \xrightarrow{\psi} B$  is exact (at  $A$ ) if and only if  $\psi$  is injective.
2. The sequence  $B \xrightarrow{\varphi} C \rightarrow 0$  is exact (at  $C$ ) if and only if  $\varphi$  is surjective.

*Proof.* The (uniquely defined) homomorphism  $0 \rightarrow A$  has image  $0$  in  $A$ . This will be the kernel of  $\psi$  if and only if  $\psi$  is injective. Similarly, the kernel of the (uniquely defined) zero homomorphism  $C \rightarrow 0$  is all of  $C$ , which is the image of  $\varphi$  if and only if  $\varphi$  is surjective.  $\square$

As a direct result of this proposition, the sequence  $0 \rightarrow A \xrightarrow{\psi} B \xrightarrow{\varphi} C \rightarrow 0$  is exact if and only if  $\psi$  is injective,  $\varphi$  is surjective, and  $\text{Im}(\psi) = \text{Ker}(\varphi)$ . An exact sequence of the form  $0 \rightarrow A \xrightarrow{\psi} B \xrightarrow{\varphi} C \rightarrow 0$  is called a *short exact sequence*.

**Definition 2.5.** The short exact sequence  $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$  of  $R$ -modules is said to be **split exact** if  $\text{Im}(f)$  is a direct summand of  $B$ .

Given two modules  $A$  and  $C$ , we can form their direct sum  $B = A \oplus C$ . Then the sequence

$$0 \rightarrow A \xrightarrow{\iota} A \oplus C \xrightarrow{\pi} C \rightarrow 0$$

where  $\iota(a) = (a, 0)$  and  $\pi(a, c) = c$  is a split exact sequence. The following proposition characterizes short exact sequences that are split exact.

**Proposition 2.6.** *Let  $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$  be a short exact sequence. Then the following are equivalent:*

1. *The sequence is split exact.*
2. *There exists  $f' \in \text{Hom}_R(B, A)$  such that  $f'f = 1_A$ .*
3. *There exists  $g' \in \text{Hom}_R(C, B)$  such that  $gg' = 1_C$ .*

*Proof.* We will show that 1. is equivalent to 2. and note that 1. and 3. are equivalent by a similar argument.

1.  $\implies$  2.

Suppose the sequence  $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$  is split exact. Then  $B = \text{Im}(f) \oplus G$  for some  $R$ -module  $G$ . Consider the sequence

$$0 \rightarrow \text{Im}(f) \xrightarrow{\iota} \text{Im}(f) \oplus G \rightarrow C \rightarrow 0$$

with  $\iota(x) = (x, 0)$ . Then there exists an  $R$ -homomorphism  $f' : B \rightarrow \text{Im}(f)$  defined by  $f'(x, y) = x$ . So for any  $x \in \text{Im}(f)$ ,

$$(f'\iota)(x) = f'(\iota(x)) = f'((x, 0)) = x.$$

Hence  $f'\iota = 1_{\text{Im}(f)}$ . Since  $\text{Im}(f) \cong A$ ,  $f'f = 1_A$  as desired.

2.  $\implies$  1.

Suppose there exists an  $R$ -homomorphism  $f' : B \rightarrow A$  such that  $f'f = 1_A$  and define a map  $\varphi : B \rightarrow A \oplus C$  by  $\varphi(b) = (f'(b), g(b))$ . Note that  $\varphi$  is an  $R$ -homomorphism. Now suppose  $\varphi(b) = (0, 0)$  for any  $b \in B$ . Then  $f'(b) = 0$  and  $g(b) = 0$ . By having exactness at  $B$ ,

$$g(b) = 0 \implies b = f(b')$$

for some  $b' \in A$ . Thus  $0 = f'(b) = f'(f(b')) = b'$  by assumption. Hence  $b = f(b') = f(0) = 0$ . Therefore  $\text{Ker}(\varphi) = \{0\}$  implying that  $\varphi$  is injective. To show that  $\varphi$  is surjective, let  $(b', b'') \in A \oplus C$ . Since  $g$  is surjective,  $b'' = g(b)$  for some  $b \in B$ . So,

$$b'' = g(b) = g(b + f(x))$$

for any  $x \in A$ . To have  $\varphi(b + f(x)) = (b', b'')$ , we need  $x \in A$  such that

$$\begin{aligned} b' &= f'(b + f(x)) \\ &= f'(b) + f'(f(x)) \\ &= f'(b) + x \end{aligned}$$

So choose  $x = b' - f'(b)$ . Then,

$$\begin{aligned} \varphi(b + f(x)) &= (f'(b + f(x)), g(b + f(x))) \\ &= (f'(b) + f'(f(x)), g(b) + g(f(x))) \\ &= (f'(b) + x, g(b) + 0) \\ &= (b', b'') \end{aligned}$$

Thus  $\varphi$  is bijective, and hence  $\text{Im}(\varphi) = A \oplus C$ , making the sequence  $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$  split exact.  $\square$

Later we will use split exact sequences to characterize both projective and injective modules.

## 2.2 Categories and Functors

**Definition 2.7.** *A category  $\mathcal{C}$  consists of a class of objects, denoted  $\text{Ob}(\mathcal{C})$ , and sets of morphisms between those objects. For every ordered pair  $A, B \in \text{Ob}(\mathcal{C})$ , there is a set  $\text{Hom}_{\mathcal{C}}(A, B)$  of morphisms from  $A$  to  $B$ . The objects and morphisms satisfy the following axioms, for any  $A, B, C, D \in \text{Ob}(\mathcal{C})$ :*

1. If  $(A, B) \neq (C, D)$ , then  $\text{Hom}_{\mathcal{C}}(A, B) \cap \text{Hom}_{\mathcal{C}}(C, D) = \emptyset$ .
2. Composition of morphisms is associative, that is  $h(gf) = (hg)f$  for every  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ ,  $g \in \text{Hom}_{\mathcal{C}}(B, C)$ , and  $h \in \text{Hom}_{\mathcal{C}}(C, D)$ .
3. Each object has an identity morphism, that is for every  $A \in \text{Ob}(\mathcal{C})$  there is a morphism  $1_A \in \text{Hom}_{\mathcal{C}}(A, A)$  such that  $f1_A = f$  for every  $f \in \text{Hom}_{\mathcal{C}}(A, B)$  and  $1_A g = g$  for every  $g \in \text{Hom}_{\mathcal{C}}(B, A)$ .

Examples of categories include sets, abelian groups, and left  $R$ -modules, with the morphisms being functions, group homomorphisms, and  $R$ -homomorphisms, respectively. We denote the category of all left  $R$ -modules by  $\mathbf{RMod}$ , the category of abelian groups by  $\mathbf{Ab}$ , and the category of sets by  $\mathbf{Sets}$ . A significant concept that is defined in terms of categories is a functor.

**Definition 2.8.** If  $\mathcal{C}$  and  $\mathcal{D}$  are categories, then we have a **covariant functor**  $F : \mathcal{C} \rightarrow \mathcal{D}$  if we have

1. a function  $\text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D})$  (denoted  $F$ )
2. functions  $\text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F(A), F(B))$  (also denoted  $F$ ) such that
  - i. if  $f \in \text{Hom}_{\mathcal{C}}(A, B)$  and  $g \in \text{Hom}_{\mathcal{C}}(B, C)$ , then  $F(gf) = F(g)F(f)$ , and
  - ii.  $F(\text{id}_A) = \text{id}_{F(A)}$  for each  $A \in \text{Ob}(\mathcal{C})$ .

If  $\mathcal{C}$  and  $\mathcal{D}$  are categories, then we have a **contravariant functor**  $F : \mathcal{C} \rightarrow \mathcal{D}$  if we have

1. a function  $\text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D})$  (denoted  $F$ )
2. functions  $\text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F(B), F(A))$  (also denoted  $F$ ) such that
  - i. if  $f \in \text{Hom}_{\mathcal{C}}(A, B)$  and  $g \in \text{Hom}_{\mathcal{C}}(B, C)$ , then  $F(gf) = F(f)F(g)$ , and

ii.  $F(id_A) = id_{F(A)}$  for each  $A \in Ob(\mathcal{C})$ .

For example, we consider the functor  $\text{Hom}(M, -): {}_R\text{Mod} \rightarrow {}_{\mathbb{Z}}\text{Mod}$ . It associates a module  ${}_R A$  with  $\text{Hom}_R(M, A)$ . It also associates a homomorphism  ${}_R A \xrightarrow{f} {}_R B$  with  $\text{Hom}(M, f): \text{Hom}(M, A) \rightarrow \text{Hom}(M, B)$  defined by  $g \in \text{Hom}(M, A) \mapsto fg \in \text{Hom}(M, B)$ . One known result is that if  $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$  is a short exact sequence, then for any  ${}_R M$  we can apply the Hom functor and the resulting sequence

$$0 \rightarrow \text{Hom}(M, A) \xrightarrow{\text{Hom}(M, f)} \text{Hom}(M, B) \xrightarrow{\text{Hom}(M, g)} \text{Hom}(M, C)$$

is exact. This means that  $\text{Hom}(M, -)$  is an *left exact functor*. We see that if  $f$  is injective, then  $\text{Hom}(M, f)$  is also injective. However,  $g$  being surjective does not imply that  $\text{Hom}(M, g)$  is also surjective. Therefore,  $\text{Hom}(M, B) \rightarrow \text{Hom}(M, C) \rightarrow 0$  is not in general exact, unless  $M$  is in fact a projective module. We will look at projective modules in the next section.

### 2.3 Projective, Injective, and Flat Modules

We now introduce the class of projective modules.

**Definition 2.9.** An  $R$ -module  $P$  is said to be **projective** if for any exact sequence  $A \xrightarrow{f} B \rightarrow 0$  of  $R$ -modules and any  $g \in \text{Hom}(P, B)$ , then  $g$  factors through  $f$ . That is, there exists some  $\pi \in \text{Hom}(P, A)$  such that  $g = f\pi$ .

$$\begin{array}{ccccc}
 & & P & & \\
 & & \downarrow g & & \\
 & \swarrow \pi & & & \\
 A & \xrightarrow{f} & B & \longrightarrow & 0
 \end{array}$$

That is,  $P$  is projective if for any exact sequence  $A \rightarrow B \rightarrow 0$ , then the sequence  $\text{Hom}(P, A) \rightarrow \text{Hom}(P, B) \rightarrow 0$  is also exact. Another way of stating this is that

whenever  $A \xrightarrow{f} B$  is surjective, it follows that  $\text{Hom}(P, A) \xrightarrow{\text{Hom}(P, f)} \text{Hom}(P, B)$  is still surjective. We will use the notation  $\mathcal{P}$  to denote the class of projective modules. Within the category of modules, we can also use free modules to describe projective modules.

**Definition 2.10.** *An  $R$ -module  $F$  is **free** if it is a direct sum of copies of  $R$ .*

It is known that for any  $R$ -module  $M$ , there exists a surjective homomorphism  $F \rightarrow M$ , where  $F$  is a free  $R$ -module. We see the connection between free modules and projective modules in the following proposition.

**Proposition 2.11.** *For any  $R$ -module  $P$ , the following are equivalent:*

1.  $P$  is projective.
2.  $\text{Hom}(P, -)$  is right exact.
3. Every exact sequence  $0 \rightarrow A \rightarrow B \rightarrow P \rightarrow 0$  is split exact.
4.  $P$  is a direct summand of a free  $R$ -module.

*Proof.* See the proof of Theorem 2.1.2 of [2]. □

From this result, we immediately deduce that every free  $R$ -module is projective. The converse, however, is not always true.

It is known that for every  $R$ -module  $M$  there exists a surjective homomorphism  $P_0 \rightarrow M$  with  $P_0$  a projective module. Using this, we can construct an exact sequence  $\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$  where each  $P_i$  is a projective module. Such an exact sequence, or complex, is called a *projective resolution* of  $M$ .

Later on, we will be interested in computing the groups  $\text{Ext}_R^i(A, B)$  for two (left)  $R$ -modules  $A, B$ . One way to compute these groups is by using a projective

resolution of  $A$ ,  $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$ . Apply the functor  $\text{Hom}(-, B)$  to the “deleted” resolution,  $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow 0$ , to obtain the complex

$$0 \rightarrow \text{Hom}(P_0, B) \xrightarrow{\alpha_0} \text{Hom}(P_1, B) \xrightarrow{\alpha_1} \text{Hom}(P_2, B) \xrightarrow{\alpha_2} \cdots$$

Then  $\text{Ext}_R^0(A, B) = \text{Ker}\alpha_0$  and  $\text{Ext}_R^i(A, B) = \frac{\text{Ker}(\alpha_i)}{\text{Im}(\alpha_{i-1})}$  for any  $i \geq 1$ . The group  $\text{Ext}_R^i(A, B)$  is called the  $i^{\text{th}}$  *cohomology group*, in this case derived from the functor  $\text{Hom}_R(-, B)$ . We often use the notation  $\text{Ext}^i$  instead of  $\text{Ext}_R^i$  when  $R$  is understood. The groups **Ext** is another example of a category.

Next, we move on to the class of injective modules.

**Definition 2.12.** An  $R$ -module  $E$  is said to be **injective** if for any exact sequence  $0 \rightarrow A \xrightarrow{i} B$ , then any  $f \in \text{Hom}(A, E)$  can be extended to some  $g \in \text{Hom}(B, E)$ . That is,  $f = gi$ .

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{i} & B \\ & & \downarrow f & \swarrow g & \\ & & E & & \end{array}$$

We will use the notation  $\mathcal{I}$  to denote the class of injective modules. We have the following result concerning injective modules.

**Proposition 2.13.** For any  $R$ -module  $E$ , the following are equivalent:

1.  $E$  is injective.
2.  $\text{Hom}(-, E)$  is right exact.
3. Any short exact sequence  $0 \rightarrow E \rightarrow U \rightarrow V \rightarrow 0$  is split exact.

*Proof.* See the proof of Theorem 3.1.2 in [2]. □

There is another characterization of injective modules over Principal Ideal Domains. First, we recall what it means for a module to be divisible.

**Definition 2.14.** A module  ${}_R I$  is **divisible** if for any nonzero divisor  $r \in R$  and for any  $y \in I$ , there exists some  $x \in I$  such that  $y = rx$ . (i.e.  $y$  is divisible by  $r$ )

This is equivalent to saying that  ${}_R I$  is divisible if  $rI = I$  for any  $r \in R$ .

**Proposition 2.15.** Let  $R$  be a Principal Ideal Domain. An  $R$ -module  $E$  is injective if and only if  $E$  is divisible.

*Proof.* See the proof of Theorem 3.1.4 in [2]. □

For example,  $\mathbb{Z}$  is not an injective  $\mathbb{Z}$ -module because  $\mathbb{Z}$  is a Principal Ideal Domain and the  $\mathbb{Z}$ -module  $\mathbb{Z}$  is not divisible. For instance,  $3\mathbb{Z} \neq \mathbb{Z}$ . However,  $\mathbb{Q}$  is divisible as a  $\mathbb{Z}$ -module since  $n\mathbb{Q} = \mathbb{Q}$  for every  $n \in \mathbb{Z} - \{0\}$ . Therefore,  $\mathbb{Q}$  is an injective  $\mathbb{Z}$ -module.

One important result concerning injective modules, which holds over any ring  $R$ , is that every  $R$ -module is a submodule of an injective  $R$ -module. In other words, any  $R$ -module can be embedded into an injective  $R$ -module. Using this notion, it follows that every  $R$ -module  $M$  has an exact sequence  $0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \dots$  where each  $E^i$  is injective. This sequence is called an *injective resolution* of  $M$ .

Injective resolutions can also be used to compute the group  $Ext^i(A, B)$ . Start with an injective resolution of  $B$ ,  $0 \rightarrow B \rightarrow E^0 \rightarrow E^1 \rightarrow \dots$ , then apply  $\text{Hom}(A, -)$  to the “deleted” resolution,  $0 \rightarrow E^0 \rightarrow E^1 \rightarrow \dots$ , to obtain the complex

$$0 \rightarrow \text{Hom}(A, E^0) \xrightarrow{\beta_0} \text{Hom}(A, E^1) \xrightarrow{\beta_1} \text{Hom}(A, E^2) \xrightarrow{\beta_2} \dots$$

Then  $Ext^i(A, B) = \frac{\text{Ker}(\beta_i)}{\text{Im}(\beta_{i-1})}$  for any  $i \geq 1$  and  $Ext^0(A, B) = \text{Ker}\beta_0$ .

The last class of modules we are looking at are flat modules. Unlike projective and injective modules, which are defined using the Hom functor, flat modules are defined in terms of the tensor product. First, we recall the following key terms.

**Definition 2.16.** Let  $M$  be a right  $R$ -module,  $N$  be a left  $R$ -module, and  $G$  be an abelian group. A map  $\sigma : M \times N \rightarrow G$  is said to be **bilinear** if

$$\sigma(x + x', y) = \sigma(x, y) + \sigma(x', y)$$

$$\sigma(x, y + y') = \sigma(x, y) + \sigma(x, y')$$

$$\sigma(xr, y) = \sigma(x, ry)$$

for all  $x, x' \in M$ ,  $y, y' \in N$ , and  $r \in R$ .

**Definition 2.17.** A map  $\sigma : M \times N \rightarrow G$  is said to be a **universal balanced map** if for every abelian group  $G'$  and bilinear map  $\varphi : M \times N \rightarrow G'$  there exists a unique map  $h : G \rightarrow G'$  such that  $\varphi = h\sigma$ .

$$\begin{array}{ccc} M \times N & \xrightarrow{\sigma} & G \\ \varphi \downarrow & \swarrow h & \\ G' & & \end{array}$$

**Definition 2.18.** A **tensor product** of a right  $R$ -module  $M$  and a left  $R$ -module  $N$  is an abelian group  $T$  together with a universal balanced map  $\sigma : M \times N \rightarrow T$ .

The usual notation for the tensor product is  $M \otimes_R N$ , or simply  $M \otimes N$  if  $R$  is understood. Additionally, we use the notation  $\sigma(x, y) = x \otimes y$ . We now have the necessary information to define flat modules.

**Definition 2.19.** An  $R$ -module  $F$  is **flat** if given any exact sequence  $0 \rightarrow A \rightarrow B$  of right  $R$ -modules, the sequence  $0 \rightarrow A \otimes F \rightarrow B \otimes F$  is still exact.

We will use the notation  $\mathcal{F}$  to denote the class of flat modules.

**Proposition 2.20.** Free modules are flat. More generally, projective modules are flat.

*Proof.* See the proof of Chapter 10 Corollary 42 in [1]. □

## 2.4 Noetherian Rings

Next, we need to recall some information about the class of Noetherian rings.

**Definition 2.21.** *A commutative ring  $R$  is said to be **Noetherian** if every ascending chain of ideals in  $R$  is finite.*

When we consider a ring  $R$  that is a left module over itself, then its  $R$ -submodules are its ideals. Therefore, for example, every Principal Ideal Domain is Noetherian. Using this notion, we can say that an  $R$ -module  $M$  is Noetherian if every ascending chain of submodules of  $M$  is finite. Since we are interested in Noetherian rings of finite Krull dimension, we recall the following definitions.

**Definition 2.22.** *A prime ideal  $P$  of a ring  $R$  is an ideal such that  $P \neq R$  and if  $ab \in P$  then either  $a \in P$  or  $b \in P$ , for all  $a, b \in R$ .*

**Definition 2.23.** *For any commutative ring  $R$  the **Krull dimension** of  $R$  is the maximum possible length of a chain  $P_0 \subset P_1 \subset \cdots \subset P_n$  of distinct prime ideals of  $R$ . The dimension of  $R$  is said to be infinite if  $R$  has arbitrarily long chains of distinct prime ideals.*

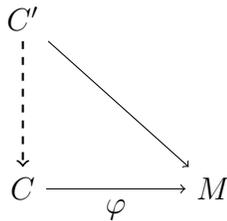
For example, a field has Krull dimension 0 and a Principal Ideal Domain that is not a field has dimension 1.

## 2.5 Covers and Precovers

Next we will discuss the concept of covers and precovers. First, we note that if  $\mathcal{K}$  denotes a class of  $R$ -modules and  $C, D$  are  $R$ -modules such that  $C \cong D$  and  $C \in \mathcal{K}$ , then  $D \in \mathcal{K}$ . Hence we assume that the classes of modules  $\mathcal{K}$  are closed under isomorphisms. Examples of classes of modules include injective, projective, and flat modules.

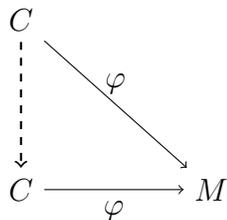
**Definition 2.24.** Let  $R$  be a ring and let  $\mathcal{K}$  be a class of  $R$ -modules. Then for an  $R$ -module  $M$ , a morphism  $\varphi : C \rightarrow M$  where  $C \in \mathcal{K}$  is called a  $\mathcal{K}$ -**cover** of  $M$  if

1. any diagram with  $C' \in \mathcal{K}$



can be completed to a commutative diagram, and

2. the diagram



can be completed only by automorphisms of  $C$ .

So, if a  $\mathcal{K}$ -cover exists, then it is unique up to isomorphism. Additionally, if  $\varphi : C \rightarrow M$  satisfies (1) but maybe not (2) in the above definition, then it is called an  $\mathcal{K}$ -**precover** of  $M$ .

If  $\mathcal{P}$  is the class of projective modules, a  $\mathcal{P}$ -(pre)cover is called a *projective (pre)cover*. If the class  $\mathcal{K}$  contains the ring  $R$ , then  $\mathcal{K}$ -precovers are surjective. We say that a class  $\mathcal{K}$  is *(pre)covering* if every  $R$ -module has a  $\mathcal{K}$ -(pre)cover.

Although our main result focuses on special precovers, which we will define in Chapter 4, the following proposition demonstrates that there is a close connection between covers and precovers.

**Proposition 2.25.** *Let  $M$  be an  $R$ -module. Then the  $\mathcal{K}$ -cover of  $M$ , if it exists, is a direct summand of any  $\mathcal{K}$ -precover of  $M$ .*

*Proof.* Let  $C \rightarrow M$  be the  $\mathcal{K}$ -cover and  $C' \rightarrow M$  be a  $\mathcal{K}$ -precover. Then we have the following commutative diagram

$$\begin{array}{ccc}
 C & & \\
 \downarrow & \searrow & \\
 C' & \longrightarrow & M \\
 \downarrow & \nearrow & \\
 C & & 
 \end{array}$$

But then  $C \rightarrow C' \rightarrow C$  is an automorphism. So  $C$  is a direct summand of  $C'$ .  $\square$

## CHAPTER 3

## GORENSTEIN INJECTIVE, PROJECTIVE, AND FLAT MODULES

## 3.1 Gorenstein Injective Modules

We can now introduce classes of Gorenstein modules, beginning with Gorenstein injective modules.

**Definition 3.1.** An  $R$ -module  $G$  is said to be **Gorenstein injective** if there exists an exact and  $\text{Hom}(\mathcal{I}, -)$  exact sequence

$$\cdots \rightarrow E_1 \rightarrow E_0 \rightarrow E_{-1} \rightarrow \cdots$$

of injective modules such that  $G = \text{Ker}(E_0 \rightarrow E_{-1})$

We use the notation  $\mathcal{GI}$  to denote the class of Gorenstein injective modules. For example, any injective  $R$ -module is Gorenstein injective. However, not every Gorenstein injective module is an injective module, so we have  $\mathcal{I} \subset \mathcal{GI}$ .

We can also give an equivalent characterization of Gorenstein injective modules using left injective resolutions. First, let us expand on the notion of precovers as we explicitly define injective precovers.

**Definition 3.2.** A homomorphism  $\varphi : G \rightarrow M$  is said to be an **injective precover** of  $M$  if  $G$  is an injective module and if for any  $\varphi' : G' \rightarrow M$  with  $G'$  an injective module, there exists  $\mu \in \text{Hom}(G', G)$  such that  $\varphi' = \varphi\mu$ .

$$\begin{array}{ccc} G' & & \\ \downarrow \mu & \searrow \varphi' & \\ G & \xrightarrow{\varphi} & M \end{array}$$

Recall that an injective precover  $\varphi$  is said to be a *injective cover* if any  $v \in \text{End}(G)$  such that  $\varphi v = \varphi$  is in fact an automorphism.

$$\begin{array}{ccc}
 G & & \\
 \downarrow v & \searrow \varphi & \\
 G & \xrightarrow{\varphi} & M
 \end{array}$$

**Proposition 3.3.** *The following statements are equivalent:*

1.  *$R$  is a left Noetherian ring.*
2. *Every left  $R$ -module  $M$  has an injective precover.*
3. *Every left  $R$ -module has an injective cover.*

*Proof.* See the proof of Theorem 5.4.1 in [2]. □

The existence of the injective (pre)covers allow us to give another definition of left injective resolutions. First, consider a left  $R$ -module  $M$  and an injective precover  $\varphi_0 : E_0 \rightarrow M$  with kernel  $k_0$ . Then consider an injective precover  $\varphi_1 : E_1 \rightarrow k_0$ . Continuing this process, we obtain a complex

$$I = \cdots \rightarrow E_2 \xrightarrow{\varphi_2} E_1 \xrightarrow{\varphi_1} E_0 \xrightarrow{\varphi_0} M \rightarrow 0$$

with all  $E_i$  being injective modules. This complex is not necessarily an exact one since an injective precover may not be a surjective map. However, for any injective left  $R$ -module  $A$ , the complex

$$Hom(A, I) = \cdots \rightarrow Hom(A, E_1) \xrightarrow{Hom(A, \alpha_1)} Hom(A, E_0) \xrightarrow{Hom(A, \alpha_0)} Hom(A, M) \rightarrow 0$$

is exact, since at each step we consider an injective precover. Such a complex  $I$  is called a *left injective resolution* of the module  $M$ .

Using left injective resolutions, the following proposition gives us another characterization of Gorenstein injective Modules.

**Proposition 3.4.** *Let  $R$  be a left Noetherian ring  $M$  be an  $R$ -module. The following are equivalent:*

1.  $M$  is Gorenstein injective.
2.  $M$  has an exact and  $\text{Hom}(\mathcal{I}, -)$  exact sequence

$$\cdots \rightarrow E_2 \rightarrow E_1 \rightarrow E_0 \rightarrow M \rightarrow 0$$

*with each  $E_i$  injective, and  $\text{Ext}^i(E, M) = 0$  for all  $i \geq 1$  and for any injective  $R$ -module  $E$ .*

*Proof.* Proof is similar to the proof of Proposition 3.12 for Gorenstein projective modules. □

The following proposition gives us some of the properties of Gorenstein injective modules.

**Proposition 3.5.** *Let  $R$  be Noetherian and  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  be an exact sequence. If  $M'$  and  $M''$  are Gorenstein injective, then so is  $M$ . If  $M'$  and  $M$  are Gorenstein injective, then so is  $M''$ . If  $M$  and  $M''$  are Gorenstein injective, then  $M'$  is Gorenstein injective if and only if  $\text{Ext}^1(E, M') = 0$  for all injective  $R$ -modules  $E$ .*

*Proof.* See the proof of Theorem 10.1.4 in [2]. □

**Definition 3.6.** *The minimal length  $n$  of a finite exact sequence of an  $R$ -module  $M$*

$$0 \rightarrow I_n \rightarrow I_{n-1} \rightarrow \cdots \rightarrow I_1 \rightarrow I_0 \rightarrow M \rightarrow 0$$

*with each  $P_i$  being Gorenstein injective is called the **Gorenstein injective dimension** of  $M$ , denoted  $G.i.d._R M$ .*

### 3.2 Gorenstein Projective Modules

Next, we consider the dual of Gorenstein injective modules, namely Gorenstein projective modules.

**Definition 3.7.** *A module  $H$  is said to be **Gorenstein projective** if there exists an exact and  $\text{Hom}(-, \mathcal{P})$  exact sequence*

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P_{-1} \rightarrow \cdots$$

*of projective modules such that  $H = \text{Ker}(P_0 \rightarrow P_{-1})$ .*

We will use the notation  $\mathcal{GP}$  to denote the class of Gorenstein projective modules. Every projective module is Gorenstein projective, but again the converse does not hold and we have  $\mathcal{P} \subset \mathcal{GP}$ . Recall the definition of Gorenstein projective dimension.

**Definition 3.8.** *The minimal length  $n$  of a finite exact sequence of an  $R$ -module  $M$*

$$0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

*with each  $P_i$  being Gorenstein projective is called the **Gorenstein projective dimension** of  $M$ , denoted  $G.p.d._R M$ .*

**Proposition 3.9.** *The projective dimension of a Gorenstein projective  $R$ -module  $G$  is either zero or infinite.*

*Proof.* See proof of Proposition 10.2.3 in [2]. □

There is also an equivalent characterization of Gorenstein projective modules that utilizes right projective resolutions as well as the concept of coherent rings. Recall that a *right projective resolution* of the module  $H$  is a complex

$$J = 0 \rightarrow H \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots$$

with all  $P^i$  projective modules and such that

$$\text{Hom}(J, Q) = 0 \rightarrow \text{Hom}(H, Q) \rightarrow \text{Hom}(P^0, Q) \rightarrow \text{Hom}(P^1, Q) \rightarrow \dots$$

is an exact complex for any projective module  $Q$ .

**Definition 3.10.** A ring  $R$  is **coherent** if every direct product of flat left  $R$ -modules is flat.

**Proposition 3.11.** Every Noetherian ring is coherent.

**Proposition 3.12.** Let  $R$  be a right coherent ring and  $H$  be an  $R$ -module. The following are equivalent:

1.  $H$  is Gorenstein projective.
2. There is an exact and  $\text{Hom}(-, \mathcal{P})$  exact sequence

$$0 \rightarrow H \rightarrow P^0 \rightarrow P^1 \rightarrow \dots$$

with each  $P^i$  projective, and  $\text{Ext}^i(H, P) = 0$  for all  $i \geq 1$ , for any projective  $R$ -module  $P$ .

*Proof.* 1.  $\implies$  2.

By definition, there is an exact sequence  $\dots \rightarrow P_1 \rightarrow P_0 \xrightarrow{f_0} P^0 \rightarrow P^1 \rightarrow P^2 \rightarrow \dots$  of projective  $R$ -modules with  $H = \text{Im}(f_0)$ . In particular, this means that  $H$  has an exact sequence  $0 \rightarrow H \rightarrow P^0 \rightarrow P^1 \rightarrow P^2 \rightarrow \dots$  with each  $P^i$  projective. The sequence  $\dots \rightarrow P_1 \rightarrow P_0 \rightarrow P^0 \rightarrow P^1 \rightarrow P^2 \rightarrow \dots$  is also  $\text{Hom}(-, \mathcal{P})$  exact. So for any projective module  $P$  we have an exact sequence

$$\dots \rightarrow \text{Hom}(P^1, P) \rightarrow \text{Hom}(P^0, P) \xrightarrow{\alpha} \text{Hom}(P_0, P) \xrightarrow{\beta} \text{Hom}(P_1, P) \rightarrow \dots$$

Thus  $\text{Im}(\alpha) = \text{Ker}(\beta)$ . But  $P_1 \rightarrow P_0 \rightarrow H \rightarrow 0$  is exact and  $\text{Hom}(-, \mathcal{P})$  is left exact. Therefore the sequence  $0 \rightarrow \text{Hom}(H, P) \rightarrow \text{Hom}(P_0, P) \xrightarrow{\beta} \text{Hom}(P_1, P)$  is

exact. This means that  $Ker(\beta) \cong Hom(H, P)$ . Thus

$$\cdots \rightarrow Hom(P^1, P) \rightarrow Hom(P^0, P) \rightarrow Hom(H, P) \rightarrow 0$$

is exact for every projective module  $P$ .

2.  $\implies$  1.

Let  $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow H \rightarrow 0$  be any projective resolution of  $H$ . Then  $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots$  is an exact sequence of projective modules.

Let  $K_0 = Ker(P_0 \rightarrow H)$ . The exact sequence  $0 \rightarrow K_0 \rightarrow P_0 \rightarrow H \rightarrow 0$  gives that for every module  $P$ , we have a long exact sequence

$$\begin{aligned} 0 \rightarrow Hom(H, P) &\rightarrow Hom(P_0, P) \rightarrow Hom(K_0, P) \\ &\rightarrow Ext^1(H, P) \rightarrow Ext^1(P_0, P) \rightarrow Ext^1(K_0, P) \\ &\rightarrow Ext^2(H, P) \rightarrow Ext^2(P_0, P) \rightarrow Ext^2(K_0, P) \\ &\rightarrow Ext^3(H, P) \rightarrow \cdots \end{aligned}$$

Since  $P_0$  is a projective module,  $Ext^i(P_0, P) = 0$  for any  $i \geq 1$  and for any module  $P$ .

If  $P$  is a projective module, then by hypothesis, we also have that  $Ext^i(H, P) = 0$  for all  $i \geq 1$ . The exact sequence given above gives that  $Ext^i(K_0, P) = 0$  for all  $i \geq 1$  and for any projective module  $P$ .

Similarly,  $Ext^i(K_j, P) = 0$  for any  $i \geq 1$  and any projective module  $P$ , where  $K_j = Ker(P_j \rightarrow P_{j-1})$ . In particular, for each  $j$  there is an exact sequence  $0 \rightarrow K_j \rightarrow P_j \rightarrow K_{j-1} \rightarrow 0$ , and this gives a long exact sequence

$$0 \rightarrow Hom(K_{j-1}, P) \rightarrow Hom(P_j, P) \rightarrow Hom(K_j, P) \rightarrow Ext^1(K_{j-1}, P) = 0$$

provided that  $P$  is projective. So each sequence  $0 \rightarrow Hom(K_{j-1}, P) \rightarrow Hom(P_j, P) \rightarrow Hom(K_j, P) \rightarrow 0$  is exact. Pasting them together, we obtain a sequence

$$0 \rightarrow Hom(H, P) \rightarrow \cdots \rightarrow Hom(P_0, P) \rightarrow Hom(P_1, P) \rightarrow \cdots .$$

By hypothesis, we also have the sequence

$$\cdots \rightarrow \text{Hom}(P^1, P) \rightarrow \text{Hom}(P^0, P) \rightarrow \text{Hom}(H, P) \rightarrow 0.$$

Splicing them together, we obtain the exact sequence

$$\cdots \rightarrow \text{Hom}(P^1, P) \rightarrow \text{Hom}(P^0, P) \rightarrow \text{Hom}(P_0, P) \rightarrow \text{Hom}(P_1, P) \rightarrow \cdots .$$

Thus,  $H$  is Gorenstein projective. □

We have the following property for Gorenstein projective modules.

**Proposition 3.13.** *Let  $R$  be right coherent and let  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  be a short exact sequence of left  $R$ -modules. If  $M'$  and  $M''$  are Gorenstein projective, then so is  $M$ . If  $M$  and  $M''$  are Gorenstein projective, then so is  $M'$ . If  $M'$  and  $M$  are Gorenstein projective, then  $M''$  is Gorenstein projective if and only if  $\text{Ext}^1(M'', Q) = 0$  for all finitely generated projective  $R$ -modules  $Q$ .*

*Proof.* See proof of Theorem 10.2.8 in [2]. □

### 3.3 Gorenstein Flat Modules

Lastly, we define Gorenstein flat modules, which were introduced by Enochs, Jenda, and Torrecillas in [3] as a generalization of flat modules.

**Definition 3.14.** *A  $R$ -module  $M$  is said to be **Gorenstein flat** if there exists an exact and  $\mathcal{I} \otimes -$  exact complex*

$$\cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F_{-1} \rightarrow \cdots$$

*of flat modules such that  $M = \text{Ker}(F_0 \rightarrow F_{-1})$ .*

We will use the notation  $\mathcal{GF}$  to denote the class of Gorenstein injective modules. It is known that every flat module is Gorenstein flat, and once again the converse does

not hold so  $\mathcal{F} \subset \mathcal{GF}$ . The following propositions reveal some properties involving Gorenstein flat modules. First, we recall the definition of Gorenstein flat dimension.

**Definition 3.15.** *The minimal length  $n$  of a finite exact sequence of an  $R$ -module  $M$*

$$0 \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

*with each  $F_i$  being Gorenstein flat is called the **Gorenstein flat dimension** of  $M$ , denoted  $G.f.d._R M$ .*

**Proposition 3.16.** *Let  $R$  be left and right coherent. Then every finitely generated Gorenstein projective  $R$ -module is Gorenstein flat.*

*Proof.* See proof of Proposition 10.3.2 in [2]. □

If we do not assume the  $R$ -modules in the previous proposition are finitely generated, then the result may not hold. We can also use this result for any Noetherian ring  $R$ , since we know that every Noetherian ring is coherent.

**Proposition 3.17.** *Suppose  $R$  is Noetherian and  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is an exact sequence of  $R$ -modules. If  $M'$  and  $M''$  are Gorenstein flat, then so is  $M$ . If  $M$  and  $M''$  are Gorenstein flat, then so is  $M'$ . If  $M'$  and  $M$  are Gorenstein flat, then  $M''$  is Gorenstein flat if and only if  $0 \rightarrow E \otimes M' \rightarrow E \otimes M$  is exact for any injective module  $E$ .*

*Proof.* See proof of Theorem 10.3.14 in [2]. □

**CHAPTER 4**  
**MAIN RESULT**

The existence of Gorenstein projective precovers over Gorenstein rings is known (Enochs-Jenda, 2000). Even more recently, the existence of precovers over commutative Noetherian rings of finite Krull dimension was proved (Murfet-Salarian, 2011). Then it was shown that if  $R$  is a right coherent and left  $n$ -perfect ring, then the class of Gorenstein projective complexes is special precovering in the category of unbounded complexes. Expanding on these notions, we will prove that the class of Gorenstein projective modules is special precovering over any left GF-closed ring  $R$  such that every Gorenstein projective module is Gorenstein flat and every Gorenstein flat module has finite Gorenstein projective dimension. We will see that this class of rings includes that of right coherent and left  $n$ -perfect rings, with the inclusion being strict.

Before we arrive at our main result, let us recall the following definitions.

**Definition 4.1.** *A ring  $R$  is **left  $n$ -perfect** if any flat left  $R$ -module has finite projective dimension. In this case, there is a nonnegative integer  $n$  such that every flat left  $R$ -module has projective dimension at most  $n$ .*

**Definition 4.2.** *A Gorenstein projective precover  $\varphi$  is said to be **special** if  $\text{Ker}(\varphi)$  is in the right orthogonal class of Gorenstein projective modules,*

$$\mathcal{GP}^\perp = \{L \mid \text{Ext}^1(G', L) = 0 \text{ for all Gorenstein projective modules } G'\}.$$

We also recall that a ring  $R$  is left GF-closed if the class of Gorenstein flat left  $R$ -modules is closed under extensions. That is, for every short exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  such that  $A$  and  $C$  are in the class of Gorenstein flat left  $R$ -modules, then it follows that  $B$  is also in this class of modules.

Lastly, we need the following proposition.

**Proposition 4.3.** *Every module of finite Gorenstein projective dimension has a special Gorenstein projective precover.*

*Proof.* See Proposition 1 in [6]. □

**Theorem 4.4 (Main Result).** *Let  $R$  be a left GF-closed ring. If every Gorenstein projective module is Gorenstein flat and every Gorenstein flat  $R$ -module has finite Gorenstein projective dimension, then the class of Gorenstein projective modules is special precovering in  ${}_R\text{Mod}$ .*

*Proof.* Let  $X$  be any left  $R$ -module. Since  $R$  is left GF-closed, the class of Gorenstein flat modules is covering in  ${}_R\text{Mod}$  (by [10]). So there exists an exact sequence  $0 \rightarrow Y \rightarrow N \rightarrow X \rightarrow 0$  with  $N$  Gorenstein flat and with  $Y \in \mathcal{GF}^\perp \subset \mathcal{GP}^\perp$  (because we have that  $\mathcal{GP} \subset \mathcal{GF}$ ). Since  $N$  has finite Gorenstein projective dimension, by Proposition 4.3, there is an exact sequence  $0 \rightarrow W \rightarrow T \rightarrow N \rightarrow 0$  with  $T$  Gorenstein projective and  $W \in \mathcal{GP}^\perp$ . Form the pull back diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & W & \xlongequal{\quad} & W & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & A & \longrightarrow & T & \longrightarrow & X \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & Y & \longrightarrow & N & \longrightarrow & X \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

The exact sequence  $0 \rightarrow W \rightarrow A \rightarrow Y \rightarrow 0$  with  $W, Y \in \mathcal{GP}^\perp$  gives  $A \in \mathcal{GP}^\perp$ . So we have an exact sequence  $0 \rightarrow A \rightarrow T \rightarrow X \rightarrow 0$  with  $T \in \mathcal{GP}$  and  $A \in \mathcal{GP}^\perp$ . It follows that  $T \rightarrow X$  is a special Gorenstein projective precover of  $X$ .  $\square$

From this result, we obtain the following corollary concerning cotorsion pairs. First, a few terms. Given a class of  $R$ -modules  $\mathcal{K}$ , we denote the class of all  $R$ -modules  $M$  such that  $\text{Ext}^1(K, M) = 0$  for every  $K \in \mathcal{K}$  by  $\mathcal{K}^\perp$ . This is the right orthogonal class of  $R$ -modules in  $\mathcal{K}$ . Similarly, the left orthogonal class of  $\mathcal{K}$ , denoted  ${}^\perp\mathcal{K}$ , the class of all (left)  $R$ -modules  $N$  such that  $\text{Ext}^1(N, K) = 0$  for every  $K \in \mathcal{K}$ .

**Definition 4.5.** *Let  $\mathcal{L}$  and  $\mathcal{C}$  be two classes of  $R$ -modules. The pair  $(\mathcal{L}, \mathcal{C})$  is a **cotorsion pair** if  $\mathcal{L}^\perp = \mathcal{C}$  and  ${}^\perp\mathcal{C} = \mathcal{L}$ . Further, a cotorsion pair  $(\mathcal{L}, \mathcal{C})$  is **complete** if for every  $R$ -module  $M$  there exists exact sequences  $0 \rightarrow C \rightarrow L \rightarrow M \rightarrow 0$  and  $0 \rightarrow M \rightarrow C' \rightarrow L' \rightarrow 0$  with  $C, C' \in \mathcal{C}$  and  $L, L' \in \mathcal{L}$ .*

We are also interested in hereditary cotorsion pairs. A cotorsion pair  $(\mathcal{L}, \mathcal{C})$  is called *hereditary* if  $\text{Ext}^i(M, N) = 0$  for all  $i \geq 1$ , for all  $M \in \mathcal{L}$  and all  $N \in \mathcal{C}$ . It is known that the following are equivalent:

1.  $(\mathcal{L}, \mathcal{C})$  is hereditary.
2. For any short exact sequence  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ , if  $M$  and  $M''$  are in the class  $\mathcal{L}$ , then so is  $M'$ .
3. For any short exact sequence  $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$ , if  $N'$  and  $N$  are in the class  $\mathcal{C}$ , then so is  $N''$ .

This brings us to a corollary of our main result.

**Corollary 4.6.** *Let  $R$  be a left GF-closed ring such that  $\mathcal{GP} \subseteq \mathcal{GF}$  and every Gorenstein flat module has finite Gorenstein projective dimension. Then  $(\mathcal{GP}, \mathcal{GP}^\perp)$  is a complete hereditary cotorsion pair.*

*Proof.* We prove first that  $(\mathcal{GP}, \mathcal{GP}^\perp)$  is a cotorsion pair. Let  $X \in {}^\perp(\mathcal{GP}^\perp)$ . By Theorem 4.4 there exists an exact sequence  $0 \rightarrow A \rightarrow B \rightarrow X \rightarrow 0$  with  $B$  Gorenstein projective and with  $A \in \mathcal{GP}^\perp$ . Then  $\text{Ext}^1(X, A) = 0$ , so the sequence is split exact. Since  $B \cong A \otimes X$  it follows that  $X$  is Gorenstein projective. Thus  ${}^\perp(\mathcal{GP}^\perp) = \mathcal{GP}$ . The pair  $(\mathcal{GP}, \mathcal{GP}^\perp)$  is complete by Theorem 4.4. Since the class of Gorenstein projective modules is projectively resolving the pair  $(\mathcal{GP}, \mathcal{GP}^\perp)$  is hereditary.  $\square$

**Lemma 4.7.** *Let  $R$  be a left  $n$ -perfect ring. If  $F$  is a flat  $R$ -module, then there exists an exact sequence  $0 \rightarrow F \rightarrow S^0 \rightarrow S^1 \rightarrow \dots \rightarrow S^n \rightarrow 0$  with all  $S^j$  flat and cotorsion modules.*

*Proof.* See proof of Lemma 1 in [6].  $\square$

**Proposition 4.8.** *Let  $R$  be a left GF-closed and left  $n$ -perfect ring. The following are equivalent:*

1.  $G.p.d._R G \leq n$  for any Gorenstein flat module  $G$ .
2.  $G.p.d._R G < \infty$  for any Gorenstein flat module  $G$ .
3.  $\text{Ext}^i(G, F) = 0$  for any Gorenstein flat module  $G$ , any flat and cotorsion module  $F$ , and all  $i \geq 1$ .

*Proof.* 1.  $\implies$  2.

Proof is immediate.

2.  $\implies$  3.

Let  $F$  be flat and cotorsion and let  $G'$  be a Gorenstein flat  $R$ -module. Then there exists a strongly Gorenstein flat module  $G$  such that  $G'$  is a direct summand of  $G$ . Since there exists an exact sequence  $0 \rightarrow G \rightarrow K \rightarrow G \rightarrow 0$  with  $K$  flat it follows that  $\text{Ext}^i(G, F) \cong \text{Ext}^1(G, F)$  for all  $i \geq 1$ . And since  $G.p.d._R G < \infty$  and  $\text{Flat} \subset \mathcal{GP}^\perp$ , there exists  $l$  such that  $\text{Ext}^j(G, F) = 0$  for any  $j \geq l+1$ . By the above,

$Ext^i(G, F) = 0$  for all  $i \geq 1$ . Since  $Ext^i(G', F)$  is a direct summand of  $Ext^i(G, F)$  it follows that  $Ext^i(G', F) = 0$  for all  $i \geq 1$ . 3.  $\implies$  1.

See proof of Proposition 2 in [6]. □

We remark that it has already been proved in [5] that the class  $\mathcal{GP}$  is special pre-covering over any right coherent and left  $n$ -perfect ring  $R$ . However, for completeness, we included a different proof of this result.

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