A Numerical Approach for Designing Unitary Space Time Codes with Large Diversity Product and Diversity Sum*

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Abstract

Unitary space-time modulation using multiple antennas promises reliable communication at high transmission rates. The basic principles are well understood and certain criteria for designing good unitary constellations have been presented. However so far no general method to design good-performing constellation with large diversity product and diversity sum for any number of transmit antennas and for any transmission rate exists.

In this paper, we define a diversity function and analyze its limiting behavior. This results in two important design criteria: the diversity product and the diversity sum. Numerical methods are derived which allows one to construct codes with excellent diversity function and excellent diversity product and sum. The numerical approach is very flexible and it allows one to construct constellations of any dimension with an arbitrary given size. This flexibility is very useful when excellent constellations with certain parameters are required for some applications.

1 Introduction and model

Consider a wireless communication system with M transmit antennas and N receive antennas operating in a Rayleigh flat-fading channel. We assume time is discrete and at each time slot, signals are transmitted simultaneously from the M transmit antennas. We can further assume that the wireless channel is quasi-static over a time block of length T.

A signal constellation $\mathcal{V} := \{\Phi_1, \ldots, \Phi_L\}$ consists of L matrices having size $T \times M$ and satisfying $T \geq M$ and $\Phi_k^* \Phi_k = I_M$. Denote by ρ the signal to noise ratio (SNR). The basic equation between the received signal R and the transmitted signal $\sqrt{T}\Phi$ is given through:

$$R = \sqrt{\frac{\rho T}{M}} \Phi H + W,$$

where the $M \times N$ matrix H accounts for the multiplicative complex Gaussian fading coefficients and the $T \times N$ matrix W accounts for the additive white Gaussian noise. It is assumed that the receiver does not know the exact values of either the entries of H or W (other than their statistical distribution). Under the assumption of above model the maximum likelihood (ML) decoder will have to compute:

$$\Phi_{ML} = \arg \max_{\Phi_l \in \{\Phi_1, \Phi_2, \cdots, \Phi_L\}} \|R^* \Phi_l\|_F$$

for each received signal R. (See [7]).

Let $\delta_m(\Phi_l^* \Phi_{l'})$ be the m-th singular value of $\Phi_l^* \Phi_{l'}$. It has been shown in [7] that the pairwise probability of mistaking Φ_l for $\Phi_{l'}$ using Maximum Likelihood Decoding satisfies:

$$P_{\Phi_l, \Phi_{l'}} \quad \leq \quad \frac{1}{2} \prod_{m=1}^{M} \left[1 + \frac{(\rho T/M)^2 (1 - \delta_m^2(\Phi_l^* \Phi_{l'}))}{4(1 + \rho T/M)} \right]^{-N}. \tag{1.1}$$

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It is a basic design objective to construct constellations $\mathcal{V} = \{\Phi_1, \ldots, \Phi_L\}$ such that the pairwise probabilities $P_{\Phi_l,\Phi_{l'}}$ are as small as possible. The main purpose of this paper is to develop numerical procedures which allow one to construct unitary constellations with excellent diversity for any set of parameters M, N, T, L and for any signal to noise ratio ρ.

2 The diversity function, the diversity sum and the diversity product

Using Chernoff's bound (1.1) we define a simplified function called the *diversity function* through:

$$\mathcal{D}(\mathcal{V}, \rho) := \max_{l \neq l'} \frac{1}{2} \prod_{m=1}^{M} \left[1 + \frac{(\rho T/M)^2}{4(1 + \rho T/M)} (1 - \delta_m^2(\Phi_l^* \Phi_{l'})) \right]^{-N}. \tag{2.1}$$

2.1 Design criterion for high SNR channel

When the SNR ρ is very large, it is the design objective to construct a constellation $\Phi_1, \Phi_2, \cdots, \Phi_L$ such that diversity product is as large as possible.

Definition 2.1. (See [6]) The *diversity product* of a unitary constellation $\mathcal{V}$ is defined as

$$\prod \mathcal{V} = \min_{l \neq l'} \left(\prod_{m=1}^{M} (1 - \delta_m (\Phi_l^* \Phi_{l'})^2) \right)^{\frac{1}{2M}}.$$

An important special case occurs when $T = 2M$. In this situation it is customary to represent all unitary matrices Φ_k in the form:

$$\Phi_k = \frac{\sqrt{2}}{2} \begin{pmatrix} I \\ \Psi_k \end{pmatrix}. \tag{2.2}$$

Note that by definition of Φ_k the matrix Ψ_k is a $M \times M$ unitary matrix. In this case the formula of the diversity product assumes the simple form:

$$\prod \mathcal{V} = \frac{1}{2} \min_{0 \leq l < l' \leq L} |\det(\Psi_l - \Psi_{l'})|^{\frac{1}{M}} \tag{2.3}$$

2.2 Design criterion for low SNR channel

When the SNR ρ is very small, it is the design objective to construct a constellation $\Phi_1, \Phi_2, \cdots, \Phi_L$ such that diversity sum is as large as possible.

Definition 2.2. The *diversity sum* of a unitary constellation $\mathcal{V}$ is defined as

$$\sum \mathcal{V} = \min_{l \neq l'} \sqrt{1 - \frac{\|\Phi_l^* \Phi_{l'}\|_F^2}{M}}.$$

For the form (2.2) the diversity sum assumes the following simple form:

$$\sum \mathcal{V} = \min_{l, l'} \frac{1}{2\sqrt{M}} \|\Psi_l - \Psi_{l'}\|_F. \tag{2.4}$$

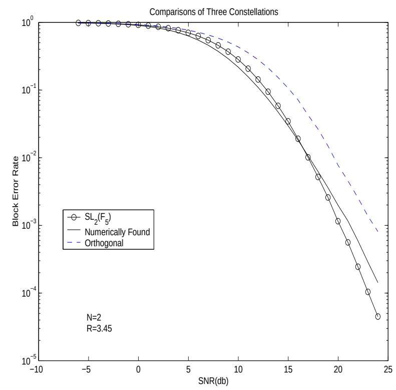
2.3 Three illustrative examples

In the following we are comparing three constellations derived by different methods. The parameters of the three constellations are summarized by:

	Orthogonal design	Numerically derived	$SL_2(\mathbb{F}_5)$
Number of elements	121	121	120
diversity sum	0.1992	0.3886	0.309
diversity product	0.1992	0.0278	0.309

We simulated all three codes and the following gives the result:

Note that the numerically designed code who has a very bad diversity product is performing very well at low SNR nevertheless due to the exceptional diversity sum. This gives an indication that the diversity sum is an important parameter as well.



3 Numerical design of unitary constellations with good diversity

3.1 Cayley transformation

Cayley transformation is one effective way to represent unitary matrices (or more precisely, a subset of unitary matrices). For more details about Cayley transformation, we refer to [5].

It is very important to have Cayley transformation in numerically designing the constellation because it makes the local topology of $U(n)$ clear. One can see that most optimization methods require us to consider the neighborhood of one element in $U(n)$.

3.2 Non-group constellation with algebraic structure

We are going to present some constellations in between general non-group constellations and group constellations. We have small number of generators in the constellations which will avoid "dimension explosion disaster" for computation, also we try to use more loose structure than group constellation which will reduce the number of targets to be optimized.

One of various examples would be the following:

$$\mathcal{V} = \{A^k B^l | A, B \in U(n), k = 0, \cdots, p, l = 0, \cdots, q\}$$

One can check this algebraic structure works for diversity function(product, sum).

3.3 Simulated Annealing Algorithm (SA)

Simulated Annealing is a method which mimics the process of melted metal getting cooled off. In the annealing process of the melted metal, first the metal is heated to melt, then the temperature is getting down gradually. The metal will get to a minimized energy state if the temperature is lowering slow enough. For more details about this algorithm, we refer to [1].

3.4 Genetic Algorithm (GA)

Genetic Algorithm [8] is a optimization algorithm proposed by J.H. Holland which emulates the species evolutionary process. GA doesn't assume much specific information about the given problem, first a group of candidates are selected in some certain way and they are encoded using binary coding in most cases, then the offsprings of the candidates will be considered (the mating process is also defined with respect to specific problem), also there might be new random candidates added in. The whole idea is "better survive", which really means some candidates with higher Fitness Evaluation Function keep staying in the group and other worse ones will get discarded. The algorithm will stop if some threshold is reached after certain number of iterations. One can further consult [2].

4 Numerical results

4.1 2 dimensional constellation design

Because we are usually dealing with a large number of target non-differential functions, we find Simulated Annealing method [9] and Genetic Algorithm [9] are more suitable for this kind of job. The following tables show some of the numerical results in 2 dimension.

Comparisons of different methods or parameters

	size	diversity product	structure
Simulated Annealing	125	0.2127	$A^kB^lC^m$
Simulated Annealing	120	0.2202	A^kB^l
Simulated Annealing	121	0.2417	A^kB^l
Brute Force	120	0.1914	A^kB^l
Genetic Algorithm	120	0.2377	N/A

SA with different structure (computation less than 3 minutes)

structure/size	36	49	64	256	400	900	10000
A^kB^l	0.3860	0.3781	0.2742	0.1025	0.0866	0.0834	0.0158
AB	0.3205	0.2659	0.2450	0.1030	0.0800	0.0579	0.0122
A^kB^k	0.3769	0.3502	0.3090	0.1651	0.1342	0.0820	0.0187

structure/size	27	64	216	343	512	729	9261
$A^kB^lC^m$	0.3418	0.2616	0.1833	0.1401	0.0632	0.1012	0.0031
ABC	0.3299	0.1832	0.1033	0.0725	0.0555	0.0430	N/A
$A^kB^kC^k$	0.4122	0.2512	0.0583	0.0206	0.0087	0.0039	N/A

Comparison of different methods or parameters

	size	diversity sum	structure
Simulated Annealing	125	0.3919	$A^kB^lC^m$
Simulated Annealing	120	0.3696	A^kB^l
Simulated Annealing	121	0.3886	A^kB^l
Brute Force	120	0.3673	A^kB^l
Genetic Algorithm	120	0.3867	N/A

SA with different structure (computation less than 3 minutes)

structure/size	36	49	64	256	400	900	10000
A^kB^l	0.5113	0.4733	0.4474	0.2875	0.2504	0.1848	0.0785
AB	0.5530	0.4240	0.3821	0.1994	0.1629	0.1064	0.0310
A^kB^k	0.5466	0.5121	0.4735	0.3088	0.2637	0.2047	0.0869

structure/size	27	64	216	343	512	729	9261
$A^kB^lC^m$	0.5400	0.4210	0.2992	0.2663	0.2099	0.2060	0.0772
ABC	0.5382	0.4497	0.2614	0.2065	0.1695	0.1447	0.0398
$A^kB^kC^k$	0.5630	0.4271	0.2864	0.2198	0.1969	0.1423	N/A

Numerical methods are extremely good for designing constellations with large diversity sum, most of the results are the best or very close to the best codes.

4.2 Constellation design for any dimension

We have the following tables to show the results of some experiments. The experiments are based on the computation on a Intel Pentium 800MHz PC and no computation lasts longer than 3 minutes.

Results on Diversity Product (SA)

number of elements	dim=2	dim=3	dim=4	dim=5
4	0.7071	0.7657	0.7388	0.6768
9	0.5701	0.5754	0.4774	0.4259
16	0.4018	0.4574	0.4651	0.3877
25	0.3443	0.3834	0.3809	0.3467
36	0.2865	0.3450	0.3501	0.3760

Results on Diversity Sum (SA)

number of elements	dim=2	dim=3	dim=4	dim=5
4	0.8147	0.8160	0.7861	0.7377
9	0.6956	0.6861	0.6539	0.6389
16	0.5908	0.6459	0.6288	0.5916
25	0.5618	0.6268	0.6190	0.5795
36	0.5286	0.6054	0.6148	0.5853

From above table, SA works fine when the cardinality of the constellation is small. When it comes to large number of constellation, SA works even better. We try SA on 10000 element 2 dimension constellation using $A^k B^l$ structure, we can have diversity sum 0.1000, while for construction in [3] , we can only have diversity sum 0.0654 with 8433 elements or diversity sum 0.0604 with 10770 elements. Also SA's results on higher dimension are showing the pretty much the same size as 2 dimension which makes us believe it is producing very good results in higher dimension as well, although we don't have many comparisons available.

We have the following tables to show the results of some experiments using *Genetic Algorithm*. The experiments are based on the computation on a Intel Pentium 800MHz PC and no computation lasts longer than 3 minutes.

Results on Diversity Product (GA)

number of elements	dim=2	optimal DP	dim=3	dim=4	dim=5
3	0.8644	$\frac{\sqrt{3}}{2}$=0.8860	0.8264	0.7305	0.6737
4	0.8051	$\sqrt{\frac{2}{3}}$=0.8165	0.7343	0.6521	0.6305
6	0.6924	unknown	0.6632	0.6154	0.5721
10	0.5768	unknown	0.5497	0.5742	0.4942

Results on Diversity Sum (GA)

number of elements	dim=2	optimal DS	dim=3	dim=4	dim=5
3	0.8601	$\frac{\sqrt{3}}{2}$=0.8860	0.8331	0.8118	0.7798
4	0.8029	$\sqrt{\frac{2}{3}}$=0.8165	0.7802	0.7757	0.7492
6	0.7443	$\sqrt{\frac{3}{5}}$=0.7746	0.7502	0.7293	0.7176
10	0.6826	$\frac{\sqrt{2}}{2}$=(0.7071)	0.6981	0.6920	0.6817

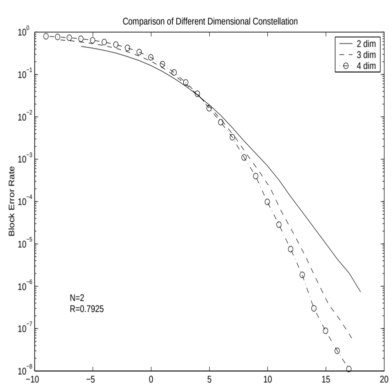
Genetic algorithm works extremely well when it comes to a small size constellation. For instance, look at 3 elements 2 dimension diversity product case. For this one can show that the optimal diversity product is $\frac{\sqrt{3}}{2} \sim 0.8660$, while our numerically designed code has a diversity sum of 0.8601. The results make us believe that for higher dimensions the diversity product (and sum) are near optimal as well.

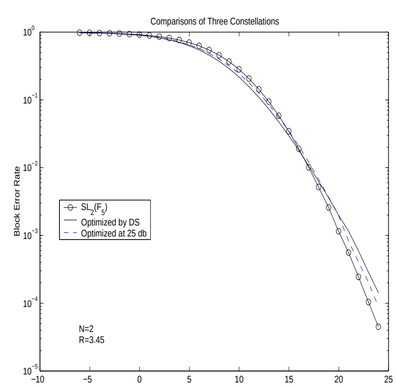
4.3 Performance of different dimension constellation

We used Genetic Algorithm to optimize the diversity product and diversity sum at the same time to derive two constellations of dimension 3 and 4 respectively. We have chosen the size in such that the rate is comparable in each case.

	size	rate	DP	DS
2 dim	3	0.7925	0.8660	0.8660
3 dim	5	0.7740	0.7183	0.7454
4 dim	9	0.7925	0.5904	0.6403

The first graph below illustrates the performance of the three different constellations. One can see how the 4-dimensional constellation really performs well at high SNR.





4.4 Constellations optimized at certain values of the diversity function

Different industrial applications require different level of reliability of the communication channel. One may want to optimize the constellation at certain Block Error Rate (BER) or Signal Noise Ratio (SNR). Algebraic methods provide results for certain specific parameters, but in general they lack a certain flexibility. As this paper shows it is always possible to numerically design excellent codes for all parameters of a reasonable size.

The second table on the previous page provides some further illustrations on the performance of numerically designed codes: We have already seen the first two constellations, the third constellation is specifically designed for SNR at 25 db using Simulated Annealing Algorithm based on structure $A^k B^l$. One can see that optimizing at the diversity at 25 db changes the shape of the curve. We would have no idea on how to algebraically construct a code optimized in the range of 25 db.

4.5 Numerical design of general form constellation

The presented numerical methods can be applied for any set of parameters M, N, T, L and optimization can in principle be done at any SNR. In the sequel we give some results only for some simple situation. We assume that $T = 2M$ and consider the following constellation:

$$\{A^k B | A \in U(M), B = \begin{pmatrix} I \\ 0 \end{pmatrix}, k = 0, 1, \cdots, l\}$$

With this algebraic structure, whenever one wants to calculate diversity product (sum) or more generally diversity function, one needs to calculate $\Phi_{l'}^* \Phi_l$, it is easily verified that they will end up being one of the following:

$$B^* A^k B \qquad k = -l, \cdots, l.$$

So this way we have a constellation with one generator and also the number of targets to be optimized are reduced. The following tables give some constellation found by SA.

size	3	4	5	6	7	8	9
rate	0.3962	0.5000	0.5905	0.6462	0.7018	0.7500	0.7925
diversity sum	0.8654	0.7901	0.7889	0.7652	0.7514	0.7422	0.7369

size	3	4	5	6	7	8	9
rate	0.3962	0.5000	0.5905	0.6462	0.7018	0.7500	0.7925
diversity product	0.8582	0.7424	0.7330	0.6450	0.6361	0.6216	0.5822

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